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Comprehensive Laminate Level Sensitivities of the Touratier Kinematic Model for Reliability Analyses and Robust Optimisation of Composite Materials and Structures

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Notation

a, b, c	summation indices
$C_{(k)}^{abcd}$	elasticity tensor
$\mathbf{D}$	vector of degrees of freedom
$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$	unit vectors of the standard Cartesian coordinate axis
$\mathbf{e}_1, \mathbf{e}_2$	tangent vectors to the shell mid-surface
$\bar{e}_1, \bar{e}_2$	tangent vectors to the deformed shell mid-surface
$\mathbf{e}_\xi$	mutually orthogonal basis vectors
ξ, η, λ	curvilinear co-ordinates of the shell mid-surface
f_1, f_2	trigonometric functions
$g_1^{(k)} - g_4^{(k)}$	laminate-level trigonometric functions
Γ	total potential energy
k	laminate number
γ_1^0, γ_2^0	transverse shear strains at the mid-surface
h	shell thickness
$\mathbf{n}_0$	unit normal vector to the shell mid-surface
$\bar{\mathbf{n}}_0$	unit normal vector to the deformed shell mid-surface
$\mathbf{P}$	vector of prescribed and concentrated forces and/or moments.
$\mathbf{r}_0$	position vector of the shell mid-surface
$\mathbf{r}$	position vector specifying a general point within the shell
$\bar{\mathbf{r}}$	vector specifying the position of a point on the deformed shell
R_1, R_2	principal radii of curvature of the un-deformed shell
θ_ξ, θ_η	rigid body rotations the initial mid-surface normal

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$\mathbf{u}$	displacement vector
$\mathbf{u}_o$	vector of mid-surface displacement
w_{ab}	strain tensor
x_o, y_o, z_o	cartesian co-ordinates of the shell mid-surface
$\bar{x}_o, \bar{y}_o, \bar{z}_o$	cartesian co-ordinates of the deformed shell mid-surface

1 Introduction

Sensitivities are an essential requirement for first and second-order reliability methods (FORM/SORM) and optimisation algorithms in which a robust solution is sought in the presence of uncertainty. Here we present comprehensive mathematical expressions that can be used for the sensitivity analysis of a laminated, curvilinear shell that can undergo large strains, rigid body rotations and transverse shear deformations. Each lamina is a fibre reinforced composite material and in all cases the material is assumed to be linear-elastic and reasonably stiff. The strengths of all the materials and the shell geometry are such that the component of strain in the through thickness direction is taken to be negligible.

A key consideration in the derivation of our model equations is the adoption of Absolute Nodal Coordinates (and their directional derivatives) instead of displacement vector components (and their directional derivates) as nodal degrees of freedom for a finite element formulation. Mikkola & Shabana (2003), and Ibrahimbegovic (1997), have argued that an Absolute Nodal Coordinate (ANC) formulation is more accurate at modelling geometrically non-linear structures that are subjected to large strains and rotations. That is relative to a classical displacement vector formulation. Dufva & Shabana (2005) have also reported that an ANC formulation is computationally more efficient at modelling these types of problems. This cited research, however, is restricted to cases where the shell was composed of a single layer.

The shallow shell theory of Beakou and Touratier (1993) (referred to here as the Touratier kinematic model) is used to implement transverse shear strains and laminations. The Touratier kinematic model has so far only been used in displacement vector based finite element formulations, see Beakou and Touratier (1993), Idlbi *et. al.* (1997), Dau *et al.* (2006). It was however shown to be numerically robust in terms of addressing the various locking instability modes that can arise in shell computations. A further advantage is that it does not require thickness correction factors.

2 Touratier Kinematic Model

Consider an arbitrary curvilinear shell. The initial geometry of a differential shell element is illustrated in Fig. 1 where the mid-surface is marked with solid curves.

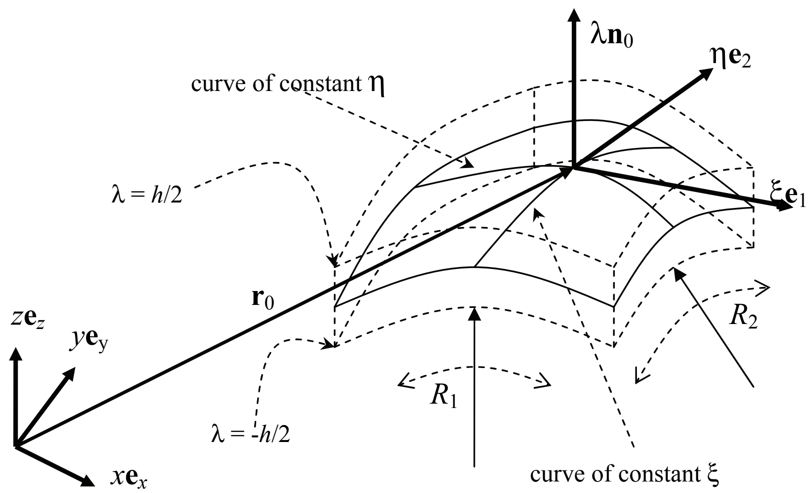

Figure 1: Absolute Nodal Coordinates (ANCs) and parameterisation of the mid-surface.

Let the vector $\mathbf{r}_0$ quantify the position of the mid-surface which can in turn be parameterised using the curvilinear co-ordinates ξ and η as follows,

$$\mathbf{r}_0 = x_0(\xi,\eta)\,\mathbf{e}_x + y_0(\xi,\eta)\,\mathbf{e}_y + z_0(\xi,\eta)\,\mathbf{e}_z. \tag{1}$$

The vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are the unit vectors of the standard Cartesian coordinate axis. The tangent vectors to the mid-surface are $\mathbf{e}_1$ and $\mathbf{e}_2$ and the unit normal vector is $\mathbf{n}_0$. Therefore,

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}_0}{\partial \xi},\ \mathbf{e}_2 = \frac{\partial \mathbf{r}_0}{\partial \eta},\ \mathbf{n}_0 = \frac{1}{|\mathbf{e}_1 \times \mathbf{e}_2|}\mathbf{e}_1 \times \mathbf{e}_2. \tag{2}$$

They are a set of base vectors that define a local coordinate system, (ξ,η,λ) where λ is the coordinate normal to the tangent plane, Let h be the plate thickness, the mid-surface is then located at $\lambda = 0$, the top surface at $\lambda = h/2$, and the bottom surface at $\lambda = -h/2$. The position vector that specifies a general point within the shell, **r**, which is not restricted to the mid-surface is decomposed into the following components,

$$\mathbf{r} = \mathbf{r}(\xi,\eta,\lambda) = \mathbf{r}_0(\xi,\eta) + \lambda\mathbf{n}_0(\xi,\eta). \tag{3}$$

In Beakou and Touratier (1993), the displacement vector (**u**) is expressed in the following form,

$$\mathbf{u} = \mathbf{u}(\mathbf{r}) = \mathbf{u}_{kl}(\mathbf{u}_0(\xi,\eta),\lambda) + \mathbf{u}_T\left(\gamma_1^0(\xi,\eta),\gamma_2^0(\xi,\eta),\lambda\right), \tag{4}$$

where γ_1^0 and γ_2^0 are the transverse shear strains at the mid-surface and $\mathbf{u}_0$ is the mid-surface displacement such that, $\mathbf{u}_0 = \mathbf{u}(\mathbf{r} = \mathbf{r}_0)$. The components of the vectors $\mathbf{u}_{kl}$ and $\mathbf{u}_T$, defined in equation (4), are given by,

$$\mathbf{u}_{kl}\cdot\mathbf{e}_\xi = \left(1+\frac{\lambda}{R_1}\right)(\mathbf{u}_0\cdot\mathbf{e}_\xi) - \frac{\lambda}{\alpha_1}\frac{\partial w}{\partial\xi},\ \mathbf{e}_\xi = \frac{1}{\alpha_1}\mathbf{e}_1,\ \alpha_1^2 = \mathbf{e}_1\cdot\mathbf{e}_1, \tag{5}$$

$$\mathbf{u}_{kl}\cdot\mathbf{e}_\eta = \left(1+\frac{\lambda}{R_2}\right)(\mathbf{u}_0\cdot\mathbf{e}_\eta) - \frac{\lambda}{\alpha_2}\frac{\partial w}{\partial\eta},\ \mathbf{e}_\eta = \frac{1}{\alpha_2}\mathbf{e}_2,\ \alpha_2^2 = \mathbf{e}_2\cdot\mathbf{e}_2, \tag{6}$$

$$\mathbf{u}_{kl}\cdot\mathbf{n}_0 = \mathbf{u}_0\cdot\mathbf{n}_0 = w(\xi,\eta),\ \mathbf{u}_T\cdot\mathbf{n}_0 = 0, \tag{7}$$

$$\mathbf{u}_T\cdot\mathbf{e}_\xi = \left[f_1(\lambda) + g_1^{(k)}(\lambda)\right]\gamma_1^0(\xi,\eta) + g_2^{(k)}(\lambda)\gamma_2^0(\xi,\eta), \tag{8}$$

$$\mathbf{u}_T\cdot\mathbf{e}_\eta = g_3^{(k)}(\lambda)\gamma_1^0(\xi,\eta) + \left[f_2(\lambda) + g_4^{(k)}(\lambda)\right]\gamma_2^0(\xi,\eta), \tag{9}$$

The Touratier kinematic model has so far been restricted to cases where the initial shell geometry was such that the unit tangent vectors, $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$, were mutually orthogonal. Referring to equations (5) and (6) and Fig. 1, R_1 and R_2 are the principal radii of curvature of the un-deformed shell. The functions f_1 and f_2 in equations (8) and (9) are trigonometric functions which are posed to provide a high order (that is in terms of the thickness coordinate λ) representations of the transverse shear deformation and so avoid the requirement for thickness correction factors. The superscript "k" used in equations (8) and (9) refers to a quantity that is defined in a specific lamina of the laminated shell such that there are a total of N layers. The expansions of f_1, f_2, $g_1^{(k)}$, $g_2^{(k)}$, $g_3^{(k)}$ and $g_4^{(k)}$ include laminate coefficients that are expressed in terms of the stiffness components of the lamina. However, these expressions are lengthy and so the reader is referred to Beakou & Touratier (1993) and Idlbi *et. al.* (1997) for their full expansions. Given that we wish to analyse a fibre-reinforced composite laminate, we take the vector $\mathbf{e}_\xi$ to be pointing in a tangential direction to the curvilinear coordinate curve that runs along the length of the fibre, thought the Touratier kinematic model is not restricted to fibre-reinforced composite laminates.

The Touratier kinematic model is used to quantify the transverse shear components of the displacement field. Each lamina is assumed to be made from an orthotropic material. The laminate constants of this model can then be used to determine the laminate level sensitivities. The geometric random variables are the thicknesses of the laminas. First ply failure of the laminated shell is assumed via a Tsai-Wu failure criterion applied to each lamina.

According to the Tsai-Wu criterion the material is deemed to have failed when:

$$\begin{aligned}&F_1\sigma_{11}+F_2\sigma_{22}+F_3\sigma_{33}+F_{11}\sigma_{11}^2+F_{22}\sigma_{22}^2+F_{33}\sigma_{33}^2+F_{44}\sigma_{23}^2+\\&F_{55}\sigma_{13}^2+F_{66}\sigma_{12}^2+2F_{12}\sigma_{11}\sigma_{22}+2F_{13}\sigma_{11}\sigma_{33}+2F_{23}\sigma_{22}\sigma_{33}\geq 1,\end{aligned} \tag{10}$$

where

$$\begin{aligned}&F_1=\frac{1}{X_T}-\frac{1}{X_C},\ F_2=\frac{1}{Y_T}-\frac{1}{Y_C},\ F_3=\frac{1}{Z_T}-\frac{1}{Z_C},\ F_{11}=\frac{1}{X_TX_C},\\&F_{22}=\frac{1}{Y_TY_C},\ F_{33}=\frac{1}{Z_TZ_C},\ F_{44}=\frac{1}{T^2},\ F_{55}=\frac{1}{R^2},\ F_{66}=\frac{1}{S^2}.\end{aligned} \tag{11}$$

and:

$X_T^{(k)}$ =tensile strength in the fibre direction;

$Y_T^{(k)}$ =tensile strength in the directions perpendicular to the fibre;

$X_C^{(k)}$ =compressive strength in the fibre direction;

$Y_C^{(k)}$ =compressive strength in the directions perpendicular to the fibre;

$S^{(k)}$ =shear strength in the curvilinear plane containing the fibres;

$R^{(k)},T^{(k)}$ =transverse shear strength components;

The other coefficients, F_{12}, F_{13} and F_{23} (called interaction coefficients) take the form,

$$F_{12}=\frac{1}{2\sigma_{b12}^2}\left\{1-\sigma_{b12}\left(F_1+F_2\right)-\sigma_{b12}^2\left(F_{11}+F_{22}\right)\right\}, \tag{12}$$

$$F_{13}=\frac{1}{2\sigma_{b13}^2}\left\{1-\sigma_{b13}\left(F_1+F_3\right)-\sigma_{b13}^2\left(F_{11}+F_{33}\right)\right\}, \tag{13}$$

$$F_{23}=\frac{1}{2\sigma_{b23}^2}\left\{1-\sigma_{b23}\left(F_2+F_3\right)-\sigma_{b23}^2\left(F_{22}+F_{33}\right)\right\}. \tag{14}$$

σ_{b12}, σ_{b13} and σ_{b23} are equibiaxial failure strengths in the curvilinear plane containing the fibres. Three equibiaxial loading, experimental set-ups would be required such that the following measurements in these respective set-ups correspond to material failure,

$$\sigma_{11} = \sigma_{22} = \sigma_{b12},\ \sigma_{11} = \sigma_{33} = \sigma_{b13},\ \sigma_{22} = \sigma_{33} = \sigma_{b23}.$$

Suitable biaxial data can be generated by means of a combined tension/torsion test. The Touratier kinematic model equations apply to a degenerate element formulation. Therefore σ_{zz} is set to zero in equation (1) and the resulting limit state function is,

$$\begin{aligned} g(\mathbf{X}) &= 1 - F_1\sigma_{11} - F_2\sigma_{22} - F_{11}\sigma_{11}^2 - F_{22}\sigma_{22}^2 - \\ &F_{44}\sigma_{23}^2 - F_{55}\sigma_{13}^2 - F_{66}\sigma_{12}^2 - 2F_{12}\sigma_{11}\sigma_{22}. \end{aligned} \tag{15}$$

