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Stable Boundary and Internal Data Reconstruction in Two-Dimensional Anisotropic Heat Conduction Cauchy Problems Using Relaxation Procedures for an Iterative MFS Algorithm

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Abstract: We investigate two algorithms involving the relaxation of either the given boundary temperatures (Dirichlet data) or the prescribed normal heat fluxes (Neumann data) on the over-specified boundary in the case of the iterative algorithm of Kozlov, Maz′ya and Fomin (1991) applied to Cauchy problems for two-dimensional steady-state anisotropic heat conduction (the Laplace-Beltrami equation). The two mixed, well-posed and direct problems corresponding to every iteration of the numerical procedure are solved using the method of fundamental solutions (MFS), in conjunction with the Tikhonov regularization method. For each direct problem considered, the optimal value of the regularization parameter is chosen according to the generalized cross-validation (GCV) criterion. The iterative MFS algorithms with relaxation are tested for over-, equally and under-determined Cauchy problems associated with the steady-state anisotropic heat conduction in various two-dimensional geometries to confirm the numerical convergence, stability, accuracy and computational efficiency of the method.

Keywords: Steady-state anisotropic heat conduction; inverse problem; Cauchy problem; iterative method of fundamental solutions (MFS); relaxation procedures; regularization.

1 Introduction

Numerous natural and man-made materials cannot be considered isotropic and the dependence of the thermal conductivity with direction has to be accounted for in the modelling of the heat transfer. More specifically, it should be mentioned that crystals, wood, sedimentary rocks, metals that have undergone heavy cold pressing,

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laminated sheets, composites, cables, heat shielding materials for space vehicles, fibre reinforced structures, and many others are examples of anisotropic materials. Composites are also of special interest to the aerospace industry because of their strength and reduced weight. Therefore, heat conduction in anisotropic materials has numerous important applications in various branches of science and engineering and hence its understanding is of great importance.

A classical and quite often encountered inverse problem in heat transfer is the so-called Cauchy problem. For such a problem, the boundary of the solution domain, the thermal conductivities and/or the heat sources are all known, while the boundary conditions are incomplete. More precisely, both the Dirichlet (temperature) and the Neumann (normal heat flux) conditions are prescribed on a part of the boundary, while on the remaining portion of the boundary no data are available. It is well known that Cauchy problems are generally ill-posed, in the sense that the existence, uniqueness and stability of their solutions are not always guaranteed, see Hadamard (1923). Consequently, a special treatment of these problems is required.

There are numerous important contributions in the literature to the theoretical and numerical solutions of the Cauchy problem associated with the steady-state heat conduction. However, most of these are related to steady-state heat conduction in isotropic solids (the Laplace equation), while just a few studies refer to steady state heat conduction in anisotropic media (the Laplace-Beltrami equation). The method of quasi-reversibility, in conjunction with a finite-difference method (FDM) and Carleman-type estimates, were employed by Klibanov and Santosa (1991) to solve this inverse problem. Kozlov, Maz′ya and Fomin (1991) proposed an alternating iterative algorithm for the stable solution of this problem, which was implemented using the boundary element method (BEM) for both isotropic and anisotropic solids by Lesnic, Elliott and Ingham (1997) and Mera, Elliott, Ingham and Lesnic (2000), respectively. Ang, Nghia and Tam (1998) reformulated the Cauchy problem as an integral equation problem and solved it by employing the Fourier transform and the Tikhonov regularization method. Reinhardt, Han and Hào (1999) proved that the standard five-point FDM approximation to the Cauchy problem for the Laplace equation satisfies some stability estimates and hence it turns out to be a well-posed problem, provided that a certain bounding requirement is fulfilled. As a result of a variational approach to the Cauchy problem, the conjugate gradient method, together with the BEM, was proposed by Hào and Lesnic (2000) in order to obtain a stable solution. Cheng, Hon, Wei and Yamamoto (2001) transformed the original problem into a moment problem by using Green's formula and they also provided an error estimate for the numerical solution. Hon and Wei (2001) converted the Cauchy problem into a classical moment problem whose numerical approximation can be achieved and also provided a convergence proof based on

Backus-Gilbert algorithm. Cimetière, Delvare, Jaoua and Pons (2001) reduced the Cauchy problem for the Laplace equation to solving a sequence of optimization problems under equality constraints using the finite element method (FEM). The minimization functional consists of two terms that measure the gap between the optimal element and the over-specified data and the gap between the optimal element and the previous optimal element (regularization term), respectively. This method was later implemented using the BEM by Delvare, Cimetière and Pons (2002). Cimetière, Delvare, Jaoua and Pons (2002) reduced the solution of harmonic Cauchy problems to the resolution of a fixed point process, while the authors implemented numerically the proposed method by employing both the BEM and the FEM. Mera, Elliott, Ingham and Lesnic (2003) implemented numerically, using the BEM, various regularization methods to solve the Cauchy steady-state heat conduction problem in an anisotropic medium. Jourhmane and Nachaoui (2004) and Jourhmane, Lesnic and Mera (2004) developed three relaxation procedures in order to increase the rate of convergence of the algorithm of Kozlov, Maz′ya and Fomin (1991), at the same time selection criteria for the variable relaxation factors having been provided. Bourgeois (2005) approached the Cauchy problem for the Laplace equation by the mixed formulation of the method of quasi-reversibility, which finally led to a $\mathscr{C}^0$ FEM. Andrieux, Baranger and Ben Abda (2006) introduced an energy-like error functional and converted the inverse problem into an optimization problem. In order to improve the reconstruction of the normal derivatives, Delvare and Cimetière (2008) extended the method originally proposed by Cimetière, Delvare, Jaoua and Pons (2001) to a higher-order one, which was implemented using the BEM. On assuming the available data to have a Fourier expansion, Liu (2008a) applied a modified indirect Trefftz method to solve the Cauchy problem for the Laplace equation. Recently, Marin (2009a,b) solved numerically the Cauchy problem in steady-state heat conduction for both isotropic and anisotropic solids, respectively, by applying, in an iterative manner, the method of fundamental solutions (MFS) for the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991).

The MFS is a simple but powerful technique that has been used to obtain highly accurate numerical approximations of solutions to linear partial differential equations. Like the BEM, the MFS is applicable when a fundamental solution of the governing partial differential equation is explicitly known. Since its introduction as a numerical method in the late 1970s by Mathon and Johnston (1977), it has been successfully applied to a large variety of physical problems, an account of which may be found in the excellent survey papers by Fairweather and Karageorghis (1998), Fairweather, Karageorghis and Martin (2003), and Golberg and Chen (1999). Since the MFS can be easily implemented and, in addition, has a very low computational cost, this meshless method becomes an ideal candidate for inverse problems as well. Consequently, the MFS, together with various regularization methods (e.g. the Tikhonov regularization method, singular value decomposition, iterative regularization methods etc.), have been used increasingly over the last decade for the numerical solution of inverse problems. For example, the Cauchy problem associated with the heat conduction equation (Dong et al. , 2007; Wei et al. , 2007; Ling and Takeuchi , 2008; Young et al. , 2008; Marin , 2008, 2009a,b), linear elasticity (Marin and Lesnic , 2004; Marin , 2005a, 2010a; Marin and Johansson , 2010b), steady-state heat conduction in functionally graded materials (Marin , 2005b), Helmholtz-type equations (Marin , 2005c; Marin and Lesnic , 2005a; Jin and Zheng , 2006; Marin , 2010b,c), Stokes problems (Chen et al. , 2005) and the biharmonic equation (Marin and Lesnic , 2005b) have been successfully addressed by using the MFS.

Recently, Marin (2009b) solved numerically the Cauchy problem in steady-state anisotropic heat conduction (the Laplace-Beltrami equation) by applying, in an iterative manner, the MFS for the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991). At every iteration, two mixed, well-posed and direct problems were solved using the MFS, in conjunction with the Tikhonov regularization method, while the optimal value of the regularization parameter was chosen automatically according to the generalized cross-validation (GCV) criterion, see e.g. Marin (2009b). Consequently, an iterative procedure, which provides the selection of the optimal regularization parameter, occurs within every step of the iterative algorithm of Kozlov, Maz′ya and Fomin (1991) and hence the computational cost of the iterative MFS-based algorithm is increased. In order to overcome this inconvenience and encouraged by the findings of Jourhmane and Nachaoui (2004) and Jourhmane, Lesnic and Mera (2004), as well as similar results obtained for the Cauchy problem associated with the Poisson equation (Jourhmane and Nachaoui , 2002), the modified Helmholtz equation (Marin , 2010c; Johansson and Marin , 2010) and the Cauchy-Navier system of elasticity (Elabib and Nachaoui , 2008; Marin and Johansson , 2010a,b), we decided to employ two relaxation procedures for the iterative MFS-based algorithm implemented by Marin (2009b) and study the influence of the relaxation parameter upon the rate of convergence of the modified method. The efficiency of these relaxation procedures is tested for over-, equally and under-determined Cauchy problems associated with the two-dimensional Laplace-Beltrami operator in simply and doubly connected domains with smooth or piecewise smooth boundaries.

2 Mathematical formulation

Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ occupied by an anisotropic solid characterised by the homogeneous, symmetric and positive-definite thermal conductivity tensor $\mathbb{K} = \left[\mathrm{K}_{ij}\right]_{1\leq i,j\leq 2}$. We also assume that Ω is bounded by a smooth or piecewise smooth curve $\partial\Omega$, such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \neq \emptyset$, $\Gamma_2 \neq \emptyset$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Let $\mathrm{H}^1(\Omega)$ be the Sobolev space of real-valued functions in Ω endowed with the standard norm. We denote by $\mathrm{H}^1_0(\Omega)$ and $\mathrm{H}^1_{\Gamma_i}(\Omega)$, $i = 1,2$, the subspaces of functions from $\mathrm{H}^1(\Omega)$ that vanish on $\partial\Omega$ and Γ_i, $i = 1,2$, respectively. The space of traces of functions from $\mathrm{H}^1(\Omega)$ to $\partial\Omega$ is denoted by $\mathrm{H}^{1/2}(\partial\Omega)$, while the restrictions of the functions belonging to the space $\mathrm{H}^{1/2}(\partial\Omega)$ to the subset $\Gamma_i \subset \partial\Omega$, $i = 1,2$, define the space $\mathrm{H}^{1/2}(\Gamma_i)$, $i = 1,2$. The set of real valued functions in $\partial\Omega$ with compact support in Γ_i, $i = 1,2$, and bounded first-order derivatives are dense in $\mathrm{H}^{1/2}(\Gamma_i)$, $i = 1,2$. Furthermore, we also define the space $\mathrm{H}^{1/2}_{00}(\Gamma_i)$, $i = 1,2$, that consists of functions from $\mathrm{H}^{1/2}(\partial\Omega)$ and vanishing on Γ_{3-i}, $i = 1,2$. The space $\mathrm{H}^{1/2}_{00}(\Gamma_i)$, $i = 1,2$, is a subspace of $\mathrm{H}^{1/2}(\partial\Omega)$ with the norm given by:

$$\|f\|_{\mathrm{H}^{1/2}_{00}(\Gamma_i)} = \left(\int_{\Gamma_i} \frac{f^2(\mathbf{x})}{\mathrm{dist}(\mathbf{x},\Gamma_i)}\, \mathrm{d}\Gamma(\mathbf{x}) + \int_{\Gamma_i}\int_{\Gamma_i} \frac{|f(\mathbf{x})-f(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^d}\, \mathrm{d}\Gamma(\mathbf{x})\, \mathrm{d}\Gamma(\mathbf{y}) \right)^{1/2}. \quad (1)$$

It should be mentioned that the space of restrictions from $\mathrm{H}^{1/2}_{00}(\Gamma_i)$ to Γ_i, $i = 1,2$, is dense in $\mathrm{H}^{1/2}(\Gamma_i)$, $i = 1,2$. Nonetheless, $\mathrm{H}^{1/2}_{00}(\Gamma_i) \neq \mathrm{H}^{1/2}(\Gamma_i)$. Finally, we denote by $\left(\mathrm{H}^{1/2}_{00}(\Gamma_i)\right)^*$ the dual space of $\mathrm{H}^{1/2}_{00}(\Gamma_i)$, $i = 1,2$.

In this paper, we refer to steady-state heat conduction applications in anisotropic homogeneous media in the absence of heat sources. Consequently, the function $\mathrm{u}(\mathbf{x})$ denotes the temperature at a point $\mathbf{x} \in \Omega$ and satisfies the heat balance equation

$$-\nabla \cdot (\mathbb{K}\nabla \mathrm{u}(\mathbf{x})) \equiv -\sum_{i,j=1}^{2} \mathrm{K}_{ij}\, \partial_i \partial_j \mathrm{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \quad (2)$$

where $\partial_i \equiv \partial/\partial \mathrm{x}_i$. In the following, we let $\mathbf{n}(\mathbf{x})$ be the unit outward normal vector at $\partial\Omega$ and $\mathrm{q}(\mathbf{x})$ be the normal heat flux at a point $\mathbf{x} \in \partial\Omega$ defined by

$$\mathrm{q}(\mathbf{x}) \equiv -\mathbf{n}(\mathbf{x}) \cdot (\mathbb{K}\nabla \mathrm{u}(\mathbf{x})) = -\sum_{i,j=1}^{2} \mathrm{n}_i(\mathbf{x})\, \mathrm{K}_{ij}\, \partial_j \mathrm{u}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \quad (3)$$

In the direct problem formulation, the knowledge of the thermal conductivity matrix $\mathbb{K}$, the location, shape and size of the entire boundary $\partial\Omega$, the temperature and/or normal heat flux on the entire boundary $\partial\Omega$ gives the corresponding Dirichlet, Neumann, or Robin conditions which enable us to determine the unknown boundary conditions, as well as the temperature distribution in the solution domain. A different and more interesting situation occurs when both the temperature and normal heat flux are prescribed on a part of the boundary, say Γ_1, whilst no boundary conditions are supplied on the remaining part of the boundary $\Gamma_2 = \partial\Omega \setminus \Gamma_1$. More precisely, we consider the following Cauchy problem for steady heat conduction in an anisotropic homogeneous medium:

$$-\nabla \cdot (\mathbb{K}\,\nabla \mathrm{u}(\mathbf{x})) = 0, \quad \mathbf{x} \in \Omega, \tag{4a}$$

$$\mathrm{u}(\mathbf{x}) = \widetilde{\mathrm{u}}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{4b}$$

$$\mathrm{q}(\mathbf{x}) = \widetilde{\mathrm{q}}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{4c}$$

where $\widetilde{\mathrm{u}} \in \mathrm{H}^{1/2}(\Gamma_1)$ and $\widetilde{\mathrm{q}} \in \left(\mathrm{H}^{1/2}_{00}(\Gamma_1)\right)^*$ are prescribed Dirichlet and Neumann boundary conditions, respectively.

