 CMC, vol.15, no.1, pp.45-66, 2010

A Quasi-Boundary Semi-Analytical Approach for Two-Dimensional Backward Heat Conduction Problems

Chih-Wen Chang[1], **Chein-Shan Liu**[2] **and Jiang-Ren Chang**[3]

Abstract: In this article, we propose a semi-analytical method to tackle the two-dimensional backward heat conduction problem (BHCP) by using a quasi-boundary idea. First, the Fourier series expansion technique is employed to calculate the temperature field $u(x,y,t)$ at any time $t < T$. Second, we consider a direct regularization by adding an extra term $\alpha u(x,\ y,\ 0)$ to reach a second-kind Fredholm integral equation for $u(x,\ y,\ 0)$. The termwise separable property of the kernel function permits us to obtain a closed-form regularized solution. Besides, a strategy to choose the regularization parameter is suggested. When several numerical examples were tested, we find that the proposed scheme is robust and applicable to the two-dimensional BHCP.

Keywords: Backward heat conduction problem, Ill-posed problem, Fredholm integral equation, Regularized solution, Fourier series

1 Introduction

In several practical application fields such as archeology, it needs to find the temperature history from the known final data. This is the so-called backward heat conduction problem (BHCP), which is a severely ill-posed problem because the solution is unstable for the given final data. For the two-dimensional homogeneous BHCP, many approaches have been studied. The regularized successive over-relaxation (SOR) inversion method and the direct SOR inversion method were proposed by Liu (2002). He mentioned that the regularized SOR approach is stable even under the influence of high noise level, but its retrieved time is only 5×10^{-3}. By contrast, the direct SOR inversion method is unstable for small disturbances. Iijima

[1] Grid Application Division, National Center for High-Performance Computing, Taichung 40763, Taiwan

[2] Department of Civil Engineering, National Taiwan University, Taipei 10617, Taiwan. Corresponding author, Tel.:+886-2-33664233. E-mail: liucs@ntu.edu.tw

[3] Department of Systems Engineering and Naval Architecture, National Taiwan Ocean University, Keelung 20224, Taiwan

(2004) established a high order lattice-free finite difference approach by employing the Taylor expansion and the Fourier transform; however, this study did not discuss its robustness when the final time data was perturbed with noises. Later, Mera (2005) claimed that the method of fundamental solutions is an efficient and accurate scheme for resolving the BHCP in one-dimensional and two-dimensional domains; nevertheless, the standard Tikhonov regularization technique with the L-curve method are still required for the numerical stability problem. Recently, Liu (2004) and Liu, Chang and Chang (2006) have used the group preserving scheme (GPS) and the backward group preserving scheme (BGPS) to tackle the BHCP, respectively. Without a priori regularization in use makes these two approaches more appealing for ill-posed problems with a final value problem. Another useful method, on the basis of the GPS [Liu (2001)], namely the Lie-group shooting method (LGSM), was further proposed to cope with boundary-value problems (BVPs). Since adding a quasi-boundary regularization at the final time condition, the BHCP, originally a final value problem, can be transformed into a BVP; accordingly, the LGSM [Chang, Liu and Chang (2007, 2008)] is used to resolve the BHCP and obtains a good result.

The new approach to be developed here will provide us a semi-analytical solution, and renders a more laconic numerical implementation than other methods to deal with the difficult backward problems. The degree of ill-posedness of the BHCP is over the sideways heat conduction problem which copes with the reconstruction of unknown boundary conditions [Chang, Liu and Chang (2005)].

Through this study, a direct regularization technique is adopted to transform the two-dimensional BHCP into a second-kind Fredholm integral equation by using the quasi-boundary method. By employing the separating kernel function and eigenfunctions expansion techniques, we can derive a closed-form solution of the second-kind Fredholm integral equation, which is a major contribution of this paper. Another one is the application of the Fredholm integral equation to develop an effective numerical scheme, whose accuracy is much better than that of other numerical methods.

This sort method of second-kind Fredholm integral equation regularization was first used by Liu (2007a) to solve a direct problem of elastic torsion in an arbitrary plane domain, where it was called a meshless regularized integral equation method. Then, Liu (2007b, 2007c) extended it to solve the Laplace direct problem in arbitrary plane domains. A similar second-kind Fredholm integral equation regularization method was used to treat the inverse problems; Liu, Chang and Chiang (2008) have applied the new method to determine the geometrical shape of a constant temperature curve, Liu (2009a) has employed this new method to solve the Robin problem in the Laplace equation, and Liu (2009b) has employed it to solve the backward heat conduction problem. Moreover, we have used this method to solve the backward in time advection-dispersion equation [Liu, Chang and Chang (2009)]. Especially, the proposed approach is time saving and easy to implement.

In the following, Section 2 describes the BHCP with a quasi-boundary regularization of its final time condition, and then we derive the second-kind Fredholm integral equation by a direct regularization in Section 3. In Section 4, we derive a closed-form solution of the second-kind Fredholm integral equation. Section 5 presents a selection principle of the regularization parameter. Numerical examples are also employed to validate the new method. A summary with some concluding remarks is given in Section 6.

2 Backward heat conduction problems

We consider a homogeneous plate of length a and width b. The plate is thin enough such that the temperature is uniformly distributed over the cross section of the plate at any time t. In many practical engineering applications we may want to retrieve all the past temperature distribution $u(x,y,t)$, for $t < T$, when the temperature is supposed to be known at a given final time T. Here, we set our problem as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2},\ 0 < x < a,\ 0 < y < b,\ 0 < t < T, \tag{1}$$

$$u(0,\ y,\ t) = u(a,\ y,\ t) = u(x,\ 0,\ t) = u(x,\ b,\ t) = 0,\ 0 \le t \le T, \tag{2}$$

$$u(x,\ y,\ T) = f(x,\ y),\ 0 \le x \le a,\ 0 \le y \le b. \tag{3}$$

This is the so-called a two-dimensional BHCP, which is known to be highly ill-posed, that is, the solution does not depend continuously on the input data $u(x,y,T)$. In fact, the rapid decay of temperature with time results in fast fading memory of initial conditions. Therefore, the numerical recovery of initial temperature from the data measured at time T is a rather difficult issue due to the effect of the noise and computational error.

One way to solve an ill-posed problem is by a perturbation of it into a well-posed one. Many perturbing techniques have been proposed, including a biharmonic regularization developed by Lattés and Lions (1969), a pseudo-parabolic regularization proposed by Showalter and Ting (1970), a stabilized quasi-reversibility proposed by Miller (1973), the method of quasi-reversibility proposed by Mel'nikova (1997), a hyperbolic regularization proposed by Ames and Cobb (1997), the Gajewski and Zacharias quasi-reversibility proposed by Huang and Zheng (2005), a quasi-boundary value method by Denche and Bessila (2005), and an optimal regularization proposed by Boussetila and Rebbani (2006). We extend the regularization of the one-dimensional BHCP of Showalter (1983) by considering a quasi-boundary-value approximation to the final value problem, that is, to supersede Eq. (3) by

$$\alpha u(x,\ y,\ 0)+u(x,\ y,\ T)=f(x,\ y). \tag{4}$$

The problems (1), (2) and (4) can be presented to be well-posed for each $\alpha > 0$.