As already stated, every layer is orthotropic and these layer thicknesses are assumed to be the only geometrical random variables. If there are a total of N layers then it follows from these assumptions along with equations (1) - (3) and (6), that the vector of random variables ($\mathbf{X}$) is,

$$\mathbf{X} = \begin{bmatrix} E_1^{(1)}, & E_2^{(1)}, & E_3^{(1)}, & \nu_{23}^{(1)}, & \nu_{13}^{(1)}, & \nu_{12}^{(1)}, & G_{23}^{(1)}, & G_{13}^{(1)}, & G_{12}^{(1)}, \\ X_T^{(1)}, & X_C^{(1)}, & Y_T^{(1)}, & Y_C^{(1)}, & R^{(1)}, & S^{(1)}, & T^{(1)}, & \sigma_{b12}^{(1)}, & h^{(1)}, \\ E_1^{(2)}, & E_2^{(2)}, & E_3^{(2)}, & \nu_{23}^{(2)}, & \nu_{13}^{(2)}, & \nu_{12}^{(2)}, & G_{23}^{(2)}, & G_{13}^{(2)}, & G_{12}^{(2)}, \\ X_T^{(2)}, & X_C^{(2)}, & Y_T^{(2)}, & Y_C^{(2)}, & R^{(2)}, & S^{(2)}, & T^{(2)}, & \sigma_{b12}^{(2)}, & h^{(2)}, \\ . \\ . \\ . \\ E_1^{(N)}, & E_2^{(N)}, & E_3^{(N)}, & \nu_{23}^{(N)}, & \nu_{13}^{(N)}, & \nu_{12}^{(N)}, & G_{23}^{(N)}, & G_{13}^{(N)}, & G_{12}^{(N)}, \\ X_T^{(N)}, & X_C^{(N)}, & Y_T^{(N)}, & Y_T^{(N)}, & R^{(N)}, & S^{(N)}, & T^{(N)}, & \sigma_{b12}^{(N)}, & h^{(N)} \end{bmatrix}^T. \tag{16}$$

$\mathbf{X}$ contains a total of $18N$ random variables where there are 18 variables assigned to each lamina. The superscripts used for the components of $\mathbf{X}$ in equation (7) are used to denote the index number of the lamina to which a given component applies; e.g. $E_1^{(2)}$ is the Young's modulus in the fibre direction for the material used to construct the 2^{nd} lamina. The vector of direction cosines used in a FORM or SORM computation ($\boldsymbol{\alpha}$) is given by,

$$\alpha = \frac{\mathbf{D}_\sigma \nabla g}{\|\mathbf{D}_\sigma \nabla g\|_2}, \tag{17}$$

where $\mathbf{D}_\sigma$ is the diagonal matrix of equivalent-Normal, standard deviations and $\nabla g$is the gradient of the limit state function. In our case g is defined in equation

(6) and the derivatives are with respect to the variables defined in equation (7). The vector of sensitivities (γ) is given by,

$$\gamma = \frac{\mathbf{D}_\lambda^{-1}\mathbf{T}\mathbf{D}_\sigma \nabla g}{\left\|\mathbf{D}_\lambda^{-1}\mathbf{T}\mathbf{D}_\sigma \nabla g\right\|_2}. \tag{18}$$

$\mathbf{T}$ is the matrix that transforms the vector of correlated, standard-Normal variables to uncorrelated, standard-Normal space. $\mathbf{D}_\lambda$ is the diagonal matrix of the square roots of the eigenvalues of the correlation matrix. The values of the elements of the correlation matrix, along with the means and standard deviations of the random variables defined in equation (16), will need to be determined from an appropriate set of experiments. This paper concentrates on the derivation of the components of the gradient vector of the limit state function.

3 Orthotropic and Monoclinic Materials

The stress tensor that applies at a general point within an orthotropic material is,

$$\begin{bmatrix}\bar{\sigma}_{11}\\ \bar{\sigma}_{22}\\ \bar{\sigma}_{33}\\ \bar{\sigma}_{23}\\ \bar{\sigma}_{13}\\ \bar{\sigma}_{12}\end{bmatrix} = \begin{bmatrix} C_{(k)}^{1111} & C_{(k)}^{1122} & C_{(k)}^{1133} & 0 & 0 & 0\\ C_{(k)}^{1122} & C_{(k)}^{2222} & C_{(k)}^{2233} & 0 & 0 & 0\\ C_{(k)}^{1133} & C_{(k)}^{2233} & C_{(k)}^{3333} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{(k)}^{2323} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{(k)}^{1313} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{(k)}^{1212}\end{bmatrix} \begin{bmatrix}\bar{w}_{11}\\ \bar{w}_{22}\\ \bar{w}_{33}\\ 2\bar{w}_{23}\\ 2\bar{w}_{13}\\ 2\bar{w}_{12}\end{bmatrix}, \tag{19}$$

$k = 1, 2, ..., N$

$$C_{(k)}^{1111} = \frac{E_2^{(k)} - \left(\nu_{23}^{(k)}\right)^2 E_3^{(k)}}{\left(E_2^{(k)}\right)^2 E_3^{(k)} \Delta_O^{(k)}},\ C_{(k)}^{2222} = \frac{E_1^{(k)} - E_3^{(k)}\left(\nu_{13}^{(k)}\right)^2}{\left(E_1^{(k)}\right)^2 E_3^{(k)} \Delta_O^{(k)}}, \tag{20}$$

$$C_{(k)}^{3333} = \frac{E_1^{(k)} - E_2^{(k)}\left(\nu_{12}^{(k)}\right)^2}{\left(E_1^{(k)}\right)^2 E_3^{(k)} \Delta_O^{(k)}},\ C_{(k)}^{1122} = \frac{\nu_{12}^{(k)} E_2^{(k)} + \nu_{13}^{(k)} \nu_{23}^{(k)} E_3^{(k)}}{E_1^{(k)} E_2^{(k)} E_3^{(k)} \Delta_O^{(k)}}, \tag{21}$$

$$C_{(k)}^{1133} = \frac{\nu_{12}^{(k)} \nu_{23}^{(k)} + \nu_{13}^{(k)}}{E_1^{(k)} E_2^{(k)} \Delta_O^{(k)}},\ C_{(k)}^{2233} = \frac{\nu_{23}^{(k)} E_1^{(k)} + \nu_{12}^{(k)} \nu_{13}^{(k)} E_2^{(k)}}{\left(E_1^{(k)}\right)^2 E_2^{(k)} \Delta_O^{(k)}}, \tag{22}$$

$$C_{(k)}^{2323} = G_{23}^{(k)},\ C_{(k)}^{1313} = G_{13}^{(k)},\ C_{(k)}^{1212} = G_{12}^{(k)}, \tag{23}$$

$$\Delta_O^{(k)} = \frac{1}{E_1^{(k)} E_2^{(k)} E_3^{(k)}} \left\| \begin{matrix} 1 & -E_2^{(k)} \nu_{12}^{(k)} / E_1^{(k)} & -E_3^{(k)} \nu_{13}^{(k)} / E_1^{(k)} \\ -\nu_{12}^{(k)} & 1 & -E_3^{(k)} \nu_{23}^{(k)} / E_2^{(k)} \\ -\nu_{13}^{(k)} & -\nu_{23}^{(k)} & 1 \end{matrix} \right\|. \tag{24}$$

The bars on top of the stress (LHS) and strain (RHS) tensor components in equation (10) denote that these are the orthotropic components that apply when the ξ-axis is aligned in the fibre direction and the λ-axis in the through thickness direction. The superscript "k" specified in equations (19)-(24) denotes that these stiffness variables apply to the material of the lamina with index number k. For our degenerate element formulation we assume that, $\sigma_{33} = 0$. Equation (19) can therefore be reduced to,

$$\begin{bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{23} \\ \bar{\sigma}_{13} \\ \bar{\sigma}_{12} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11}^{(k)} & \bar{Q}_{12}^{(k)} & 0 & 0 & 0 \\ \bar{Q}_{12}^{(k)} & \bar{Q}_{22}^{(k)} & 0 & 0 & 0 \\ 0 & 0 & \bar{Q}_{44}^{(k)} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{55}^{(k)} & 0 \\ 0 & 0 & 0 & 0 & \bar{Q}_{66}^{(k)} \end{bmatrix} \begin{bmatrix} \bar{w}_{11} \\ \bar{w}_{22} \\ 2\bar{w}_{23} \\ 2\bar{w}_{13} \\ 2\bar{w}_{12} \end{bmatrix}, \tag{25}$$

$$\bar{Q}_{11}^{(k)} = C_{(k)}^{1111} - \frac{\left[C_{(k)}^{1133}\right]^2}{C_{(k)}^{3333}}, \ \bar{Q}_{22}^{(k)} = C_{(k)}^{2222} - \frac{\left[C_{(k)}^{2233}\right]^2}{C_{(k)}^{3333}}, \tag{26}$$

$$\bar{Q}_{12}^{(k)} = C_{(k)}^{1122} - \frac{C_{(k)}^{1133} C_{(k)}^{2233}}{C_{(k)}^{3333}}, \ \bar{Q}_{66}^{(k)} = C_{(k)}^{1212} - \frac{\left[C_{(k)}^{3312}\right]^2}{C_{(k)}^{3333}}, \tag{27}$$

$$\bar{Q}_{44}^{(k)} = C_{(k)}^{2323} = G_{23}^{(k)}, \ \bar{Q}_{55}^{(k)} = C_{(k)}^{1313} = G_{13}^{(k)}. \tag{28}$$

In the Touratier kinematic model only the mid-surface layer, $k = k_0$, is defined with all the axes of the model equations aligned in the principal directions. The co-ordinate system used to derive the laminate constants of the Touratier kinematic model has its origin located on the mid-surface and one of the axes is aligned in the fibre direction (ξ). None of these axes are therefore necessarily constrained to be aligned in the in-plane principal directions of any of the other layers and so the stiffness components that apply to these layers will be monoclinic, i.e.

$$\mathbf{Q}_M^{(k)} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & 0 & 0 & Q_{16}^{(k)} \\ Q_{12}^{(k)} & Q_{22}^{(k)} & 0 & 0 & Q_{26}^{(k)} \\ 0 & 0 & Q_{44}^{(k)} & Q_{45}^{(k)} & 0 \\ 0 & 0 & Q_{45}^{(k)} & Q_{55}^{(k)} & 0 \\ Q_{16}^{(k)} & Q_{26}^{(k)} & 0 & 0 & Q_{66}^{(k)} \end{bmatrix}, \tag{29}$$

where,

$$\mathbf{Q}_M^{(k)} = \mathbf{T}_{OM}^{(k)} \mathbf{Q}_O^{(k)} \left(\mathbf{T}_{OM}^{(k)} \right)^T , \tag{30}$$

$$\mathbf{Q}_O^{(k)} = \begin{bmatrix} \bar{Q}_{11}^{(k)} & \bar{Q}_{12}^{(k)} & 0 & 0 & 0 \\ \bar{Q}_{12}^{(k)} & \bar{Q}_{22}^{(k)} & 0 & 0 & 0 \\ 0 & 0 & \bar{Q}_{44}^{(k)} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{55}^{(k)} & 0 \\ 0 & 0 & 0 & 0 & \bar{Q}_{66}^{(k)} \end{bmatrix} ,$$

$$\mathbf{T}_{OM}^{(k)} = \begin{bmatrix} T_{11}^{(k)} & T_{12}^{(k)} & 0 & 0 & T_{16}^{(k)} \\ T_{21}^{(k)} & T_{22}^{(k)} & 0 & 0 & T_{26}^{(k)} \\ 0 & 0 & T_{44}^{(k)} & T_{45}^{(k)} & 0 \\ 0 & 0 & T_{54}^{(k)} & T_{55}^{(k)} & 0 \\ T_{61}^{(k)} & T_{62}^{(k)} & 0 & 0 & T_{66}^{(k)} \end{bmatrix} .$$

The matrix $\mathbf{T}_{OM}^{(k)}$ is the matrix that transforms the components of the degenerate stiffness tensor,$\mathbf{Q}_O^{(k)}$, from orthotropic to monoclinic components as specified in equation (30). The full expansions of the components of $\mathbf{T}_{OM}^{(k)}$ are documented by Arciniega and Reddy (2007), they are combinations of trigonometric functions of the angle necessary ($\theta_f^{(k)}$ say) to rotate the mid-surface fibre axis in order to be aligned with the fibres in the layer with index number k – see Fig. 2. The mid-surface base vectors are,

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial \xi}, \; \mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial \eta}, \tag{31}$$

where $\mathbf{e}_1$ is the tangent vector parallel to the mid-surface fibres, see Beakou and Touratier (1993). If $\mathbf{e}_1^{(k)}$ and $\mathbf{e}_2^{(k)}$ are the respective basis vectors that are parallel and perpendicular to the fibres of the lamina with index number "k" then the components of the transformation matrix $\mathbf{T}_{QM}^{(k)}$ from equation (30) are,

$$T_{11}^{(k)} = (c_{11})^2 , \; T_{12}^{(k)} = (c_{12})^2 , \; T_{16} = 2c_{11}c_{12}, \tag{32}$$

$$T_{21}^{(k)} = (c_{21})^2 , \; T_{22}^{(k)} = (c_{22})^2 , \; T_{26}^{(k)} = 2c_{21}c_{22}, \tag{33}$$

$$T_{44}^{(k)} = c_{22},\ T_{45}^{(k)} = c_{21},\ T_{54}^{(k)} = c_{12},\ T_{55}^{(k)} = c_{11}, \tag{34}$$

$$T_{61}^{(k)} = c_{11}c_{21},\ T_{62}^{(k)} = c_{12}c_{22},\ T_{66}^{(k)} = c_{11}c_{22} + c_{21}c_{12}, \tag{35}$$

where,

$$c_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1^{(k)} = \frac{\cos\left(\theta_f^{(k)}\right)}{\sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1}},\ c_{21} = \mathbf{e}_2 \cdot \mathbf{e}_1^{(k)} = \frac{\sin\left(\theta_f^{(k)}\right)}{\sqrt{\mathbf{e}_2 \cdot \mathbf{e}_2}},$$

$$c_{12} = \mathbf{e}_1 \cdot \mathbf{e}_2^{(k)} = -\frac{\sin\left(\theta_f^{(k)}\right)}{\sqrt{\mathbf{e}_1 \cdot \mathbf{e}_1}},\ c_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2^{(k)} = \frac{\cos\left(\theta_f^{(k)}\right)}{\sqrt{\mathbf{e}_2 \cdot \mathbf{e}_2}}.$$

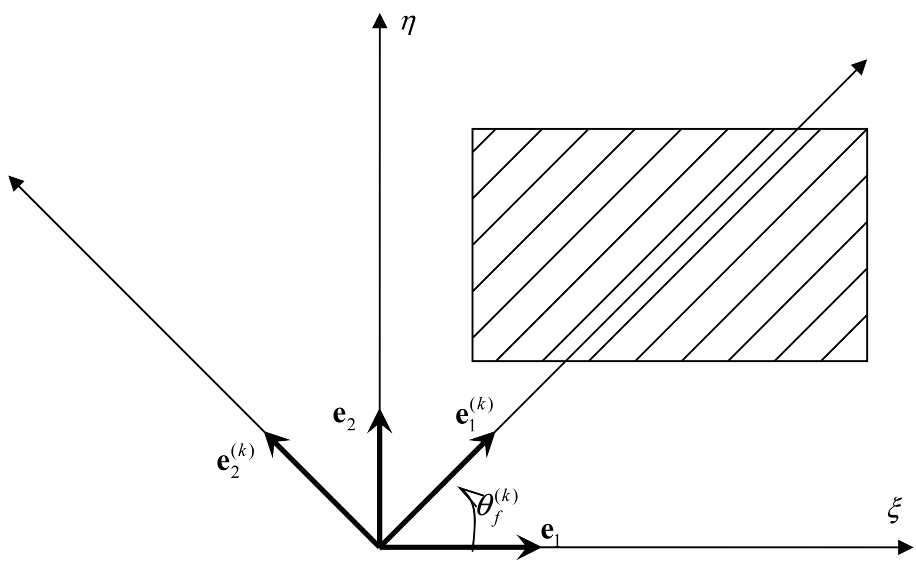

Figure 2: Fibre alignment of non, mid-surface layers.