If we denote by $|\Gamma_i|$, $i = 1,2$, the measure of the boundary $\Gamma_i \subset \partial\Omega$ then a necessary condition for the Cauchy problem given by Eqs. (4a)–(4c) to be identifiable is that $|\Gamma_1| > 0$, see Isakov (2006). However, in the discretised version of the aforementioned Cauchy problem, the corresponding identifiability condition reduces to $|\Gamma_1| \geq |\Gamma_2|$, see e.g. Lesnic, Elliott and Ingham (1997). This inverse problem is much more difficult to solve both analytically and numerically than the direct problem, since the solution does not satisfy the general conditions of well-posedness. Although the problem may have a unique solution, it is well known that this solution is unstable with respect to small perturbations into the data on Γ_1, see Hadamard (1923). Thus the problem is ill-posed and, therefore, regularization methods are required in order to solve accurately the inverse problem (4a)–(4c) associated with the steady-state anisotropic heat conduction.

3 Alternating iterative algorithms with relaxation

In the following, we present two alternating iterative algorithms with relaxation, originally proposed by Jourhmane, Lesnic and Mera (2004), which aim to reduce the computational time of the alternating iterative algorithm introduced by Kozlov, Maz′ya and Fomin (1991) for the simultaneous and stable reconstruction of the unknown temperature $\mathrm{u}\big|_{\Gamma_2}$ and normal heat flux $\mathrm{q}\big|_{\Gamma_2}$.

Alternating iterative algorithm with relaxation I:

Step 1. (i) If $k = 1$ then specify an initial guess for the normal heat flux on Γ_2, namely $\mathrm{q}^{(1)} \in \left(\mathrm{H}^{1/2}_{00}(\Gamma_2)\right)^*$.

(ii) If $k \geq 2$ then solve the following mixed, well-posed, direct problem:

$$-\nabla \cdot \left(\mathbb{K}\,\nabla \mathrm{u}^{(2k-1)}(\mathbf{x})\right) = 0, \quad \mathbf{x} \in \Omega, \tag{5a}$$

$$\mathrm{q}^{(2k-1)}(\mathbf{x}) = \widetilde{\mathrm{q}}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{5b}$$

$$\mathrm{u}^{(2k-1)}(\mathbf{x}) = \mathrm{u}^{(2k-2)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2, \tag{5c}$$

to determine $\mathrm{u}^{(2k-1)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $\mathrm{q}^{(2k-1)}(\mathbf{x}) \equiv -\mathbf{n}(\mathbf{x}) \cdot \left(\mathbb{K}\,\nabla \mathrm{u}^{(2k-1)}(\mathbf{x})\right)$, $\mathbf{x} \in \Gamma_2$.

Step 2. Update the unknown Neumann data on Γ_2 by setting

$$\xi^{(k)}(\mathbf{x}) = \begin{cases} \mathrm{q}^{(2k-1)}(\mathbf{x}) & \text{for } k = 1 \\ \omega\, \mathrm{q}^{(2k-1)}(\mathbf{x}) + (1-\omega)\, \xi^{(k-1)}(\mathbf{x}) & \text{for } k \geq 2 \end{cases}, \quad \mathbf{x} \in \Gamma_2, \tag{6}$$

where $\omega \in (0,2)$ is a fixed relaxation factor.

Having constructed the approximation $\mathrm{u}^{(2k-1)}$, $k \geq 1$, the following mixed, well-posed, direct problem:

$$-\nabla \cdot \left(\mathbb{K}\,\nabla \mathrm{u}^{(2k)}(\mathbf{x})\right) = 0, \quad \mathbf{x} \in \Omega, \tag{7a}$$

$$\mathrm{u}^{(2k)}(\mathbf{x}) = \widetilde{\mathrm{u}}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{7b}$$

$$\mathrm{q}^{(2k)}(\mathbf{x}) = \xi^{(k)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2, \tag{7c}$$

is solved in order to determine $\mathrm{u}^{(2k)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $\mathrm{u}^{(2k)}(\mathbf{x})$, $\mathbf{x} \in \Gamma_2$.

Step 3. Repeat steps 1 and 2 until a prescribed stopping criterion is satisfied.

Remark 3.1 *The value* $\omega = 1$ *in Eq.* (6) *corresponds to the alternating iterative algorithm introduced by* Kozlov, Maz′ya and Fomin (1991) *with an initial guess for the Neumann data, while the values* $\omega \in (0,1)$ *and* $\omega \in (1,2)$ *in Eq.* (6) *correspond to the alternating iterative algorithm introduced by* Kozlov, Maz′ya and Fomin (1991) *with an initial guess for the Neumann data and a constant under- and over-relaxation factor, respectively.*

Alternating iterative algorithm with relaxation II:

Step 1. (i) If $k = 1$ then specify an initial guess for the boundary temperature on Γ_2, namely $\mathrm{u}^{(1)} \in \mathrm{H}^{1/2}(\Gamma_2)$.

(ii) If $k \geq 2$ then solve the following mixed, well-posed, direct problem:

$$-\nabla \cdot \left(\mathbb{K}\,\nabla \mathrm{u}^{(2k-1)}(\mathbf{x})\right) = 0, \quad \mathbf{x} \in \Omega, \tag{8a}$$

$$\mathrm{u}^{(2k-1)}(\mathbf{x}) = \widetilde{\mathrm{u}}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{8b}$$

$$\mathrm{q}^{(2k-1)}(\mathbf{x}) = \mathrm{q}^{(2k-2)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2, \tag{8c}$$

to determine $\mathrm{u}^{(2k-1)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $\mathrm{u}^{(2k-1)}(\mathbf{x})$, $\mathbf{x} \in \Gamma_2$.

Step 2. Update the unknown Dirichlet data on Γ_2 by setting

$$\eta^{(k)}(\mathbf{x}) = \begin{cases} \mathrm{u}^{(2k-1)}(\mathbf{x}) & \text{for } k=1 \\ \omega\, \mathrm{u}^{(2k-1)}(\mathbf{x}) + (1-\omega)\,\eta^{(k-1)}(\mathbf{x}) & \text{for } k \geq 2 \end{cases}, \quad \mathbf{x} \in \Gamma_2, \tag{9}$$

where $\omega \in (0,2)$ is a fixed relaxation factor.
Having constructed the approximation $\mathrm{u}^{(2k-1)}$, $k \geq 1$, the following mixed, well-posed, direct problem:

$$-\nabla \cdot \left(\mathbb{K}\, \nabla \mathrm{u}^{(2k)}(\mathbf{x}) \right) = 0, \quad \mathbf{x} \in \Omega, \tag{10a}$$

$$\mathrm{q}^{(2k)}(\mathbf{x}) = \widetilde{\mathrm{q}}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{10b}$$

$$\mathrm{u}^{(2k)}(\mathbf{x}) = \eta^{(k)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2, \tag{10c}$$

is solved in order to determine $\mathrm{u}^{(2k)}(\mathbf{x})$, $\mathbf{x} \in \Omega$, and $\mathrm{q}^{(2k)}(\mathbf{x}) \equiv -\mathbf{n}(\mathbf{x}) \cdot \left(\mathbb{K}\, \nabla \mathrm{u}^{(2k)}(\mathbf{x}) \right)$, $\mathbf{x} \in \Gamma_2$.

Step 3. Repeat steps 1 and 2 until a prescribed stopping criterion is satisfied.

Remark 3.2 *The value* $\omega = 1$ *in Eq.* (9) *corresponds to the alternating iterative algorithm introduced by* Kozlov, Maz′ya and Fomin (1991) *with an initial guess for the Dirichlet data, while the values* $\omega \in (0,1)$ *and* $\omega \in (1,2)$ *in Eq.* (9) *correspond to the alternating iterative algorithm introduced by* Kozlov, Maz′ya and Fomin (1991) *with an initial guess for the Dirichlet data and a constant under- and over-relaxation factor, respectively.*

The convergence of the alternating iterative algorithm with relaxation II presented above can be recast in the following convergence theorem, with the mention that a similar result can also be obtained for the alternating iterative algorithm with relaxation I:

Theorem 3.1 *Let* $\widetilde{\mathrm{u}} \in \mathrm{H}^{1/2}(\Gamma_1)$ *and* $\widetilde{\mathrm{q}} \in \left(\mathrm{H}_{00}^{1/2}(\Gamma_1) \right)^*$*, and assume that the Cauchy problem* (4a)–(4c) *has a solution* $\mathrm{u} \in \mathrm{H}^1(\Omega)$*. Let* $\mathrm{u}^{(k)}$ *be the k-th approximate solution in the alternating procedure II described above. Then there exists a number* $1 < b \leq 2$ *such that when the relaxation parameter* ω *is chosen with* $1 \leq \omega \leq b$*, then*

$$\lim_{k \to \infty} \left\| \mathrm{u} - \mathrm{u}^{(k)} \right\|_{\mathrm{H}^1(\Omega)} = 0 \tag{11}$$

for any initial data element $\eta^{(1)} \in \mathrm{H}^{1/2}(\Gamma_2)$.

The proof for this theorem in the case of the proposed relaxation algorithms associated with the Cauchy problem for the steady-state anisotropic heat conduction is similar to that for the corresponding relaxation algorithms for the Cauchy problem in elasticity, see Marin and Johansson (2010a). The proof given by Marin and Johansson (2010a) is based on the reformulation of the Cauchy problem (4a)–(4c) as a fixed point operator equation with a self-adjoint, injective, positive definite and non-expansive operator, while the scheme is shown to be a fixed point iteration for that equation. An alternative proof for the convergence result can also be found in Jourhmane, Lesnic and Mera (2004). As reported by Marin and Johansson (2010a) for Cauchy problems associated with the Navier-Lamé system of elasticity, it was also found for two-dimensional steady-state anisotropic heat conduction Cauchy problems that a relaxation factor $\omega > 2$ cannot be employed since the iterative process becomes divergent in such a situation.

4 The method of fundamental solutions

The fundamental solution G of the heat balance equation (2) or (4a) for two-dimensional steady heat conduction in anisotropic homogeneous media is given by, see e.g. Fairweather and Karageorghis (1998)

$$G(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{2\pi\sqrt{\det \mathbb{K}}} \log\left(\frac{1}{\mathrm{R}}\right), \quad \mathbf{x} \in \overline{\Omega}, \quad \boldsymbol{\xi} \in \mathbb{R}^2 \setminus \overline{\Omega}, \tag{12}$$

where $\mathrm{R} = \sqrt{(\mathbf{x} - \boldsymbol{\xi}) \cdot \mathbb{K}^{-1} (\mathbf{x} - \boldsymbol{\xi})}$ and $\boldsymbol{\xi}$ is a singularity or source point. The main idea of the MFS consists of the approximation of the temperature in the solution domain by a linear combination of fundamental solutions with respect to M singularities $\boldsymbol{\xi}^{(j)}, j = 1, \ldots, M$, in the form

$$\mathrm{u}(\mathbf{x}) \approx \mathrm{u}_M(\mathbf{c}, \boldsymbol{\xi}; \mathbf{x}) = \sum_{j=1}^{M} \mathrm{c}_j \, G(\mathbf{x}, \boldsymbol{\xi}^{(j)}), \quad \mathbf{x} \in \overline{\Omega}, \tag{13}$$

where $\mathbf{c} = [\mathrm{c}_1, \ldots, \mathrm{c}_M]^{\mathsf{T}}$ and $\boldsymbol{\xi} \in \mathbb{R}^{2M}$ is a vector containing the coordinates of the singularities $\boldsymbol{\xi}^{(j)}$, $j = 1, \ldots, M$. On taking into account the definitions of the normal heat flux (3) and the fundamental solution (12) then the normal heat flux, through a curve defined by the outward unit normal vector $\mathbf{n}(\mathbf{x})$, can be approximated on the boundary $\partial\Omega$ by

$$\mathrm{q}(\mathbf{x}) \approx \mathrm{q}_M(\mathbf{c}, \boldsymbol{\xi}; \mathbf{x}) = \sum_{j=1}^{M} \mathrm{c}_j \, H(\mathbf{x}, \boldsymbol{\xi}^{(j)}), \quad \mathbf{x} \in \partial\Omega, \tag{14}$$

where

$$H(\mathbf{x},\boldsymbol{\xi}) = -\mathbf{n}(\mathbf{x})\cdot(\mathbb{K}\,\nabla_{\mathbf{x}}G(\mathbf{x},\boldsymbol{\xi})) = \frac{1}{2\pi\mathrm{R}\sqrt{\det\mathbb{K}}}\left[\frac{\mathbf{x}-\boldsymbol{\xi}}{\mathrm{R}}\cdot\mathbf{n}(\mathbf{x})\right], \tag{15}$$

$\mathbf{x}\in\partial\Omega, \quad \boldsymbol{\xi}\in\mathbb{R}^2\setminus\overline{\Omega}$.

Next, we select N_1 MFS collocation points $\mathbf{x}^{(i)}, i=1,\ldots,N_1$, on the boundary Γ_1 and N_2 MFS collocation points $\mathbf{x}^{(N_1+i)}, i=1,\ldots,N_2$, on the boundary Γ_2, such that the total number of MFS collocation points used to discretise the boundary $\partial\Omega$ of the solution domain Ω is given by $N=N_1+N_2$.

According to the MFS approximations (13) and (14), the discretised versions of the the boundary value problems (5a)–(5c) and (7a)–(7c) recast as

$$\mathbf{A}^{(1)}\,\mathbf{c}^{(2k-1)} = \mathbf{b}^{(2k-1)}, \quad k\geq 2, \tag{16}$$

and

$$\mathbf{A}^{(2)}\,\mathbf{c}^{(2k)} = \mathbf{b}^{(2k)}, \quad k\geq 1, \tag{17}$$

respectively. Here the components of the MFS matrices and right-hand side vectors corresponding to Eqs. (16) and (17) are given by

$$\mathrm{A}_{ij}^{(1)} = \begin{cases} H(\mathbf{x}^{(i)},\boldsymbol{\xi}^{(j)}), & i=1,\ldots,N_1, \quad j=1,\ldots,M, \\ G(\mathbf{x}^{(i)},\boldsymbol{\xi}^{(j)}), & i=N_1+1,\ldots,N_1+N_2, \quad j=1,\ldots,M, \end{cases} \tag{18a}$$

$$\mathrm{b}_i^{(2k-1)} = \begin{cases} \widetilde{\mathrm{q}}(\mathbf{x}^{(i)}), & i=1,\ldots,N_1, \\ \mathrm{u}^{(2k-2)}(\mathbf{x}^{(i)}), & i=N_1+1,\ldots,N_1+N_2, \end{cases} \tag{18b}$$

and

$$\mathrm{A}_{ij}^{(2)} = \begin{cases} G(\mathbf{x}^{(i)},\boldsymbol{\xi}^{(j)}), & i=1,\ldots,N_1, \quad j=1,\ldots,M, \\ H(\mathbf{x}^{(i)},\boldsymbol{\xi}^{(j)}), & i=N_1+1,\ldots,N_1+N_2, \quad j=1,\ldots,M, \end{cases} \tag{19a}$$

$$\mathrm{b}_i^{(2k)} = \begin{cases} \widetilde{\mathrm{u}}(\mathbf{x}^{(i)}), & i=1,\ldots,N_1, \\ \xi^{(k)}(\mathbf{x}^{(i)}), & i=N_1+1,\ldots,N_1+N_2, \end{cases} \tag{19b}$$

respectively.