3 The Fredholm integral equation

By using the technique for separation of variables, we can easily write a series expansion of $u(x,y,t)$ satisfying Eqs. (1) and (2):

$$u(x,\ y,\ t)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} d_{kj}\exp[-(k^2/a^2+j^2/b^2)\pi^2 t]\sin\frac{k\pi x}{a}\sin\frac{j\pi y}{b}, \tag{5}$$

where d_{kj} are coefficients to be determined. By imposing the two-point boundary condition (4) on the above equation, we obtain

$$\begin{aligned} u(x,\ y,\ T)&=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} d_{kj}\exp[-(k^2/a^2+j^2/b^2)\pi^2 T]\sin\frac{k\pi x}{a}\sin\frac{j\pi y}{b}\\ &=f(x,\ y)-\alpha u(x,\ y,\ 0). \end{aligned} \tag{6}$$

Fixing any $t<T$ and applying the eigenfunctions expansion to Eq. (5), we have

$$d_{kj}=\frac{4\exp[-(k^2/a^2+j^2/b^2)\pi^2 t]}{ab}\int_0^b\int_0^a \sin\frac{k\pi\xi}{a}\sin\frac{j\pi\varphi}{b}u(\xi,\ \varphi,\ t)d\xi d\varphi. \tag{7}$$

Substituting Eq. (7) for d_{kj} into Eq. (6) and presuming that the order of summation and integral can be interchanged, it follows that

$$\begin{aligned} \left(K_{xy}^{T-t}u(\cdot,\ \cdot,\ t)\right)(x,\ y):&=\int_0^b\int_0^a K(x,\ \xi;\ y,\ \varphi;\ T-t)u(\xi,\ \varphi,\ t)d\xi d\varphi\\ &=f(x,\ y)-\alpha u(x,\ y,\ 0), \end{aligned} \tag{8}$$

where

$$\begin{aligned} &K(x,\ \xi;\ y,\ \varphi;\ t)=\\ &\quad\frac{4}{ab}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\exp[-(k^2/a^2+j^2/b^2)\pi^2 t]\sin\frac{k\pi x}{a}\sin\frac{k\pi\xi}{a}\sin\frac{j\pi y}{b}\sin\frac{j\pi\varphi}{b} \end{aligned} \tag{9}$$

is a kernel function, α is a regularization parameter, and K_{xy}^{T-t} is an integral operator generated from $K(x,\ \xi;\ y,\ \varphi;\ T-t)$. Corresponding to the kernel $K(x,\ \xi;\ y,\ \varphi;\ t)$, the operator is denoted by K_{xy}^{t}.

To retrieve the initial temperature $u(x,y,0)$, we have to solve the two-dimensional second-kind Fredholm integral equation:

$$\alpha u(x,\ y,\ 0)+\int_0^b\int_0^a K(x,\ \xi;\ y,\ \varphi;\ T)u(\xi,\ \varphi,\ 0)d\xi d\varphi = f(x,\ y), \tag{10}$$

which is obtained from Eq. (8) by taking $t=0$. Taking $x=\eta$ and $y=\omega$ in Eq. (10), we can get

$$\alpha u(\eta,\ \omega,\ 0)+\int_0^b\int_0^a K(\eta,\ \xi;\ \omega,\ \varphi;\ T)u(\xi,\ \varphi,\ 0)d\xi d\varphi = f(\eta,\ \omega), \tag{11}$$

and applying the operator K_{xy}^{t} on the above equation and noting that

$$\left(K_{xy}^{t}u(\cdot,\ \cdot,\ 0)\right)\ (x,\ y)=\int_0^b\int_0^a K(x,\eta;\ y,\omega;\ t)u(\eta,\ \omega,\ 0)d\eta d\omega = u(x,\ y,\ t),$$
$$\left(K_{xy}^{t}K_{\eta\omega}^{T}u(\cdot,\ \cdot,\ 0)\right)\ (x,\ y)=\left(K_{xy}^{T}K_{\eta\omega}^{t}u(\cdot,\ \cdot,\ 0)\right)\ (x,\ y),$$

we have

$$\alpha u(x,\ y,\ t)+\int_0^b\int_0^a K(x,\ \xi;\ y,\ \varphi;\ T)u(\xi,\ \varphi,\ t)d\xi d\varphi = F(x,\ y,\ t)$$

$$=\int_0^b\int_0^a K(x,\ \xi;\ y,\ \varphi;\ t)f(\xi,\ \varphi)d\xi d\varphi. \tag{12}$$

This equation was extended from Ames, Clark, Epperson and Oppenheimer (1998) to the two-dimensional case, and the numerical implementation has been carried out only for the one-dimensional case.

4 A closed-form solution

However, we begin from Eq. (10) by a different approach, rather than Eq. (12), since Eq. (10) is simpler than Eq. (12). We suppose that the kernel function in Eq. (10) can be approximated by m and n terms with

$$K(x,\ \xi;\ y,\ \varphi;\ T)=$$
$$\frac{4}{ab}\sum_{j=1}^{n}\sum_{k=1}^{m}\exp[-(k^2/a^2+j^2/b^2)\pi^2T]\sin\frac{k\pi x}{a}\sin\frac{k\pi\xi}{a}\sin\frac{j\pi y}{b}\sin\frac{j\pi\varphi}{b} \tag{13}$$

owing to $T > 0$. The above kernel is termwise separable, which is also called the degenerate kernel or the Pincherle-Goursat kernel [Tricomi (1985)].

By inspection of Eq. (13), we can have

$$K(x,\ \xi;\ y,\ \varphi;\ T) = \mathbf{P}(x,\ y;\ T)\cdot\mathbf{Q}(\xi,\varphi), \tag{14}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are nm-vectors given by

$$\mathbf{P} := \frac{4}{ab}\begin{bmatrix} \exp(-\rho_{11}^2\pi^2 T)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \\ \exp(-\rho_{21}^2\pi^2 T)\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b} \\ \vdots \\ \exp(-\rho_{m1}^2\pi^2 T)\sin\frac{m\pi x}{a}\sin\frac{\pi y}{b} \\ \exp(-\rho_{12}^2\pi^2 T)\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b} \\ \exp(-\rho_{22}^2\pi^2 T)\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b} \\ \vdots \\ \exp(-\rho_{m2}^2\pi^2 T)\sin\frac{m\pi x}{a}\sin\frac{2\pi y}{b} \\ \vdots \\ \exp(-\rho_{1n}^2\pi^2 T)\sin\frac{\pi x}{a}\sin\frac{n\pi y}{b} \\ \exp(-\rho_{2n}^2\pi^2 T)\sin\frac{2\pi x}{a}\sin\frac{n\pi y}{b} \\ \vdots \\ \exp(-\rho_{mn}^2\pi^2 T)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \end{bmatrix},\ \mathbf{Q} := \begin{bmatrix} \sin\frac{\pi\xi}{a}\sin\frac{\pi\varphi}{b} \\ \sin\frac{2\pi\xi}{a}\sin\frac{\pi\varphi}{b} \\ \vdots \\ \sin\frac{m\pi\xi}{a}\sin\frac{\pi\varphi}{b} \\ \sin\frac{\pi\xi}{a}\sin\frac{2\pi\varphi}{b} \\ \sin\frac{2\pi\xi}{a}\sin\frac{2\pi\varphi}{b} \\ \vdots \\ \sin\frac{m\pi\xi}{a}\sin\frac{2\pi\varphi}{b} \\ \vdots \\ \sin\frac{\pi\xi}{a}\sin\frac{n\pi\varphi}{b} \\ \sin\frac{2\pi\xi}{a}\sin\frac{n\pi\varphi}{b} \\ \vdots \\ \sin\frac{m\pi\xi}{a}\sin\frac{n\pi\varphi}{b} \end{bmatrix}, \tag{15}$$

where $\rho_{kj}^2 = k^2/a^2 + j^2/b^2$, $k = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ and the dot between $\mathbf{P}$ and $\mathbf{Q}$ denotes the inner product, which is sometimes written as $\mathbf{P}^T\mathbf{Q}$, where the superscript T signifies the transpose. With the aid of Eq. (14), Eq. (10) can be written as

$$\alpha u(x,\ y,\ 0) + \int_0^b\int_0^a \mathbf{P}^T(x,\ y)\mathbf{Q}(\xi,\ \varphi)u(\xi,\ \varphi,\ 0)d\xi d\varphi = f(x,\ y), \tag{16}$$

where we abridge the parameter T in $\mathbf{P}$ for clarity. Let us define

$$\mathbf{c} := \int_0^b\int_0^a \mathbf{Q}(\xi,\ \varphi)u(\xi,\ \varphi,\ 0)d\xi d\varphi \tag{17}$$

to be an unknown vector with dimensions mn.