The stress at a general point within the laminated shell is given by,

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & 0 & 0 & Q_{16}^{(k)}\\ Q_{12}^{(k)} & Q_{22}^{(k)} & 0 & 0 & Q_{26}^{(k)}\\ 0 & 0 & Q_{44}^{(k)} & Q_{45}^{(k)} & 0\\ 0 & 0 & Q_{45}^{(k)} & Q_{55}^{(k)} & 0\\ Q_{16}^{(k)} & Q_{26}^{(k)} & 0 & 0 & Q_{66}^{(k)} \end{bmatrix} \begin{bmatrix} w_{11}\\ w_{22}\\ w_{23}\\ w_{13}\\ w_{12}\end{bmatrix},\ k = 1,2,...N. \tag{36}$$

The expansions of the strain tensor components, w_{11}, w_{22}, w_{23}, w_{13} and w_{12} are (Shaw *et al* (2010)),

$$w_{aa} = \frac{1}{2}\left(\frac{\partial \bar{\mathbf{r}}_0}{\partial x^a}\cdot\frac{\partial \bar{\mathbf{r}}_0}{\partial x^a} - \frac{\partial \mathbf{r}_0}{\partial x^a}\cdot\frac{\partial \mathbf{r}_0}{\partial x^a}\right) + \lambda\left(\frac{\partial \bar{\mathbf{r}}_0}{\partial x^a}\cdot\frac{\partial \bar{\mathbf{n}}_0}{\partial x^a} - \frac{\partial \mathbf{r}_0}{\partial x^a}\cdot\frac{\partial \mathbf{n}_0}{\partial x^a}\right) + \frac{\lambda^2}{2}\left(\frac{\partial \bar{\mathbf{n}}_0}{\partial x^a}\cdot\frac{\partial \bar{\mathbf{n}}_0}{\partial x^a} - \frac{\partial \mathbf{n}_0}{\partial x^a}\cdot\frac{\partial \mathbf{n}_0}{\partial x^a}\right) + \frac{\partial \bar{\mathbf{r}}_0}{\partial x^a}\cdot\frac{\partial \mathbf{u}_{ts}}{\partial x^a} + \lambda\frac{\partial \bar{\mathbf{n}}_0}{\partial x^a}\cdot\frac{\partial \mathbf{u}_{ts}}{\partial x^a} + \frac{1}{2}\frac{\partial \mathbf{u}_{ts}}{\partial x^a}\cdot\frac{\partial \mathbf{u}_{ts}}{\partial x^a},\ a = 1,2,\ \text{where there is no summation on } a, \tag{37}$$

$$w_{12} = \frac{\partial \bar{\mathbf{r}}_0}{\partial \xi}\cdot\frac{\partial \bar{\mathbf{r}}_0}{\partial \eta} - \frac{\partial \mathbf{r}_0}{\partial \xi}\cdot\frac{\partial \mathbf{r}_0}{\partial \eta} + \lambda\left(\frac{\partial \bar{\mathbf{r}}_0}{\partial \xi}\cdot\frac{\partial \bar{\mathbf{n}}_0}{\partial \eta} + \frac{\partial \bar{\mathbf{r}}_0}{\partial \eta}\cdot\frac{\partial \bar{\mathbf{n}}_0}{\partial \xi} - \frac{\partial \mathbf{r}_0}{\partial \xi}\cdot\frac{\partial \mathbf{n}_0}{\partial \eta} - \frac{\partial \mathbf{r}_0}{\partial \eta}\cdot\frac{\partial \mathbf{n}_0}{\partial \xi}\right) + \lambda^2\left(\frac{\partial \bar{\mathbf{n}}_0}{\partial \xi}\cdot\frac{\partial \bar{\mathbf{n}}_0}{\partial \eta} - \frac{\partial \mathbf{n}_0}{\partial \xi}\cdot\frac{\partial \mathbf{n}_0}{\partial \eta}\right) + \frac{\partial \bar{\mathbf{r}}_0}{\partial \xi}\cdot\frac{\partial \mathbf{u}_{ts}}{\partial \eta} + \frac{\partial \bar{\mathbf{r}}_0}{\partial \eta}\cdot\frac{\partial \mathbf{u}_{ts}}{\partial \xi} + \lambda\left(\frac{\partial \bar{\mathbf{n}}_0}{\partial \xi}\cdot\frac{\partial \mathbf{u}_{ts}}{\partial \eta} + \frac{\partial \bar{\mathbf{n}}_0}{\partial \eta}\cdot\frac{\partial \mathbf{u}_{ts}}{\partial \xi}\right) + \frac{\partial \mathbf{u}_{ts}}{\partial \xi}\cdot\frac{\partial \mathbf{u}_{ts}}{\partial \eta}, \tag{38}$$

$$w_{a3} = \lambda\left(\bar{\mathbf{n}}_0\cdot\frac{\partial \bar{\mathbf{n}}_0}{\partial x^a} - \mathbf{n}_0\cdot\frac{\partial \mathbf{n}_0}{\partial x^a}\right) + \bar{\mathbf{n}}_0\cdot\frac{\partial \mathbf{u}_{ts}}{\partial x^a} + \frac{\partial \bar{\mathbf{r}}_0}{\partial x^a}\cdot\frac{\partial \mathbf{u}_{ts}}{\partial \lambda} + \lambda\frac{\partial \bar{\mathbf{n}}_0}{\partial x^a}\cdot\frac{\partial \mathbf{u}_{ts}}{\partial \lambda} + \frac{\partial \mathbf{u}_{ts}}{\partial x^a}\cdot\frac{\partial \mathbf{u}_{ts}}{\partial \lambda},\ a = 1,2, \tag{39}$$

Referring to the random variables specified in equation (16), the transverse shear component of the displacement vector ($\mathbf{u}_{ts}$) is dependent on the random transverse shear components of stiffness and the lamina thicknesses as follows,

$$\mathbf{u}_{ts} = \mathbf{u}_{ts}\left(G_{23}^{(1)}, G_{13}^{(1)}, h^{(1)}, G_{23}^{(2)}, G_{13}^{(2)}, h^{(2)}, \ldots, G_{23}^{(N)}, G_{13}^{(N)}, h^{(N)}\right). \tag{40}$$

The mid-surface position vectors before and after deformation, $\mathbf{r}_0$ and $\bar{\mathbf{r}}_0$ respectively, along with the initial and the deformed normal vectors to the mid-surface, $\mathbf{n}_0$ and $\bar{\mathbf{n}}_0$ respectively, are all independent of the random variables of equation (16). It follows from equations (25)-(36) that,

$$Q_{11}^{(k)} = Q_{11}^{(k)}\left(E_1^{(k)}, E_2^{(k)}, E_3^{(k)}, \nu_{23}^{(k)}, \nu_{13}^{(k)}, \nu_{12}^{(k)}, G_{12}^{(k)}\right), \tag{41}$$

$$Q_{22}^{(k)} = Q_{22}^{(k)}\left(E_1^{(k)}, E_2^{(k)}, E_3^{(k)}, \nu_{23}^{(k)}, \nu_{13}^{(k)}, \nu_{12}^{(k)}, G_{12}^{(k)}\right), \tag{42}$$

$$Q_{12}^{(k)} = Q_{12}^{(k)}\left(E_1^{(k)}, E_2^{(k)}, E_3^{(k)}, \nu_{23}^{(k)}, \nu_{13}^{(k)}, \nu_{12}^{(k)}, G_{12}^{(k)}\right), \tag{43}$$

$$Q_{16}^{(k)} = Q_{16}^{(k)}\left(E_1^{(k)}, E_2^{(k)}, E_3^{(k)}, \nu_{23}^{(k)}, \nu_{13}^{(k)}, \nu_{12}^{(k)}, G_{12}^{(k)}\right), \tag{44}$$

$$Q_{26}^{(k)} = Q_{26}^{(k)}\left(E_1^{(k)}, E_2^{(k)}, E_3^{(k)}, \nu_{23}^{(k)}, \nu_{13}^{(k)}, \nu_{12}^{(k)}, G_{12}^{(k)}\right), \tag{45}$$

$$Q_{66}^{(k)} = Q_{66}^{(k)}\left(E_1^{(k)}, E_2^{(k)}, E_3^{(k)}, \nu_{23}^{(k)}, \nu_{13}^{(k)}, \nu_{12}^{(k)}, G_{12}^{(k)}\right), \tag{46}$$

$$Q_{44}^{(k)} = Q_{44}^{(k)}\left(G_{23}^{(k)}, G_{13}^{(k)}\right),\ Q_{55}^{(k)} = Q_{55}^{(k)}\left(G_{23}^{(k)}, G_{13}^{(k)}\right),\ Q_{45}^{(k)} = Q_{45}^{(k)}\left(G_{23}^{(k)}, G_{13}^{(k)}\right), \tag{47}$$

4 Random Strength Variables

The remainder of the random variables listed in equation (16) are strength components ($X_T^{(k)}$, $Y_T^{(k)}$, $X_C^{(k)}$, $Y_C^{(k)}$, $S^{(k)}$, $R^{(k)}$, $T^{(k)}$, $\sigma_{b12}^{(k)}$).
As for the stiffness variables, the superscript "k" specified in the above variables denotes that these strength values apply to the material of the lamina with index number k where, k = 1, 2, ..., N. The limit state function of equation (15) is derived from a Tsai-Wu criterion that applies to a single material. Equation (15) needs to rewritten to take account of the fact that the shell is laminated. The material properties will vary from layer to layer and so we need to ensure that the correct strength components, i.e. those of the material in which the limit state function evaluation point is located, are substituted into the Tsai-Wu failure criterion. Therefore,

$$\begin{aligned} g(\mathbf{X}) = 1 - F_1^{(k)}\sigma_{11} - F_2^{(k)}\sigma_{22} - F_{11}^{(k)}\sigma_{11}^2 - F_{22}^{(k)}\sigma_{22}^2 - \\ F_{44}^{(k)}\sigma_{23}^2 - F_{55}^{(k)}\sigma_{13}^2 - F_{66}^{(k)}\sigma_{12}^2 - 2F_{12}^{(k)}\sigma_{11}\sigma_{22}, \end{aligned} \tag{48}$$

where "k" is the index number of the lamina in which the thickness coordinate (λ) of the limit state function evaluation point is located and,

$$\begin{aligned} & F_1^{(k)} = \frac{1}{X_T^{(k)}} - \frac{1}{X_C^{(k)}},\ F_2^{(k)} = \frac{1}{Y_T^{(k)}} - \frac{1}{Y_C^{(k)}},\ F_3^{(k)} = \frac{1}{Z_T^{(k)}} - \frac{1}{Z_C^{(k)}}, \\ & F_{11}^{(k)} = \frac{1}{X_T^{(k)}X_C^{(k)}},\ F_{22}^{(k)} = \frac{1}{Y_T^{(k)}Y_C^{(k)}},\ F_{33}^{(k)} = \frac{1}{Z_T^{(k)}Z_C^{(k)}}, \\ & F_{44}^{(k)} = \frac{1}{\left(T^{(k)}\right)^2},\ F_{55}^{(k)} = \frac{1}{\left(R^{(k)}\right)^2},\ F_{66}^{(k)} = \frac{1}{\left(S^{(k)}\right)^2}, \\ & F_{12}^{(k)} = \frac{1}{2\left(\sigma_{b12}^{(k)}\right)^2}\left\{1 - \sigma_{b12}^{(k)}\left(F_1^{(k)} + F_2^{(k)}\right) - \left(\sigma_{b12}^{(k)}\right)^2\left(F_{11}^{(k)} + F_{22}^{(k)}\right)\right\}. \end{aligned} \tag{49}$$

It follows that the derivatives with respect to these random strength variables are,

$$\frac{\partial g}{\partial X_T^{(k)}} = \frac{\sigma_{11}\left\{\sigma_{b12}^{(k)}\left(\sigma_{11} + X_C^{(k)}\right) - \sigma_{22}\left(\sigma_{b12}^{(k)} + X_C^{(k)}\right)\right\}}{\sigma_{b12}^{(k)}\left(X_T^{(k)}\right)^2 X_C^{(k)}}, \tag{50}$$

$$\frac{\partial g}{\partial Y_T^{(k)}} = \frac{\sigma_{22}\left\{\sigma_{b12}^{(k)}\left(\sigma_{22}+Y_C^{(k)}\right)-\sigma_{11}\left(\sigma_{b12}^{(k)}+Y_C^{(k)}\right)\right\}}{\left(Y_T^{(k)}\right)^2 Y_C^{(k)}\sigma_{b12}^{(k)}}, \tag{51}$$

$$\frac{\partial g}{\partial X_C^{(k)}} = \frac{\sigma_{11}\left\{\sigma_{22}\left(X_T^{(k)}-\sigma_{b12}^{(k)}\right)+\sigma_{b12}^{(k)}\left(\sigma_{11}-X_T^{(k)}\right)\right\}}{\left(X_C^{(k)}\right)^2 X_T^{(k)}\sigma_{b12}^{(k)}}, \tag{52}$$

$$\frac{\partial g}{\partial Y_C^{(k)}} = \frac{\sigma_{22}\left\{\sigma_{11}\left(Y_T^{(k)}-\sigma_{b12}^{(k)}\right)+\sigma_{b12}^{(k)}\left(\sigma_{22}-Y_T^{(k)}\right)\right\}}{\left(Y_C^{(k)}\right)^2 Y_T^{(k)}\sigma_{b12}^{(k)}}, \tag{53}$$

$$\frac{\partial g}{\partial S^{(k)}} = \frac{2\sigma_{12}^2}{\left(S^{(k)}\right)^3},\ \frac{\partial g}{\partial R^{(k)}} = \frac{2\sigma_{13}^2}{\left(S^{(k)}\right)^3},\ \frac{\partial g}{\partial R^{(k)}} = \frac{2\sigma_{23}^2}{\left(T^{(k)}\right)^3}. \tag{54}$$

$$\begin{aligned}\frac{\partial g}{\partial \sigma_{b12}^{(k)}} &= \left\{\left[X_T^{(k)}X_C^{(k)}\left(Y_T^{(k)}-Y_C^{(k)}\right)+Y_T^{(k)}Y_C^{(k)}\left(X_T^{(k)}-X_C^{(k)}\right)\right]\sigma_{b12}^{(k)}+\right.\\ &2X_T^{(k)}X_C^{(k)}Y_T^{(k)}Y_C^{(k)}\Big\}\,\sigma_{11}\sigma_{22}\Big/\left\{X_T^{(k)}X_C^{(k)}Y_T^{(k)}Y_C^{(k)}\left(\sigma_{b12}^{(k)}\right)^3\right\},\end{aligned} \tag{55}$$

