

# A Fictitious Time Integration Method for Multi-Dimensional Backward Wave Problems

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**Abstract:** We address a new numerical approach to deal with these multi-dimensional backward wave problems (BWPs) in this study. A fictitious time  $\tau$  is utilized to transform the dependent variable  $u(x, y, z, t)$  into a new one by  $(1+\tau)u(x, y, z, t) =: v(x, y, z, t, \tau)$ , such that the original wave equation is written as a new hyperbolic type partial differential equation in the space of  $(x, y, z, t, \tau)$ . Besides, a fictitious viscous damping coefficient can be employed to strengthen the stability of numerical integration of the discretized equations by using a group preserving scheme. Several numerical instances demonstrate that the present scheme can be utilized to retrieve the initial wave very well. Even though the noisy final data are very large, the fictitious time integration method is also robust against disturbance.

**Keywords:** Backward wave problem, Wave equation, Strongly ill-posed problem, Fictitious time integration method (FTIM), Group preserving scheme (GPS)

## 1 Introduction

Wave problems that emerge from engineering fields are often classified as forward wave problems and backward wave problems. There are lots of algorithms for resolving those forward wave problems (e.g., the wave propagation in water bodies, sound wave propagation in medium and stress wave in elastic solids, and so forth). Recently, the method of fundamental solutions (MFS) was been used in the propagation of elastic waves around thin structures [Godinho, Tadeu and Amado Mendes (2007)]. Later, Gu, Young and Fan (2008) employed the Eulerian-Lagrangian method of fundamental solutions (ELMFS) and the D'Alembert formula to solve the one-dimensional (1-D) wave equation. However, their scheme cannot tackle those multi-dimensional wave problems. Then, on the basis of the Houbolt finite difference scheme, the method of particular solutions and the MFS, six numerical examples can be resolved in Young, Gu and Fan (2009), but their approach cannot

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directly deal with the wave problem and cost much time in three-dimensional case. After that, Gu, Young and Fan (2009) proposed the ELMFS to solve the 1-D wave equation, and they claimed that the algorithm is accurate; however, it also cannot directly resolve the wave problem.

For these backward wave problems (BWPs), Lesnic (2002) has addressed the Adomain decomposition method to resolve the BWP. He explained that for the forward problem the convergence of the Adomain decomposition method is faster than that for the backward problem. Furthermore, the BWP as mentioned by Ames and Straughan (1997) has important applications in optimal control theory and geophysics. Recently, Liu (2010) has utilized the separating characteristic property of kernel function and eigenfunctions expansion techniques to derive a semi-analytical solution of 1-D BWP and obtained good results. After that, Chang and Liu (2010) extended the backward group preserving scheme to tackle these multi-dimensional BWPs and acquired accurate results without employing a priori regularization.

This current study is summarized as follows. Those multi-dimensional BWPs are described in the next section. In Section 3, we introduce a fictitious time coordinate  $\tau$  by transforming the dependent variable  $u(x, y, z, t)$  into a new one by  $(1 + \tau)u(x, y, z, t) := v(x, y, z, t, \tau)$ , such that the original equation is mathematically equivalently written as another different hyperbolic equation, where adding a fictitious viscous damping coefficient to enhance the stability of numerical integration of the discretized equations, and offering a concise depiction of GPS for ordinary differential equations (ODEs). Section 4 displays three instances to demonstrate the accuracy of the proposed scheme. The most significant conclusions from the present research are summarized in Section 5.

## 2 Multi-dimensional backward wave problems

The multi-dimensional homogeneous BWP we ponder is respectively given by the following equations:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u \quad \text{in } \Omega, \quad (1)$$

$$u = u_B \quad \text{on } \Gamma_B, \quad (2)$$

$$u = u_F \quad \text{on } \Gamma_F, \quad (3)$$

where  $u$  is the displacement of a rectangular volume and  $c$  is the velocity of a propagation wave. We take a bounded domain  $D$  in  $\mathcal{R}^j$ ,  $j = 1, 2, 3$  and a spacetime domain  $\Omega = D \times (0, T)$  in  $\mathcal{R}^{j+1}$  for a final time  $T > 0$ , and write two surfaces  $\Gamma_B = \partial D \times [0, T]$  and  $\Gamma_F = \partial D \times \{T\}$  of the boundary  $\partial\Omega$ .  $\Delta$  represents the  $j$ -dimensional Laplacian operator. While Eqs. (1)-(3) constitute a  $j$ -dimensional backward wave

problem for a given boundary data  $u_B: \Gamma_B \mapsto \mathcal{R}$  and a final data  $u_F: \Gamma_F \mapsto \mathcal{R}$ . For the existence and uniqueness of solution of BWP, Bourgin and Duffin (1939), and Abdul-Latif and Diaz (1971) have proposed a geometric proof of the uniqueness of the one-dimensional wave equation under the Dirichlet conditions. Fox and Pucci (1958) have addressed the existence issue of this problem in detail. Levine and Vessella (1985) have concerned about the existence and continuous on data and treated the problem in Hilbert space.

### 3 A fictitious time integration approach

#### 3.1 Transformation into a different evolutionary PDE and semi-discretization

First, we propose the following transformation:

$$v(x, y, z, t, \tau) = (1 + \tau)u(x, y, z, t), \tag{4}$$

and introduce a fictitious viscosity damping coefficient  $\nu > 0$  in Eq. (1):

$$0 = \nu \frac{\partial^2 u}{\partial t^2} - \nu c^2 \Delta u. \tag{5}$$

Multiplying the above equation by  $1 + \tau$  and employing Eq. (4), we have

$$0 = \nu \frac{\partial^2 v}{\partial t^2} - \nu c^2 \Delta v. \tag{6}$$

Recalling that  $\partial v / \partial \tau = u(x, y, z, t)$  by Eq. (4), and adding it on both the sides of the above equation we acquire

$$\frac{\partial v}{\partial \tau} = \nu \frac{\partial^2 v}{\partial t^2} - \nu c^2 \Delta v + u. \tag{7}$$

Finally by employing  $u = v/(1 + \tau)$ , we can change Eqs. (1)-(3) into another parabolic type PDE:

$$\frac{\partial v}{\partial \tau} = \nu \frac{\partial^2 v}{\partial t^2} - \nu c^2 \Delta v + \frac{v}{1 + \tau} \text{ in } \tilde{\Omega}, \tag{8}$$

$$v = v_B \text{ on } \tilde{\Gamma}_B, \tag{9}$$

$$v = v_F \text{ on } \tilde{\Gamma}_F, \tag{10}$$

where  $\tilde{\Omega} = \Omega \otimes [0, \tau_f]$ ,  $\tilde{\Gamma}_B = \Gamma_B \otimes [0, \tau_f]$ ,  $\tilde{\Gamma}_F = \Gamma_F \otimes [0, \tau_f]$  and  $\tau_f$  is a selected value. In Eq. (8),  $\tau$  is a fictitious time, being different to the real time  $t$ . Even though we raise these dimensions of Eq. (8) one higher than Eq. (1), several advantages can be obtained as to be shown below.