Multiplying Eq. (16) by $\mathbf{Q}(x,y)$, and integrating it, we can obtain

$$\alpha\int_0^b\int_0^a \mathbf{Q}(x,\ y)u(x,\ y,\ 0)dxdy+\int_0^b\int_0^a \mathbf{Q}(x,\ y)\mathbf{P}^T(x,\ y)dxdy \\ \times\int_0^b\int_0^a \mathbf{Q}(\xi,\ \varphi)u(\xi,\ \varphi,\ 0)d\xi d\varphi=\int_0^b\int_0^a f(x,\ y)\mathbf{Q}(x,\ y)dxdy. \quad (18)$$

By definition (17) we thus have

$$\left(\alpha\mathbf{I}_{nm}+\int_0^b\int_0^a \mathbf{Q}(\xi,\ \varphi)\mathbf{P}^T(\xi,\ \varphi)d\xi d\varphi\right)\mathbf{c}:=\int_0^b\int_0^a f(\xi,\ \varphi)\mathbf{Q}(\xi,\ \varphi)d\xi d\varphi, \quad (19)$$

where $\mathbf{I}_{nm}$ denotes an identity matrix of order mn. Solving Eq. (19) one has

$$\mathbf{c}=\left(\alpha\mathbf{I}_{nm}+\int_0^b\int_0^a \mathbf{Q}(\xi,\ \varphi)\mathbf{P}^T(\xi,\ \varphi)d\xi d\varphi\right)^{-1}\int_0^b\int_0^a f(\xi,\ \varphi)\mathbf{Q}(\xi,\ \varphi)d\xi d\varphi. \quad (20)$$

On the other hand, from Eq. (16) we obtain

$$\alpha u(x,\ y,\ 0)=f(x,\ y)-\mathbf{P}(x,\ y)\cdot\mathbf{c}. \quad (21)$$

Inserting Eq. (20) into the above equation, we get

$$\alpha u(x,\ y,\ 0)=f(x,\ y)-\mathbf{P}(x,\ y)\cdot\left(\alpha\mathbf{I}_{nm}+\int_0^b\int_0^a \mathbf{Q}(\xi,\ \varphi)\mathbf{P}^T(\xi,\ \varphi)d\xi d\varphi\right)^{-1} \\ \int_0^b\int_0^a f(\xi,\ \varphi)\mathbf{Q}(\xi,\ \varphi)d\xi d\varphi. \quad (22)$$

Because of the orthogonality of

$$\int_0^b\int_0^a \sin\frac{j\pi\xi}{a}\sin\frac{k\pi\xi}{a}\sin\frac{m\pi\varphi}{b}\sin\frac{n\pi\varphi}{b}d\xi d\varphi=\frac{ab}{4}\delta_{jk}\delta_{mn}, \quad (23)$$

where δ_{jk} and δ_{mn} are the Kronecker delta, the $nm\times nm$ matrix can be written as

$$\int_0^b\int_0^a \mathbf{Q}(\xi,\ \varphi)\mathbf{P}^T(\xi,\ \varphi)d\xi d\varphi= \\ diag[\exp(-\rho_{11}^2\pi^2T),\exp(-\rho_{21}^2\pi^2T),\ldots,\exp(-\rho_{m1}^2\pi^2T), \\ \exp(-\rho_{12}^2\pi^2T),\exp(-\rho_{22}^2\pi^2T),\ldots,\exp(-\rho_{m2}^2\pi^2T),\ldots, \\ \exp(-\rho_{1n}^2\pi^2T),\exp(-\rho_{2n}^2\pi^2T),\ldots,\exp(-\rho_{mn}^2\pi^2T)], \quad (24)$$

where diag means that the matrix is a diagonal matrix. Inserting Eq. (24) into Eq. (22), we hence obtain

$$u(x,\ y,\ 0)=\frac{1}{\alpha}f(x,\ y)-\frac{1}{\alpha}\mathbf{P}^T(x,\ y)$$

$$diag\left[\frac{1}{\alpha+\exp(-\rho_{11}^2\pi^2T)},\frac{1}{\alpha+\exp(-\rho_{21}^2\pi^2T)},\cdots,\right.$$

$$\frac{1}{\alpha+\exp(-\rho_{m1}^2\pi^2T)},\frac{1}{\alpha+\exp(-\rho_{12}^2\pi^2T)},\frac{1}{\alpha+\exp(-\rho_{22}^2\pi^2T)},\cdots,$$

$$\frac{1}{\alpha+\exp(-\rho_{m2}^2\pi^2T)},\cdots,\frac{1}{\alpha+\exp(-\rho_{1n}^2\pi^2T)},\frac{1}{\alpha+\exp(-\rho_{2n}^2\pi^2T)},\cdots,$$

$$\left.\frac{1}{\alpha+\exp(-\rho_{mn}^2\pi^2T)}\right]\int_0^b\int_0^a f(\xi,\ \varphi)\mathbf{Q}(\xi,\ \varphi)d\xi d\varphi. \tag{25}$$

Using Eq. (15) for **P** and **Q**, we can obtain

$$u(x,\ y,\ 0)=\frac{1}{\alpha}f(x,\ y)-\frac{4}{\alpha ab}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{\exp[-(k^2/a^2+j^2/b^2)\pi^2T]}{\alpha+\exp[-(k^2/a^2+j^2/b^2)\pi^2T]}$$

$$\times\int_0^b\int_0^a\sin\frac{k\pi x}{a}\sin\frac{k\pi\xi}{a}\sin\frac{j\pi y}{b}\sin\frac{j\pi\varphi}{b}f(\xi,\ \varphi)d\xi d\varphi, \tag{26}$$

where the summation upper bound m and n can be replaced by ∞ since our argument is independent of m and n. For a given $f(x,y)$, through some integrals one may employ the above equation to calculate $u(x,y,0)$.

If $u(x,y,0)$ is given, we can calculate $u(x,y,t)$ at any time $t<T$ by

$$u^\alpha(x,\ y,\ t)=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}d_{kj}^\alpha\exp[-(k^2/a^2+j^2/b^2)\pi^2t]\sin\frac{k\pi x}{a}\sin\frac{j\pi y}{b}, \tag{27}$$

where

$$d_{kj}^\alpha=\frac{4}{ab}\int_0^b\int_0^a\sin\frac{k\pi\xi}{a}\sin\frac{j\pi\varphi}{b}u(\xi,\ \varphi,\ 0)d\xi d\varphi. \tag{28}$$

Inserting Eq. (26) into the above equation and using the orthogonality equation (23), one obtains

$$d_{kj}^\alpha=\frac{4}{ab\{\alpha+\exp[-(k^2/a^2+j^2/b^2)\pi^2T]\}}\int_0^b\int_0^a\sin\frac{k\pi\xi}{a}\sin\frac{j\pi\varphi}{b}f(\xi,\ \varphi)d\xi d\varphi. \tag{29}$$

Eqs. (27) and (29) compose an analytical solution of the two-dimensional BHCP. To distinguish it from the exact solution $u(x,\ y,\ t)$,we have employed the symbol $u^\alpha(x,\ y,\ t)$ to indicate that it is a regularization solution.