5 Top-Level Stiffness and Thickness Derivatives

The Tsai-Wu coefficients $F_1^{(k)}$, $F_2^{(k)}$, $F_{11}^{(k)}$, $F_{22}^{(k)}$, $F_{44}^{(k)}$, $F_{55}^{(k)}$, $F_{66}^{(k)}$ and $F_{12}^{(k)}$ are independent of the stiffness components and the laminate thickness. Differentiating the degenerate limit state function of equation (48) with respect to the stiffness and geometric random variables we get,

$$\begin{aligned}\frac{\partial g}{\partial Z_l^{(j)}} &= -\left(F_1^{(k)}+2F_{11}^{(k)}\sigma_{11}+2F_{12}^{(k)}\sigma_{22}\right)\frac{\partial \sigma_{11}}{\partial Z_l^{(j)}}\\ &-\left(F_2^{(k)}+2F_{22}^{(k)}\sigma_{22}+2F_{12}^{(k)}\sigma_{11}\right)\frac{\partial \sigma_{22}}{\partial Z_l^{(j)}}-2F_{66}\sigma_{12}\frac{\partial \sigma_{12}}{\partial Z_l^{(j)}},\\ &j=1,2,...,N,\ k=1,2,...N,\ l=1,2,...,7,\end{aligned} \tag{56}$$

where

$$Z_1^{(k)}=E_1^{(k)},\ Z_2^{(k)}=E_2^{(k)},\ Z_3^{(k)}=E_3^{(k)},\ Z_4^{(k)}=\nu_{23}^{(k)},$$
$$Z_5^{(k)}=\nu_{13}^{(k)},\ Z_6^{(k)}=\nu_{12}^{(k)},Z_7^{(k)}=G_{12}^{(k)},$$

$$\frac{\partial g}{\partial G_{a3}^{(j)}}=-\left(F_1^{(k)}+2F_{11}^{(k)}\sigma_{11}+2F_{12}^{(k)}\sigma_{22}\right)\frac{\partial\sigma_{11}}{\partial G_{a3}^{(j)}}$$
$$-\left(F_2^{(k)}+2F_{22}^{(k)}\sigma_{22}+2F_{12}^{(k)}\sigma_{11}\right)\frac{\partial\sigma_{22}}{\partial G_{a3}^{(j)}}$$
$$-2F_{44}^{(k)}\sigma_{23}\frac{\partial\sigma_{23}}{\partial G_{a3}^{(j)}}-2F_{55}^{(k)}\sigma_{13}\frac{\partial\sigma_{13}}{\partial G_{a3}^{(j)}}-2F_{66}^{(k)}\sigma_{12}\frac{\partial\sigma_{12}}{\partial G_{a3}^{(j)}},\tag{57}$$
$$j=1,2,...,N,\ k=1,2,...,N,\ \mathrm{a}=1,2,$$

$$\frac{\partial g}{\partial h^{(j)}}=-\left(F_1^{(k)}+2F_{11}^{(k)}\sigma_{11}+2F_{12}^{(k)}\sigma_{22}\right)\frac{\partial\sigma_{11}}{\partial h^{(j)}}$$
$$-\left(F_2^{(k)}+2F_{22}^{(k)}\sigma_{22}+2F_{12}^{(k)}\sigma_{11}\right)\frac{\partial\sigma_{22}}{\partial h^{(j)}}$$
$$-2F_{44}^{(k)}\sigma_{23}\frac{\partial\sigma_{23}}{\partial h^{(j)}}-2F_{55}^{(k)}\sigma_{13}\frac{\partial\sigma_{13}}{\partial h^{(j)}}-2F_{66}^{(k)}\sigma_{12}\frac{\partial\sigma_{12}}{\partial h^{(j)}},\tag{58}$$
$$j=1,2,...,N,\ k=1,2,...,N,$$

The stress tensor derivatives in equations (56)-(58) can be expanded using equations (36)-(47) to give,

$$\frac{\partial\sigma_{11}}{\partial Z_l^{(j)}}=\begin{cases}w_{11}\frac{\partial Q_{11}^{(k)}}{\partial Z_l^{(k)}}+w_{22}\frac{\partial Q_{12}^{(k)}}{\partial Z_l^{(k)}}+2w_{12}\frac{\partial Q_{16}^{(k)}}{\partial Z_l^{(k)}}, & j=k,\\ 0, & j\neq k,\end{cases}\tag{59}$$

where j, k = 1, 2, . . . , N, and l = 1, 2, . . . , 7, as in equation (56).

$$\frac{\partial\sigma_{11}}{\partial G_{a3}^{(j)}}=Q_{11}^{(k)}\frac{\partial w_{11}}{\partial G_{a3}^{(j)}}+Q_{12}^{(k)}\frac{\partial w_{22}}{\partial G_{a3}^{(j)}}+2Q_{16}^{(k)}\frac{\partial w_{12}}{\partial G_{a3}^{(j)}},\ j,k=1,2,...,N,\ a=1,2,\tag{60}$$

$$\frac{\partial\sigma_{11}}{\partial h^{(j)}}=Q_{11}^{(k)}\frac{\partial w_{11}}{\partial h^{(j)}}+Q_{12}^{(k)}\frac{\partial w_{22}}{\partial h^{(j)}}+2Q_{16}^{(k)}\frac{\partial w_{12}}{\partial h^{(j)}},\ j,k=1,2,...,N,\tag{61}$$

$$\frac{\partial\sigma_{23}}{\partial G_{a3}^{(j)}}=\begin{cases}2Q_{44}^{(k)}\frac{\partial w_{23}}{\partial G_{a3}^{(k)}}+2w_{23}\frac{\partial Q_{44}^{(k)}}{\partial G_{a3}^{(k)}}+2Q_{45}^{(k)}\frac{\partial w_{13}}{\partial G_{a3}^{(k)}}+2w_{13}\frac{\partial Q_{45}^{(k)}}{\partial G_{a3}^{(k)}}, & j=k,\\ 2Q_{44}^{(k)}\frac{\partial w_{23}}{\partial G_{a3}^{(j)}}+2Q_{45}^{(k)}\frac{\partial w_{13}}{\partial G_{a3}^{(j)}}, & j\neq k,\end{cases}\tag{62}$$

$$\frac{\partial\sigma_{23}}{\partial h^{(j)}}=2Q_{44}^{(k)}\frac{\partial w_{23}}{\partial h^{(j)}}+2Q_{45}^{(k)}\frac{\partial w_{13}}{\partial h^{(j)}},\ \frac{\partial\sigma_{23}}{\partial Z_l^{(k)}}=0,\ k,j=1,2,...,N,\ \forall l,\tag{63}$$

The equivalent expressions for σ_{12}, σ_{22}, and $\sigma_{13}$are given in Appendix A, Equations (A1) – (A8).

6 Stiffness Derivatives

By differentiating both sides of equation (30) the following relationships between the monoclinic and orthotropic coordinate forms the derivatives of the normal and in-plane stiffness terms, quoted in equations (59) are obtained as,

$$\frac{\partial Q_{11}^{(k)}}{\partial Z_l^{(k)}} = \left(T_{11}^{(k)}\right)^2 \frac{\partial \bar{Q}_{11}^{(k)}}{\partial Z_l^{(k)}} + 2T_{11}^{(k)}T_{12}^{(k)} \frac{\partial \bar{Q}_{12}^{(k)}}{\partial Z_l^{(k)}} + \left(T_{12}^{(k)}\right)^2 \frac{\partial \bar{Q}_{22}^{(k)}}{\partial Z_l^{(k)}},\ l = 1,2,...,6, \tag{64}$$

$Z_1^{(k)} = E_1^{(k)},\ Z_2^{(k)} = E_2^{(k)},\ Z_3^{(k)} = E_3^{(k)},$
$Z_4^{(k)} = \nu_{23}^{(k)},\ Z_5^{(k)} = \nu_{13}^{(k)},\ Z_6^{(k)} = \nu_{12}^{(k)},$

$$\frac{\partial Q_{11}^{(k)}}{\partial G_{12}^{(k)}} = \left(T_{16}^{(k)}\right)^2,\ \frac{\partial Q_{22}^{(k)}}{\partial G_{12}^{(k)}} = \left(T_{26}^{(k)}\right)^2,\ \frac{\partial Q_{12}^{(k)}}{\partial G_{12}^{(k)}} = T_{16}^{(k)}T_{26}^{(k)}, \tag{65}$$

$$\frac{\partial Q_{16}^{(k)}}{\partial G_{12}^{(k)}} = T_{16}^{(k)}T_{66}^{(k)},\ \frac{\partial Q_{26}^{(k)}}{\partial G_{12}^{(k)}} = T_{26}^{(k)}T_{66}^{(k)},\ \frac{\partial Q_{12}^{(k)}}{\partial G_{12}^{(k)}} = \left(T_{66}^{(k)}\right)^2. \tag{66}$$

Terms associated with Q_{12}, Q_{16}, Q_{22}, Q_{26}, and Q_{66} are provided in Appendix A (Equations (A9) – (A13)).

The degenerate element forms of the orthotropic stiffness components defined in equation (25) are related to the 3D elasticity tensor components through equations (20)-(24). The transverse shear component derivatives are obtained as, for example,

$$\frac{\partial Q_{44}^{(k)}}{\partial G_{23}^{(k)}} = \left(T_{44}^{(k)}\right)^2,\ \frac{\partial Q_{44}^{(k)}}{\partial G_{13}^{(k)}} = \left(T_{45}^{(k)}\right)^2, \tag{67}$$

It also follows that,

$$\frac{\partial \bar{Q}_{11}^{(k)}}{\partial Z_l^{(k)}} = \frac{\partial C_{(k)}^{1111}}{\partial Z_l^{(k)}} - \frac{2C_{(k)}^{1133}}{C_{(k)}^{3333}} \frac{\partial C_{(k)}^{1133}}{\partial Z_l^{(k)}} + \left(\frac{C_{(k)}^{1133}}{C_{(k)}^{3333}}\right)^2 \frac{\partial C_{(k)}^{3333}}{\partial Z_l^{(k)}},\ l = 1,2,...,6, \tag{68}$$

where,

$Z_1^{(k)} = E_1^{(k)},\ Z_2^{(k)} = E_2^{(k)},\ Z_3^{(k)} = E_3^{(k)},$
$Z_4^{(k)} = \nu_{23}^{(k)},\ Z_5^{(k)} = \nu_{13}^{(k)},\ Z_6^{(k)} = \nu_{12}^{(k)}.$

The derivatives of Q_{55}, Q_{45} and Q_{12},Q_{22} given in Appendix A, Equations (A14), (A15) and (A16), (A17) respectively.

The completed expansions of the derivatives,

$$\frac{\partial C^{1111}_{(k)}}{\partial Z^{(k)}_l},\ \frac{\partial C^{2222}_{(k)}}{\partial Z^{(k)}_l},\ \frac{\partial C^{3333}_{(k)}}{\partial Z^{(k)}_l},\ \frac{\partial C^{1122}_{(k)}}{\partial Z^{(k)}_l},\ \frac{\partial C^{1133}_{(k)}}{\partial Z^{(k)}_l},\ \frac{\partial C^{2233}_{(k)}}{\partial Z^{(k)}_l},$$

are readily obtained using the definitions provided in equations (20)-(23).

7 Strain tensor component derivatives

The derivatives of the strain tensor components w_{11}, w_{22} and w_{12}, that appear in Equations (60)-(63) with (A1)-(A8) can be determined from equations (37)-(39) as,

$$\frac{\partial w_{aa}}{\partial Y^{(j)}_l} = \left(\frac{\partial \bar{\mathbf{r}}_0}{\partial x^a} + \lambda \frac{\partial \bar{\mathbf{n}}_0}{\partial x^a} + \frac{\partial \mathbf{u}_{ts}}{\partial x^a}\right) \cdot \frac{\partial^2 \mathbf{u}_{ts}}{\partial Y^{(j)}_l \partial x^a},$$
$$a = 1,2,\ x^1 = \xi,\ x^2 = \eta,\ l = 1,2,3,\ j = 1,2,...,N, \tag{69}$$

$$\frac{\partial w_{12}}{\partial Y^{(j)}_l} = \left(\frac{\partial \bar{\mathbf{r}}_0}{\partial \eta} + \lambda \frac{\partial \bar{\mathbf{n}}_0}{\partial \eta} + \frac{\partial \mathbf{u}_{ts}}{\partial \eta}\right) \cdot \frac{\partial^2 \mathbf{u}_{ts}}{\partial Y^{(j)}_l \partial \eta} +$$
$$\left(\frac{\partial \bar{\mathbf{r}}_0}{\partial \xi} + \lambda \frac{\partial \bar{\mathbf{n}}_0}{\partial \xi} + \frac{\partial \mathbf{u}_{ts}}{\partial \xi}\right) \cdot \frac{\partial^2 \mathbf{u}_{ts}}{\partial Y^{(j)}_l \partial \xi}, \tag{70}$$
$$l = 1,2,3,\ j = 1,2,...,N,$$

$$\frac{\partial w_{a3}}{\partial Y^{(j)}_l} = \left(\bar{\mathbf{n}}_0 + \frac{\partial \mathbf{u}_{ts}}{\partial \lambda}\right) \cdot \frac{\partial^2 \mathbf{u}_{ts}}{\partial Y^{(j)}_l \partial x^a} +$$
$$\left(\frac{\partial \bar{\mathbf{r}}_0}{\partial x^a} + \lambda \frac{\partial \bar{\mathbf{n}}_0}{\partial x^a} + \frac{\partial \mathbf{u}_{ts}}{\partial x^a}\right) \cdot \frac{\partial^2 \mathbf{u}_{ts}}{\partial Y^{(j)}_l \partial \lambda}, \tag{71}$$
$$a = 1,2,\ x^1 = \xi,\ x^2 = \eta,\ l = 1,2,3,\ j = 1,2,...,N,$$

where

$$Y^{(j)}_1 = G^{(j)}_{23},\ Y^{(j)}_2 = G^{(j)}_{13},\ Y^{(j)}_3 = h^{(j)}. \tag{72}$$

In the remaining equations the indices a, l, and k take the same values as specified in equation (69) unless otherwise stated. The transverse shear component of the displacement vector ($\mathbf{u}_{ts}$) can be expressed as

$$\mathbf{u}_{ts} = u^{(\xi)}\hat{\mathbf{e}}_\xi + u^{(\eta)}\hat{\mathbf{e}}_\eta. \tag{73}$$