Each of Eqs. (16) and (17) represents a system of N linear algebraic equations with M unknowns, namely the MFS coefficients $\mathbf{c}^{(2k-1)} = \left[\mathrm{c}_1^{(2k-1)},\ldots,\mathrm{c}_M^{(2k-1)}\right]^{\mathsf{T}}$ and $\mathbf{c}^{(2k)} = \left[\mathrm{c}_1^{(2k)},\ldots,\mathrm{c}_M^{(2k)}\right]^{\mathsf{T}}$, respectively. It should be noted that in order to uniquely determine the solutions $\mathbf{c}^{(2k-1)} \in \mathbb{R}^M$ and $\mathbf{c}^{(2k)} \in \mathbb{R}^M$ to the systems of linear algebraic equations (16) and (17), respectively, the number N of MFS boundary collocation points on the boundary $\partial\Omega$ and the number M of singularities must satisfy the inequality $M \leq N$. However, the systems of linear algebraic equations (16) and (17) cannot be solved by direct methods, such as the least-squares method, since such an approach would produce a highly unstable solution in the case of noisy Cauchy data on Γ_1.

In order to implement the MFS, the location of the singularities has to be determined and this is usually achieved by considering either the static or the dynamic approach. In the static approach, the singularities are pre-assigned and kept fixed throughout the solution process, whilst in the dynamic approach, the singularities and the unknown coefficients are determined simultaneously during the solution process, see Fairweather and Karageorghis (1998). Thus the dynamic approach transforms the inverse problem into a more difficult nonlinear ill-posed problem which is also computationally much more expensive. The advantages and disadvantages of the MFS with respect to the location of the fictitious sources are described at length in Heise (1978) and Burgess and Maharejin (1984). Recently, Gorzelańczyk and Kołodziej (2008) thoroughly investigated the performance of the MFS with respect to the shape of the pseudo-boundary on which the source points are situated, proving that, for the same number of boundary collocation points and sources, more accurate results are obtained if the shape of the pseudo-boundary is similar to that of the boundary of the solution domain. Therefore, we employ the static approach in our computations, at the same time accounting for the findings of Gorzelańczyk and Kołodziej (2008).

5 The Tikhonov regularization method

Since the right-hand sides of the systems of linear algebraic equations (16) and (17) are in general polluted by noise, the retrieval of accurate and stable solutions to Eqs. (16) and (17) is very important for obtaining physically meaningful numerical results. For perturbed right-hand sides in Eqs. (16) and (17), the direct inversion of these equations or, equivalently, a least-squares minimization applied to Eqs. (16) and (17) will fail to produce stable, accurate and physically meaningful numerical solutions. It is the purpose of this section to present a classical regularization procedure for obtaining such solutions to the systems of linear algebraic equations (16) and (17), as well as details regarding the optimal choice of the regularization parameter.

Several regularization techniques used for the stable solution of systems of linear and nonlinear algebraic equations are available in the literature, such as the singular value decomposition (Hansen , 1998), the Tikhonov regularization method

(Tikhonov and Arsenin , 1986), the fictitious time integration method (FTIM) (Liu, 2008b) and various iterative methods (Kunisch and Zou , 1998). In this study, we employ the Tikhonov regularization method. Consider the following system of linear algebraic equations

$$\mathbf{A}\,\mathbf{c} = \mathbf{b}, \tag{20}$$

where $N \geq M$, $\mathbf{A} \in \mathbb{R}^{N\times M}$, $\mathbf{c} \in \mathbb{R}^{M}$ and $\mathbf{b} \in \mathbb{R}^{N}$. Note that Eq. (20) may describe each of the MFS systems of linear equations (16) and (17), provided that

$$\mathbf{A} = \mathbf{A}^{(1)}, \qquad \mathbf{c} = \mathbf{c}^{(2k-1)}, \qquad \mathbf{b} = \mathbf{b}^{(2k-1)}, \qquad k \geq 2, \tag{21}$$

and

$$\mathbf{A} = \mathbf{A}^{(2)}, \qquad \mathbf{c} = \mathbf{c}^{(2k)}, \qquad \mathbf{b} = \mathbf{b}^{(2k)}, \qquad k \geq 1, \tag{22}$$

respectively. The Tikhonov regularized solution to the generically written system of linear algebraic equations (20) is sought as, see Tikhonov and Arsenin (1986)

$$\mathbf{c}_\lambda \in \mathbb{R}^M : \quad \mathscr{F}_\lambda\left(\mathbf{c}_\lambda\right) = \min_{\mathbf{c}\in\mathbb{R}^M} \mathscr{F}_\lambda\left(\mathbf{c}\right), \tag{23}$$

where $\mathscr{F}_\lambda$ represents the Tikhonov regularization functional given by, see Tikhonov and Arsenin (1986)

$$\mathscr{F}_\lambda(\cdot) : \mathbb{R}^M \longrightarrow [0,\infty), \quad \mathscr{F}_\lambda\left(\mathbf{c}\right) = \left\|\mathbf{A}\,\mathbf{c} - \mathbf{b}\right\|^2 + \lambda^2\left\|\mathbf{c}\right\|^2, \tag{24}$$

and $\lambda > 0$ is the regularization parameter to be prescribed. Formally, the Tikhonov regularized solution $\mathbf{c}_\lambda$ of the problem (20) is given as the solution of the normal equation

$$\left(\mathbf{A}^{\mathsf{T}}\mathbf{A} + \lambda^2\mathbf{I}_M\right)\mathbf{c} = \mathbf{A}^{\mathsf{T}}\,\mathbf{b}, \tag{25}$$

where $\mathbf{I}_M \in \mathbb{R}^{M\times M}$ is the identity matrix, namely

$$\mathbf{c}_\lambda = \mathbf{A}^{\dagger}\,\mathbf{b}, \quad \mathbf{A}^{\dagger} \equiv \left(\mathbf{A}^{\mathsf{T}}\mathbf{A} + \lambda^2\mathbf{I}_M\right)^{-1}\mathbf{A}^{\mathsf{T}}. \tag{26}$$

To summarize, the Tikhonov regularization method solves a constrained minimization problem using a smoothness norm in order to provide a stable solution which fits the data and also has a minimum structure.

The performance of regularization methods depends crucially on the suitable choice of the regularization parameter. One extensively studied criterion is the discrepancy principle (Morozov , 1966). Although this criterion is mathematically rigorous, it requires a reliable estimation of the amount of noise added into the data which may not be available in practical problems. Heuristic approaches are preferable in the case when no *a priori* information about the noise is available. For the Tikhonov regularization method, several heuristic approaches have been proposed, including the L-curve criterion (Hansen , 1998) and the GCV (Wahba , 1977). In this paper, we employ the GCV criterion to determine the optimal regularization parameter, λ_{opt}, for the Tikhonov regularization method, namely

$$\lambda_{\mathrm{opt}}: \quad \mathscr{G}\left(\lambda_{\mathrm{opt}}\right) = \min_{\lambda>0} \mathscr{G}(\lambda). \tag{27}$$

Here

$$\mathscr{G}(\cdot):(0,\infty) \longrightarrow [0,\infty), \quad \mathscr{G}(\lambda) = \frac{\left\|\mathbf{A}\,\mathbf{c}_\lambda - \mathbf{b}^\epsilon\right\|^2}{\left[\mathrm{trace}\left(\mathbf{I}_N - \mathbf{A}\,\mathbf{A}^\dagger\right)\right]^2}, \tag{28}$$

where $\mathbf{c}_\lambda$ is given by Eq. (26) with $\mathbf{b} = \mathbf{b}^\epsilon$, while $\mathbf{b}^\epsilon$ is a perturbation of the right-hand side, $\mathbf{b}$, of Eq. (20) such that $\left\|\mathbf{b}^\epsilon - \mathbf{b}\right\| \le \epsilon$.

6 Numerical results

In this section, we present the performance of the proposed numerical method, namely the alternating iterative MFS algorithms with relaxation presented in Section 3. To do so, we solve numerically the Cauchy problem given by Eqs. (4a)–(4c) for the steady-state anisotropic heat conduction in the two-dimensional geometries described below and analyse the numerical convergence and stability of this procedure, as well as the influence of the constant relaxation parameter, ω.

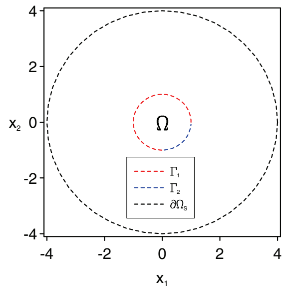
6.1 Examples

Example 1. (*Simply connected domain with a smooth boundary, see Fig.* 1(*a*)) We consider the following analytical solutions for the temperature and normal heat flux

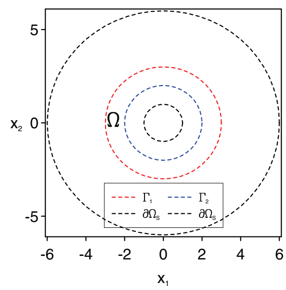
$$\mathrm{u}^{(\mathrm{an})}(\mathbf{x}) = \mathrm{x}_1^2 - 4\mathrm{x}_1\mathrm{x}_2 + \mathrm{x}_2^2, \quad \mathbf{x} = (\mathrm{x}_1,\mathrm{x}_2) \in \overline{\Omega}, \tag{29a}$$

$$\mathrm{q}^{(\mathrm{an})}(\mathbf{x}) = 3\left[\mathrm{x}_2\,\mathrm{n}_1(\mathbf{x}) + \mathrm{x}_1\,\mathrm{n}_2(\mathbf{x})\right], \quad \mathbf{x} = (\mathrm{x}_1,\mathrm{x}_2) \in \partial\Omega, \tag{29b}$$

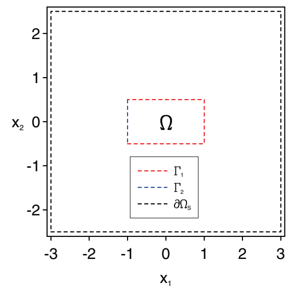
respectively, in the unit disk $\Omega = \left\{\mathbf{x} \in \mathbb{R}^2 \,\middle|\, \rho(\mathbf{x}) < \mathrm{r}\right\}$, where $\rho(\mathbf{x}) = \sqrt{\mathrm{x}_1^2 + \mathrm{x}_2^2}$ is the radial polar coordinate of $\mathbf{x}$ and $\mathrm{r} = 1.0$. Here $\mathrm{K}_{11} = 1.0$, $\mathrm{K}_{12} = \mathrm{K}_{21} = 0.5$, $\mathrm{K}_{22} = 1.0$, $\Gamma_1 = \left\{\mathbf{x} \in \partial\Omega \,\middle|\, 0 \le \theta(\mathbf{x}) \le 3\pi/2\right\}$ and $\Gamma_2 = \{\mathbf{x} \in \partial\Omega \,|\, 3\pi/2 < \theta(\mathbf{x}) < 2\pi\}$, where $\theta(\mathbf{x})$ is the angular polar coordinate of $\mathbf{x}$.



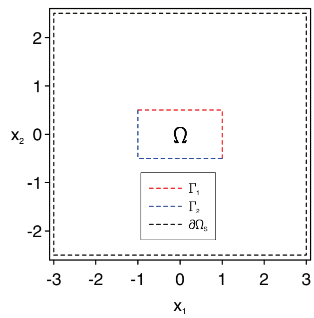
(a) Example 1:
Disk, $|\Gamma_1|/|\Gamma_2| = 3$



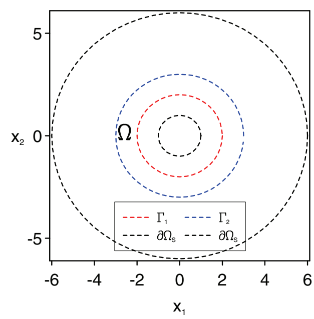
(b) Example 2:
Annulus, $|\Gamma_1|/|\Gamma_2| = 3/2$



(c) Example 3:
Rectangle, $|\Gamma_1|/|\Gamma_2| = 5$



(d) Example 3:
Rectangle, $|\Gamma_1|/|\Gamma_2| = 1$



(e) Example 4:
Annulus, $|\Gamma_1|/|\Gamma_2| = 2/3$

Figure 1: Schematic diagram of the domain, Ω, over-determined boundary, Γ_1 (- - -), under-determined boundary, Γ_2 (- - -), and pseudo-boundary, $\partial\Omega_S$ (- - -), for the inverse problems investigated, namely (a) Example 1 (disk), $|\Gamma_1|/|\Gamma_2| = 3$, (b) Example 2 (annulus), $|\Gamma_1|/|\Gamma_2| = 3/2$, (c) Example 3 (rectangle), $|\Gamma_1|/|\Gamma_2| = 5$, (d) Example 3 (rectangle), $|\Gamma_1|/|\Gamma_2| = 1$, and (e) Example 4 (annulus), $|\Gamma_1|/|\Gamma_2| = 2/3$, respectively.