There is maybe another selection of Eq. (4) by using  $v = p(\tau)u$ , where we require that  $p(\tau)$  be a monotonically increasing function of  $\tau$ ,  $p(0) = 1$  and  $p(\inf) = \inf$ . However, when  $p(\tau)$  is more complex than  $1 + \tau$  the resulting partial differential equation (PDE) is more complicated than Eq. (8), and currently there seems no good reason to determine a more complex  $p(\tau)$ . However, other choices are possible if they can supply better result than the present one.

Applying a semi-discrete procedure to Eq. (8), yields a coupled system of ODEs:

$$\begin{aligned} \dot{v}_{i,j,k,l} = & \frac{v}{(\Delta t)^2} [v_{i,j,k,l+2} - 2v_{i,j,k,l+1} + v_{i,j,k,l}] - c^2 \left\{ \frac{v}{(\Delta x)^2} [v_{i+1,j,k,l} - 2v_{i,j,k,l} + v_{i-1,j,k,l}] \right. \\ & \left. + \frac{v}{(\Delta y)^2} [v_{i,j+1,k,l} - 2v_{i,j,k,l} + v_{i,j-1,k,l}] + \frac{v}{(\Delta z)^2} [v_{i,j,k+1,l} - 2v_{i,j,k,l} + v_{i,j,k-1,l}] \right\} + \frac{v_{i,j,k,l}}{1 + \tau}, \end{aligned} \quad (11)$$

where  $\Delta x$ ,  $\Delta y$  and  $\Delta z$  are uniform spatial lengths in  $x$ ,  $y$  and  $z$  directions,  $\Delta t$  is a time stepsize,  $v_{i,j,k,l}(\tau) = v(x_i, y_j, z_k, t_\ell, \tau)$ , and  $\dot{v}$  denotes the differential of  $v$  with respect to  $\tau$ .

When one employs a suitable numerical integrator to integrate Eq. (11), a sequence of  $v_{i,j,k,l}$  can be obtained. Given a stopping criterion, as shown below, to terminate the fictitious time stepping solution, we can gain the solution of  $v$  at a fictitious time  $\tau_f$ , and calculating  $u$  by Eq. (4), we can acquire the solution of  $u$  in a fully space-time region. Consequently, we call this novel approach a fictitious time integration method (FTIM).

The above notion by adding a fictitious time was first proposed by Liu (2008a) to resolve an inverse Sturm-Liouville problem by transforming an ODE into a PDE. Later, Liu (2008b, 2008c) and Liu, Chang, Chang and Chen (2008) extended this concept to develop new schemes for estimating parameters in the inverse vibration problems. More recently, Liu and his coworkers have utilized the FTIM to tackle many problems; see, e.g., Liu and Atluri (2008a) [nonlinear algebraic equations], Liu and Atluri (2008b) [discretized inverse sturm-Liouville problems], Liu (2008d) [nonlinear complementarity problems], Liu (2008e) [quasi-linear elliptic boundary value problems], Liu and Atluri (2008c) [mixed complementarity problems], Liu (2009a) [ $m$ -point boundary value problems], Liu (2009b) [delay ordinary differential equations], Liu (2009c) [a quasilinear elliptic boundary value problem with an arbitrary plane domain], Liu and Atluri (2009) [numerical solution of the Fredholm integral equation and numerical differentiation of noisy data, and its relation to the filter theory], Ku, Yeih, Liu and Chi (2009) [highly nonlinear boundary value problems], Chang and Liu (2009) [backward advection-dispersion equation], and Chi, Yeih and Liu (2009) [Cauchy problem of Laplace equation].

### 3.2 GPS for differential equations system

We can write Eq. (11) as a vector form:

$$\dot{\mathbf{v}} = \mathbf{f}(\mathbf{v}, \tau), \quad \mathbf{v} \in \mathcal{R}^n, \quad \tau \in R, \tag{12}$$

where  $\mathbf{v}$  is an  $n$ -dimensional state vector, and  $\mathbf{f} \in \mathcal{R}^n$  is a vector-valued function of  $\mathbf{v}$  and  $\tau$ .

The GPS can preserve the internal symmetry group of the considered ODE system. For nonlinear differential equations systems, Liu (2001) has embedded them into the augmented dynamical systems, which concern with not only the evolution of state variables but also the evolution of the magnitude of state variables vector.

We can embed Eq. (12) into the following  $n+1$ -dimensional augmented dynamical system:

$$\frac{d}{d\tau} \begin{bmatrix} \mathbf{v} \\ \|\mathbf{v}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n \times n} & \frac{f(\mathbf{v}, \tau)}{\|\mathbf{v}\|} \\ \frac{f^T(\mathbf{v}, \tau)}{\|\mathbf{v}\|} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \|\mathbf{v}\| \end{bmatrix}. \tag{13}$$

Here, we assume  $\|\mathbf{v}\| > 0$  and hence, the above system is well-defined.