It therefore follows that,

$$\frac{\partial^2 \mathbf{u}_{ts}}{\partial Y_l^{(j)} \partial x^a} = \frac{\partial u^{(\xi)}}{\partial Y_l^{(j)}} \frac{\partial \hat{\mathbf{e}}_\xi}{\partial x^a} + \frac{\partial u^{(\eta)}}{\partial Y_l^{(j)}} \frac{\partial \hat{\mathbf{e}}_\eta}{\partial x^a} + \frac{\partial^2 u^{(\xi)}}{\partial Y_l^{(j)} \partial x^a} \hat{\mathbf{e}}_\xi + \frac{\partial^2 u^{(\eta)}}{\partial Y_l^{(j)} \partial x^a} \hat{\mathbf{e}}_\eta, \tag{74}$$

$$\frac{\partial^2 \mathbf{u}_{ts}}{\partial Y_l^{(j)} \partial \lambda} = \frac{\partial^2 u^{(\xi)}}{\partial Y_l^{(j)} \partial \lambda} \hat{\mathbf{e}}_\xi + \frac{\partial^2 u^{(\eta)}}{\partial Y_l^{(j)} \partial \lambda} \hat{\mathbf{e}}_\eta. \tag{75}$$

Given that the angles θ_ξ and θ_η are the rotations about the ξ and η axes necessary to align the initial mid-surface normal with the corresponding normal to the deformed surface, then let,

$$s_\xi = \sin\left(\theta_\xi\right),\; c_\xi = \cos\left(\theta_\xi\right),\; s_\eta = \sin\left(\theta_\eta\right),\; c_\eta = \cos\left(\theta_\eta\right). \tag{76}$$

Using equations (14)-(18) of Shaw *et. al.* (2010),

$$\begin{aligned}\frac{\partial \hat{\mathbf{e}}_\xi}{\partial x^a} = &\frac{\partial c_\eta}{\partial x^a} \mathbf{e}_\xi + \left(s_\xi \frac{\partial s_\eta}{\partial x^a} + s_\eta \frac{\partial s_\xi}{\partial x^a} \right) \mathbf{e}_\eta - \left(c_\xi \frac{\partial s_\eta}{\partial x^a} + s_\eta \frac{\partial c_\xi}{\partial x^a} \right) \mathbf{n}_0 + \\ &c_\eta \frac{\partial \mathbf{e}_\xi}{\partial x^a} + s_\xi s_\eta \frac{\partial \mathbf{e}_\eta}{\partial x^a} - c_\xi s_\eta \frac{\partial \mathbf{n}_0}{\partial x^a},\end{aligned} \tag{77}$$

$$\frac{\partial \hat{\mathbf{e}}_\eta}{\partial x^a} = \frac{\partial c_\xi}{\partial x^a} \mathbf{e}_\eta + \frac{\partial s_\xi}{\partial x^a} \mathbf{n}_0 + c_\xi \frac{\partial \mathbf{e}_\eta}{\partial x^a} + s_\xi \frac{\partial \mathbf{n}_0}{\partial x^a}, \tag{78}$$

$$\mathbf{e}_\xi = \frac{1}{|\mathbf{e}_1|} \mathbf{e}_1,\; \mathbf{e}_\eta = \frac{1}{|\mathbf{e}_2|} \mathbf{e}_2, \text{ see equation (22)}, \tag{79}$$

$$\begin{aligned}\frac{\partial s_\xi}{\partial x^a} = &\left\{ \bar{\mathbf{n}}_0 \cdot \left(\left[1 - \left(\bar{\mathbf{n}}_0 \cdot \mathbf{e}_\xi \right)^2 \right] \frac{\partial \mathbf{e}_\eta}{\partial x^a} + \left[\bar{\mathbf{n}}_0 \cdot \mathbf{e}_\xi \right] \left[\bar{\mathbf{n}}_0 \cdot \frac{\partial \mathbf{e}_\xi}{\partial x^a} + \mathbf{e}_\xi \cdot \frac{\partial \bar{\mathbf{n}}_0}{\partial x^a} \right] \mathbf{e}_\eta \right) + \right. \\ &\left. \left[1 - \left(\bar{\mathbf{n}}_0 \cdot \mathbf{e}_\xi \right)^2 \right] \mathbf{e}_\eta \cdot \frac{\partial \bar{\mathbf{n}}_0}{\partial x^a} \right\} / \left[1 - \left(\bar{\mathbf{n}}_0 \cdot \mathbf{e}_\xi \right)^2 \right]^{3/2},\end{aligned} \tag{80}$$

$$\begin{aligned}\frac{\partial c_\xi}{\partial x^a} = &\left\{ \bar{\mathbf{n}}_0 \cdot \left(\left[1 - \left(\bar{\mathbf{n}}_0 \cdot \mathbf{e}_\xi \right)^2 \right] \frac{\partial \mathbf{n}_0}{\partial x^a} + \left[\bar{\mathbf{n}}_0 \cdot \mathbf{e}_\xi \right] \left[\bar{\mathbf{n}}_0 \cdot \frac{\partial \mathbf{e}_\xi}{\partial x^a} + \mathbf{e}_\xi \cdot \frac{\partial \bar{\mathbf{n}}_0}{\partial x^a} \right] \mathbf{n}_0 \right) + \right. \\ &\left. \left[1 - \left(\bar{\mathbf{n}}_0 \cdot \mathbf{e}_\xi \right)^2 \right] \mathbf{n}_0 \cdot \frac{\partial \bar{\mathbf{n}}_0}{\partial x^a} \right\} / \left[1 - \left(\bar{\mathbf{n}}_0 \cdot \mathbf{e}_\xi \right)^2 \right]^{3/2},\end{aligned} \tag{81}$$

$$\begin{aligned}\frac{\partial c_\eta}{\partial x^a} = &-\left(\bar{\mathbf{n}}_0 \cdot \mathbf{e}_\eta \right) \frac{\partial s_\xi}{\partial x^a} - s_\xi \left[\bar{\mathbf{n}}_0 \cdot \frac{\partial \mathbf{e}_\eta}{\partial x^a} + \mathbf{e}_\eta \cdot \frac{\partial \bar{\mathbf{n}}_0}{\partial x^a} \right] + \\ &\left(\bar{\mathbf{n}}_0 \cdot \mathbf{n}_0 \right) \frac{\partial c_\xi}{\partial x^a} + c_\xi \left[\bar{\mathbf{n}}_0 \cdot \frac{\partial \mathbf{n}_0}{\partial x^a} + \mathbf{n}_0 \cdot \frac{\partial \bar{\mathbf{n}}_0}{\partial x^a} \right],\end{aligned} \tag{82}$$

$$\frac{\partial s_\eta}{\partial x^a} = \bar{\mathbf{n}}_0 \cdot \frac{\partial \mathbf{e}_\xi}{\partial x^a} + \mathbf{e}_\xi \cdot \frac{\partial \bar{\mathbf{n}}_0}{\partial x^a}. \tag{83}$$

8 Expansion of displacement vector derivatives

Combining equation (73) with equations (8), (9) and (19) of Shaw *et. al.* (2010) gives,

$$\frac{\partial u^{(\xi)}}{\partial x^a} = \left[f_1 + g_1^{(k)}\right]\frac{\partial \gamma_1^0}{\partial x^a} + g_2^{(k)}\frac{\partial \gamma_2^0}{\partial x^a}, \tag{84}$$

$$\frac{\partial u^{(\xi)}}{\partial \lambda} = \gamma_1^0\left[\frac{\partial f_1}{\partial \lambda} + \frac{\partial g_1^{(k)}}{\partial \lambda}\right] + \gamma_2^0\frac{\partial g_2^{(k)}}{\partial \lambda}, \tag{85}$$

$$\frac{\partial u^{(\eta)}}{\partial x^a} = g_3^{(k)}\frac{\partial \gamma_1^0}{\partial x^a} + \left[f_2 + g_4^{(k)}\right]\frac{\partial \gamma_2^0}{\partial x^a}, \tag{86}$$

$$\frac{\partial u^{(\eta)}}{\partial \lambda} = \gamma_1^0\frac{\partial g_3^{(k)}}{\partial \lambda} + \gamma_2^0\left[\frac{\partial f_2}{\partial \lambda} + \frac{\partial g_4^{(k)}}{\partial \lambda}\right]. \tag{87}$$

Referring to equation (6) of Beakou and Touratier (1993), the trigonometric functions proposed to quantify the profile of the displacement field are,

$$f_1 = f(\lambda) - \frac{h}{\pi}b_{55}\frac{df}{d\lambda},\ f_2 = f(\lambda) - \frac{h}{\pi}b_{44}\frac{df}{d\lambda}, \tag{88}$$

where,

$$f(\lambda) = \frac{h}{\pi}\sin\left[\frac{\pi\lambda}{h}\right], \tag{89}$$

and,

$$g_i^{(k)}(\lambda) = a_i^{(k)}\lambda + d_i^{(k)},\ i = 1,2,3,4,\ k = 1,2,...,N. \tag{90}$$

The parameters $a_i^{(k)}$, b_{55}, b_{44}, and $d_i^{(k)}$ are laminate level constants of the same form such that, for example,

$$a_i^{(k)} = a_i^{(k)}\left(G_{23}^{(1)}, G_{13}^{(1)}, h^{(1)}, G_{23}^{(2)}, G_{13}^{(2)}, h^{(2)}, ..., G_{23}^{(N)}, G_{13}^{(N)}, h^{(N)}\right), \tag{91}$$

and similarly for b_{55}, b_{44}, and $d_i^{(k)}$.

Expansions of the displacement vector derivatives of equations (74) and (75), with equations (84)-(87), are,

$$\frac{\partial u^{(\xi)}}{\partial Y_l^{(j)}} = \gamma_1^0\left[\frac{\partial f_1}{\partial Y_l^{(j)}} + \frac{\partial g_1^{(k)}}{\partial Y_l^{(j)}}\right] + \gamma_2^0\frac{\partial g_2^{(k)}}{\partial Y_l^{(j)}}, \tag{92}$$

$$\frac{\partial^2 u^{(\xi)}}{\partial Y_l^{(j)} \partial x^a} = \left[\frac{\partial f_1}{\partial Y_l^{(j)}} + \frac{\partial g_1^{(k)}}{\partial Y_l^{(j)}} \right] \frac{\partial \gamma_1^0}{\partial x^a} + \frac{\partial g_2^{(k)}}{\partial Y_l^{(j)}} \frac{\partial \gamma_2^0}{\partial x^a}, \tag{93}$$

for example. The remaining equivalent derivatives are given in Appendix A, Equations (A18)-(A21).

9 Shape function and laminate level stiffness derivatives

9.1 Shape function derivatives

The shape function derivatives required by equations (92)-(93) with (A18)-A(21) are obtained using equation (88) such that,

$$\frac{\partial f_1}{\partial Y_l^{(j)}} = -\frac{h}{\pi} \cos\left[\frac{\pi\lambda}{h}\right] \times \frac{\partial b_{55}}{\partial Y_l^{(j)}}, \; \frac{\partial f_2}{\partial Y_l^{(j)}} = -\frac{h}{\pi} \cos\left[\frac{\pi\lambda}{h}\right] \times \frac{\partial b_{44}}{\partial Y_l^{(j)}}, \; l = 1,2, \tag{94}$$

$$\frac{\partial^2 f_1}{\partial Y_l^{(j)} \partial \lambda} = \sin\left[\frac{\pi\lambda}{h}\right] \times \frac{\partial b_{55}}{\partial Y_l^{(j)}}, \; \frac{\partial^2 f_2}{\partial Y_l^{(j)} \partial \lambda} = \sin\left[\frac{\pi\lambda}{h}\right] \times \frac{\partial b_{44}}{\partial Y_l^{(j)}}, \; l = 1,2, \tag{95}$$

$$\frac{\partial g_i^{(k)}}{\partial \lambda} = a_i^{(k)}, \; \frac{\partial g_i^{(k)}}{\partial Y_l^{(j)}} = \lambda \frac{\partial a_i^{(k)}}{\partial Y_l^{(j)}} + \frac{\partial d_i^{(k)}}{\partial Y_l^{(j)}}, \; \frac{\partial^2 g_i^{(k)}}{\partial Y_l^{(j)} \partial \lambda} = \frac{\partial a_i^{(k)}}{\partial Y_l^{(j)}}, \; l = 1,2,3. \tag{96}$$

with additional equivalent forms provided in Appendix A, Equations (A22)-(A25). It follows from equations (11) and (12) of Beakou and Touratier (1993) that,

$$a_1^{(k)} = S_{55}^{(k)} a_{55}^{(k)} + S_{45}^{(k)} a_{45}^{(k)}, \; a_2^{(k)} = S_{55}^{(k)} a_{45}^{(k)} + S_{45}^{(k)} a_{44}^{(k)}, \tag{97}$$

$$a_3^{(k)} = S_{44}^{(k)} a_{45}^{(k)} + S_{45}^{(k)} a_{55}^{(k)}, \; a_4^{(k)} = S_{44}^{(k)} a_{44}^{(k)} + S_{45}^{(k)} a_{45}^{(k)}. \tag{98}$$

Referring to equation (29) here and equation (10) of Beakou and Touratier (1993),

$$\begin{gathered} S_{44}^{(k)} = \frac{Q_{55}^{(k)}}{Q_{44}^{(k)} Q_{55}^{(k)} - \left(Q_{45}^{(k)}\right)^2}, \; S_{45}^{(k)} = \frac{Q_{45}^{(k)}}{Q_{44}^{(k)} Q_{55}^{(k)} - \left(Q_{45}^{(k)}\right)^2}, \\ S_{55}^{(k)} = \frac{Q_{44}^{(k)}}{Q_{44}^{(k)} Q_{55}^{(k)} - \left(Q_{45}^{(k)}\right)^2}. \end{gathered} \tag{99}$$

Differentiating equations (97) and (98) gives,

$$\frac{\partial a_1^{(k)}}{\partial h^{(j)}} = S_{55}^{(k)} \frac{\partial a_{55}^{(k)}}{\partial h^{(j)}} + S_{45}^{(k)} \frac{\partial a_{45}^{(k)}}{\partial h^{(j)}}, \; j,k = 1,2,...N, \tag{100}$$

with the derivatives with respect to $G_{1,2}^{(j,k)}$ given in Appendix A, Equations (A26), (A27).