Example 2. (*Doubly connected domain with a smooth boundary, see Fig.* 1(*b*)) Consider the following analytical solutions for the temperature and normal heat flux

$$u^{(an)}(\mathbf{x}) = \frac{1}{5}x_1^3 - x_1^2 x_2 + x_1 x_2^2 + \frac{1}{3}x_2^3, \quad \mathbf{x} = (x_1, x_2) \in \overline{\Omega}, \tag{30a}$$

$$q^{(an)}(\mathbf{x}) = -\left(x_1^2 - 6x_1x_2 + 6x_2^2\right) n_1(\mathbf{x}) - \left(\frac{1}{5}x_1^2 - 2x_1x_2 + 3x_2^2\right) n_2(\mathbf{x}), \tag{30b}$$

$\mathbf{x} = (x_1, x_2) \in \partial\Omega,$

respectively, in the annular domain $\Omega = \left\{ \mathbf{x} \in \mathbb{R}^2 \,\middle|\, \mathrm{r_{int}} < \rho(\mathbf{x}) < \mathrm{r_{out}} \right\}$, where $\mathrm{r_{int}} = 2.0$ and $\mathrm{r_{out}} = 3.0$. Here $\mathrm{K}_{11} = 5.0$, $\mathrm{K}_{12} = \mathrm{K}_{21} = 2.0$, $\mathrm{K}_{22} = 1.0$, $\Gamma_1 = \Gamma_{\mathrm{out}} = \left\{ \mathbf{x} \in \partial\Omega \,\middle|\, \rho(\mathbf{x}) = \mathrm{r_{out}} \right\}$ and $\Gamma_2 = \Gamma_{\mathrm{int}} = \left\{ \mathbf{x} \in \partial\Omega \,\middle|\, \rho(\mathbf{x}) = \mathrm{r_{int}} \right\}$ such that $|\Gamma_1|/|\Gamma_2| = 3/2$.

Example 3. (*Simply connected domain with a piecewise smooth boundary, see Figs.* $1(c)$ *and* $1(d)$) We consider the rectangle $\Omega = (-\mathrm{r},\mathrm{r}) \times \left(-\mathrm{r}/2, \mathrm{r}/2\right)$, where $\mathrm{r} = 1.0$, and the same analytical solutions for the temperature and normal heat flux on the boundary $\partial\Omega$ as those corresponding to Example 2, namely Eqs. (30a) and (30b), respectively, with $\mathrm{K}_{11} = 5.0$, $\mathrm{K}_{12} = \mathrm{K}_{21} = 2.0$, $\mathrm{K}_{22} = 1.0$. Here two situations are considered, namely (i) $\Gamma_1 = [-\mathrm{r},\mathrm{r}] \times \left\{\pm\mathrm{r}/2\right\} \cup \{\mathrm{r}\} \times \left(-\mathrm{r}/2, \mathrm{r}/2\right)$ and $\Gamma_2 = \{-\mathrm{r}\} \times \left(-\mathrm{r}/2, \mathrm{r}/2\right)$ such that $|\Gamma_1|/|\Gamma_2| = 5$; and (ii) $\Gamma_1 = \{\mathrm{r}\} \times \left[-\mathrm{r}/2, \mathrm{r}/2\right] \cup [-\mathrm{r},\mathrm{r}] \times \left\{\mathrm{r}/2\right\}$ and $\Gamma_2 = \{-\mathrm{r}\} \times (-\mathrm{r}/2, \mathrm{r}/2) \cup [-\mathrm{r},\mathrm{r}) \times \{-\mathrm{r}/2\}$ such that $|\Gamma_1|/|\Gamma_2| = 1$.

Example 4. (*Doubly connected domain with a smooth boundary, see Fig.* $1(e)$) Consider the temperature distribution in the annular domain $\Omega = \{\mathbf{x} \in \mathbb{R}^2 \,|\, \mathrm{r_{int}} < \rho(\mathbf{x}) < \mathrm{r_{out}}\}$ ($\mathrm{r_{int}} = 2.0$ and $\mathrm{r_{out}} = 3.0$) and the corresponding normal heat fluxes on the boundary associated with the following exact boundary temperatures

$$\mathrm{u}^{(\mathrm{an})}(\mathbf{x}) = \mathrm{u}_0, \quad \mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \Gamma_{\mathrm{out}} = \left\{ \mathbf{x} \in \partial\Omega \,\middle|\, \rho(\mathbf{x}) = \mathrm{r_{out}} \right\}, \tag{31a}$$

$$\mathrm{u}^{(\mathrm{an})}(\mathbf{x}) = \mathrm{u}_0 \left(1 - \mathrm{x}_1 / \mathrm{r_{int}}\right), \quad \mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \Gamma_{\mathrm{int}} = \left\{ \mathbf{x} \in \partial\Omega \,\middle|\, \rho(\mathbf{x}) = \mathrm{r_{int}} \right\}. \tag{31b}$$

Here $\mathrm{u}_0 = 1.0$, $\mathrm{K}_{11} = 2.0$, $\mathrm{K}_{12} = \mathrm{K}_{21} = 0.5$, $\mathrm{K}_{22} = 1.0$, $\Gamma_1 = \Gamma_{\mathrm{int}}$ and $\Gamma_2 = \Gamma_{\mathrm{out}}$ such that $|\Gamma_1|/|\Gamma_2| = 2/3$.

Although not presented herein, it is reported that the numerical results obtained for the unknown temperature and normal heat flux on the boundary Γ_2 are convergent with respect to increasing the distance d_{S} between the physical boundary $\partial\Omega$ and the pseudo-boundary $\partial\Omega_{\mathrm{S}}$. However, it should be noted that for the geometries considered herein the value $d_{\mathrm{S}} = 1.0$ was found to be sufficiently large such that any further increase of the distance between the singularities and the boundary of the solution domain did not significantly improve the accuracy of the numerical solutions for the examples tested in this paper.

The Cauchy problems investigated in this paper have been solved using the uniform distribution of both the MFS boundary collocation points $\mathbf{x}^{(i)}$, $i = 1, \ldots, N$, and the singularities $\boldsymbol{\xi}^{(j)}$, $j = 1, \ldots, M$. Furthermore, the numbers of MFS boundary collocation points N_1 and N_2 corresponding to the over- and under-specified boundaries Γ_1 and Γ_2, respectively, the number of singularities M and the distance d_{S} between the physical boundary $\partial\Omega$ and the pseudo-boundary $\partial\Omega_{\mathrm{S}}$ on which the singularities are situated, were set to:

(i) $N_1/3 = N_2 = N/4 = 20$, $M = N/2$ and $d_{\mathrm{S}} = 3.0$, for Example 1;

(ii) $N_1/3 = N_2/2 = N/5 = 20$, $M = N_1 + N_2/2 = 4N/5$, while $d_S = 1.0$ and $d_S = 3.0$ for the inner and outer boundaries, respectively, in the case of Example 2;

(iii) $N_1 = 72$, $N_2 = 14$ $(N_1 = N_2 = 43)$, $M = N_1 + N_2 = N$ and $d_S = 2.0$, for Example 3 with $|\Gamma_1|/|\Gamma_2| = 5$ $(|\Gamma_1|/|\Gamma_2| = 1)$;

(iv) $N_1/2 = N_2/3 = N/5 = 20$, $M = N_1 + N_2/2 = 4N/5$, while $d_S = 1.0$ and $d_S = 3.0$ for the inner and outer boundaries, respectively, in the case of Example 4.

It should be noted that all of the numerical computations related to the Cauchy problems analysed in this paper have been performed using a set of uniformly distributed collocation points $\mathbf{z}^{(\ell)}$, $\ell = 1, \ldots, L$, on the boundary $\partial\Omega$, where the numbers of collocation points L_1 and L_2 $(L_1 + L_2 = L)$ corresponding to the over- and under-specified boundaries Γ_1 and Γ_2, respectively, were set to:

(i) $L_1/3 = L_2 = L/4 = 50$ for Example 1;

(ii) $L_1/3 = L_2/2 = L/5 = 60$ in the case of Example 2;

(iii) $L_1/5 = L_2 = L/6 = 50$ for Example 3;

(iv) $L_1/2 = L_2/3 = L/5 = 60$ in the case of Example 4.

It is important to mention that the two mixed, well-posed, direct problems (5a)–(5c) and (7a)–(7c), and (8a)–(8c) and (10a)–(10c), corresponding to every iteration of the algorithm with relaxation I and II, respectively, have been solved numerically using the aforementioned MFS discretisations, in conjunction with the Tikhonov regularization method presented in Section 5. The optimal value of the regularization parameter, λ_{opt}, corresponding to every direct problem considered in the algorithms with relaxation I and II has automatically been chosen according to the GCV criterion (27) and (28). For the inverse problems investigated in this paper, as well as the alternating iterative algorithms I and II, we have taken homogeneous initial guesses for both the normal heat flux, $\mathrm{q}^{(1)}$, and the temperature, $\mathrm{u}^{(1)}$. Moreover, all numerical computations have been performed in FORTRAN 90 in double precision on a 3.00 GHz Intel Pentium 4 machine.

6.2 Results obtained with exact data: Convergence of the algorithms

If L_i collocation points, $\mathbf{z}^{(\ell)}$, $\ell = 1, \ldots, L_i$, are considered on the boundary $\Gamma_i \subset \partial\Omega$ then the *root mean square error* (RMS error) associated with the real valued function $f(\cdot) : \Gamma_i \longrightarrow \mathbb{R}$ on Γ_i is defined by

$$\mathrm{RMS}_{\Gamma_\mathrm{i}}(f) = \sqrt{\frac{1}{L_i}\sum_{\ell=1}^{L_i} f\left(\mathbf{z}^{(\ell)}\right)^2}, \tag{32}$$

In order to investigate the convergence of the algorithm, at every iteration, $k \geq 1$, we evaluate the following accuracy errors corresponding to the temperature and normal heat flux on the under-specified boundary, Γ_2, which are defined as *relative RMS errors*, i.e.

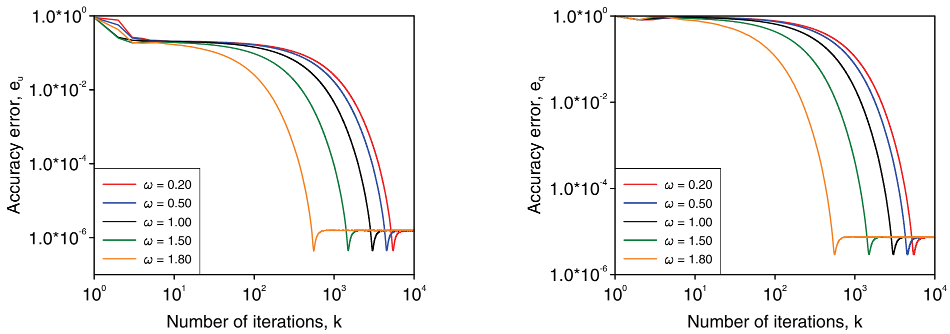
$$\mathrm{e_u}(k) = \begin{cases} \dfrac{\mathrm{RMS}_{\Gamma_2}\left(\mathrm{u}^{(2k)} - \mathrm{u}^{(\mathrm{an})}\right)}{\mathrm{RMS}_{\Gamma_2}\left(\mathrm{u}^{(\mathrm{an})}\right)} & \text{for the alternating iterative algorithm with relaxation I} \\[2ex] \dfrac{\mathrm{RMS}_{\Gamma_2}\left(\mathrm{u}^{(2k-1)} - \mathrm{u}^{(\mathrm{an})}\right)}{\mathrm{RMS}_{\Gamma_2}\left(\mathrm{u}^{(\mathrm{an})}\right)} & \text{for the alternating iterative algorithm with relaxation II} \end{cases} \tag{33a}$$

and

$$\mathrm{e_q}(k) = \begin{cases} \dfrac{\mathrm{RMS}_{\Gamma_2}\left(\mathrm{q}^{(2k-1)} - \mathrm{q}^{(\mathrm{an})}\right)}{\mathrm{RMS}_{\Gamma_2}\left(\mathrm{q}^{(\mathrm{an})}\right)} & \text{for the alternating iterative algorithm with relaxation I} \\[2ex] \dfrac{\mathrm{RMS}_{\Gamma_2}\left(\mathrm{q}^{(2k)} - \mathrm{q}^{(\mathrm{an})}\right)}{\mathrm{RMS}_{\Gamma_2}\left(\mathrm{q}^{(\mathrm{an})}\right)} & \text{for the alternating iterative algorithm with relaxation II.} \end{cases} \tag{33b}$$

Here $\mathrm{u}^{(2k)}$ $\left(\mathrm{u}^{(2k-1)}\right)$ and $\mathrm{q}^{(2k-1)}$ $\left(\mathrm{q}^{(2k)}\right)$ are the temperature and normal heat flux on the boundary Γ_2 retrieved after k iterations using the alternating iterative algorithm with relaxation I (II), respectively, with the mention that each iteration consists of solving two direct well-posed mixed boundary value problems, namely Eqs. (5a)–(5c) and (7a)–(7c) for the alternating iterative algorithm with relaxation I (Eqs. (8a)–(8c) and (10a)–(10c) for the alternating iterative algorithm with relaxation II).

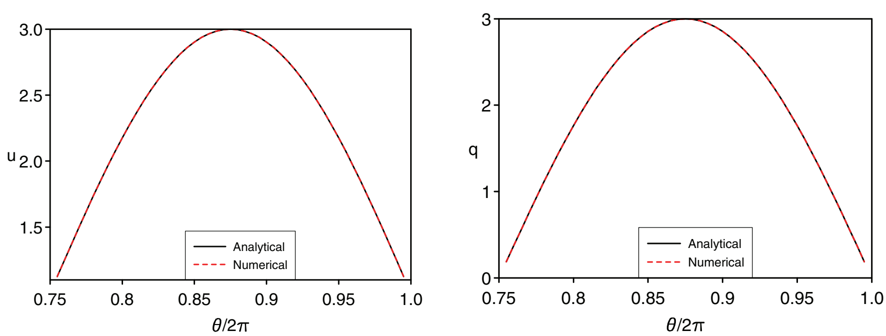
Figs. 2(a) and 2(b) show, on a logarithmic scale, the accuracy errors $\mathrm{e_u}$ and $\mathrm{e_q}$, as functions of the number of iterations, k, obtained using the alternating iterative algorithm II, exact Cauchy data and various values of the relaxation parameter ω,



(a) Example 1: $\mathrm{e_u}$ for $p_\mathrm{u} = p_\mathrm{q} = 0\%$ and algorithm II

(b) Example 1: $\mathrm{e_q}$ for $p_\mathrm{u} = p_\mathrm{q} = 0\%$ and algorithm II

Figure 2: The accuracy errors (a) $\mathrm{e_u}$, and (b) $\mathrm{e_q}$, as functions of the number of iterations, k, obtained using the alternating iterative algorithm with relaxation II, exact Cauchy data on Γ_1 and various values of the relaxation parameter, ω, namely $\omega \in \{0.20, 0.50, 1.00, 1.50, 1.80\}$, for Example 1.