It is obvious that the first row in Eq. (13) is the same as the original Eq. (12), but the inclusion of the second row in Eq. (13) gives us a Minkowskian structure of the augmented state variables of  $\mathbf{X} := (\mathbf{v}^T, \|\mathbf{v}\|)^T$ , which satisfies the cone condition:

$$\mathbf{X}^T \mathbf{g} \mathbf{X} = 0, \tag{14}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & -1 \end{bmatrix} \tag{15}$$

is a Minkowski metric.  $\mathbf{I}_n$  is the identity matrix of order  $n$ , and the superscript T denotes the transpose. In terms of  $(\mathbf{x}^T, \|x\|)$ , Eq. (14) holds, as

$$\mathbf{X}^T \mathbf{g} \mathbf{X} = \mathbf{v} \cdot \mathbf{v} - \|\mathbf{v}\|^2 = \|\mathbf{v}\|^2 - \|\mathbf{v}\|^2 = 0, \tag{16}$$

where the dot between two  $n$ -dimensional vectors represents their Euclidean inner product. The cone condition is therefore the most natural constraint that we can impose on the dynamical system (13).

Consequently, we have an  $n+1$ -dimensional augmented system:

$$\dot{\mathbf{X}} = \mathbf{A} \mathbf{X} \tag{17}$$

with a constraint (14), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n \times n} & \frac{\mathbf{f}(\mathbf{v}, \tau)}{\|\mathbf{v}\|} \\ \frac{\mathbf{f}^T(\mathbf{v}, \tau)}{\|\mathbf{v}\|} & 0 \end{bmatrix} \tag{18}$$

is an element of the Lie algebra  $so(n,1)$  of the proper orthochronous Lorentz group  $SO_o(n,1)$ , satisfying

$$\mathbf{A}^T \mathbf{g} + \mathbf{g} \mathbf{A} = \mathbf{0}. \tag{19}$$

This fact prompts us to employ the group preserving scheme, and its discretized mapping  $\mathbf{G}$  exactly preserves the following properties:

$$\mathbf{G}^T \mathbf{g} \mathbf{G} = \mathbf{g}, \tag{20}$$

$$\det \mathbf{G} = 1, \tag{21}$$

$$G_0^0 > 0, \tag{22}$$

where  $G_0^0$  is the 00th component of  $\mathbf{G}$ . Such  $\mathbf{G}$  is an element of  $SO_o(n,1)$ .

Although the dimension of the new system rises by one, it has been shown that the new system has the advantage of admitting a GPS given as follows [Liu (2001)]:

$$\mathbf{X}_{\ell+1} = \mathbf{G}(\ell) \mathbf{X}_\ell, \tag{23}$$

where  $\mathbf{X}_\ell$  represents the numerical evaluation of  $\mathbf{X}$  at the discrete time  $\tau_\ell$ , and  $\mathbf{G}(\ell) \in SO_o(n,1)$  is the group evaluation at time  $\tau_\ell$ .

To give a step by step numerical scheme, we suppose that  $\mathbf{A}(\ell)$  in Eq. (19) is a constant matrix, taking its value at the  $\ell$ -thstep. An exponential mapping of  $\mathbf{A}(\ell)$  for the interval  $\tau_\ell \leq \tau < \tau_\ell + \Delta\tau$ , when the time parameter  $\tau$  in Eq. (18) is approximately fixed as  $\tau = \tau_\ell$ , admits:

$$\mathbf{G}(\ell) = \exp[\Delta\tau \mathbf{A}(\ell)] = \begin{bmatrix} \mathbf{I}_n + \frac{(a_\ell - 1)}{\|\mathbf{f}_\ell\|^2} \mathbf{f}_\ell \mathbf{f}_\ell^T & \frac{b_\ell \mathbf{f}_\ell}{\|\mathbf{f}_\ell\|} \\ \frac{b_\ell \mathbf{f}_\ell^T}{\|\mathbf{f}_\ell\|} & a_\ell \end{bmatrix}, \tag{24}$$

where

$$a_\ell := \cosh\left(\frac{\Delta\tau \|\mathbf{f}_\ell\|}{\|\mathbf{v}_\ell\|}\right), \quad b_\ell := \sinh\left(\frac{\Delta\tau \|\mathbf{f}_\ell\|}{\|\mathbf{v}_\ell\|}\right). \tag{25}$$

For saving notation, we employ  $\mathbf{f}_\ell = \mathbf{f}(\mathbf{v}_\ell, \tau_\ell)$ . Substituting Eq. (24) for  $\mathbf{G}(\ell)$  into Eq. (23) and taking its first row, we obtain

$$\mathbf{v}_{\ell+1} = \mathbf{v}_\ell + \frac{(a_\ell - 1) \mathbf{f}_\ell \cdot \mathbf{v}_\ell + b_\ell \|\mathbf{v}_\ell\| \|\mathbf{f}_\ell\|}{\|\mathbf{f}_\ell\|^2} \mathbf{f}_\ell = \mathbf{v}_\ell + \eta_\ell \mathbf{f}_\ell, \tag{26}$$

where  $\eta_\ell$  is an adaptive factor. From  $\mathbf{f}_\ell \cdot \mathbf{v}_\ell \geq -\|\mathbf{f}_\ell\| \|\mathbf{v}_\ell\|$ , we can prove that

$$\eta_\ell \geq \left[ 1 - \exp\left(-\frac{\Delta\tau \|\mathbf{f}_\ell\|}{\|\mathbf{v}_\ell\|}\right) \right] \frac{\|\mathbf{v}_\ell\|}{\|\mathbf{f}_\ell\|} > 0, \quad \forall \Delta\tau > 0. \quad (27)$$

This scheme is group properties preserved for all  $\Delta\tau > 0$ , and is called the group preserving scheme.

### 3.3 The convergent criterion

We use the above GPS to integrate Eq. (11) from  $\tau = 0$  to a chose fictitious final time  $\tau_f$ . In the numerical integration procedure, we can test the convergence of  $v_{i,j,k,\ell}$  at the  $q$ - and  $q + 1$ -steps by

$$\sqrt{\sum_{\ell=1}^{m_1} \sum_{i,j,k=1}^m [v_{i,j,k,\ell}^{q+1} - v_{i,j,k,\ell}^q]^2} \leq \varepsilon, \quad (28)$$

where  $\varepsilon$  is a selected criterion,  $m_1$  is the number of subintervals in time direction, and  $m$  is the number of grid points in each spatial direction, assuming the same. If at a time  $\tau_0 \leq \tau_f$  the above criterion is satisfied, then the solution of  $u$  is given by

$$u_{i,j,k,\ell} = \frac{v_{i,j,k,\ell}(\tau_0)}{1 + \tau_0}. \quad (29)$$

Practically, if a suitable  $\tau_f$  is selected, we discover that the numerical solution is also approached very well to the true solution, even the above convergence criterion is not satisfied. The coefficient  $v$  introduced in Eq. (11) can enhance the stability of numerical integration.