Similarly,

$$\frac{\partial a_2^{(k)}}{\partial h^{(j)}} = S_{55}^{(k)} \frac{\partial a_{45}^{(k)}}{\partial h^{(j)}} + S_{45}^{(k)} \frac{\partial a_{44}^{(k)}}{\partial h^{(j)}},\ j,k = 1,2,...N, \tag{101}$$

$$\frac{\partial a_2^{(k)}}{\partial G_l^{(j)}} = S_{55}^{(k)} \frac{\partial a_{45}^{(k)}}{\partial G_l^{(j)}} + S_{45}^{(k)} \frac{\partial a_{44}^{(k)}}{\partial G_l^{(j)}},\ j \neq k, \tag{102}$$

$$\frac{\partial a_2^{(k)}}{\partial G_l^{(k)}} = S_{55}^{(k)} \frac{\partial a_{45}^{(k)}}{\partial G_l^{(k)}} + a_{45}^{(k)} \frac{\partial S_{55}^{(k)}}{\partial G_l^{(k)}} + S_{45}^{(k)} \frac{\partial a_{44}^{(k)}}{\partial G_l^{(k)}} + a_{44}^{(k)} \frac{\partial S_{45}^{(k)}}{\partial G_l^{(k)}}, \tag{103}$$

The derivatives of $S_{44}^{(k)}$, $S_{45}^{(k)}$ and $S_{55}^{(k)}$ which are required by equations (103), (A30), & (A33) are obtained from equation (99), as, for example,

$$\frac{\partial S_{44}^{(k)}}{\partial G_l^{(k)}} = \frac{-\left\{ \left(Q_{45}^{(k)}\right)^2 \frac{\partial Q_{55}^{(k)}}{\partial G_l^{(k)}} + \left(Q_{55}^{(k)}\right)^2 \frac{\partial Q_{44}^{(k)}}{\partial G_l^{(k)}} - 2Q_{55}^{(k)} Q_{45}^{(k)} \frac{\partial Q_{45}^{(k)}}{\partial G_l^{(k)}} \right\}}{\left(Q_{44}^{(k)} Q_{55}^{(k)} - \left[Q_{45}^{(k)}\right]^2\right)^2}, \tag{104}$$

with the derivatives of $S_{45}^{(k)}$ and $S_{55}^{(k)}$ given in Appendix A, Equations (A34), (A35). Using definitions from equations (27) and (28) the derivatives of the monoclinic, transverse shear stiffness components are given in Appendix A Equation A36, and,

$$\frac{\partial Q_{44}^{(k)}}{\partial G_{23}^{(k)}} = \left(T_{44}^{(k)}\right)^2,\ \frac{\partial Q_{44}^{(k)}}{\partial G_{13}^{(k)}} = \left(T_{45}^{(k)}\right)^2,\ \frac{\partial Q_{55}^{(k)}}{\partial G_{23}^{(k)}} = \left(T_{54}^{(k)}\right)^2, \tag{105}$$

Referring to equations (97) and (98), the expansions of a_{44} and a_{55} are,

$$a_{ii}^{(k)} = Q_{ii}^{(1)} b_{ii} + \sum_{m=2}^{k} \left(Q_{ii}^{(m-1)} - Q_{ii}^{(m)}\right) \left(\cos\left[\frac{\pi\lambda_m}{h}\right] + b_{55} \sin\left[\frac{\pi\lambda_m}{h}\right]\right), \tag{106}$$
$$ii = 44, 55,\ k = 1,2,...N,$$

and the expansion of $a_{45}$is,

$$a_{45}^{(k)} = \sum_{m=2}^{k} \left(Q_{45}^{(m-1)} + Q_{45}^{(m)}\right) \cos\left[\frac{\pi\lambda_m}{h}\right],\ k = 1,2,...,N. \tag{107}$$

It follows that, for example,

$$\frac{\partial a_{ii}^{(k)}}{\partial G_l^{(j)}} = \frac{\partial a_{ii}^{(k)}}{\partial G_l^{(k)}} = \left\{ Q_{ii}^{(1)} + \sum_{m=2}^{k} \left(Q_{ii}^{(m-1)} - Q_{ii}^{(m)} \right) \sin\left[\frac{\pi\lambda_m}{h}\right] \right\} \frac{\partial b_{ii}}{\partial G_l^{(k)}} - \left(\cos\left[\frac{\pi\lambda_k}{h}\right] + b_{ii} \sin\left[\frac{\pi\lambda_k}{h}\right] \right) \frac{\partial Q_{ii}^{(k)}}{\partial G_l^{(k)}},\ j = k, \tag{108}$$

with $ii = 44, 55$.

Similarly,

$$\frac{\partial a_{45}^{(k)}}{\partial G_l^{(j)}} = \frac{\partial a_{45}^{(k)}}{\partial G_l^{(k)}} = -\cos\left[\frac{\pi\lambda_k}{h}\right] \times \frac{\partial Q_{45}^{(k)}}{\partial G_l^{(k)}},\ j = k, \tag{109}$$

The expressions for the remaining combinations of j and k relating to Equations (108) and (109) are provided on Appendix A, Equations (A37)-(A42).

The expansions of the laminate level coefficients b_{44} and $b_{55}$of equation (108) are,

$$b_{ii} = \frac{C_{ii}}{A_{ii}},\ ii = 44, 55, \tag{110}$$

where

$$A_{ii} = Q_{ii}^{(1)} + \sum_{k=1}^{N-1} \left(Q_{ii}^{(k)} - Q_{ii}^{(k+1)} \right) \sin\left[\frac{\pi\lambda_{k+1}}{h}\right] + Q_{ii}^{(N)},\ ii = 44, 55, \tag{111}$$

$$C_{ii} = \sum_{k=1}^{N-1} \left(Q_{ii}^{(k)} - Q_{ii}^{(k+1)} \right) \cos\left[\frac{\pi\lambda_{k+1}}{h}\right],\ ii = 44, 55. \tag{112}$$

Differentiating b_{44} and $b_{55}$with respect to the transverse shear moduli completes the chain of derivatives in equations (108) (with (A37)-(A40)) to give,

$$\frac{\partial b_{ii}}{\partial G_l^{(j)}} = \frac{A_{ii} \frac{\partial C_{ii}}{\partial G_l^{(j)}} - C_{ii} \frac{\partial A_{ii}}{\partial G_l^{(j)}}}{A_{ii}^2},\ l = 1, 2,\ j = 1, 2, ..., N, \tag{113}$$

where,

$$\frac{\partial A_{ii}}{\partial G_l^{(j)}} = \frac{\partial A_{ii}}{\partial Q_{ii}^{(j)}} \frac{\partial Q_{ii}^{(j)}}{\partial G_l^{(j)}},\ \frac{\partial C_{ii}}{\partial G_l^{(j)}} = \frac{\partial C_{ii}}{\partial Q_{ii}^{(j)}} \frac{\partial Q_{ii}^{(j)}}{\partial G_l^{(j)}}, \tag{114}$$

and, for example,

$$\frac{\partial A_{ii}}{\partial Q_{ii}^{(1)}} = 1 + \sin\left[\frac{\pi\lambda_2}{h}\right],\ \frac{\partial A_{ii}}{\partial Q_{ii}^{(N)}} 1 - \sin\left[\frac{\pi\lambda_N}{h}\right], \tag{115}$$

The remaining components are provided in Appendix A, Equations (A43)-(A45).

9.2 Laminate level derivatives

The laminate thickness h is given by,

$$h=\sum_{k=1}^{N}h^{(k)},\ \lambda_1=-\frac{h}{2},\ \lambda_{N+1}=\frac{h}{2},\ h^{(k)}=\lambda_{k+1}-\lambda_k. \tag{116}$$

It therefore follows that,

$$\frac{\partial h}{\partial h^{(k)}}=1,\ \frac{\partial \lambda_m}{\partial h^{(k)}}=\begin{cases}-\frac{1}{2}, & k\leq m,\ m=1,2,...,N+1,\\ \frac{1}{2}, & m>k,\ k=1,2,...,N,\end{cases} \tag{117}$$

The derivatives of a_{44} and a_{55} from equations (106) with respect to the lamina thicknesses ($h^{(k)}$- see equation (115)) are,

$$\frac{\partial a_{ii}^{(k)}}{\partial h^{(j)}}=Q_{ii}^{(1)}\frac{\partial b_{ii}}{\partial h^{(j)}}+\sum_{m=2}^{k}\left(Q_{ii}^{(m-1)}-Q_{ii}^{(m)}\right)\left\{\frac{\pi}{h^2}\left(h\frac{\partial \lambda_m}{\partial h^{(j)}}-\lambda_m\right)\times\right.$$
$$\left.\left(b_{ii}\cos\left[\frac{\pi\lambda_m}{h}\right]-\sin\left[\frac{\pi\lambda_m}{h}\right]\right)+\sin\left[\frac{\pi\lambda_m}{h}\right]\times\frac{\partial b_{ii}}{\partial h^{(j)}}\right\}, \tag{118}$$
$$ii=44,55,\ j,k=1,2,...,N.$$

Similarly the corresponding derivatives of a_{45} are,

$$\frac{\partial a_{45}^{(k)}}{\partial h^{(j)}}=\sum_{m=2}^{k}\left(Q_{45}^{(m)}-Q_{45}^{(m-1)}\right)\sin\left[\frac{\pi\lambda_m}{h}\right]\times\frac{\pi\left(h\frac{\partial\lambda_m}{\partial h^{(j)}}-\lambda_m\right)}{h^2},\ j,k=1,2,...N. \tag{119}$$

The following expansions of the derivates of b_{ii} (Equation (110)) can be determined from equations (111) and (112),

$$\frac{\partial b_{ii}}{\partial h^{(j)}}=\frac{A_{ii}\frac{\partial C_{ii}}{\partial h^{(j)}}-C_{ii}\frac{\partial A_{ii}}{\partial h^{(j)}}}{A_{ii}^2},\ j=1,2,...,N, \tag{120}$$

where

$$\frac{\partial A_{ii}}{\partial h^{(j)}}=\sum_{k=1}^{N-1}\left\{\frac{\pi}{h^2}\left(Q_{ii}^{(k)}-Q_{ii}^{(k+1)}\right)\left(h\frac{\partial\lambda_{k+1}}{\partial h^{(j)}}-\lambda_{k+1}\right)\cos\left[\frac{\pi\lambda_{k+1}}{h}\right]\right\}, \tag{121}$$
$$ii=44,55,\ j=1,2,...,N,$$

$$\frac{\partial C_{ii}}{\partial h^{(j)}} = \sum_{k=1}^{N-1} \left\{ \frac{\pi}{h^2} \left(Q_{ii}^{(k+1)} - Q_{ii}^{(k)} \right) \left(h \frac{\partial \lambda_{k+1}}{\partial h^{(j)}} - \lambda_{k+1} \right) \sin \left[\frac{\pi \lambda_{k+1}}{h} \right] \right\} \tag{122}$$
$ii = 44, 55, \ j = 1, 2, ..., N.$

The chains for the derivatives of a_1, a_2, a_3 and a_4 in equation (96), with respect to all the $Y_l^{(j)}$ $(l = 1,2,3, \ j = 1,2,...,N)$ specified in equation (72) is now complete. The derivatives of the laminate level coefficients $d_i^{(k)}$ are required to complete the overall chain of shape function derivates, (see equations (94) and (95). The expansions of $d_i^{(k)}$ that are derived in Beakou & Touratier (1993) are,

$$\left(d_1^{(k_0)}, d_2^{(k_0)}, d_3^{(k_0)}, d_4^{(k_0)} \right) = \left(\frac{h}{\pi} b_{55}, 0, 0, \frac{h}{\pi} b_{44} \right), \tag{123}$$

where $k = k_0$ is the index number of the lamina within which the mid-surface is located and,

$$d_i^{(k)} = d_i^{(k_0)} + \sum_{m=k_0+1}^{k} \lambda_m \left(a_i^{(m-1)} - a_i^{(m)} \right), \tag{124}$$
$i = 1,2,3,4, \ k = k_0+1, k_0+2, ..., N,$

$$d_i^{(k)} = d_i^{(k_0)} - \sum_{m=k}^{m=k_0-1} \lambda_{m+1} \left(a_i^{(m)} - a_i^{(m+1)} \right), \tag{125}$$
$i = 1,2,3,4, \ k = k_0-1, k_0-2, ..., 1.$

Given equations (123)-(125) then,

$$\frac{\partial d_i^{(k)}}{\partial Y_l^{(j)}} = \frac{\partial d_i^{(k_0)}}{\partial Y_l^{(j)}} + \sum_{m=k_0+1}^{k} \lambda_m \left(\frac{\partial a_i^{(m-1)}}{\partial Y_l^{(j)}} - \frac{\partial a_i^{(m)}}{\partial Y_l^{(j)}} \right), \tag{126}$$
$i = 1,2,3,4, \ l = 1,2, \ j = 1,2,...,N,$
$k = k_0+1, k_0+2, ..., N,$

$$\frac{\partial d_1^{(k_0)}}{\partial Y_l^{(j)}} = \frac{h}{\pi} \frac{\partial b_{55}}{\partial Y_l^{(j)}}, \ \frac{\partial d_2^{(k_0)}}{\partial Y_l^{(j)}} = \frac{\partial d_3^{(k_0)}}{\partial Y_l^{(j)}} = 0, \ \frac{\partial d_4^{(k_0)}}{\partial Y_l^{(j)}} = \frac{h}{\pi} \frac{\partial b_{44}}{\partial Y_l^{(j)}}, \tag{127}$$
$l = 1,2, \ j = 1,2,...,N,$

$$\frac{\partial d_i^{(k)}}{\partial Y_3^{(j)}} = \frac{\partial d_i^{(k)}}{\partial h^{(j)}} = \frac{\partial d_i^{(k_0)}}{\partial h^{(j)}} - \sum_{m=k}^{k_0-1} \left\{ \lambda_{m+1} \left(\frac{\partial a_i^{(m)}}{\partial h^{(j)}} - \frac{\partial a_i^{(m+1)}}{\partial h^{(j)}} \right) + \left(a_i^{(m)} - a_i^{(m+1)} \right) \frac{\partial \lambda_{m+1}}{\partial h^{(j)}} \right\}, \ i = 1,2,3,4, \ j = 1,2,...,N, \tag{128}$$
$k = k_0-1, k_0-2, ..., 1,$

$$\frac{\partial d_1^{(k_0)}}{\partial h^{(j)}} = \frac{1}{\pi}\left(h\frac{\partial b_{55}}{\partial h^{(j)}} + b_{55}\right), \ \frac{\partial d_2^{(k_0)}}{\partial h^{(j)}} = \frac{\partial d_3^{(k_0)}}{\partial h^{(j)}} = 0,$$
$$\frac{\partial d_4^{(k_0)}}{\partial h^{(j)}} = \frac{1}{\pi}\left(h\frac{\partial b_{44}}{\partial h^{(j)}} + b_{44}\right), \ j = 1,2,...,N. \tag{129}$$

with the equivalent symmetric laminate terms given in Equations (A46)-(A47).

The expansions of the derivatives of b_{55} and b_{44} in equations (127), (129) are given by Equations (120) and (127). The expansions of the derivatives of a_1, a_2, a_3, and a_4 in equations (125), (A46), (127) and (A47) are specified by equations (100) and (96).

10 Finite difference verification

The geometry of the deformed laminate must be known in order to compute all the derivatives with respect to all the random variables listed in the random vector **X** of equation (16). The set of derivative evaluations required to complete the laminate level sensitivity analysis includes the various strain tensor and displacement vector component derivatives are specified in equations (69)-(93).

As an example, laminate level shape function derivatives (derived in section 6) are verified with respect to the transverse shear components of stiffness ($G_{13}^{(k)}$ and $G_{23}^{(k)}$), and with respect to the lamina thicknesses ($h^{(k)}$) for the non-symmetric laminate detailed in Table 1. These are mathematical idealizations are used to test the analytical derivatives of f_1 and $a_1^{(k)}$ with respect to $G_{13}^{(j)}$ (see equations (94)-(95)) along with the derivatives of $a_{55}^{(k)}$ and $a_{45}^{(k)}$ with respect to $h^{(j)}$ (see equation (119)).

Table 1: Assumed laminate properties.