5 Selection of the regularization parameter α and numerical examples

Up to this point, we have not yet specified how to choose the regularization parameter α. Suppose that f has the following Fourier sine series expansion:

$$f(x,\ y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} d_{kj}^{*} \sin\frac{k\pi x}{a} \sin\frac{j\pi y}{b}, \tag{30}$$

where

$$d_{kj}^{*} = \frac{4}{ab}\int_0^b\int_0^a \sin\frac{k\pi\xi}{a}\sin\frac{j\pi\varphi}{b} f(\xi,\ \varphi)d\xi d\varphi. \tag{31}$$

Substituting Eq. (30) into Eq. (26), we obtain

$$u^{\alpha}(x,\ y,\ 0) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{\exp[-(k^2/a^2+j^2/b^2)\pi^2 T]}{\alpha+\exp[-(k^2/a^2+j^2/b^2)\pi^2 T]} d_{kj}^{*} \exp[(k^2/a^2+j^2/b^2)\pi^2 T] \sin\frac{k\pi x}{a}\sin\frac{j\pi y}{b}, \tag{32}$$

where we indicate that

$$\frac{\exp[-(k^2/a^2+j^2/b^2)\pi^2 T]}{\alpha+\exp[-(k^2/a^2+j^2/b^2)\pi^2 T]} = \frac{1}{1+\alpha\exp[(k^2/a^2+j^2/b^2)\pi^2 T]}.$$

For a better numerical solution, we need to set

$$\alpha\exp[(k^2/a^2+j^2/b^2)\pi^2 T] = \alpha_0 \ll 1.$$

On the other hand, the term $\exp[-(k^2/a^2+j^2/b^2)\pi^2 T]/(\alpha+\exp[-(k^2/a^2+j^2/b^2)\pi^2 T])$ in Eq. (32) will be very small when k, j and/or T are large, which may result in a large numerical error. Hence, we obtain an approximation

$$\frac{\exp[-(k^2/a^2+j^2/b^2)\pi^2 T]}{\alpha+\exp[-(k^2/a^2+j^2/b^2)\pi^2 T]} = \frac{1}{1+\alpha_0} = 1-\alpha_0+\alpha_0^2-\alpha_0^3+\dots.$$

When the terms with order higher than one are truncated, we reach a good approximation of $u(x,y,0)$ by

$$u^{\alpha_0}(x,\ y,\ 0) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(1-\alpha_0) d_{kj}^{*}\sin\frac{k\pi x}{a}\sin\frac{j\pi y}{b}. \tag{33}$$

The regularization parameter α_0 is a small number, and k and j represent a number of the finite terms in the numerical examples. In doing so, we can filter out the noise induced by the higher-modes in Eq. (32).

We will apply the quasi-boundary approach on the calculations of BHCP through numerical examples. We are interested in the stability of our method when the input final measured data are polluted by random noise. We can assess the stability by increasing the different levels of random noise on the final data:

$$\hat{f}(x_i, y_j) = f(x_i, y_j) + sR(i, j), \tag{34}$$

where $f(x_i, y_j)$ is the exact data. We use the function RANDOM_NUMBER given in Fortran to generate the noisy data $R(i, j)$, which are random numbers in [-1, 1], and s means the level of noise. Then, the noisy data $\hat{f}(x_i, y_j)$ are employed in the calculations.

5.1 Example 1

Let us consider the first example of two-dimensional BHCP:

$$u_t = u_{xx} + u_{yy}, \ -\pi < x < \pi, \ -\pi < y < \pi, 0 < t < T, \tag{35}$$

with the boundary conditions

$$u(-\pi, y, t) = u(\pi, y, t) = u(x, -\pi, t) = u(x, \pi, t) = 0, \tag{36}$$

and the final time condition

$$u(x, y, T) = e^{-2\beta^2 T} sin(\beta x) sin(\beta y). \tag{37}$$

The exact solution is given by

$$u(x, y, t) = e^{-2\beta^2 t} sin(\beta x) sin(\beta y), \tag{38}$$

where $\beta \in N$ is a positive integer.

In Fig. 1(a), we show the errors of numerical solutions obtained from the quasi-boundary semi-analytical approach for the case of $\beta = 1$. $T = 1$ is used in this comparison, where the grid lengths $\Delta x = \Delta y = 2\pi/40$ are employed. At the point $x = -\pi + 60\pi/40$ the error is plotted with respect to y by a solid line, and at the point $y = -\pi + 66\pi/40$ the error is plotted with respect to x by a dashed line. Nevertheless, the errors are much smaller than that calculated by Iijima (2004), Liu (2004), Liu, Chang and Chang (2006) and Chang, Liu and Chang (2008) as displayed in Figure 4 therein.

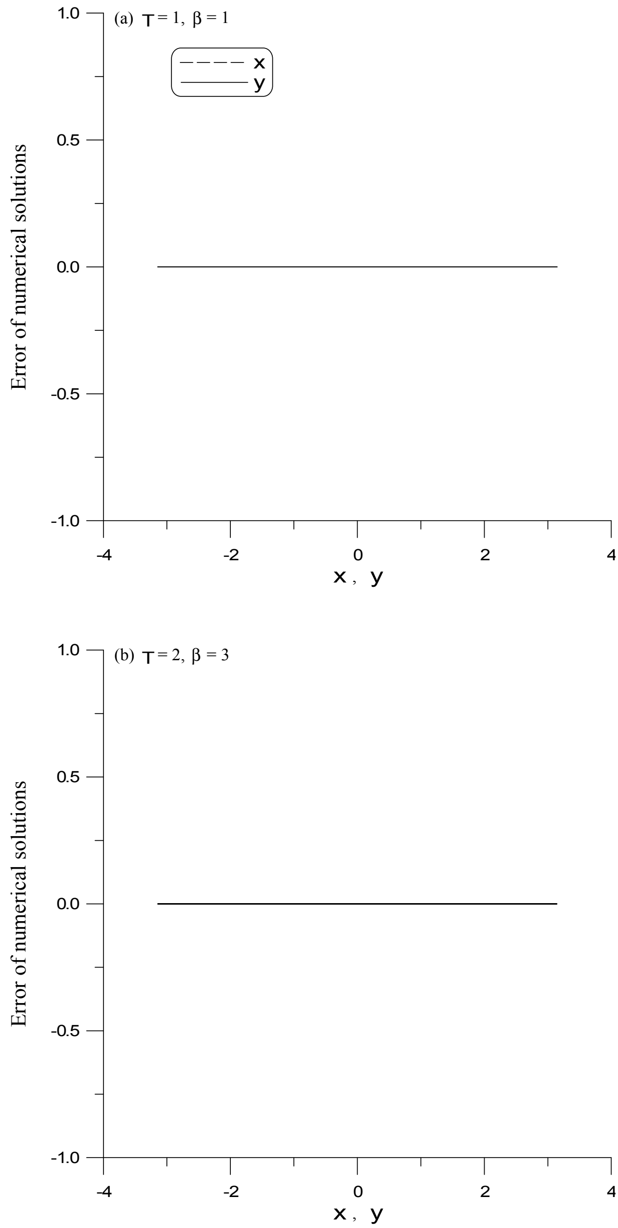

Figure 1: The errors of semi-analytical solutions for Example 1 are shown in (a) with $T = 1$ and $\beta = 1$, and in (b) with $T = 2$ and $\beta = 3$.