In particular, we would stress that the present approach is a new FTIM, which can calculate the hyperbolic PDE very effectively and stably. We give three numerical experiments to show some merits of the proposed FTIM in the next section.

## 4 Numerical examples

We will apply the FTIM to the calculations of BWP through numerical instances. We would like to realize the stability of our algorism when the input final measured data are contaminated by random noise for different problems. We can evaluate the stability by increasing the different levels of random disturbance in the final data:

$$\hat{\mathbf{v}}_F = \mathbf{v}_F + s[2R(i) - 1], \quad (30)$$

where  $\mathbf{v}_F$  are the final exact data. We employ the function RANDOM\_NUMBER given in Fortran to generate the noisy data  $R(i)$ , which are random numbers in  $[-1, 1]$ , and  $s$  means the level of absolute noise. Then, the final noisy data  $\hat{\mathbf{v}}_F$  are used in the calculations.

#### 4.1 Example 1

The following one-dimensional BWP is considered:

$$u_{tt} = u_{xx}, \quad 0 < x < \pi, \quad 0 < t < T, \quad (31)$$

with the boundary conditions

$$u(0, t) = u(\pi, t) = 0, \quad (32)$$

and the final time condition

$$u(x, T) = \sin(T)\sin(x). \quad (33)$$

The data to be retrieved are given by

$$u(x, t) = \sin(t)\sin(x), \quad 0 \leq t < T. \quad (34)$$

A straightforward derivation according to the concept of FTIM results in

$$\dot{v}_{i,\ell} = \frac{\nu}{(\Delta t)^2} [v_{i,\ell+2} - 2v_{i,\ell+1} + v_{i,\ell}] - \frac{\nu c^2}{(\Delta x)^2} [v_{i+1,\ell} - 2v_{i,\ell} + v_{i-1,\ell}] + \frac{v_{i,\ell}}{1 + \tau}, \quad (35)$$

where

$$v_{0,\ell}(\tau) = v_{1,\ell}(\tau) = 0, \quad (36)$$

$$v_{i,m}(\tau) = (1 + \tau)\sin(T)\sin(x_i). \quad (37)$$

Let us investigate some ill-posed cases of this benchmark BWP, where  $T = 90$  are utilized such that when the final data are in the order of  $O(10^{-1})$ , we attempt to employ the FTIM to recover the desired initial data of  $\sin(0.1)\sin(x)$ , which are in the order of  $O(10^{-4})$ . Under the following parameters:  $m = 20$ ,  $m_1 = 20$ ,  $\varepsilon = 10^{-8}$ ,  $\nu = 0.0089155$ , a fictitious terminal time  $\tau_f = 2.4$ , and starting from an initial value of  $v_{i,\ell} = 0.1\sin(90)\sin(x_i)$  because we know the final data in Eq. (33), we resolve this problem by our scheme with a fictitious time stepsize  $\Delta\tau = 0.24$ . Fig. 1 displays the numerical results and numerical errors, and the maximum error is about  $10^{-8}$ . Upon comparing with the numerical results in [Lesnic (2002)] with that from the decomposition method (see Figures 3 and 4 of the above cited paper) and Chang and Liu (2010) with the BGPS (see Figures 2 and 3 of the above cited paper), we can say that the FTIM does not need to use the regularization technique and still acquire good results.

In Fig. 2, we demonstrate the numerical results and numerical errors for a final time  $T = 90$  and with a noise of  $s = 1$ . The maximum error as shown is in the

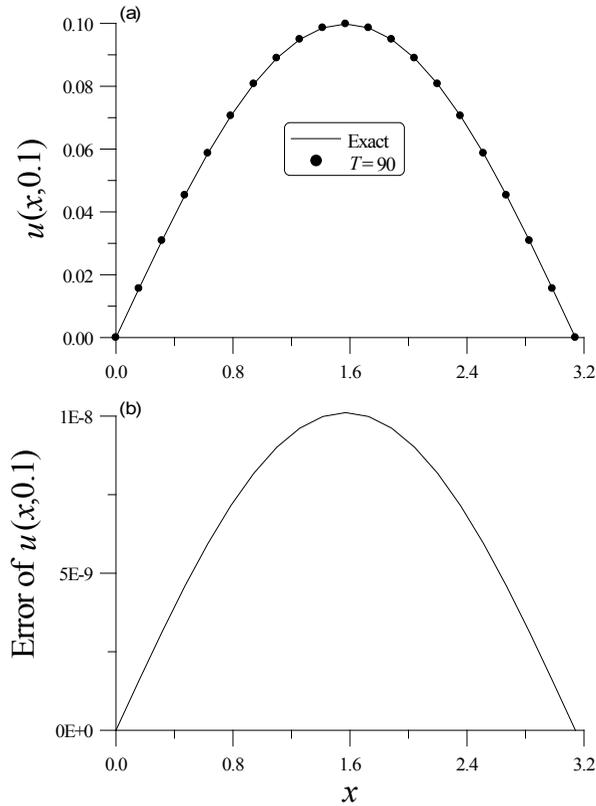


Figure 1: Comparisons of the exact solutions and numerical solutions for Example 1 with  $T = 90$ , and the corresponding numerical errors.

order of  $10^{-8}$  even for a larger disturbance with  $s = 1$ . The present results are also better than that calculated by Chang and Liu (2010), of which the maximum error is  $8.38 \times 10^{-3}$ , under a noise of  $s = 0.1$ . To the authors' best knowledge, there has been no open report that the numerical methods can calculate this ill-posed BWP very well as the FTIM.