(a) Example 1: $\mathrm{u}|_{\Gamma_2}$ for $p_\mathrm{u} = p_\mathrm{q} = 0\%$, $\omega =$ 1.90 and algorithm II

(b) Example 1: $\mathrm{q}|_{\Gamma_2}$ for $p_\mathrm{u} = p_\mathrm{q} = 0\%$, $\omega =$ 1.90 and algorithm II

Figure 3: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm II, exact Cauchy data on Γ_1, $\omega = 1.80$ and $k = 1000$ iterations, for Example 1.

in the case of Example 1. It can be seen from these figures that, for all values of the relaxation parameter used in this paper, both errors $\mathrm{e_u}$ and $\mathrm{e_q}$ keep decreasing until a specific number of iterations, after which they slightly increase until the convergence rate of the aforementioned accuracy errors becomes very slow so that they reach a plateau. As expected, for each value of the relaxation parameter employed, $\mathrm{e_u}(k) < \mathrm{e_q}(k)$ for all $k \geq 1$, i.e. temperatures are more accurate than normal heat fluxes; also, the larger the parameter ω, the lower the number of iterations and, consequently, the computational time are required for obtaining accurate numerical results for both the temperature and the normal heat flux on Γ_2. Therefore, choosing $\omega \in (1,2)$ in the alternating iterative algorithms I and II results in a significant reduction of the number of iterations as compared with the corresponding original alternating iterative algorithms proposed by Kozlov, Maz′ya and Fomin (1991), i.e. for $\omega = 1$. Furthermore, it can also be noticed from Figs. $2(a)$ and $2(b)$ that, for exact Cauchy data on Γ_1 and all the values of the relaxation factor employed, the errors in the numerical temperature, $\mathrm{e_u}$, and normal heat flux, $\mathrm{e_q}$, retrieved on Γ_2 corresponding to the plateau region mentioned above have almost the same values, i.e. the numerical results obtained using various values of the constant relaxation parameter, ω, are consistent.

The same conclusions can be drawn from Figs. $3(a)$ and $3(b)$, which illustrate the analytical and numerical temperature and normal heat flux, respectively, obtained with $\omega = 1.80$ after $k = 1000$ iterations. From Figs. 2 and 3, it can be concluded that the alternating iterative algorithm with relaxation II described in Section 3 provides excellent approximations for the unknown Cauchy data on Γ_2 and is convergent with respect to increasing the number of iterations, k, if exact Cauchy data are prescribed on the over-specified boundary Γ_1. Although not presented, it is reported that similar results have been obtained for Examples $2-4$ and all admissible values of the relaxation parameter, as well as the alternating iterative algorithm with relaxation I applied to all examples investigated in this study.

6.3 Regularizing stopping criterion

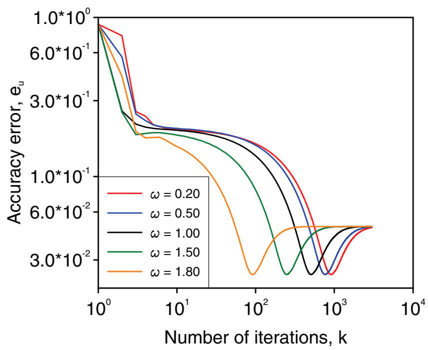
Once the convergence of the numerical solution to the exact solution, with respect to number of iterations performed, k, has been established, we investigate the stability of the numerical solution for the examples considered. In what follows, the temperature, $\mathrm{u}\big|_{\Gamma_1} = \mathrm{u}^{(\mathrm{an})}\big|_{\Gamma_1}$, and/or the normal heat flux, $\mathrm{q}\big|_{\Gamma_1} = \mathrm{q}^{(\mathrm{an})}\big|_{\Gamma_1}$, on the over-specified boundary have been perturbed as

$$\widetilde{\mathrm{u}}^{\epsilon}\big|_{\Gamma_1} = \mathrm{u}\big|_{\Gamma_1} + \delta\mathrm{u}, \quad \delta\mathrm{u} = \mathtt{G05DDF}(0,\sigma_{\mathrm{u}}), \quad \sigma_{\mathrm{u}} = \max_{\Gamma_1} |\mathrm{u}| \times (p_{\mathrm{u}}/100)\,, \tag{34}$$

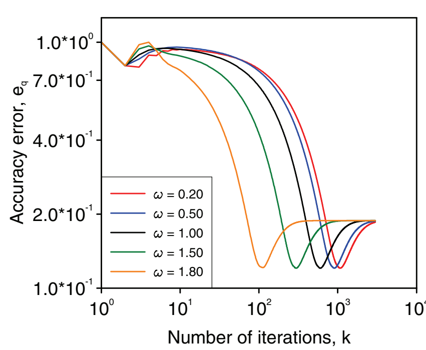
and

$$\widetilde{q}^{\epsilon}\big|_{\Gamma_1} = q\big|_{\Gamma_1} + \delta q, \quad \delta q = \texttt{G05DDF}(0, \sigma_q), \quad \sigma_q = \max_{\Gamma_1} |q| \times \left(p_q/100\right), \tag{35}$$

respectively. Here δu and δq are Gaussian random variables with mean zero and standard deviations σ_u and σ_q, respectively, generated by the NAG subroutine G05DDF (NAG Library Mark 21 , 2007), while $p_u\%$ and $p_q\%$ are the percentages of additive noise included into the input boundary temperature, $u\big|_{\Gamma_1}$, and normal heat flux, $q\big|_{\Gamma_1}$, respectively, in order to simulate the inherent measurement errors.



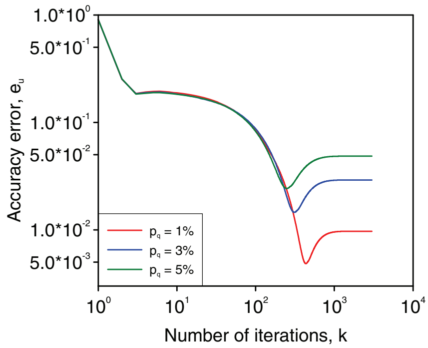
(a) Example 1: Accuracy error e_u



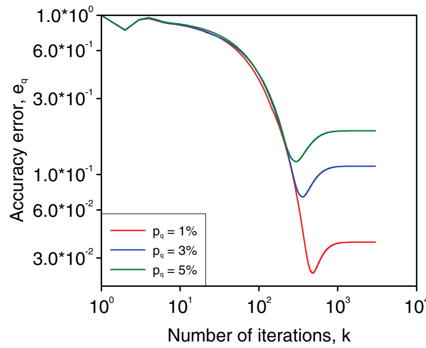
(b) Example 1: Accuracy error e_q

Figure 4: The accuracy errors (a) e_u, and (b) e_q, as functions of the number of iterations, k, obtained using the alternating iterative algorithm I, $p_q = 5\%$ noise added into the Neumann data on Γ_1 and various values of the relaxation parameter, ω, namely $\omega \in \{0.20, 0.50, 1.00, 1.50, 1.80\}$, for Example 1.

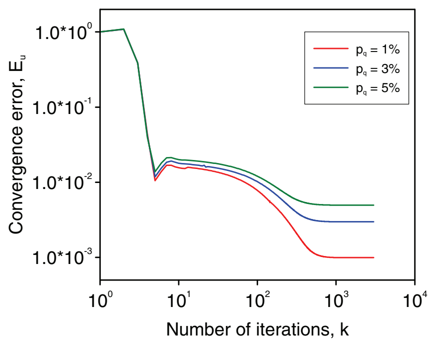
The evolution of the accuracy errors, e_u and e_q, as functions of the number of iterations, k, obtained using the alternating iterative algorithm I, $p_q = 5\%$ and $\omega \in \{0.20, 0.50, 1.00, 1.50, 1.80\}$, for Example 1, are displayed, on a logarithmic scale, in Figs. 4(a) and 4(b), respectively. From these figures it can be noted that the number of iterations required for both errors e_u and e_q to attain their corresponding minimum values (i.e. to obtain the optimal numerical solution to the Cauchy problem) decreases with respect to increasing the value of the relaxation parameter, ω. Similar to the case of exact Cauchy data on Γ_1, the inaccuracies in the numerical solutions for both the temperature and normal heat flux on the under-determined boundary Γ_2, obtained using various values of the relaxation parameter, have almost the same value, therefore emphasising the consistency of the proposed MFS-based iterative algorithms with relaxation with respect to the values of the relaxation parameter, $\omega \in (0, 2)$.



(a) Example 1: Accuracy error $\mathrm{e_u}$



(b) Example 1: Accuracy error $\mathrm{e_q}$



(c) Example 1: Convergence error $\mathrm{E_u}$

Figure 5: The accuracy errors (a) $\mathrm{e_u}$, and (b) $\mathrm{e_q}$, and (c) the convergence error, $\mathrm{E_u}$, as functions of the number of iterations, k, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into the Neumann data on Γ_1, namely $p_{\mathrm{q}} \in \{1\%, 3\%, 5\%\}$, for Example 1.

Figs. 5(a) and 5(b) present, on a logarithmic scale, the accuracy errors $\mathrm{e_u}$ and $\mathrm{e_q}$, respectively, as functions of the number of iterations, k, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and $p_{\mathrm{q}} \in \{1\%, 3\%, 5\%\}$, for the Cauchy problem given by Example 1. From these figures it can be seen that, for each fixed value of p_{q}, the errors in predicting the temperature and normal heat flux on the under-specified boundary Γ_2 decrease up to a certain iteration number and after that they start increasing. If the iterative process is continued beyond this point then the numerical solutions lose their smoothness and become highly oscillatory and unbounded, i.e. unstable. Therefore, a regularizing stopping criterion must be used in order to cease the iterative process at the point where the errors in the numerical solutions start increasing.

To define the stopping criterion required for regularizing/stabilizing the iterative methods analysed in this paper, after each iteration, k, we evaluate the following convergence error which is associated with the temperature on the over-specified boundary, Γ_1, namely

$$\mathrm{E_u}(k) = \begin{cases} \dfrac{\mathrm{RMS}_{\Gamma_1}\left(\mathrm{u}^{(2k-1)} - \widetilde{\mathrm{u}}^{\epsilon}\right)}{\mathrm{RMS}_{\Gamma_1}\left(\widetilde{\mathrm{u}}^{\epsilon}\right)} & \text{for the alternating iterative algorithm with relaxation I} \\[2ex] \dfrac{\mathrm{RMS}_{\Gamma_1}\left(\mathrm{u}^{(2k)} - \widetilde{\mathrm{u}}^{\epsilon}\right)}{\mathrm{RMS}_{\Gamma_1}\left(\widetilde{\mathrm{u}}^{\epsilon}\right)} & \text{for the alternating iterative algorithm with relaxation II.} \end{cases} \tag{36}$$

Here $\mathrm{u}^{(2k-1)}$ $\left(\mathrm{u}^{(2k)}\right)$ is the temperature on the boundary Γ_1, retrieved numerically after k iterations by solving the well-posed mixed direct boundary value problem (5a)–(5c) [(10a)–(10c)], in the case of the alternating iterative algorithm I (II), while $\widetilde{\mathrm{u}}^{\epsilon}$ is the perturbed Dirichlet data (boundary temperature) on the over-specified boundary Γ_1, as given by Eq. (34). This error $\mathrm{E_u}$ should tend to zero as the sequences $\left\{\mathrm{u}^{(2k-1)}\right\}_{k\geq 1}$ and $\left\{\mathrm{u}^{(2k)}\right\}_{k\geq 1}$ tend to the analytical solution, $\mathrm{u}^{(\mathrm{an})}$, in the space $\mathrm{H}^1(\Omega)$ and hence it is expected to provide an appropriate stopping criterion. Indeed, if we investigate the error $\mathrm{E_u}$ obtained at each iteration for Example 1, using the alternating iterative algorithm I, $\omega = 1.50$ and $p_\mathrm{q} \in \{1\%, 3\%, 5\%\}$, we obtain the curves graphically represented in Fig. 5(c). By comparing Figs. 5(a)–(c), it can be noticed that the convergence error $\mathrm{E_u}$, as well as the accuracy errors $\mathrm{e_u}$ and $\mathrm{e_q}$, attain their corresponding minimum at around the same number of iterations. Therefore, for noisy Cauchy data a natural stopping criterion terminates the MFS iterative algorithms with relaxation I and II at the optimal number of iterations, k_{opt}, given by:

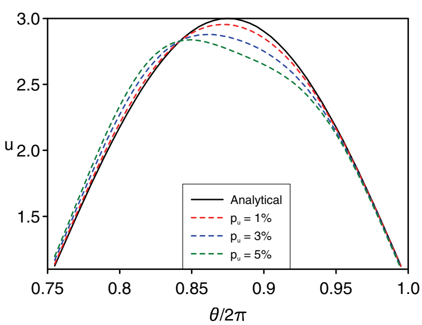
$$k_{\mathrm{opt}}: \quad \mathrm{E_u}(k_{\mathrm{opt}}) = \min_{k\geq 1} \mathrm{E_u}(k). \tag{37}$$

Although not illustrated, it is important to mention that similar results and conclusions have been obtained for the other examples considered, $\omega \in (0,2)$ and the MFS-based iterative algorithm with relaxation II.

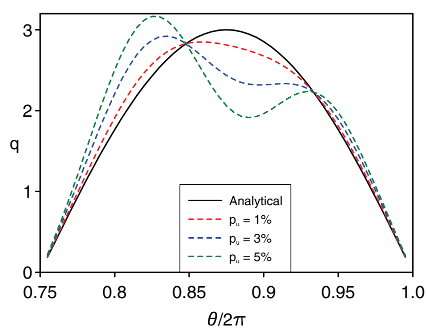
As mentioned in the previous section, for exact data the iterative process is convergent with respect to increasing the number of iterations, k, see e.g. Fig. 2 for Example 1, and hence a stopping criterion is not necessary in this case. However, even in this situation the errors $\mathrm{E_u}$, $\mathrm{e_u}$ and $\mathrm{e_q}$ have a similar behaviour and, consequently, the error $\mathrm{E_u}$ may be used to stop the iterative process at the point where the rate of convergence is very small and no substantial improvement in the numerical solution is obtained even if the iterative process is continued. Therefore, it can be concluded that the regularizing stopping criterion (37) proposed for the MFS-based iterative algorithms with relaxation I and II is very efficient in locating the point where the errors start increasing and the iterative process should be stopped.

6.4 Results obtained with noisy data: Stability of the algorithms

Based on the stopping criterion (37) described in Section 6.3, the analytical and numerical values for the temperature, u, and normal heat flux , q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and $p_{\mathrm{u}} \in \{1\%, 3\%, 5\%\}$, for Example 1, are illustrated in Figs. $6(a)$ and $6(b)$, respectively. From these figures it can be seen that the numerical solution is a stable approximation for the exact solution, free of unbounded and rapid oscillations. It should also be noted from Figs. $6(a)$ and $6(b)$ that the numerical solution converges to the exact solution as $p_{\mathrm{u}} \longrightarrow 0$.