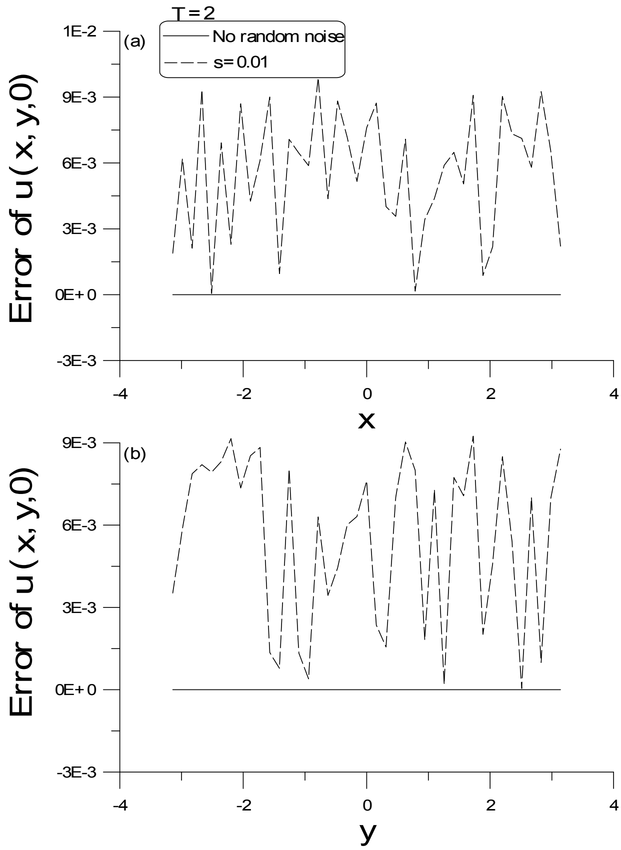

Figure 2: The numerical errors of semi-analytical solutions with and without random noise effect for Example 1 are plotted in (a) with respect to x at fixed $y = 2\pi/3$, and in (b) with respect to y at fixed $x = \pi/2$.

We will give a more ill-posed example than the above one by using the quasi-boundary semi-analytical approach. Let $\beta = 3$, $T = 2$, and the final data is very small in the order of $O(10^{-16})$. However, we can employ this method to retrieve the desired initial data $\sin\beta x \sin\beta y$, which is in the order of $O(1)$. In Fig. 1(b), the errors of numerical solutions calculated by the proposed approach with $\Delta x = \Delta y = 2\pi/40$.

In Fig. 2, we compare the numerical errors with $T = 2$ and $\beta = 1$ for two cases: one without the random noise and another one with the random noise in the level of $s = 0.01$. In Figs. 3(a)-(c), we represent the exact solution and numerical solutions sequentially. Even under the noise the numerical solution displayed in Fig. 3(c)

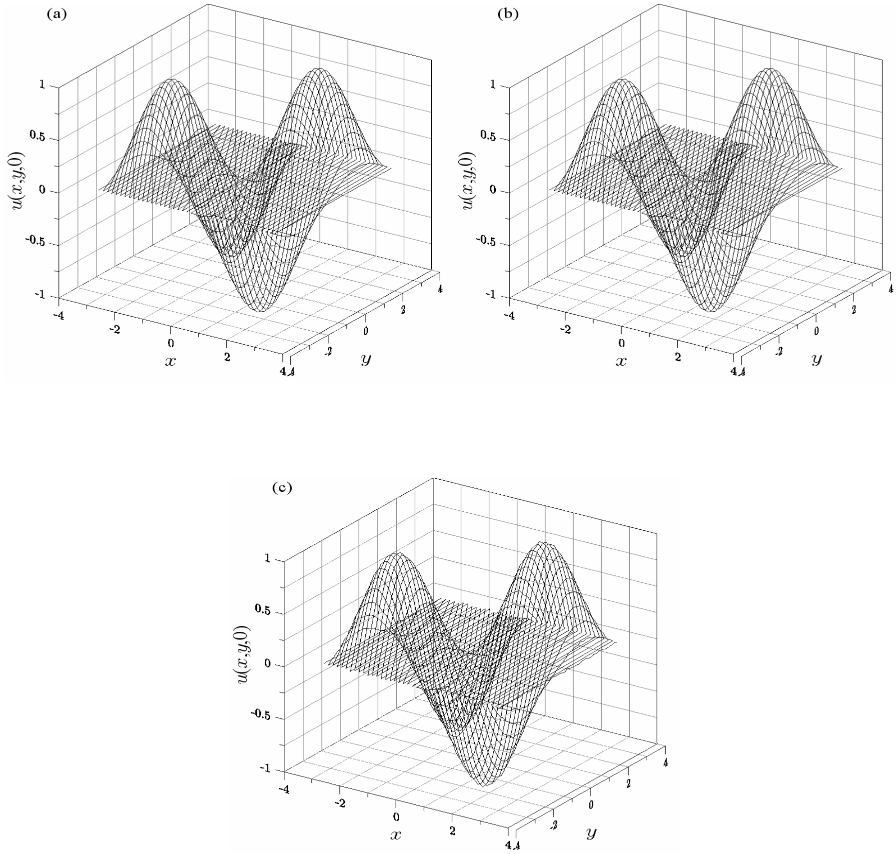

Figure 3: The exact solution for Example 1 of two-dimensional BHCP are shown in (a), in (b) the semi-analytical solution without random noise effect, and in (c) the semi-analytical solution with random noise.

is a good estimation to the exact initial data as shown in Fig. 3(a). In addition, we should emphasize that in all the calculations, we can use $\alpha_0 = 0$ without any difficulty since Eq. (33) is still applicable.

5.2 Example 2

Let us consider the second example of two-dimensional BHCP:

$$u_t = u_{xx} + u_{yy},\ 0 < x < 1, 0 < y < 1, 0 < t < T, \tag{39}$$

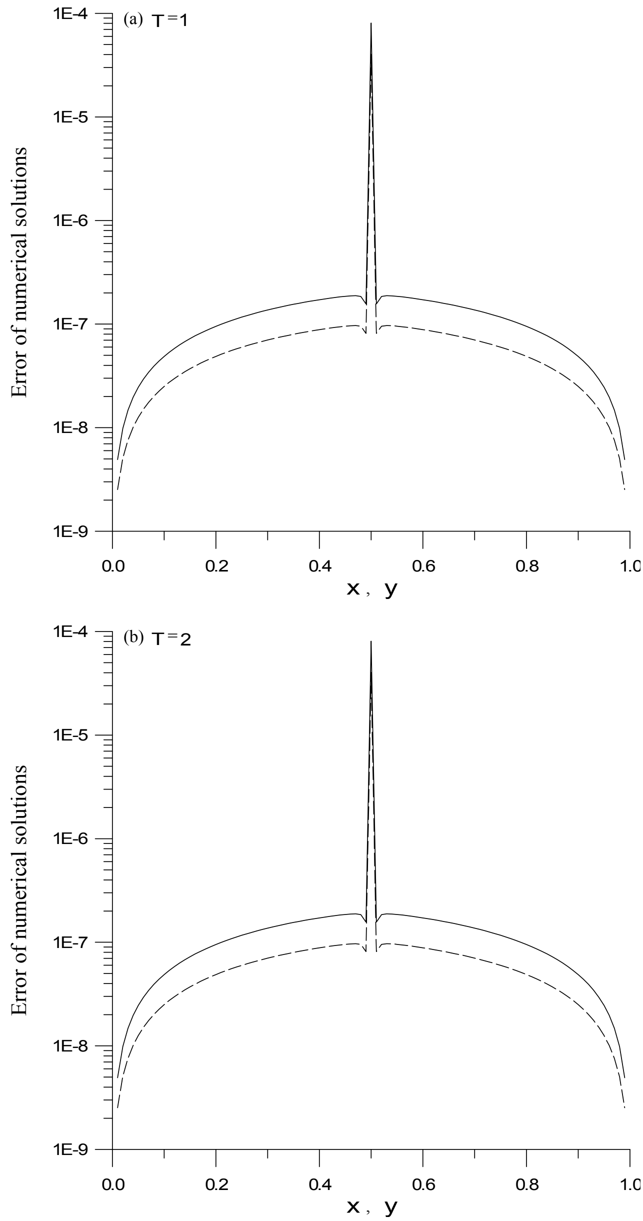

Figure 4: The errors of semi-analytical solutions for Example 1 are shown in (a) with $T = 1$, and in (b) with $T = 2$.

with the boundary conditions

$$u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0, \tag{40}$$

and the initial condition

$$u(x,y,0) = \begin{cases} 4xy, & \text{for } 0 \le x \le 0.5,\ 0 \le y \le 0.5, \\ 4y(1-x), & \text{for } 0.5 \le x \le 1,\ 0 \le y \le 0.5, \\ 4x(1-y), & \text{for } 0 \le x \le 0.5,\ 0.5 \le y \le 1, \\ 4(1-x)(1-y), & \text{for } 0.5 \le x \le 1,\ 0.5 \le y \le 1. \end{cases} \tag{41}$$

The exact solution is given by

$$u(x,y,t) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{64(-1)^k(-1)^j}{ab\pi^4(2k+1)^2(2j+1)^2} \exp[-(k^2/a^2 + j^2/b^2)\pi^2 t] \sin\left[\frac{(2k+1)\pi x}{a}\right] \sin\left[\frac{(2j+1)\pi y}{b}\right]. \tag{42}$$

The backward numerical solution is subjected to the final condition at time T:

$$f(x,y) = u(x,y,T) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{64(-1)^k(-1)^j}{ab\pi^4(2k+1)^2(2j+1)^2} \exp[-(k^2/a^2 + j^2/b^2)\pi^2 T] \sin\left[\frac{(2k+1)\pi x}{a}\right] \sin\left[\frac{(2j+1)\pi y}{b}\right]. \tag{43}$$

The difficulty of this problem is stemmed from that we employ a smooth final data to retrieve a non-smooth initial data.

Let $a = b = 1$ and insert Eq. (42) for $f(x,y)$ into Eq. (29) to obtain

$$d_{kj}^{\alpha} = \frac{1}{\{\alpha + \exp[-(k^2 + j^2)\pi^2 T]\}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{64(-1)^m(-1)^n \delta_{k,(2m+1)}\delta_{j,(2n+1)}}{\pi^4(2m+1)^2(2n+1)^2} \exp\{-[(2m+1)^2 + (2n+1)^2]\pi^2 T\}. \tag{44}$$

Inserting it into Eq. (27), we have

$$u^{\alpha}(x,y,t) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{\{\alpha + \exp[-(k^2 + j^2)\pi^2 T]\}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{64(-1)^m(-1)^n \delta_{k,(2m+1)}\delta_{j,(2n+1)}}{\pi^4(2m+1)^2(2n+1)^2} \exp\{-[(2m+1)^2 + (2n+1)^2]\pi^2 T\} \exp[-(k^2 + j^2)\pi^2 t] \sin(k\pi x)\sin(j\pi y). \tag{45}$$

Interchanging the order of summation and using the δ property, we obtain

$$u^{\alpha}(x,y,t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{64(-1)^m(-1)^n}{\pi^4(2m+1)^2(2n+1)^2}\frac{\exp\{-[(2m+1)^2+(2n+1)^2]\pi^2T\}}{\{\alpha+\exp\{-[(2m+1)^2+(2n+1)^2]\pi^2T\}\}}$$
$$\times\exp\{-[(2m+1)^2+(2n+1)^2]\pi^2t\}\sin[(2m+1)\pi x]\sin[(2n+1)\pi y]. \quad (46)$$

It gives

$$u^{\alpha}(x,y,0)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{64(-1)^m(-1)^n}{\pi^4(2m+1)^2(2n+1)^2}\frac{\exp\{-[(2m+1)^2+(2n+1)^2]\pi^2T\}}{\{\alpha+\exp\{-[(2m+1)^2+(2n+1)^2]\pi^2T\}\}}$$
$$\times\sin[(2m+1)\pi x]\sin[(2n+1)\pi y]. \quad (47)$$

The term

$$\frac{\exp\{-[(2m+1)^2+(2n+1)^2]\pi^2T\}}{\{\alpha+\exp\{-[(2m+1)^2+(2n+1)^2]\pi^2T\}\}}=1-\alpha_0$$

is already derived at the first of this section. Therefore, we get

$$u^{\alpha_0}(x,y,0)=(1-\alpha_0)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{64(-1)^m(-1)^n}{\pi^4(2m+1)^2(2n+1)^2}\sin[(2m+1)\pi x]\sin[(2n+1)\pi y]. \quad (48)$$

Thus, we use this solution to compare that in Eq. (41). In practice, the data is obtained by taking the sum of the first one thousand terms, which guarantees the convergence of the series.

In Fig. 4(a), we show the errors of numerical solutions obtained from the quasi-boundary semi-analytical approach for the case. T = 1 is employed in this comparison, where the grid lengths $\Delta x=\Delta y=0.01$ are used. At the point $x=0.1$ the error is plotted with respect to y by a dashed line, and at the point $y=0.8$ the error is plotted with respect to x by a solid line. The latter one is larger than the former one since the point $y=0.8$ is near to the boundary. Then, we give a more ill-posed example than the above one by using the current approach at T = 2. The errors of numerical solutions were shown in Fig. 4(b).

In Fig. 5, we compare the numerical errors with T = 2 for two cases: one without the relative random noise and another one with the relative random noise in the level of s = 0.01. In Figs. 6(a)-(c), we show the exact solution and numerical solutions sequentially. Even under the noise the numerical solution shown in Fig. 6(c) is a good estimation to the exact initial data as represented in Fig. 6(a). Besides, the maximum error estimation is shown in Fig. 7.