Layer	Lamina Thickness (m)	Fibre Orientation (Degrees)	$G_{13}^{(k)}$(MPa)	$G_{23}^{(k)}$(MPa)
$k=1$	$h^{(1)} = 6\times10^{-3}$	90	5000	3000
$k=2$	$h^{(2)} = 4\times10^{-3}$	45	5000	3000
$k=3$	$h^{(3)} = 5\times10^{-3}$	0	5000	3000
$k=4$	$h^{(4)} = 4\times10^{-3}$	72	5000	3000
$k=5$	$h^{(5)} = 6\times10^{-3}$	37	5000	3000

The derivative of the shape functionf_1, (Equation (88)), and $a_1^{(k)}$ (Equation (97)) with respect to the transverse shear modulus $G_{13}^{(j)}$ is tested with the data from table 2. The corresponding sample results are listed in tables 2(a) and 2(b) respectively.

Table 2a: Finite difference verification of $\frac{\partial f_1}{\partial G_{13}^{(j)}}(\lambda)$ from Equation (94).

Thickness Coordinate (λ)	Lamina Index (k)	Random Variable Index (j)	Analytical Solution	Finite-Difference Comparison (see equation (130))	Step-size δx_i
-0.0095	1	1	0	0	10^{-5}
-0.0095	1	2	-4.2848e-008	-4.2848e-008	10^{-3}
-0.0095	1	5	1.6685e-007	1.6685e-007	10^{-3}
-0.0045	2	3	4.3557e-008	4.3557e-008	10^{-3}
-0.0045	2	4	2.4422e-008	2.4422e-008	10^{-3}
0.0	3	5	4.5324e-007	4.5324e-007	10^{-3}
0.0095	5	2	-4.2848e-008	-4.2848e-008	10^{-3}
0.0095	5	3	1.8991e-008	1.8991e-008	10^{-3}

Table 2b: Finite difference verification of $\frac{\partial a_1^{(k)}}{\partial G_{13}^{(j)}}$ from Equation (A27).

Thickness Coordinate (λ)	Lamina Index (k)	Random Variable Index (j)	Analytical Solution	Finite-Difference Comparison (see equation (130))	Step-size δx_i
-0.0095	1	2	1.4627e-005	1.4627e-005	10^{-3}
-0.0095	1	4	-3.6347e-006	-3.6347e-006	10^{-3}
-0.0095	1	5	-5.6955e-005	-5.6955e-005	10^{-3}
-0.0045	2	2	-0.00018513	-0.00018513	10^{-3}
0.0	3	3	-0.00053532	-0.00053532	10^{-3}
0.0	3	4	-4.8003e-006	-4.8003e-006	10^{-3}
0.0	3	5	-7.5219e-005	-7.5219e-005	10^{-3}
0.0095	5	5	-0.00032519	-0.00032519	10^{-3}

Table 2c: Finite difference verification of $\frac{\partial a_{55}^{(k)}}{\partial h^{(j)}}$ from equation (118).

Thickness Coordinate (λ)	Lamina Index (k)	Random Variable Index (j)	Analytical Solution	Finite-Difference Comparison (see equation (130))	Step-size δx_i
-0.0095	1	1	57532.7133	57532.7133	10^{-7}
-0.0095	1	2	4198.5576	4198.5576	10^{-7}
-0.0095	1	3	-14054.8182	-14054.8182	10^{-7}
-0.0095	1	4	-69753.2736	-69753.2736	10^{-7}
-0.0095	1	5	-2117.2208	-2117.2208	10^{-7}
-0.0045	2	1	124194.359	124194.359	10^{-7}
-0.0045	2	5	-25508.4026	-25508.4026	10^{-7}
0.0095	5	5	-6730.1735	-6730.1735	10^{-7}

Table 2d: Finite difference verification of $\frac{\partial a_{45}^{(k)}}{\partial h^{(j)}}$ from equation (119).

Thickness Coordinate (λ)	Lamina Index (k)	Random Variable Index (j)	Analytical Solution	Finite-Difference Comparison (see equation (130))	Step-size δx_i
-0.0095	1	5	0	0	10^{-7}
-0.0045	2	1	69619.7235	69619.7235	10^{-7}
-0.0045	2	2	-21985.1759	-21985.1759	10^{-7}
0.0	3	1	46320.3911	46320.3911	10^{-7}
0.0	3	2	-45284.5083	-45284.5083	10^{-7}
0.0045	4	1	37190.3884	37190.3884	10^{-7}
0.0095	5	4	-968.2288	-968.2288	10^{-7}
0.0095	5	5	33244.0432	33244.0432	10^{-7}

Similarly, the derivatives of the laminate level constants $a_{55}^{(k)}$ and $a_{45}^{(k)}$, specified in equations (106) and (107) respectively, with respect to a given lamina thickness $h^{(j)}$ are examined with the corresponding sample results listed tables 2(c) and 2(d) respectively. For verification of the proposed analytical sensitivities, the following standard, second order finite difference scheme is used here,

$$y = y(\mathbf{X}), \ \frac{\partial y}{\partial x_i} = \frac{y(x_i + \delta x_i) - y(x_i - \delta x_i)}{2\delta x_i} + O\left([\delta x_i]^2\right), \tag{130}$$

where $y(\mathbf{X})$ is some scalar function of the vector $\mathbf{X}$ and x_i is the i^{th} component of $\mathbf{X}$. The step size is δx_i.

Referring to tables 3(a)-3(d), excellent agreement is obtained between the analytical solutions derived here and the corresponding numerical solution to a level corresponding to a benchmark. A comparison with experimental test data is possible based on the constituent components or complete laminate data and the mathematical formulation presented in this paper. In assessing the validity of the analytical sensitivities against experimental data it will be necessary to consider both aleatoric and epistemic uncertainties. Nevertheless, the accuracy achieved even in using deterministic values would be expected to be good. Experimental comparisons were not included in the scope of the work presented in this paper.

11 Conclusions

The derivation of the analytical sensitivities of a high fidelity kinematic model for the analysis of composite laminated plates, shells, and structures subject to uncertainty have been presented. Verification of the sensitivities has been demonstrated via comparisons with finite difference approximations.

The application of these sensitivities is intended for both optimisation and reliability analyses. To this end, the analytical sensitivities derived in this paper are available via the corresponding author.

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Appendix A – Supplimentary Definitions

Top-Level Stiffness & Thickness Derivatives

$$\frac{\partial\sigma_{22}}{\partial Z_l^{(j)}} = \begin{cases} w_{11}\frac{\partial Q_{12}^{(k)}}{\partial Z_l^{(k)}} + w_{22}\frac{\partial Q_{22}^{(k)}}{\partial Z_l^{(k)}} + 2w_{12}\frac{\partial Q_{26}^{(k)}}{\partial Z_l^{(k)}},\ j=k, \\ 0,\ j\neq k, \end{cases} \tag{A1}$$

$$\frac{\partial\sigma_{12}}{\partial Z_l^{(j)}} = \begin{cases} w_{11}\frac{\partial Q_{16}^{(k)}}{\partial Z_l^{(k)}} + w_{22}\frac{\partial Q_{26}^{(k)}}{\partial Z_l^{(k)}} + 2w_{12}\frac{\partial Q_{66}^{(k)}}{\partial Z_l^{(k)}},\ j=k, \\ 0,\ j\neq k, \end{cases} \tag{A2}$$

$$\frac{\partial\sigma_{22}}{\partial G_{a3}^{(j)}} = Q_{12}^{(k)}\frac{\partial w_{11}}{\partial G_{a3}^{(j)}} + Q_{22}^{(k)}\frac{\partial w_{22}}{\partial G_{a3}^{(j)}} + 2Q_{26}^{(k)}\frac{\partial w_{12}}{\partial G_{a3}^{(j)}},\ j,k=1,2,...,N,\ a=1,2, \tag{A3}$$

$$\frac{\partial\sigma_{12}}{\partial G_{a3}^{(j)}} = Q_{16}^{(k)}\frac{\partial w_{11}}{\partial G_{a3}^{(j)}} + Q_{26}^{(k)}\frac{\partial w_{22}}{\partial G_{a3}^{(j)}} + 2Q_{66}^{(k)}\frac{\partial w_{12}}{\partial G_{a3}^{(j)}},\ j,k=1,2,...,N,\ a=1,2, \tag{A4}$$

$$\frac{\partial \sigma_{22}}{\partial h^{(j)}} = Q_{12}^{(k)} \frac{\partial w_{11}}{\partial h^{(j)}} + Q_{22}^{(k)} \frac{\partial w_{22}}{\partial h^{(j)}} + 2Q_{26}^{(k)} \frac{\partial w_{12}}{\partial h^{(j)}},\ j,k = 1,2,...,N, \tag{A5}$$

$$\frac{\partial \sigma_{12}}{\partial h^{(j)}} = Q_{16}^{(k)} \frac{\partial w_{11}}{\partial h^{(j)}} + Q_{26}^{(k)} \frac{\partial w_{22}}{\partial h^{(j)}} + 2Q_{66}^{(k)} \frac{\partial w_{12}}{\partial h^{(j)}},\ j,k = 1,2,...,N, \tag{A6}$$

$$\frac{\partial \sigma_{13}}{\partial G_{a3}^{(j)}} = \begin{cases} 2Q_{45}^{(k)} \frac{\partial w_{23}}{\partial G_{a3}^{(k)}} + 2w_{23} \frac{\partial Q_{45}^{(k)}}{\partial G_{a3}^{(k)}} + 2Q_{55}^{(k)} \frac{\partial w_{13}}{\partial G_{a3}^{(k)}} + 2w_{13} \frac{\partial Q_{55}^{(k)}}{\partial G_{a3}^{(k)}},\ j = k, \\ 2Q_{45}^{(k)} \frac{\partial w_{23}}{\partial G_{a3}^{(j)}} + 2Q_{55}^{(k)} \frac{\partial w_{13}}{\partial G_{a3}^{(j)}},\ j \neq k, \end{cases} \tag{A7}$$

$$\frac{\partial \sigma_{13}}{\partial h^{(j)}} = 2Q_{45}^{(k)} \frac{\partial w_{23}}{\partial h^{(j)}} + 2Q_{55}^{(k)} \frac{\partial w_{13}}{\partial h^{(j)}},\ \frac{\partial \sigma_{13}}{\partial Z_l^{(k)}} = 0,\ k,j = 1,2,...,N,\ \forall l, \tag{A8}$$

Stiffness Derivatives

$$\frac{\partial Q_{22}^{(k)}}{\partial Z_l^{(k)}} = \left(T_{21}^{(k)}\right)^2 \frac{\partial \bar{Q}_{11}^{(k)}}{\partial Z_l^{(k)}} + 2T_{21}^{(k)} T_{22}^{(k)} \frac{\partial \bar{Q}_{12}^{(k)}}{\partial Z_l^{(k)}} + \left(T_{22}^{(k)}\right)^2 \frac{\partial \bar{Q}_{22}^{(k)}}{\partial Z_l^{(k)}},\ l = 1,2,...,6, \tag{A9}$$

$$\frac{\partial Q_{12}^{(k)}}{\partial Z_l^{(k)}} = T_{11}^{(k)} T_{21}^{(k)} \frac{\partial \bar{Q}_{11}^{(k)}}{\partial Z_l^{(k)}} + \left(T_{11}^{(k)} T_{22}^{(k)} + T_{12}^{(k)} T_{21}^{(k)}\right) \frac{\partial \bar{Q}_{12}^{(k)}}{\partial Z_l^{(k)}} + T_{12}^{(k)} T_{22}^{(k)} \frac{\partial \bar{Q}_{22}^{(k)}}{\partial Z_l^{(k)}},\ l = 1,2,...,6, \tag{A10}$$

$$\frac{\partial Q_{16}^{(k)}}{\partial Z_l^{(k)}} = T_{11}^{(k)} T_{61}^{(k)} \frac{\partial \bar{Q}_{11}^{(k)}}{\partial Z_l^{(k)}} + \left(T_{11}^{(k)} T_{62}^{(k)} + T_{12}^{(k)} T_{61}^{(k)}\right) \frac{\partial \bar{Q}_{12}^{(k)}}{\partial Z_l^{(k)}} + T_{12}^{(k)} T_{62}^{(k)} \frac{\partial \bar{Q}_{22}^{(k)}}{\partial Z_l^{(k)}},\ l = 1,2,...,6, \tag{A11}$$

$$\frac{\partial Q_{26}^{(k)}}{\partial Z_l^{(k)}} = T_{21}^{(k)} T_{61}^{(k)} \frac{\partial \bar{Q}_{11}^{(k)}}{\partial Z_l^{(k)}} + \left(T_{21}^{(k)} T_{62}^{(k)} + T_{22}^{(k)} T_{61}^{(k)}\right) \frac{\partial \bar{Q}_{12}^{(k)}}{\partial Z_l^{(k)}} + T_{22}^{(k)} T_{62}^{(k)} \frac{\partial \bar{Q}_{22}^{(k)}}{\partial Z_l^{(k)}},\ l = 1,2,...,6, \tag{A12}$$

$$\frac{\partial Q_{66}^{(k)}}{\partial Z_l^{(k)}} = \left(T_{61}^{(k)}\right)^2 \frac{\partial \bar{Q}_{11}^{(k)}}{\partial Z_l^{(k)}} + 2T_{61}^{(k)}T_{62}^{(k)} \frac{\partial \bar{Q}_{12}^{(k)}}{\partial Z_l^{(k)}} + \left(T_{62}^{(k)}\right)^2 \frac{\partial \bar{Q}_{22}^{(k)}}{\partial Z_l^{(k)}},\ l = 1,2,...,6, \quad \text{(A13)}$$

$$\frac{\partial Q_{55}^{(k)}}{\partial G_{23}^{(k)}} = \left(T_{54}^{(k)}\right)^2,\ \frac{\partial Q_{55}^{(k)}}{\partial G_{13}^{(k)}} = \left(T_{55}^{(k)}\right)^2, \quad \text{(A14)}$$

$$\frac{\partial Q_{45}^{(k)}}{\partial G_{23}^{(k)}} = T_{44}^{(k)}T_{54}^{(k)},\ \frac{\partial Q_{45}^{(k)}}{\partial G_{13}^{(k)}} = T_{45}^{(k)}T_{55}^{(k)}, \quad \text{(A15)}$$

$$\frac{\partial \bar{Q}_{22}^{(k)}}{\partial Z_l^{(k)}} = \frac{\partial C_{(k)}^{2222}}{\partial Z_l^{(k)}} - \frac{2C_{(k)}^{2233}}{C_{(k)}^{3333}} \frac{\partial C_{(k)}^{2233}}{\partial Z_l^{(k)}} + \left(\frac{C_{(k)}^{2233}}{C_{(k)}^{3333}}\right)^2 \frac{\partial C_{(k)}^{3333}}{\partial Z_l^{(k)}},\ l = 1,2,...,6, \quad \text{(A16)}$$