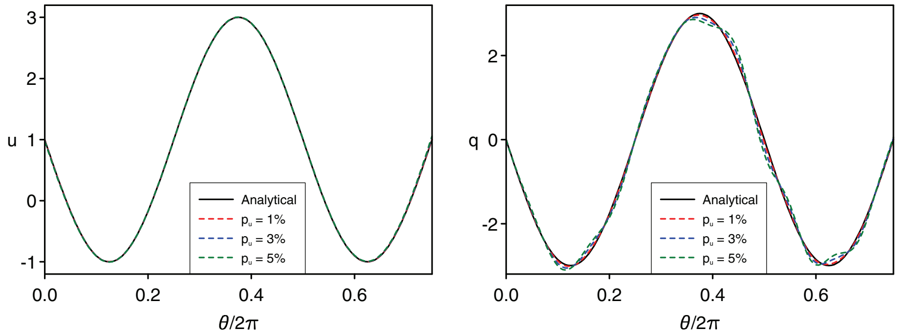
(a) Example 1: Temperatures on Γ_2



(b) Example 1: Normal heat fluxes on Γ_2

Figure 6: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into the Dirichlet data on Γ_1, namely $p_{\mathrm{u}} \in \{1\%, 3\%, 5\%\}$, for Example 1.

The values of the optimal iteration number, k_{opt}, the corresponding accuracy errors, $e_{\mathrm{u}}(k_{\mathrm{opt}})$ and $e_{\mathrm{q}}(k_{\mathrm{opt}})$, and the CPU time, obtained using the alternating iterative algorithm I, the stopping criterion (37), various levels of noise added into the Dirichlet data on Γ_1 and various values of the relaxation parameter, $\omega \in (0,2)$, for the Cauchy problem given by Example 1, are presented in Table 1. The following major conclusions can be drawn from this table:



(a) Example 1: Temperatures on Γ_1 (b) Example 1: Normal heat fluxes on Γ_1

Figure 7: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the over-specified boundary Γ_1, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into the Dirichlet data on Γ_1, namely $p_u \in \{1\%, 3\%, 5\%\}$, for Example 1.

(i) For all fixed values of the relaxation parameter $\omega \in (0,2)$, both accuracy errors $e_u(k_{opt})$ and $e_q(k_{opt})$ decrease as p_u decreases (i.e. the algorithm I is stable with respect to decreasing the level of noise added into the Dirichlet data on Γ_1), while the optimal number of iterations k_{opt} and, consequently, the CPU time required for the alternating iterative algorithm I to reach the numerical solutions for the unknown temperature and normal heat flux on Γ_1 increase as p_u decreases;

(ii) For all fixed amounts of noise added into the temperature on the over-specified boundary Γ_1, $p_u \in \{1\%, 3\%, 5\%\}$, the accuracy errors $e_u(k_{opt})$ and $e_q(k_{opt})$ have the same values regardless the value of the relaxation parameter $\omega \in (0,2)$, while the optimal number of iterations, k_{opt} and the CPU time required for the alternating iterative algorithm I to reach the numerical solutions for the unknown temperature and normal heat flux on Γ_1 decrease as $\omega \longrightarrow 2$, i.e. as more over-relaxation is introduced in the algorithm I.

In order to assess the performance of the alternating iterative algorithm I with under-, no and over-relaxation, we exemplify by considering Example 1 with $p_u = 1\%$: In this case, the CPU times needed for the alternating iterative algorithm I with $\omega = 0.50$ (under-relaxation), $\omega = 1.00$ (no relaxation) and $\omega = 1.50$ (over-relaxation) to reach the numerical solutions for the temperature and normal heat flux on Γ_2 were found to be 5785.39, 3455.76 and 1674.59 s, respectively, while the corresponding values for the optimal iteration number required, k_{opt}, were found to be 4136, 2755 and 1367, respectively. This means that, to attain the numerical solutions for the unknown Dirichlet and Neumann data on Γ_2, the alternating iterative algorithm I with over-relaxation ($\omega = 1.50$) requires a reduction in the number of iterations performed and CPU time by approximately 50% and 67% with respect to those corresponding to the standard iterative algorithm I as proposed by Kozlov, Maz′ya and Fomin (1991), i.e. without relaxation ($\omega = 1.00$), and the alternating iterative algorithm I with under-relaxation ($\omega = 0.50$), respectively.

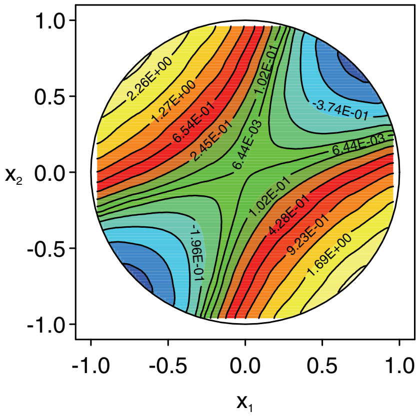
It should be emphasised that the alternating iterative algorithm I provides excellent numerical approximation for both the boundary temperature and the normal heat flux on the over-determined boundary Γ_1. This fact can be clearly seen from Figs. 7(a) and 7(b), which present a comparison between the analytical and numerical values for the given Cauchy data on Γ_1, $\omega = 1.50$ and $p_{\mathrm{u}} \in \{1\%, 3\%, 5\%\}$, in the case of Example 1. Similar results have been obtained for the other inverse problems analysed and therefore they are not illustrated herein. Also, very good reconstructions of the temperature distribution in the domain Ω have been retrieved for Example 1, when using the MFS-based iterative algorithm I, $\omega = 1.50$ and $p_{\mathrm{u}} \in \{1\%, 3\%, 5\%\}$, and these are illustrated in Figs. 8(a)–(d).

Similar conclusions to those obtained from Figs. 6 and 8 can be drawn from Figs. 9 and 10, which show the numerical values for the temperature and normal heat flux obtained on the under-specified boundary Γ_2 and the temperature distribution in the domain Ω, respectively, in comparison with their analytical counterparts, using the alternating iterative algorithm I, the regularizing stopping criterion (37), $\omega = 1.50$ and $p_{\mathrm{q}} \in \{1\%, 3\%, 5\%\}$, for Example 1. By comparing Figs. 6, 8 – 10, it can be observed that the alternating iterative algorithm I is more sensitive to noise added into the temperature $\mathrm{u}\big|_{\Gamma_1}$ than to perturbations of the normal heat flux $\mathrm{q}\big|_{\Gamma_1}$, in the case of Example 1.

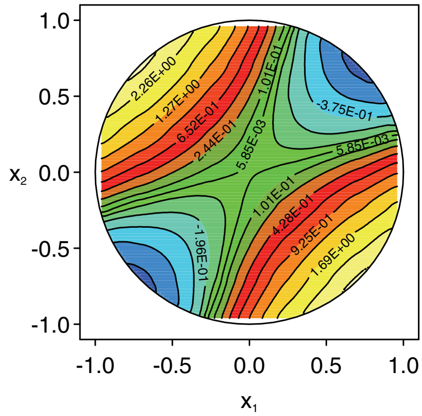
Table 2 tabulates the values of the optimal iteration number, k_{opt}, according to the stopping criterion (37), the corresponding accuracy errors given by Eqs. (33a) and (33b), and the CPU time, obtained using the alternating iterative algorithm I, various levels of noise added into the Neumann data on Γ_1 and various values of the relaxation parameter, $\omega \in (0, 2)$, for the Cauchy problem given by Example 1. From Tables 1 and 2 it can be noticed that the sensitivity of the alternating iterative algorithm I with respect to noisy Dirichlet and Neumann data on Γ_1, for Example 1, results in the following:

(i) More inaccurate numerical results for both $\mathrm{u}\big|_{\Gamma_2}$ and $\mathrm{q}\big|_{\Gamma_2}$ are obtained for perturbed temperature on Γ_1 than for noisy normal heat flux on Γ_1;

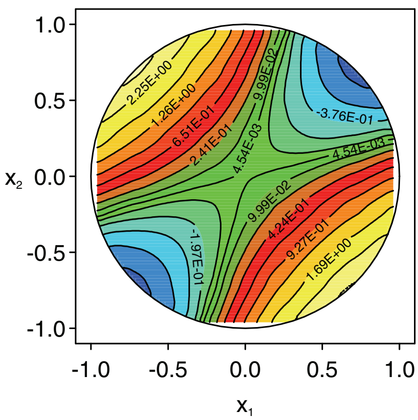
(ii) The optimal number of iterations, k_{opt}, and hence the CPU time required for the alternating iterative algorithm I to reach the numerical solutions for the



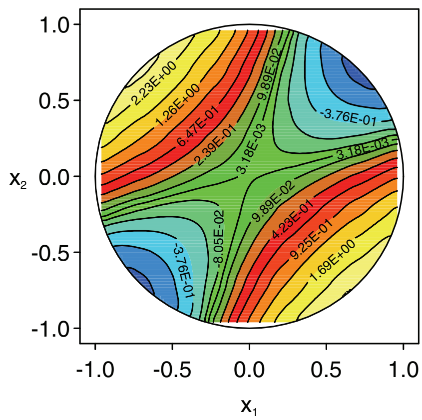
(a) Example 1: $\mathrm{u}^{(\mathrm{an})}|_{\Omega}$



(b) Example 1: $\mathrm{u}^{(\mathrm{num})}|_{\Omega}$, $p_{\mathrm{u}} = 1\%$

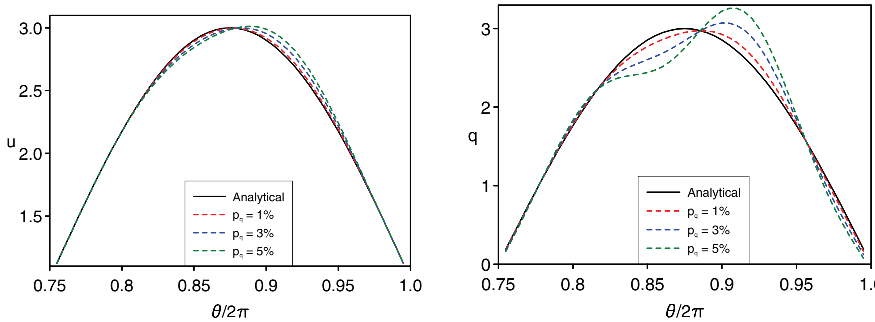


(c) Example 1: $\mathrm{u}^{(\mathrm{num})}|_{\Omega}$, $p_{\mathrm{u}} = 3\%$



(d) Example 1: $\mathrm{u}^{(\mathrm{num})}|_{\Omega}$, $p_{\mathrm{u}} = 5\%$

Figure 8: (a) The analytical, $\mathrm{u}^{(\mathrm{an})}|_{\Omega}$, and numerical internal temperatures, $\mathrm{u}^{(\mathrm{num})}|_{\Omega}$, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into the Dirichlet data on Γ_1, namely (b) $p_{\mathrm{u}} = 1\%$, (c) $p_{\mathrm{u}} = 3\%$, and (d) $p_{\mathrm{u}} = 5\%$, for Example 1.



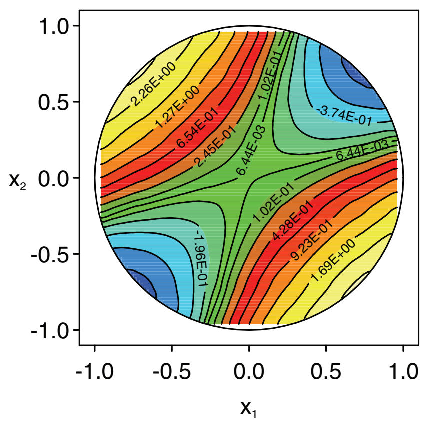
(a) Example 1: Temperatures on Γ_2 (b) Example 1: Normal heat fluxes on Γ_2

Figure 9: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into the Neumann data on Γ_1, namely $p_{\mathrm{q}} \in \{1\%, 3\%, 5\%\}$, for Example 1 with $|\Gamma_1|/|\Gamma_2| = 3$.

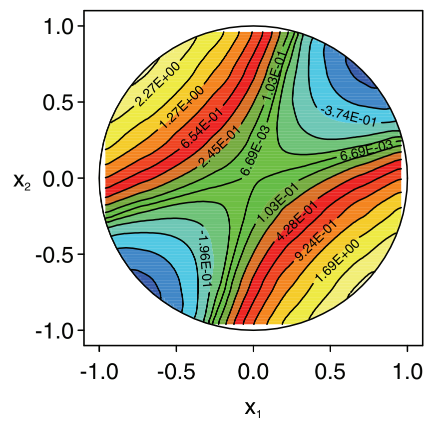
unknown temperature and normal heat flux on Γ_2 for perturbed temperature on Γ_1 are larger that those corresponding to noisy normal heat flux on Γ_1.

Accurate, convergent and stable numerical results for both the temperature and the normal heat flux on Γ_2 have also been obtained when using the MFS-based iterative algorithm II, in the case of steady-state anisotropic heat conduction Cauchy problems in doubly connected domain with a smooth boundary. Figs. $11(a)$ and $11(b)$ illustrate the analytical and numerical temperatures and normal heat fluxes on Γ_2, respectively, retrieved using the iterative algorithm II, $\omega = 1.50$ and $p_{\mathrm{u}} \in \{1\%, 3\%, 5\%\}$, for Example 2 (i.e. annular domain). Similar stable results have also been obtained for the reconstruction of the internal temperature, $\mathrm{u}\big|_{\Omega}$, and these are displayed in Fig. 12.

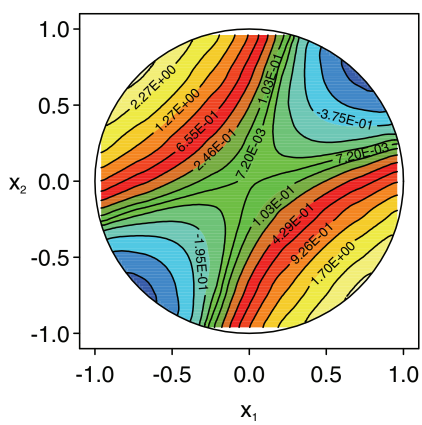
The proposed MFS-alternating iterative algorithm II, in conjunction with the stopping criterion (37), works equally well also for the Cauchy problem (4a)–(4c) associated with the Laplace-Beltrami equation in a simply connected convex two-dimensional domain with a piecewise smooth boundary, such as the rectangular domain considered in Example 3. Figs. $13(a)$ and $13(b)$ show the numerical results for the temperature and normal heat flux on the boundary Γ_2, respectively, obtained using the stopping criterion (37), $\omega = 1.50$, $p_{\mathrm{q}} \in \{1\%, 3\%, 5\%\}$ and $|\Gamma_1|/|\Gamma_2| = 5$ (i.e. over-determined Cauchy problem), in comparison with their corresponding analytical values, in the case of Example 3, while the analytical and numerically reconstructed temperature distributions in the domain Ω are presented in Figs. 14(a)–(d), respectively.