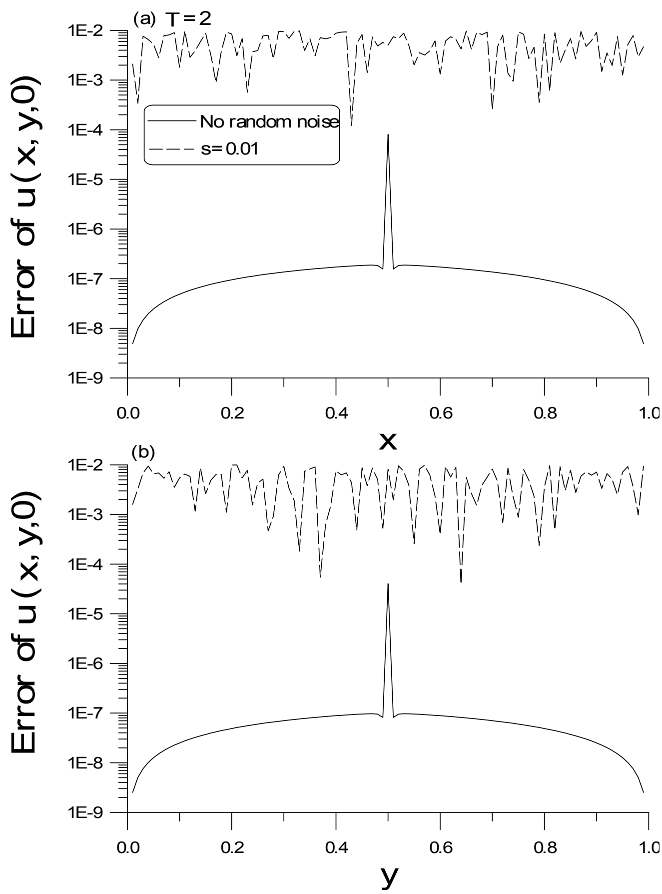

Figure 5: The numerical errors of semi-analytical solutions with and without random noise effect for Example 2 are plotted in (a) with respect to x at fixed $y = 0.8$, and in (b) with respect to y at fixed $x = 0.1$.

5.3 Example 3

In the previous numerical example, we employed closed-form solutions as the inputs of the final time data. In practice, we cannot easily obtain the closed-form solution of the BHCP as that shown in Eq. (33) when the available data $f(x,y)$ is not in a closed-form; many numerical schemes can be used to evaluate the data. Instead of directly using Eqs. (27) and (29) as a semi-analytical solution of the BHCP where the data $f(x,y)$ can be in a discretized form, we apply the trapezoidal rule to perform the integral. Here, the final data of Example 2 is calculated by the

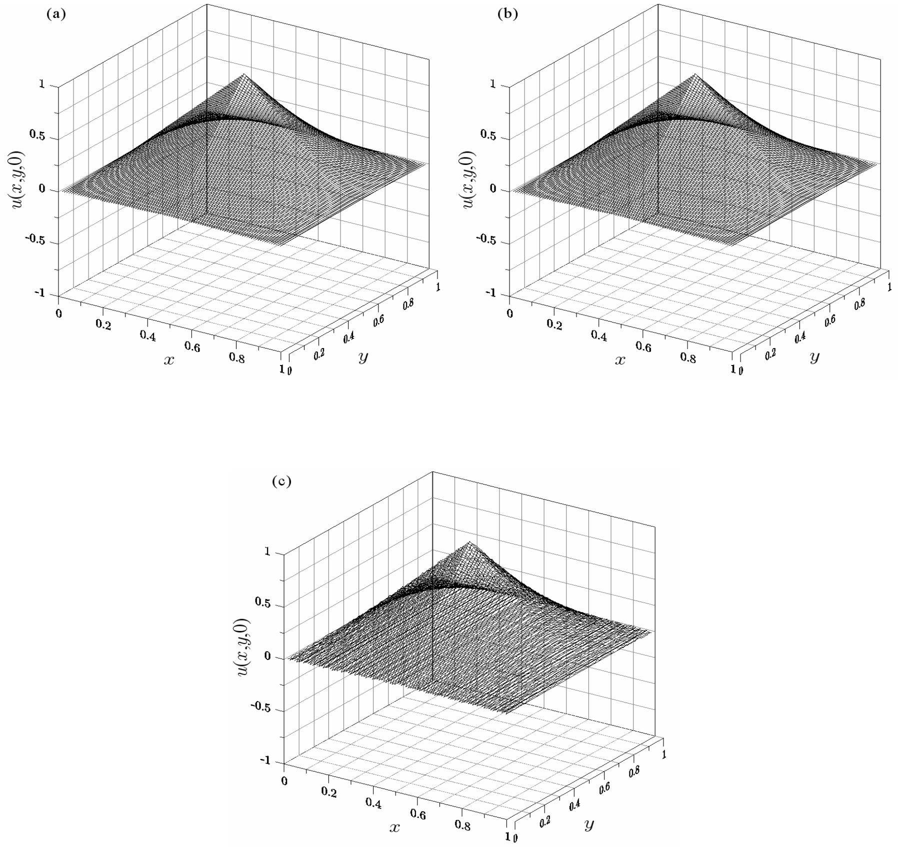

Figure 6: The exact solution for Example 2 of two-dimensional BHCP are shown in (a), in (b) the semi-analytical solution without random noise effect, and in (c) the semi-analytical solution with random noise.

GPS with a time increment $\Delta t = 2.5 \times 10^{-5}$:

$$u(x,y,T) = f(x,y), \ 0 \le t < T. \tag{49}$$

Let $a = b = 1$, and substitute Eq. (49) for $f(x,y)$ into Eq. (29) to obtain

$$d_{kj}^{\alpha} = \frac{4}{\{\alpha + \exp[-(k^2+j^2)\pi^2 T]\}} \int_0^1 \int_0^1 \sin(k\pi\xi)\sin(j\pi\varphi) f(\xi,\ \varphi) d\xi d\varphi. \tag{50}$$

Substituting Eq. (50) into Eq. (27), we have

$$u^{\alpha}(x,\ y,\ t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{kj}^{\alpha} \exp[-(m^2+n^2)\pi^2 t] \sin(m\pi x)\sin(n\pi y), \tag{51}$$

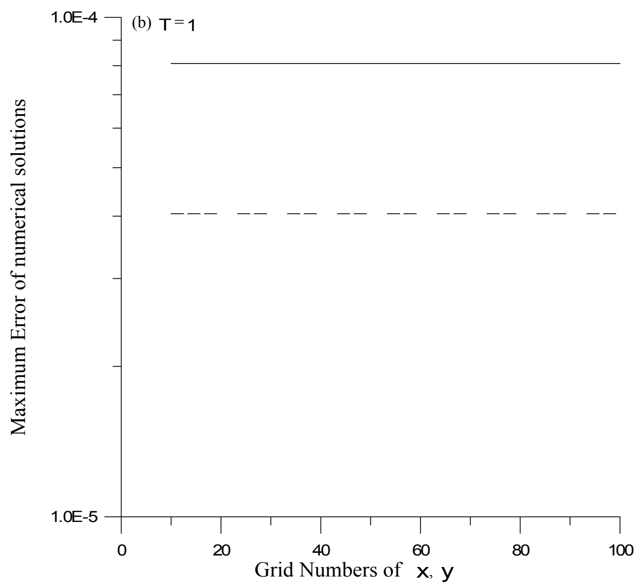

Figure 7: The maximum error as a function of the number of grid points for the final time of $T = 1$.

which gives

$$u^{\alpha}(x,\ y,\ 0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{kj}^{\alpha}\sin(m\pi x)\sin(n\pi y). \tag{52}$$

In Fig. 8, we display the exact solutions and numerical solutions for a fixed $T = 0.001$ with $\Delta x = \Delta y = 0.1$ and $\alpha = 0$. At the point $y = 0.8$ the error is plotted with respect to x in Fig. 8(a), and at the point $x = 0.1$ the error is plotted with respect to y in Fig. 8(b). The accuracy is quite better.