#### 4.2 Example 2

Let us further consider the two-dimensional BWP:

$$u_{tt} = 2(u_{xx} + u_{yy}), \quad 0 < x < \pi, \quad 0 < y < \pi, \quad 0 < t < T, \quad (38)$$

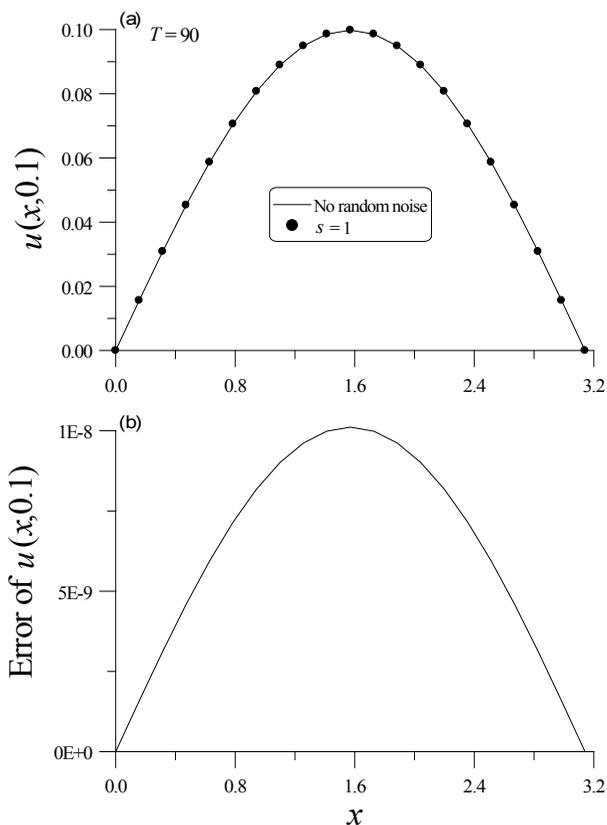


Figure 2: Comparisons of numerical solutions for Example 1 with different levels of noise  $s = 0, 1$ , and the corresponding numerical errors.

with the boundary conditions

$$\begin{aligned} u(0, y, t) &= y, & u(\pi, y, t) &= \pi + y, \\ u(x, 0, t) &= x, & u(x, \pi, t) &= \pi + x, \end{aligned} \tag{39}$$

and the final time condition

$$u(x, y, T) = x + y + \sin x \sin y \cos(2T). \tag{40}$$

The exact solution is given by

$$u(x, y, t) = x + y + \sin x \sin y \cos(2t), \quad 0 \leq t < T. \tag{41}$$

A derivation according to the FTIM leads to

$$\dot{v}_{i,j,\ell} = \frac{\mathbf{v}}{(\Delta t)^2} [v_{i,j,\ell+2} - 2v_{i,j,\ell+1} + v_{i,j,\ell}] - \frac{\mathbf{v}c^2}{(\Delta x)^2} [v_{i+1,j,\ell} - 2v_{i,j,\ell} + v_{i-1,j,\ell}]$$

$$-\frac{vc^2}{(\Delta y)^2} [v_{i,j+1,\ell} - 2v_{i,j,k,\ell} + v_{i,j-1,\ell}] + \frac{v_{i,j,\ell}}{1 + \tau}. \tag{42}$$

Under the following parameters:  $m = 20$ ,  $m_1 = 20$ ,  $\varepsilon = 10^{-6}$ ,  $v = 0.3931001$ ,  $T = 135$ ,  $\tau_f = 10^{-2}$ , and starting from an initial value of  $v_{i,j,\ell} = x_i + y_j + \sin x_i \sin y_j \cdot \cos(270)$ , we integrate Eq. (42) by using the GPS with a fictitious time stepsize  $\Delta\tau = 10^{-3}$ .

Fig. 3 shows the errors of numerical solutions obtained from the FTIM for the case of  $T = 135$ , and the final data is in the order of  $O(1)$ . However, we can utilize the FTIM to recover the desired initial data  $x + y + \sin x \sin y$ . At the point  $x = 2\pi/20$ , the error is plotted with respect to  $y$  by a dashed line, and at the point  $y = 9\pi/20$ , the error is plotted with respect to  $x$  by a solid line. The latter one is smaller than the former one because the point  $y = 9\pi/20$  is near the boundary. However, the errors are smaller than that calculated by Chang and Liu (2010) as shown in Figure 4 therein. For this difficult problem, the FTIM proposed here is still good with a maximum error  $4.4 \times 10^{-5}$ .

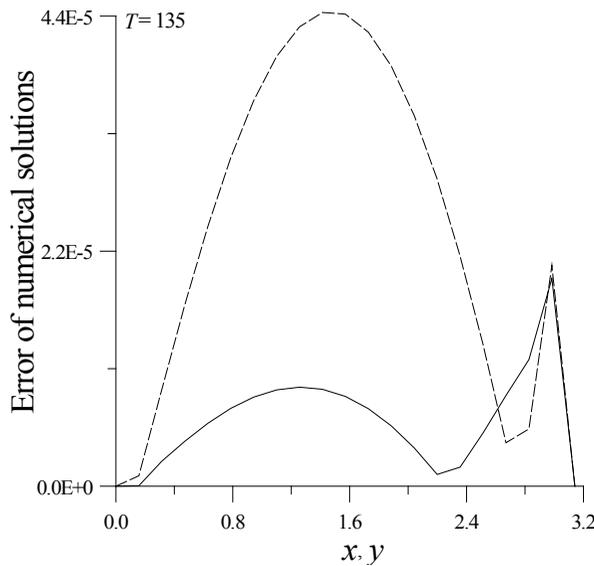


Figure 3: The errors of FTIM solutions for Example 2 with  $T = 135$ .

Recently, the numerical results have been calculated by Chang and Liu (2010), of which the final time was 0.5 and the maximum error was  $2.7 \times 10^{-4}$ , under a noise of  $s = 0.01$ . In Fig. 4, we compare the numerical errors with  $T = 135$  for two cases: one without the random noise and the other with the absolute random noise

in the level of  $s = 1$ . The exact solutions and numerical solutions are plotted in Figs. 5(a)-(c) sequentially. Even under the large noise, the numerical solution indicated in Fig. 5(c) is a good approximation to the exact initial data as displayed in Fig. 5(a).