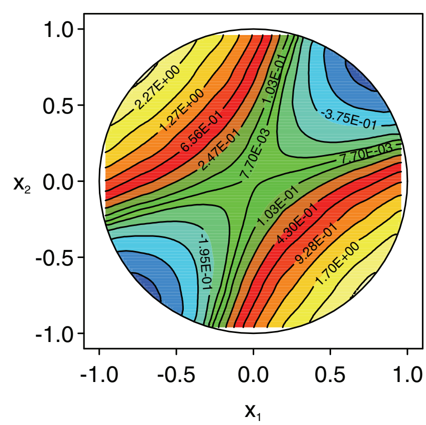
(a) Example 1: $\mathrm{u}^{(\mathrm{an})}|_{\Omega}$



(b) Example 1: $\mathrm{u}^{(\mathrm{num})}|_{\Omega}$, $p_{\mathrm{q}} = 1\%$



(c) Example 1: $\mathrm{u}^{(\mathrm{num})}|_{\Omega}$, $p_{\mathrm{q}} = 3\%$



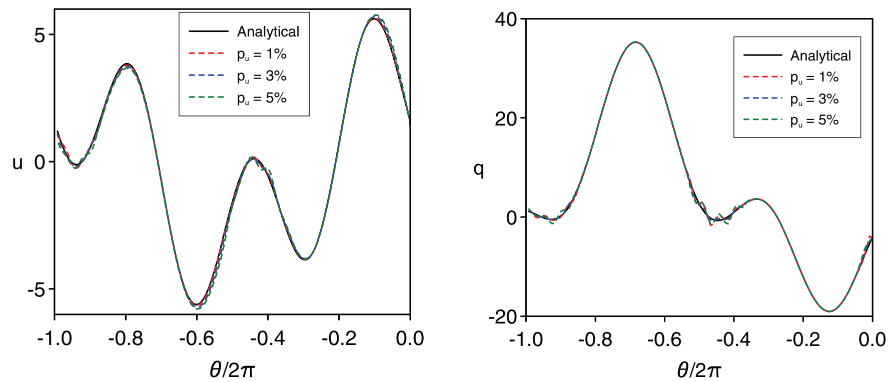
(d) Example 1: $\mathrm{u}^{(\mathrm{num})}|_{\Omega}$, $p_{\mathrm{q}} = 5\%$

Figure 10: (a) The analytical, $\mathrm{u}^{(\mathrm{an})}|_{\Omega}$, and numerical internal temperatures, $\mathrm{u}^{(\mathrm{num})}|_{\Omega}$, obtained using the alternating iterative algorithm I, $\omega = 1.50$ and various levels of noise added into the Neumann data on Γ_1, namely (b) $p_{\mathrm{q}} = 1\%$, (c) $p_{\mathrm{q}} = 3\%$, and (d) $p_{\mathrm{q}} = 5\%$, for Example 1 with $|\Gamma_1|/|\Gamma_2| = 3$.

Table 1: The values of the optimal iteration number, k_{opt}, the corresponding accuracy errors, $e_{\text{u}}(k_{\text{opt}})$ and $e_{\text{q}}(k_{\text{opt}})$, and the computational time, obtained using the alternating iterative algorithm I, the regularizing stopping criterion (37), various amounts of noise added into $\text{u}\big|_{\Gamma_1}$, i.e. $p_{\text{u}} \in \{1\%, 3\%, 5\%\}$ and $p_{\text{q}} = 0\%$, and various values for the relaxation parameter, ω, for the Cauchy problem given by Example 1.

ω	p_{u}	p_{q}	k_{opt}	$e_{\text{u}}(k_{\text{opt}})$	$e_{\text{q}}(k_{\text{opt}})$	CPU time [s]
0.20	1%	0%	4958	0.12065×10^{-1}	0.51310×10^{-1}	6845.51
	3%	0%	4716	0.36192×10^{-1}	0.15391×10^{0}	6029.39
	5%	0%	4483	0.60320×10^{-1}	0.25652×10^{0}	5228.04
0.50	1%	0%	4136	0.12065×10^{-1}	0.51310×10^{-1}	5785.39
	3%	0%	3934	0.36192×10^{-1}	0.15391×10^{0}	5306.01
	5%	0%	3740	0.60320×10^{-1}	0.25652×10^{0}	4327.90
1.00	1%	0%	2755	0.12065×10^{-1}	0.51310×10^{-1}	3455.76
	3%	0%	2620	0.36192×10^{-1}	0.15391×10^{0}	3269.54
	5%	0%	2490	0.60320×10^{-1}	0.25652×10^{0}	2868.35
1.50	1%	0%	1367	0.12065×10^{-1}	0.51310×10^{-1}	1674.59
	3%	0%	1299	0.36192×10^{-1}	0.15391×10^{0}	1560.15
	5%	0%	1234	0.60320×10^{-1}	0.25652×10^{0}	1396.48
1.80	1%	0%	508	0.12065×10^{-1}	0.51310×10^{-1}	603.45
	3%	0%	483	0.36192×10^{-1}	0.15391×10^{0}	572.96
	5%	0%	458	0.60320×10^{-1}	0.25652×10^{0}	523.65

Next, we investigate the sensitivity of the alternating iterative algorithm with relaxation II with respect to the measure of the over-specified boundary and hence the robustness of the proposed numerical method. To do so, we consider again the geometry described in Example 3 with $\Gamma_1 = \{\text{r}\} \times [-\text{r}/2, \text{r}/2] \cup [-\text{r}, \text{r}] \times \{\text{r}/2\}$ and $\Gamma_2 = \{-\text{r}\} \times (-\text{r}/2, \text{r}/2) \cup [-\text{r}, \text{r}) \times \{-\text{r}/2\}$ such that $|\Gamma_1|/|\Gamma_2| = 1$ (i.e. equally determined Cauchy problem). The analytical and numerical values for the temperature, u, and normal heat flux, q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm II, $\omega = 1.50$, $p_{\text{q}} \in \{1\%, 3\%, 5\%\}$ and $|\Gamma_1|/|\Gamma_2| = 1$, for Example 3, are illustrated in Figs. 15(a) and 15(b), respectively, while the analytical and numerically reconstructed internal temperatures, $\text{u}\big|_{\Omega}$, are presented in Figs. 16(a)–(d). By comparing Figs. 13 – 16, it can be noticed that, as expected, more inaccurate boundary and internal data reconstructions are obtained when the Cauchy data are available on a smaller part of the boundary, i.e. for lower



(a) Example 2: Temperatures on Γ_2 (b) Example 2: Normal heat fluxes on Γ_2

Figure 11: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm II, $\omega = 1.50$ and various levels of noise added into the Dirichlet data on Γ_1, namely $p_{\mathrm{u}} \in \{1\%, 3\%, 5\%\}$, for Example 2 with $|\Gamma_1|/|\Gamma_2| = 3/2$.

values of the ratio $|\Gamma_1|/|\Gamma_2|$. However, good approximations for both the boundary, $\mathrm{u}\big|_{\Gamma_2}$, and the internal temperatures, $\mathrm{u}\big|_{\Omega}$, and reasonable reconstructions of the normal heat fluxes on Γ_2 have been obtained in the case of the equally determined Cauchy problem associated with Example 3, see Figs. 15 and 16.

Finally, we investigate the robustness of the proposed MFS-based iterative algorithms with relaxation for the most difficult Cauchy problem as described by Example 4. In this case, the difficulty of the problem is dual, namely (i) no analytical solution to the direct problem is available (the Neumann data have been obtained by solving numerically the Dirichlet problem for the Laplace-Beltrami equation), and, more importantly (ii) the inverse problem is under-determined ($|\Gamma_1|/|\Gamma_2| = 2/3$). Nonetheless, even for this very severe inverse problem the alternating iterative algorithm II with over-relaxation ($\omega = 1.50$), in conjunction with the stopping criterion (37), produces very good numerical approximations for both the boundary temperature, $\mathrm{u}\big|_{\Gamma_2}$, and the normal heat flux, $\mathrm{q}\big|_{\Gamma_2}$, from perturbed Cauchy data on Γ_1 ($p_{\mathrm{u}} \in \{1\%, 3\%, 5\%\}$ and $p_{\mathrm{q}} = 0\%$), as shown in Figs. 17(a) and 17(a), respectively. Similar accurate, stable and convergent results have been obtained for the internal temperature, $\mathrm{u}\big|_{\Omega}$, and these are illustrated in Fig. 18.

From the numerical results presented in this section, it can be concluded that the stopping criterion developed in Section 6.3 has a regularizing effect and the numerical solutions for both the temperature and the normal heat flux on the under-determined boundary, Γ_2, obtained by the iterative MFS algorithms described in

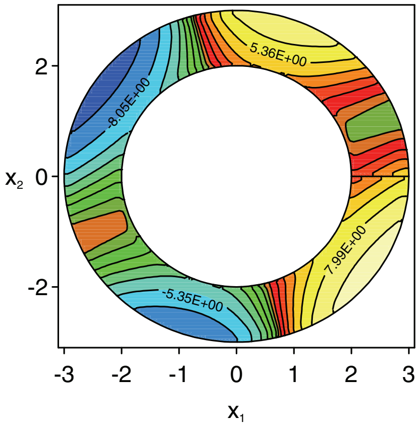
Table 2: The values of the optimal iteration number, $k_{\rm opt}$, the corresponding accuracy errors, $e_{\rm u}(k_{\rm opt})$ and $e_{\rm q}(k_{\rm opt})$, and the computational time, obtained using the alternating iterative algorithm I, the regularizing stopping criterion (37), various amounts of noise added into $\mathrm{q}\big|_{\Gamma_1}$, i.e. $p_{\rm u} = 0\%$ and $p_{\rm q} \in \{1\%, 3\%, 5\%\}$, and various values for the relaxation parameter, ω, for the Cauchy problem given by Example 1.

ω	$p_{\rm u}$	$p_{\rm q}$	$k_{\rm opt}$	$e_{\rm u}(k_{\rm opt})$	$e_{\rm q}(k_{\rm opt})$	CPU time [s]
0.20	0%	1%	1584	0.48522×10^{-2}	0.26140×10^{-1}	1883.54
	0%	3%	1147	0.14558×10^{-1}	0.78556×10^{-1}	1302.14
	0%	5%	905	0.24264×10^{-1}	0.13092×10^{0}	1088.95
0.50	0%	1%	1323	0.48522×10^{-2}	0.26164×10^{-1}	1549.07
	0%	3%	960	0.14558×10^{-1}	0.78571×10^{-1}	1284.92
	0%	5%	763	0.24264×10^{-1}	0.13091×10^{0}	871.51
1.00	0%	1%	880	0.48522×10^{-2}	0.26163×10^{-1}	1106.43
	0%	3%	638	0.14558×10^{-1}	0.78511×10^{-1}	733.87
	0%	5%	506	0.24264×10^{-1}	0.13097×10^{0}	661.15
1.50	0%	1%	435	0.48522×10^{-2}	0.26152×10^{-1}	585.96
	0%	3%	310	0.14558×10^{-1}	0.78789×10^{-1}	445.68
	0%	5%	249	0.24264×10^{-1}	0.13084×10^{0}	355.50
1.80	0%	1%	162	0.48522×10^{-2}	0.26349×10^{-1}	187.09
	0%	3%	116	0.14558×10^{-1}	0.79429×10^{-1}	156.56
	0%	5%	92	0.24264×10^{-1}	0.13120×10^{0}	115.96

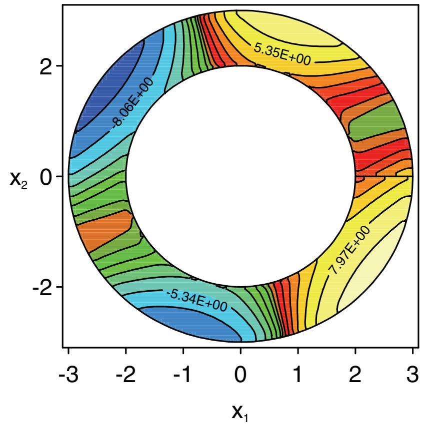
this paper is convergent and stable with respect to increasing the number of iterations and decreasing the level of noise added into the Cauchy input data, respectively.

7 Conclusions

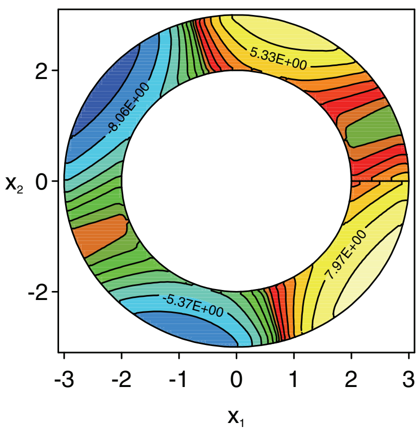
In this paper, we proposed two algorithms involving the relaxation of either the given Dirichlet data (temperature) or the prescribed Neumann data (normal heat flux) on the over-specified boundary in the case of the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991) applied to two-dimensional steady-state anisotropic heat conduction Cauchy problems. The two mixed, well-posed and direct problems corresponding to each iteration of the numerical procedure were solved using a meshless method, namely the MFS, in conjunction with the Tikhonov regularization method. For each direct problem considered, the optimal



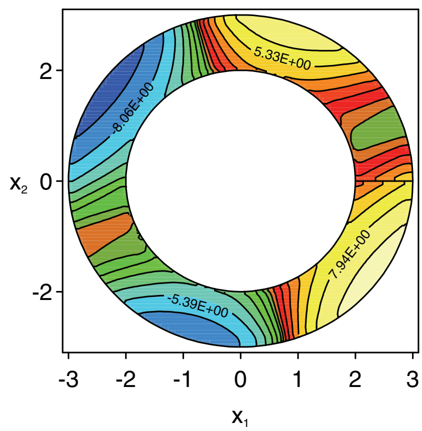
(a) Example 2: $u^{(an)}|_{\Omega}$



(b) Example 2: $u^{(num)}|_{\Omega}$, $p_u = 1\%$



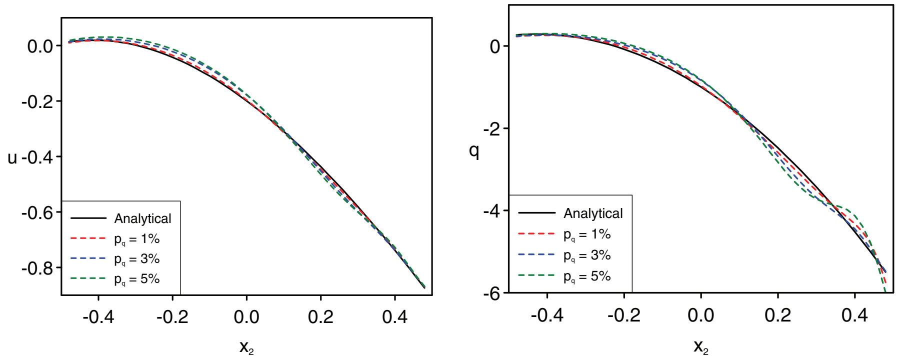
(c) Example 2: $u^{(num)}|_{\Omega}$, $p_u = 3\%$



(d) Example 2: $u^{(num)}|_{\Omega}$, $p_u = 5\%$

Figure 12: (a) The analytical, $u^{(an)}|_{\Omega}$, and numerical internal temperatures, $u^{(num)}|_{\Omega}$, obtained using the alternating iterative algorithm II, $\omega = 1.50$ and various levels of noise added into the Dirichlet data on Γ_1, namely (b) $p_u = 1\%$, (c) $p_u = 3\%$, and (d) $p_u = 5\%$, for Example 2 with $|\Gamma_1|/|\Gamma_2| = 3/2$.