6 Conclusions

In this study, we have transformed the two-dimensional BHCP into a second-kind two-dimensional Fredholm integral equation through a direct regularization technique and a quasi-boundary concept. By using the Fourier series expansion technique and a termwise separable property of kernel function, an analytical solution of the regularized type for approximating the exact solution is shown. The influence of regularization parameter on the perturbed solution is clarified. Several numerical experiments have represented that the proposed method can retrieve all initial data very well, even though the final data are very small or noised by a large disturbance,

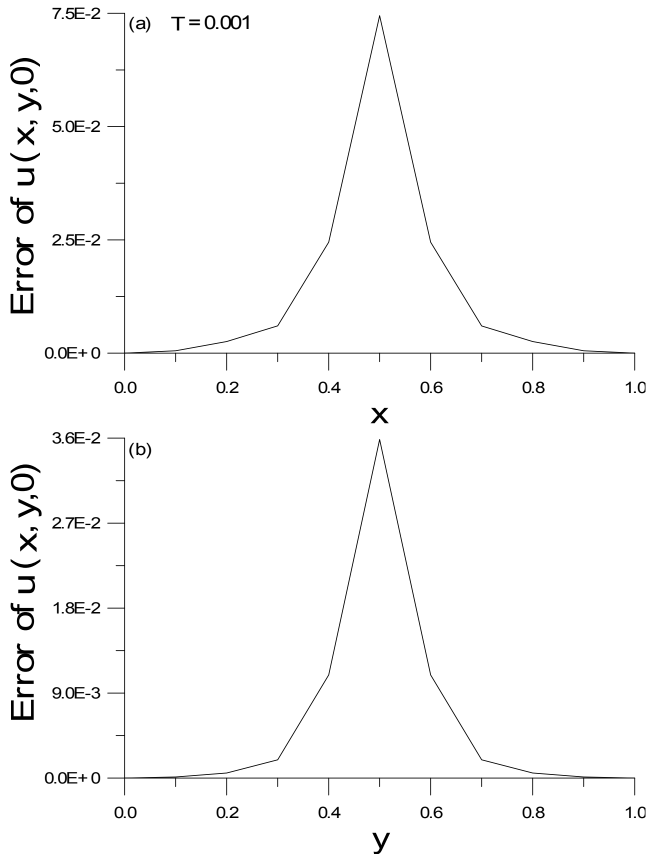

Figure 8: For Example 5 we compare the exact solutions and numerical solutions with $T = 1$ in (a), and with $T = 0.01$ in (b).

and the initial data to be recovered are not smooth. Hence, the current approach is advocated to deal with the two-dimensional BHCPs.

Acknowledgement: The corresponding author would like to express his thanks to the National Science Council, ROC for their financial support under Grant Numbers, NSC 97-2221-E-002-264-MY3.

References

Ames, K. A.; Cobb, S. S. (1997): Continuous dependence on modeling for related Cauchy problems of a class of evolution equations. *J. Math. Anal. Appl.*, vol. 215, pp. 15–31.

Ames, K. A.; Clark, G. W.; Epperson, J. F.; Oppenheimer, S. F. (1998): A comparison of regularizations for an ill-posed problem. *Math. Comput.*, vol. 67, pp. 1451–1471.

Boussetila, N.; Rebbani, F. (2006): Optimal regularization method for ill-posed Cauchy problems. *Electronic J. Diff. Equations*, vol. 2006, pp. 1–15.

Chang, C.-W.; Liu, C.-S.; Chang, J.-R. (2005): A group preserving scheme for inverse heat conduction problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 10, pp. 13–38.

Chang, C.-W.; Liu, C.-S.; Chang, J.-R. (2008): A new shooting method for quasi-boundary regularization of multi-dimensional backward heat conduction problems. *J. Chinese Insti. Eng.*, vol. 32, pp. 307–318.

Chang, J.-R.; Liu, C.-S.; Chang, C.-W. (2007): A new shooting method for quasi-boundary regularization of backward heat conduction problems. *Int. J. Heat Mass Transfer*, vol. 50, pp. 2325–2332.

Denche, M.; Bessila, K. (2005): A modified quasi-boundary value method for ill-posed problems. *J. Math. Anal. Appl.*, vol. 301, pp. 419–426.

Huang, Y.; Zheng, Q. (2005): Regularization for a class of ill-posed Cauchy problems. *Proc. A. M. S.*, vol. 133, pp. 3005–3012.

Iijima, K. (2004): Numerical solution of backward heat conduction problems by a high order lattice-free finite difference method. *J. Chinese Inst. Engineers*, vol. 27, pp. 611–620.

Lattés, R.; Lions, J. L. (1969): The Method of Quasireversibility, Applications to Partial Differential Equations. Elsevier, New York.

Liu, C.-S. (2001): Cone of non-linear dynamical system and group preserving schemes. *Int. J. Non-Linear Mech.*, vol. 36, pp. 1047–1068.

Liu, C.-S. (2004): Group preserving scheme for backward heat conduction problems. *Int. J. Heat Mass Transfer*, vol. 47, pp. 2567–2576.

Liu, C.-S. (2007a): Elastic torsion bar with arbitrary cross-section using the Fredholm integral equations. *CMC: Computers, Materials & Continua*, vol. 5, pp. 31-42.

Liu, C.-S. (2007b): A meshless regularized integral equation method for Laplace equation in arbitrary interior or exterior plane domains. *CMES: Computer Modeling in Engineering & Sciences*, vol. 19, pp. 99-109.

Liu, C.-S. (2007c): A MRIEM for solving the Laplace equation in the doubly-connected domain. *CMES: Computer Modeling in Engineering & Sciences*, vol.

19, pp. 145-161.

Liu, C.-S. (2009a): Solving the inverse problems of Laplace equation to determine the Robin coefficient/cracks' position inside a disk. *CMES: Computer Modeling in Engineering & Sciences*, vol. 40, pp. 1-28.

Liu, C.-S. (2009b): A new method for Fredholm integral equations of 1D backward heat conduction problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 47, pp. 1-21.

Liu, C.-S.; Chang, C.-W.; Chang, J.-R. (2006): Past cone dynamics and backward group preserving schemes for backward heat conduction problems. *CMES: Computer Modeling in Engineering & Sciences*, vol. 12, pp. 67–81.

Liu, C.-S.; Chang, C.-W.; Chang, J.-R. (2009): The backward group preserving scheme for 1D backward in time advection-dispersion equation. *CMC: Computer, Materials & Continua*, vol. 9, pp. 111–135.

Liu, C.-S.; Chang, C. W.; Chiang, C. Y. (2008): A regularized integral equation method for the inverse geometry heat conduction problem. *Int. J. Heat Mass Transfer*, vol. 51, pp. 5380-5388.

Liu, J. (2002): Numerical Solution of Forward and Backward Problem for 2-D Heat Conduction Equation. *J. Comput. Appl. Math.*, vol. 145, pp. 459–482.

Mel'nikova, I. V. (1992): Regularization of ill-posed differential problems. *Siberian Math. J.*, vol. 33, pp. 289–298.

Mera, N. S. (2005): The method of fundamental solutions for the backward heat conduction problem. *Inv. Prob. Sci. Eng.*, vol. 13, pp. 65–78.

Miller, K. (1973): Stabilized quasireversibility and other nearly best possible methods for non-well posed problems. Lecture Notes in Math. Springer-Verlag, Berlin, vol. 316, pp. 161–176.

Showalter, R. E. (1983): Cauchy problem for hyper-parabolic partial differential equations. In Trends in the Theory and Practice of Non-Linear Analysis, Elsevier, Amsterdam, pp. 421–425.

Showalter, R. E.; Ting, T. W. (1970): Pseudo-parabolic partial differential equations. *SIAM J. Math. Anal.*, vol. 1, pp. 1–26.

Tricomi, F. G. (1985): Integral Equations. Dover Publications, New York.