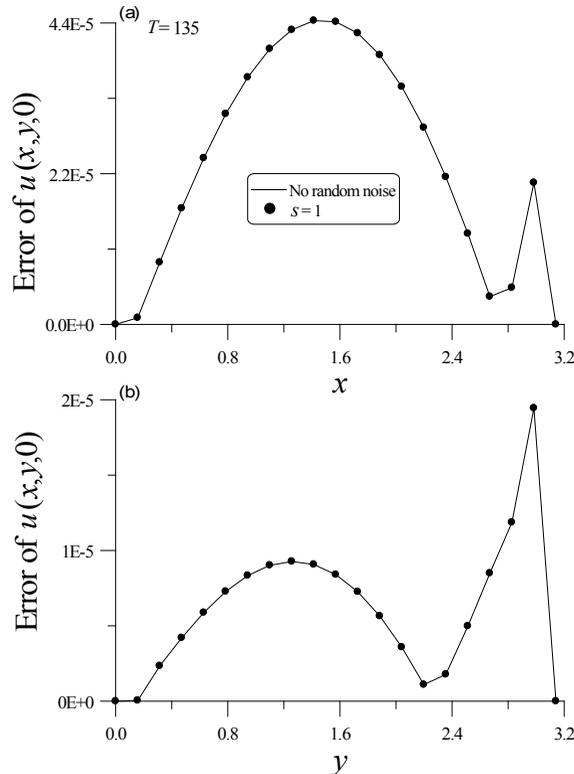


Figure 4: The numerical errors of FTIM solutions with and without random noise effect for Example 2 are plotted in (a) with respect to  $x$  at fixed  $y = \pi/2$ , and in (b) with respect to  $y$  at fixed  $x = \pi/6$ .

### 4.3 Example 3

Let us ponder a three-dimensional BWP:

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}, \quad 0 < x < \pi, \quad 0 < y < \pi, \quad 0 < z < \pi, \quad 0 < t < T, \quad (43)$$

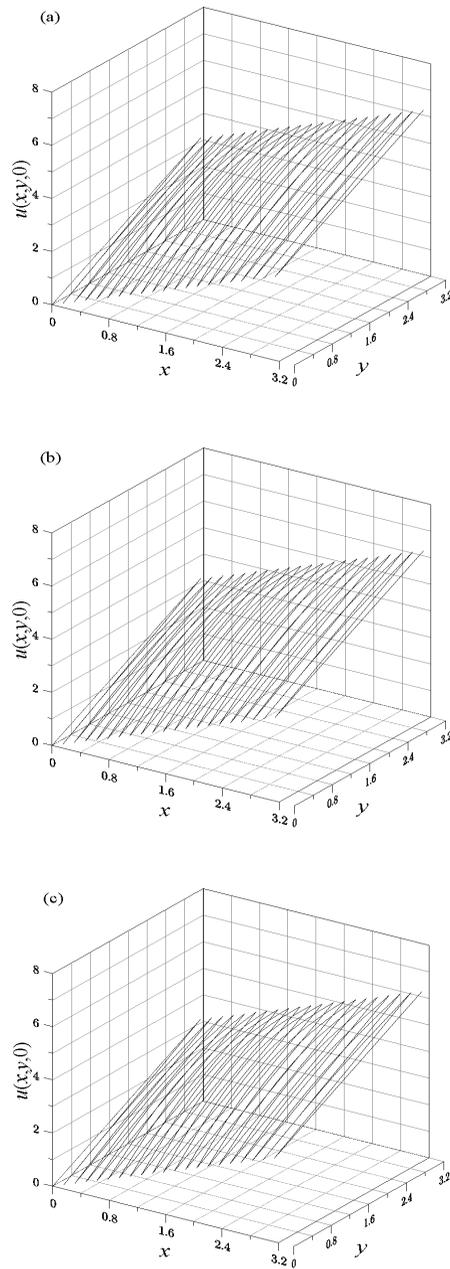


Figure 5: The exact solution for Example 2 of two-dimensional BWP with  $T = 135$  are shown in (a), in (b) the FTIM solution without random noise effect, and in (c) the FTIM solution with random noise.

with the boundary conditions

$$\begin{aligned} u(0, y, z, t) &= \exp(y + z + t), & u(\pi, y, z, t) &= \exp(\pi + y + z + t), \\ u(x, 0, z, t) &= \exp(x + z + t), & u(x, \pi, z, t) &= \exp(\pi + x + z + t), \\ u(x, y, 0, t) &= \exp(x + y + t), & u(x, y, \pi, t) &= \exp(\pi + x + y + t), \end{aligned} \quad (44)$$

and the final time condition

$$u(x, y, z, T) = \exp(x + y + z + T). \quad (45)$$

The exact solution is given by

$$u(x, y, z, t) = \exp(x + y + z + t). \quad (46)$$

Under the following parameters:  $m = 20$ ,  $m_1 = 20$ ,  $\varepsilon = 10^{-6}$ ,  $\nu = 10^{-5}$ ,  $T = 10$ , and a fictitious terminal time  $\tau_f = 10^{-7}$ , and starting from an initial value of  $v_{i,j,k,\ell} = 4.54 \times 10^{-5} \cdot \exp(x_i + y_j + z_k + 10)$ , we tackle this problem by our approach with a fictitious time stepsize  $\Delta\tau = 10^{-8}$ . Nevertheless, we can employ the FTIM to retrieve the desired initial data  $\exp(x + y + z)$ , which are in the order of  $O(10^5)$ . In Fig. 6, we compare the numerical errors with  $T = 10$  for two cases: one without the random noise and the other with the random noise in the level of  $s = 1$ . Besides, the exact solutions and numerical solutions are plotted in Figs. 7(a)-(c) sequentially. Even under the noise, the numerical solution displayed in Fig. 7(c) is a good approximation to the exact initial data as displayed in Fig. 7(a). Moreover, the errors are much smaller than those calculated by Chang and Liu (2010) as shown in Figure 7 therein.