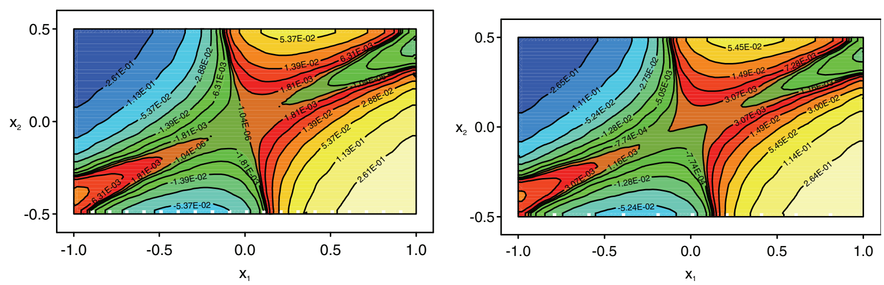
value of the regularization parameter was selected according to the GCV criterion. An efficient regularizing stopping criterion which ceases the iterative procedure at the point where the accumulation of noise becomes dominant and the errors in predicting the exact solutions increase, was also presented. The MFS-based iterative algorithms with relaxation were tested for over-, equally and under-determined Cauchy problems associated with the Laplace-Beltrami operator in simply and dou-



(a) Example 3: Temperatures on $\{-1\} \times (-0.5, 0.5) = \Gamma_2$

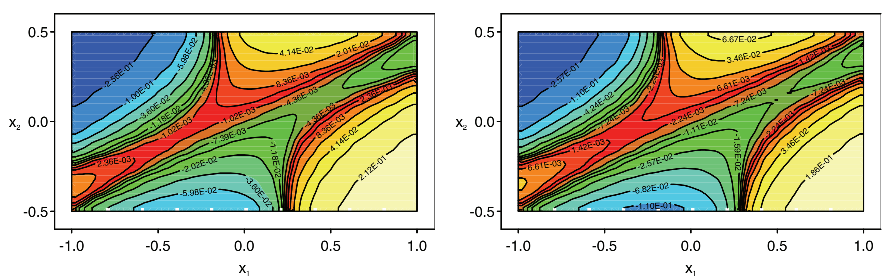
(b) Example 3: Normal heat fluxes on $\{-1\} \times (-0.5, 0.5) = \Gamma_2$

Figure 13: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary $\{-1\} \times (-0.5, 0.5) = \Gamma_2$, obtained using the alternating iterative algorithm II, $\omega = 1.50$ and various levels of noise added into the Neumann data on Γ_1, namely $p_q \in \{1\%, 3\%, 5\%\}$, for Example 3 with $|\Gamma_1|/|\Gamma_2| = 5$.



(a) Example 3: $u^{(an)}|_{\Omega}$

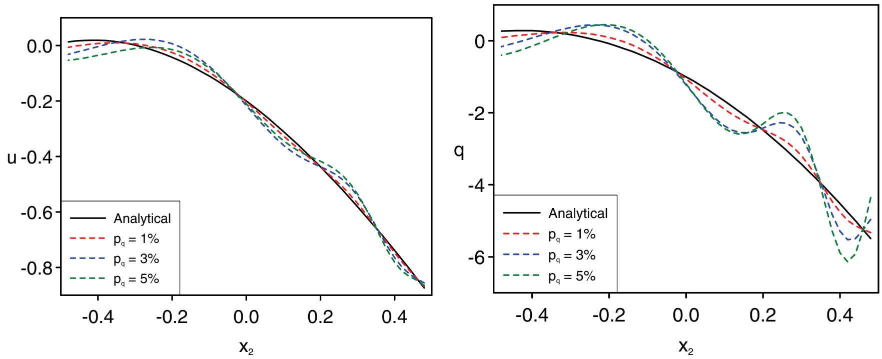
(b) Example 3: $u^{(num)}|_{\Omega}$, $p_u = 1\%$, $|\Gamma_1|/|\Gamma_2| = 5$



(c) Example 3: $u^{(num)}|_{\Omega}$, $p_u = 3\%$, $|\Gamma_1|/|\Gamma_2| = 5$

(d) Example 3: $u^{(num)}|_{\Omega}$, $p_u = 5\%$, $|\Gamma_1|/|\Gamma_2| = 5$

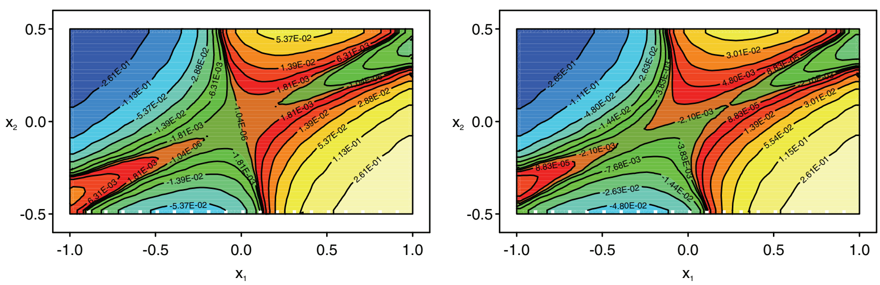
Figure 14: (a) The analytical, $u^{(an)}|_{\Omega}$, and numerical internal temperatures, $u^{(num)}|_{\Omega}$, obtained using the alternating iterative algorithm II, $\omega = 1.50$ and various levels of noise added into the Neumann data on Γ_1, namely (b) $p_q = 1\%$, (c) $p_q = 3\%$, and (d) $p_q = 5\%$, for Example 3 with $|\Gamma_1|/|\Gamma_2| = 5$.



(a) Example 3: Temperatures on $\{-1\} \times (-0.5,0.5) \subset \Gamma_2$

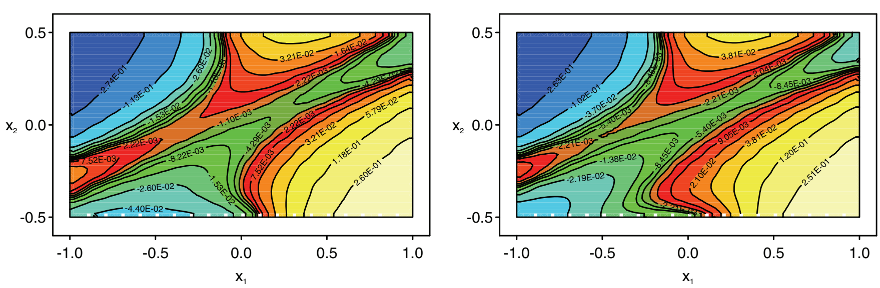
(b) Example 3: Normal heat fluxes on $\{-1\} \times (-0.5,0.5) \subset \Gamma_2$

Figure 15: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary $\{-1\} \times (-0.5,0.5) \subset \Gamma_2$, obtained using the alternating iterative algorithm II, $\omega = 1.50$ and various levels of noise added into the Neumann data on Γ_1, namely $p_q \in \{1\%, 3\%, 5\%\}$, for Example 3 with $|\Gamma_1|/|\Gamma_2| = 1$.



(a) Example 3: $u^{(an)}|_{\Omega}$

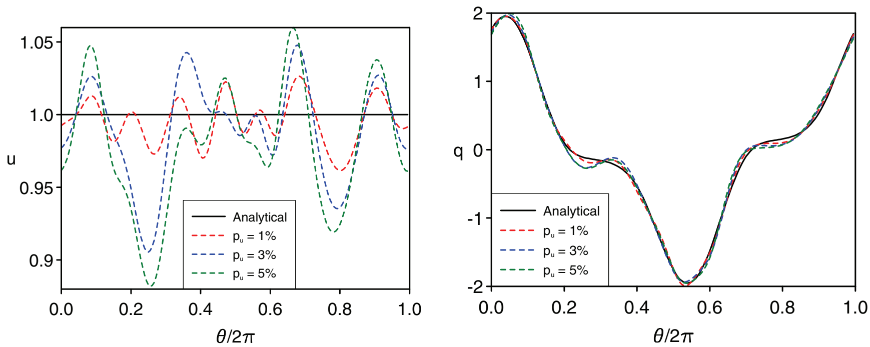
(b) Example 3: $u^{(num)}|_{\Omega}$, $p_u = 1\%$, $|\Gamma_1|/|\Gamma_2| = 1$



(c) Example 3: $u^{(num)}|_{\Omega}$, $p_u = 3\%$, $|\Gamma_1|/|\Gamma_2| = 1$

(d) Example 3: $u^{(num)}|_{\Omega}$, $p_u = 5\%$, $|\Gamma_1|/|\Gamma_2| = 1$

Figure 16: (a) The analytical, $u^{(an)}|_{\Omega}$, and numerical internal temperatures, $u^{(num)}|_{\Omega}$, obtained using the alternating iterative algorithm II, $\omega = 1.50$ and various levels of noise added into the Neumann data on Γ_1, namely (b) $p_q = 1\%$, (c) $p_q = 3\%$, and (d) $p_q = 5\%$, for Example 3 with , $|\Gamma_1|/|\Gamma_2| = 1$.



(a) Example 4: Temperatures on Γ_2 (b) Example 4: Normal heat fluxes on Γ_2

Figure 17: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using the alternating iterative algorithm II, $\omega = 1.50$ and various levels of noise added into the Dirichlet data on Γ_1, namely $p_{\mathrm{u}} \in \{1\%, 3\%, 5\%\}$, for Example 4 with $|\Gamma_1|/|\Gamma_2| = 2/3$.

bly connected with smooth or piecewise smooth boundaries. The numerical results obtained in this paper showed the numerical stability, convergence, accuracy, consistency and computational efficiency of the proposed methods. One possible disadvantage of the MFS-based iterative algorithms is related to the optimal choice of the regularization parameter associated with the Tikhonov regularization method which requires, at each step of the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991), additional iterations with respect to the regularization parameter. However, this inconvenience was overcome by employing the relaxation procedures presented in this study, emphasizing at the same time the computational efficiency of the relaxation procedures applied to the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991).

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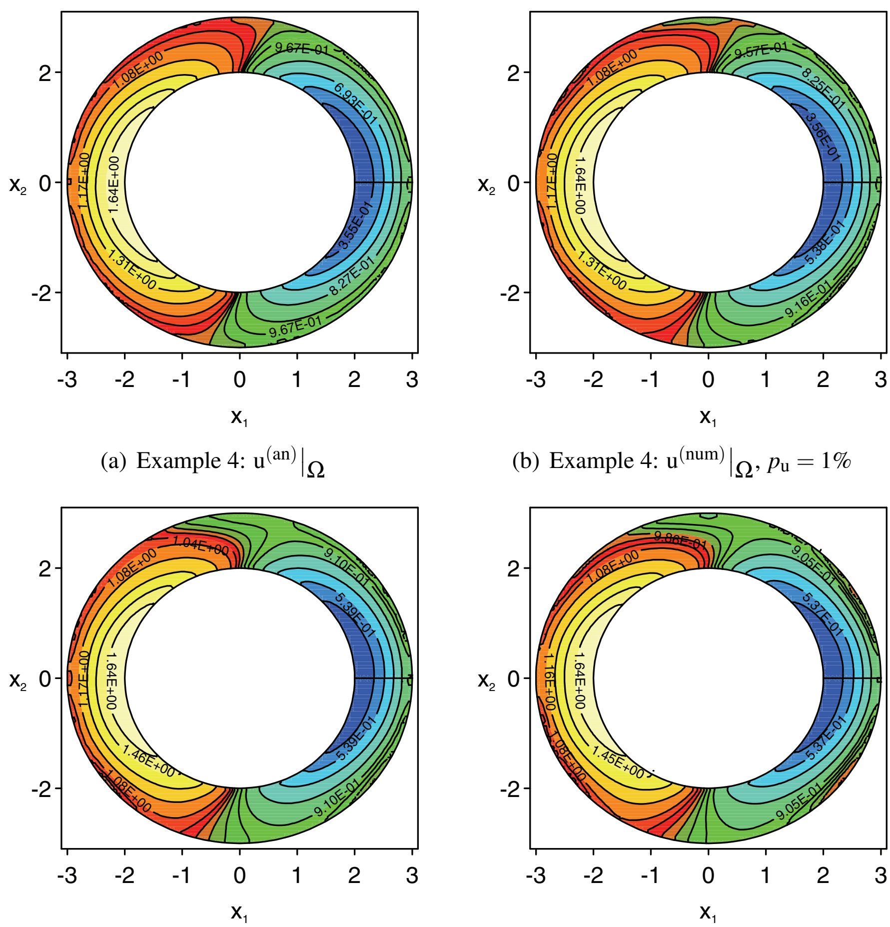

(a) Example 4: $u^{(an)}|_{\Omega}$ (b) Example 4: $u^{(num)}|_{\Omega}$, $p_u = 1\%$

(c) Example 4: $u^{(num)}|_{\Omega}$, $p_u = 3\%$ (d) Example 4: $u^{(num)}|_{\Omega}$, $p_u = 5\%$

Figure 18: (a) The analytical, $u^{(an)}|_{\Omega}$, and numerical internal temperatures, $u^{(num)}|_{\Omega}$, obtained using the alternating iterative algorithm II, $\omega = 1.50$ and various levels of noise added into the Dirichlet data on Γ_1, namely (b) $p_u = 1\%$, (c) $p_u = 3\%$, and (d) $p_u = 5\%$, for Example 4 with $|\Gamma_1|/|\Gamma_2| = 2/3$.

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