$$\frac{\partial \bar{Q}_{12}^{(k)}}{\partial Z_l^{(k)}} = \frac{1}{\left(C_{(k)}^{3333}\right)^2} \left\{ \left(C_{(k)}^{3333}\right)^2 \frac{\partial C_{(k)}^{1122}}{\partial Z_l^{(k)}} - C_{(k)}^{2233}C_{(k)}^{3333} \frac{\partial C_{(k)}^{1133}}{\partial Z_l^{(k)}} - \right.$$
$$\left. C_{(k)}^{1133}C_{(k)}^{3333} \frac{\partial C_{(k)}^{2233}}{\partial Z_l^{(k)}} + C_{(k)}^{1133}C_{(k)}^{2233} \frac{\partial C_{(k)}^{3333}}{\partial Z_l^{(k)}} \right\},\ l = 1,2,...,6, \quad \text{(A17)}$$

$$\frac{\partial u^{(\eta)}}{\partial Y_l^{(j)}} = \gamma_1^0 \frac{\partial g_3^{(k)}}{\partial Y_l^{(j)}} + \gamma_2^0 \left[\frac{\partial f_2}{\partial Y_l^{(j)}} + \frac{\partial g_4^{(k)}}{\partial Y_l^{(j)}} \right], \quad \text{(A18)}$$

$$\frac{\partial^2 u^{(\xi)}}{\partial Y_l^{(j)} \partial \lambda} = \gamma_1^0 \left[\frac{\partial^2 f_1}{\partial Y_l^{(j)} \partial \lambda} + \frac{\partial^2 g_1^{(k)}}{\partial Y_l^{(j)} \partial \lambda} \right] + \gamma_2^0 \frac{\partial^2 g_2^{(k)}}{\partial Y_l^{(j)} \partial \lambda}, \quad \text{(A19)}$$

$$\frac{\partial^2 u^{(\eta)}}{\partial Y_l^{(j)} \partial x^a} = \frac{\partial g_3^{(k)}}{\partial Y_l^{(j)}} \frac{\partial \gamma_1^0}{\partial x^a} + \left[\frac{\partial f_2}{\partial Y_l^{(j)}} + \frac{\partial g_4^{(k)}}{\partial Y_l^{(j)}} \right] \frac{\partial \gamma_2^0}{\partial x^a}, \quad \text{(A20)}$$

$$\frac{\partial^2 u^{(\eta)}}{\partial Y_l^{(j)} \partial \lambda} = \gamma_1^0 \frac{\partial^2 g_3^{(k)}}{\partial Y_l^{(j)} \partial \lambda} + \gamma_2^0 \left[\frac{\partial^2 f_2}{\partial Y_l^{(j)} \partial \lambda} + \frac{\partial^2 g_4^{(k)}}{\partial Y_l^{(j)} \partial \lambda} \right]. \quad \text{(A21)}$$

Stiffness Derivatives

$$\frac{\partial f_1}{\partial h^{(j)}} = \frac{1}{\pi}\left\{\sin\left[\frac{\pi\lambda}{h}\right] - \left(b_{55} + h\frac{\partial b_{55}}{\partial h^{(j)}}\right)\cos\left[\frac{\pi\lambda}{h}\right]\right\} - \frac{\lambda}{h}\left(\cos\left[\frac{\pi\lambda}{h}\right] + b_{55}\sin\left[\frac{\pi\lambda}{h}\right]\right),\ l = 3,\ Y_3^{(j)} = h^{(j)}, \tag{A22}$$

$$\frac{\partial f_2}{\partial h^{(j)}} = \frac{1}{\pi}\left\{\sin\left[\frac{\pi\lambda}{h}\right] - \left(b_{44} + h\frac{\partial b_{44}}{\partial h^{(j)}}\right)\cos\left[\frac{\pi\lambda}{h}\right]\right\} - \frac{\lambda}{h}\left(\cos\left[\frac{\pi\lambda}{h}\right] + b_{44}\sin\left[\frac{\pi\lambda}{h}\right]\right),\ l = 3,\ Y_3^{(j)} = h^{(j)}, \tag{A23}$$

$$\frac{\partial^2 f_1}{\partial h^{(j)}\partial\lambda} = \frac{\pi\lambda}{h^2}\left(\sin\left[\frac{\pi\lambda}{h}\right] - b_{55}\cos\left[\frac{\pi\lambda}{h}\right]\right) + \sin\left[\frac{\pi\lambda}{h}\right] \times \frac{\partial b_{55}}{\partial h^{(j)}}, \tag{A24}$$

$$\frac{\partial^2 f_2}{\partial h^{(j)}\partial\lambda} = \frac{\pi\lambda}{h^2}\left(\sin\left[\frac{\pi\lambda}{h}\right] - b_{44}\cos\left[\frac{\pi\lambda}{h}\right]\right) + \sin\left[\frac{\pi\lambda}{h}\right] \times \frac{\partial b_{44}}{\partial h^{(j)}}. \tag{A25}$$

$$\frac{\partial a_1^{(k)}}{\partial G_l^{(j)}} = S_{55}^{(k)}\frac{\partial a_{55}^{(k)}}{\partial G_l^{(j)}} + S_{45}^{(k)}\frac{\partial a_{45}^{(k)}}{\partial G_l^{(j)}},\ j \neq k, \tag{A26}$$

$$\frac{\partial a_1^{(k)}}{\partial G_l^{(k)}} = S_{55}^{(k)}\frac{\partial a_{55}^{(k)}}{\partial G_l^{(k)}} + a_{55}^{(k)}\frac{\partial S_{55}^{(k)}}{\partial G_l^{(k)}} + S_{45}^{(k)}\frac{\partial a_{45}^{(k)}}{\partial G_l^{(k)}} + a_{45}^{(k)}\frac{\partial S_{45}^{(k)}}{\partial G_l^{(k)}}, \tag{A27}$$

in which, $G_1^{(k)} = G_{23}^{(k)},\ G_2^{(k)} = G_{13}^{(k)},\ G_1^{(j)} = G_{23}^{(j)},\ G_2^{(j)} = G_{13}^{(j)},\ l = 1,2.$

$$\frac{\partial a_3^{(k)}}{\partial h^{(j)}} = S_{44}^{(k)}\frac{\partial a_{45}^{(k)}}{\partial h^{(j)}} + S_{45}^{(k)}\frac{\partial a_{55}^{(k)}}{\partial h^{(j)}},\ j,k = 1,2,...N, \tag{A28}$$

$$\frac{\partial a_3^{(k)}}{\partial G_l^{(j)}} = S_{44}^{(k)}\frac{\partial a_{45}^{(k)}}{\partial G_l^{(j)}} + S_{45}^{(k)}\frac{\partial a_{55}^{(k)}}{\partial G_l^{(j)}},\ j \neq k, \tag{A29}$$

$$\frac{\partial a_3^{(k)}}{\partial G_l^{(k)}} = S_{44}^{(k)} \frac{\partial a_{45}^{(k)}}{\partial G_l^{(k)}} + a_{45}^{(k)} \frac{\partial S_{44}^{(k)}}{\partial G_l^{(k)}} + S_{45}^{(k)} \frac{\partial a_{55}^{(k)}}{\partial G_l^{(k)}} + a_{55}^{(k)} \frac{\partial S_{45}^{(k)}}{\partial G_l^{(k)}}, \tag{A30}$$

$$\frac{\partial a_4^{(k)}}{\partial h^{(j)}} = S_{44}^{(k)} \frac{\partial a_{44}^{(k)}}{\partial h^{(j)}} + S_{45}^{(k)} \frac{\partial a_{45}^{(k)}}{\partial h^{(j)}}, \; j,k = 1,2,...N, \tag{A31}$$

$$\frac{\partial a_4^{(k)}}{\partial G_l^{(j)}} = S_{44}^{(k)} \frac{\partial a_{44}^{(k)}}{\partial G_l^{(j)}} + S_{45}^{(k)} \frac{\partial a_{45}^{(k)}}{\partial G_l^{(j)}}, \; j \neq k, \tag{A32}$$

$$\frac{\partial a_4^{(k)}}{\partial G_l^{(k)}} = S_{44}^{(k)} \frac{\partial a_{44}^{(k)}}{\partial G_l^{(k)}} + a_{44}^{(k)} \frac{\partial S_{44}^{(k)}}{\partial G_l^{(k)}} + S_{45}^{(k)} \frac{\partial a_{45}^{(k)}}{\partial G_l^{(k)}} + a_{45}^{(k)} \frac{\partial S_{45}^{(k)}}{\partial G_l^{(k)}}. \tag{A33}$$

$$\begin{aligned} \frac{\partial S_{45}^{(k)}}{\partial G_l^{(k)}} = -\Bigg\{ & Q_{44}^{(k)} Q_{55}^{(k)} \frac{\partial Q_{45}^{(k)}}{\partial G_l^{(k)}} + \left(Q_{45}^{(k)}\right)^2 \frac{\partial Q_{45}^{(k)}}{\partial G_l^{(k)}} - Q_{45}^{(k)} Q_{55}^{(k)} \frac{\partial Q_{44}^{(k)}}{\partial G_l^{(k)}} \\ & -Q_{45}^{(k)} Q_{44}^{(k)} \frac{\partial Q_{55}^{(k)}}{\partial G_l^{(k)}} \Bigg\} \Big/ \left(Q_{44}^{(k)} Q_{55}^{(k)} - \left[Q_{45}^{(k)} \right]^2 \right)^2, \end{aligned} \tag{A34}$$

$$\frac{\partial S_{55}^{(k)}}{\partial G_l^{(k)}} = \frac{-\left\{ \left(Q_{45}^{(k)}\right)^2 \frac{\partial Q_{44}^{(k)}}{\partial G_l^{(k)}} + \left(Q_{44}^{(k)}\right)^2 \frac{\partial Q_{55}^{(k)}}{\partial G_l^{(k)}} - 2 Q_{44}^{(k)} Q_{45}^{(k)} \frac{\partial Q_{45}^{(k)}}{\partial G_l^{(k)}} \right\}}{\left(Q_{44}^{(k)} Q_{55}^{(k)} - \left[Q_{45}^{(k)} \right]^2 \right)^2}, \tag{A35}$$

$$\frac{\partial Q_{55}^{(k)}}{\partial G_{13}^{(k)}} = \left(T_{55}^{(k)}\right)^2, \; \frac{\partial Q_{45}^{(k)}}{\partial G_{23}^{(k)}} = T_{44}^{(k)} T_{54}^{(k)}, \; \frac{\partial Q_{45}^{(k)}}{\partial G_{13}^{(k)}} = T_{45}^{(k)} T_{55}^{(k)}. \tag{A36}$$

$$\frac{\partial a_{ii}^{(k)}}{\partial G_l^{(j)}} = b_{ii} \frac{\partial Q_{ii}^{(1)}}{\partial G_l^{(1)}} + Q_{ii}^{(1)} \frac{\partial b_{ii}}{\partial G_l^{(1)}}, \; k = j = 1, \; l = 1,2, \; ii = 44,55, \tag{A37}$$

$$\frac{\partial a_{ii}^{(k)}}{\partial G_l^{(j)}} = Q_{ii}^{(1)} \frac{\partial b_{ii}}{\partial G_l^{(j)}},\ k = 1,\ j > 1, \tag{A38}$$

$$\frac{\partial a_{ii}^{(k)}}{\partial G_l^{(j)}} = \left\{ Q_{ii}^{(1)} + \sum_{m=2}^{k} \left(Q_{ii}^{(m-1)} - Q_{ii}^{(m)} \right) \sin\left[\frac{\pi\lambda_m}{h}\right] \right\} \frac{\partial b_{ii}}{\partial G_l^{(j)}},\ k > 1,\ j > k, \tag{A39}$$

and for all other combinations of j and k,

$$\frac{\partial a_{ii}^{(k)}}{\partial G_l^{(j)}} = \left\{ \cos\left[\frac{\pi\lambda_{j+1}}{h}\right] - \cos\left[\frac{\pi\lambda_j}{h}\right] + b_{ii}\left(\sin\left[\frac{\pi\lambda_{j+1}}{h}\right] - \sin\left[\frac{\pi\lambda_j}{h}\right] \right) \right\} \frac{\partial Q_{ii}^{(j)}}{\partial G_l^{(j)}} + \left\{ Q_{ii}^{(1)} + \sum_{m=2}^{k} \left(Q_{ii}^{(m-1)} - Q_{ii}^{(m)} \right) \sin\left[\frac{\pi\lambda_m}{h}\right] \right\} \frac{\partial b_{ii}}{\partial G_l^{(j)}}. \tag{A40}$$

$$\frac{\partial a_{45}^{(k)}}{\partial G_l^{(j)}} = 0,\ k = j = 1,\ k = 1 \text{ and } j > 1,\ j > k, \tag{A41}$$

and for all other combinations of j and k,

$$\frac{\partial a_{45}^{(k)}}{\partial G_l^{(j)}} = \left\{ \cos\left[\frac{\pi\lambda_{j+1}}{h}\right] - \cos\left[\frac{\pi\lambda_j}{h}\right] \right\} \frac{\partial Q_{45}^{(j)}}{\partial G_l^{(j)}}. \tag{A42}$$

$$\frac{\partial A_{ii}}{\partial Q_{ii}^{(j)}} = \sin\left[\frac{\pi\lambda_{j+1}}{h}\right] - \sin\left[\frac{\pi\lambda_j}{h}\right],\ j = 2, 3, ..., N-1, \tag{A43}$$

$$\frac{\partial C_{ii}}{\partial Q_{ii}^{(1)}} = \cos\left[\frac{\pi\lambda_2}{h}\right],\ \frac{\partial C_{ii}}{\partial Q_{ii}^{(N)}} = -\cos\left[\frac{\pi\lambda_N}{h}\right], \tag{A44}$$

$$\frac{\partial C_{ii}}{\partial Q_{ii}^{(j)}} = \cos\left[\frac{\pi\lambda_{j+1}}{h}\right] - \cos\left[\frac{\pi\lambda_j}{h}\right],\ j = 2, 3, ..., N-1. \tag{A45}$$

Laminate level Derivatives

$$\frac{\partial d_i^{(k)}}{\partial Y_l^{(j)}} = \frac{\partial d_i^{(k_0)}}{\partial Y_l^{(j)}} - \sum_{m=k}^{k_0-1} \lambda_{m+1} \left(\frac{\partial a_i^{(m)}}{\partial Y_l^{(j)}} - \frac{\partial a_i^{(m+1)}}{\partial Y_l^{(j)}} \right),$$
$$i = 1,2,3,4,\ l = 1,2,\ j = 1,2,...,N,$$
$$k = k_0 - 1, k_0 - 2, ..., 1, \tag{A46}$$

$$\frac{\partial d_i^{(k)}}{\partial Y_3^{(j)}} = \frac{\partial d_i^{(k)}}{\partial h^{(j)}} = \frac{\partial d_i^{(k_0)}}{\partial h^{(j)}} + \sum_{m=k_0+1}^{k} \left\{ \lambda_m \left(\frac{\partial a_i^{(m-1)}}{\partial h^{(j)}} - \frac{\partial a_i^{(m)}}{\partial h^{(j)}} \right) + \right.$$
$$\left. \left(a_i^{(m-1)} - a_i^{(m)} \right) \frac{\partial \lambda_m}{\partial h^{(j)}} \right\},\ i = 1,2,3,4,\ j = 1,2,...,N,$$
$$k = k_0 + 1, k_0 + 2, ..., N, \tag{A47}$$