To offer a rigorous examination of the FTIM when applied this scheme to this example, we let  $T = 15$ , and the final data is very large in the order of  $O(10^{10})$ . Under the following parameters:  $m = 20$ ,  $m_1 = 20$ ,  $\varepsilon = 10^{-9}$ ,  $\nu = 10^{-5}$ ,  $\tau_f = 10^{-11}$ , and starting from an initial value of  $v_{i,j,k,\ell} = 3.059 \times 10^{-7} \cdot \exp(x_i + y_j + z_k + 10)$ , we resolve this problem by our approach with a fictitious time stepsize  $\Delta\tau = 10^{-12}$ . In Fig. 8, we compare the numerical errors with  $T = 15$ , where the grid lengths are used  $\Delta x = \Delta y = \Delta z = \pi/20$ . At the point  $x = \pi/5$  the error is plotted with respect to  $y$  and  $z$  by a dashed line, at the point  $y = \pi/4$  the error is plotted with respect to  $x$  and  $z$  by a solid line, and at the point  $z = \pi/3$  the error is plotted with respect to  $x$  and  $y$  by a dotted line. The latter one is smaller than the former two because the point  $z = \pi/3$  is near the boundary. This experiment is a hard BWP to examine the numerical performance of our novel numerical algorithm. Nevertheless, to the authors' best knowledge, there has been no report that numerical methods can calculate this ill-posed three-dimensional BWP very well as our scheme.

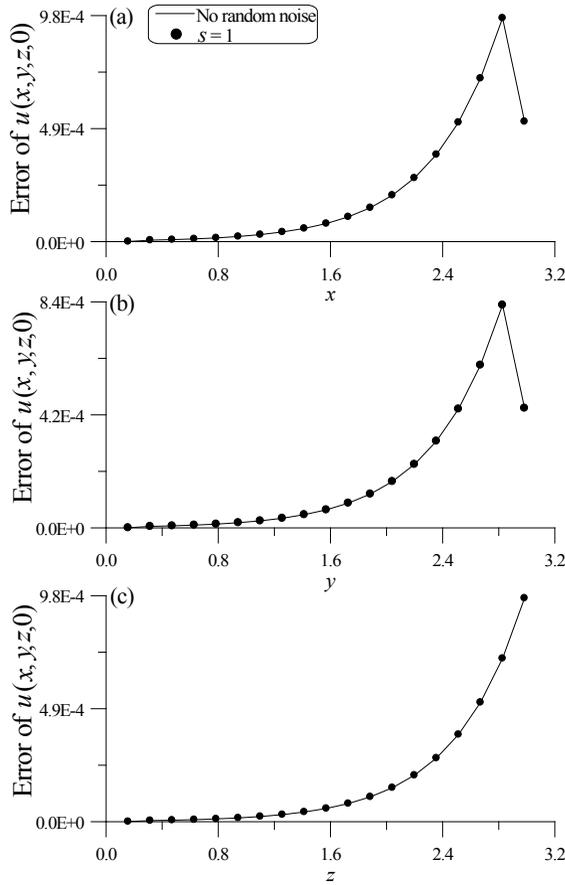


Figure 6: The numerical errors of FTIM solutions with and without random noise effect for Example 3 are plotted in (a) with respect to  $x$  at fixed  $y = \pi/4$  and  $z = \pi/3$ , (b) with respect to  $y$  at fixed  $x = \pi/5$  and  $z = \pi/3$ , and (c) with respect to  $z$  at fixed  $x = \pi/5$  and  $y = \pi/4$ .

## 5 Conclusions

We have transformed the original hyperbolic equation into another hyperbolic type equation by introducing a fictitious time variable, and adding a fictitious viscous damping coefficient to enhance the stability of numerical integration of the discretized equations by using the GPS. Besides, we merely required spending a little of computational time for integrating the discretized equations to obtain solutions. By utilizing the FTIM, we can calculate the solution and recover the initial data

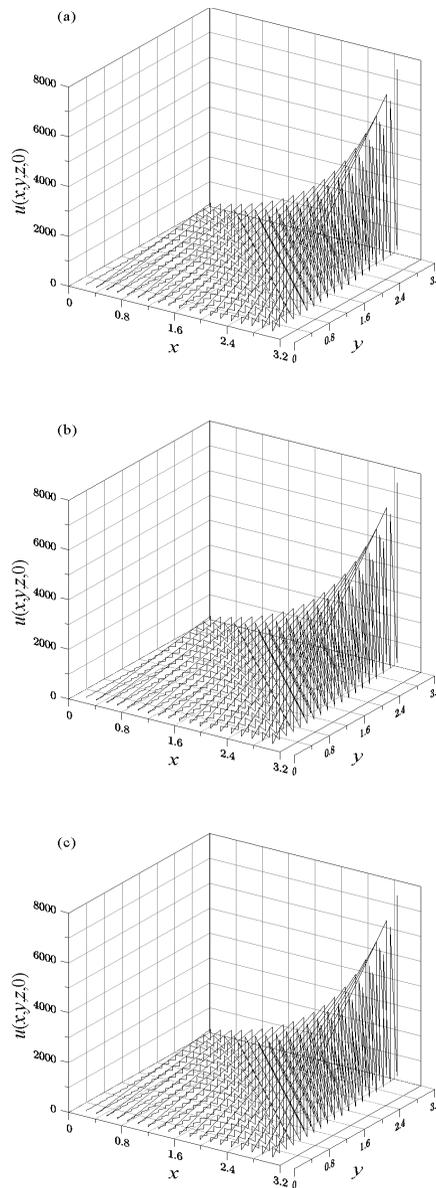


Figure 7: The exact solution for Example 3 of three-dimensional BWP with  $T = 10$  are shown in (a), in (b) the FTIM solution without random noise effect, and in (c) the FTIM solution with random noise.

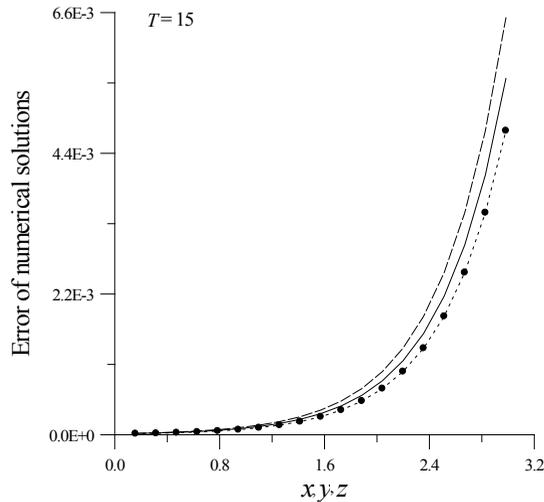


Figure 8: The errors of FTIM solutions for Example 3 with  $T = 15$ .

very well with a high order accuracy. Three numerical experiments of the BWP are worked out, which exhibit that our proposed scheme is applicable to the seriously ill-posed BWP. The numerical errors of our algorithm are in the order of  $O(10^{-3})$ - $O(10^{-9})$ . Furthermore, those effects are very pivotal in the computations of three-dimensional BWP. Hence, it can be concluded that the FTIM is very stable, accurate, effective, and insensitive to the perturbation on final data.

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## References

- Abdul-Latif, A. I.; Diaz, J. B.** (1971): Dirichlet, Neumann, and mixed boundary value problems for the wave equation  $u_{xx}-u_{yy} = 0$  for a rectangle. *Appl. Anal.*, vol. 1, pp. 1-12.
- Ames, K. A.; Straughan, B.** (1997): *Non-standard and Improperly Posed Problems*, Academic Press, New York.
- Bourgin, D. G.; Duffin, R.** (1939): The Dirichlet problem for the vibrating string equation. *Bull. Amer. Math. Soc.*, vol. 45, pp. 851-858.
- Chang, C.-W.; Liu, C.-S.** (2009): A fictitious time integration method for back-

ward advection-dispersion equation. *CMES: Computer Modeling in Engineering & Sciences*, vol. 51, pp. 261-276.

**Chang, C.-W.; Liu, C.-S.** (2010): A backward group preserving scheme for multi-dimensional backward wave problems. *CMC: Computers, Materials & Continua*, vol. 19, pp. 17-36.

**Chi, C.-C.; Yeih, W.; Liu, C.-S.** (2009): A novel method for solving the Cauchy problem of Laplace equation using the fictitious time integration method. *CMES: Computer Modeling in Engineering & Sciences*, vol. 47, pp. 167-190.

**Fox, D.; Pucci, C.** (1958): The Dirichlet problem for the wave equation. *Ann. Mat. Pura Appl.*, vol. 46, pp. 155-182.

**Godinho, L.; Tadeu, A.; Amado Mendes, P.** (2007): Wave propagation around thin structures using the MFS. *CMC: Computers Materials & Continua*, vol. 5, pp. 117-128.

**Gu, M. H.; Young, D. L.; Fan, C. M.** (2008): The meshless method for one-dimensional hyperbolic equation. *J. Aeronaut Astronaut Aviat (A)*, vol. 40, pp. 63-72.

**Gu, M. H.; Young, D. L.; Fan, C. M.** (2009): The method of fundamental solution for one-dimensional wave equations. *CMC: Computers, Materials & Continua*, vol. 11, pp. 185-208.

**Ku, C.-Y.; Yeih, W.; Liu, C.-S.; Chi, C.C.** (2009): Applications of the fictitious time integration method using a new time-like function. *CMES: Computer Modeling in Engineering & Sciences*, vol. 43, pp. 173-190.

**Lesnic, D.** (2002): The decomposition method for forward and backward time-dependent problems. *J. Comp. Appl. Math.*, vol. 147, pp. 27-39.

**Levine, H. A.; Vessella, S.** (1985): Stabilization and regularization for solutions of an ill-posed problem for the wave equation. *Math. Meth. Appl. Sci.*, vol. 7, pp. 202-209.

**Liu, C.-S.** (2001): Cone of non-linear dynamical system and group preserving schemes. *Int. J. of Non-Linear Mech.*, vol. 36, pp. 1047-1068.

**Liu, C.-S.** (2008a): Solving an inverse Sturm-Liouville problem by a Lie-group method. *Bound. Value Prob.*, vol. 2008, Article ID 749865.

**Liu, C.-S.** (2008b): Identifying time-dependent damping and stiffness functions by a simple and yet accurate method. *J. Sound Vib.*, vol. 318, pp. 148-165.

**Liu, C.-S.** (2008c): A Lie-group shooting method for simultaneously estimating the time-dependent damping and stiffness coefficients. *CMES: Computer Modeling in Engineering & Science*, vol. 27, pp. 137-149.

**Liu, C.-S.** (2008d): A time-marching algorithm for solving non-linear obstacle problems with the aid of an NCP-function. *CMC: Computers, Materials & Continua*, vol. 8, pp. 53-65.

**Liu, C.-S.** (2008e): A fictitious time integration method for two-dimensional quasi-linear elliptic boundary value problems. *CMES: Computer Modeling in Engineering & Science*, vol. 33, pp. 179-198.

**Liu, C.-S.; Atluri, S. N.** (2008a): A novel time integration method for solving a large system of non-linear algebraic equations. *CMES: Computer Modeling in Engineering & Science*, vol. 31, pp. 71-83.

**Liu, C.-S.; Atluri, S. N.** (2008b): A novel fictitious time integration method for solving the discretized inverse Sturm-Liouville problems, for specified eigenvalues. *CMES: Computer Modeling in Engineering & Science*, vol. 36, pp. 261-286.

**Liu, C.-S.; Atluri, S. N.** (2008c): A novel time integration method (FTIM) for solving mixed complementarity problems with applications to non-linear optimization. *CMES: Computer Modeling in Engineering & Science*, vol. 34, pp. 155-178.

**Liu, C.-S.; Atluri, S. N.** (2009): A fictitious time integration method for the numerical solution of the Fredholm integral equation and for numerical differentiation of noisy data, and its relation to the filter theory. *CMES: Computer Modeling in Engineering & Science*, vol. 41, pp. 243-261.

**Liu, C.-S.** (2010): An analytical method for computing the one-dimensional backward wave problem. *CMC: Computers, Materials & Continua*, vol. 13, pp. 219-234.

**Young, D. L.; Gu, M. H.; Fan, C. M.** (2009): The time-marching method of fundamental solutions for wave equations. *Eng. Anal. Bound. Elem.*, vol. 33, pp. 1411-1425.

