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Heat Transfer in Composite Beams using Combined Cellular Automaton and Fibre Model

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Abstract: A simple cellular automaton (CA) scheme is proposed to simulate heat conduction in anisotropic domains. The CA is built on random nodes rather than an irregular grid. The local rule used in the CA is defined by physical concepts instead of differential equations. The accuracy of the proposed approach is verified by classical examples. As an application of the proposed method, the CA approach is incorporated into fibre model which is widely used in finite element analysis to calculate the temperature distribution on the cross-section of composite beams. Numerical examples demonstrate that the proposed scheme can be conveniently applied to finite element analysis.

Keywords: Cellular automaton, fibre model, heat conduction

1 Introduction

An accurate prediction of temperature distribution across a section for a given fire condition plays a key role in the simulation of thermal effect in structural analyses [Tan and Yuan (2008); Tan and Yuan (2009)]. In finite element modelling, fibre model is frequently applied to beam and column members since it offers a good balance between simplicity and accuracy [Spacone, Filippou and Taucer (1996a); Spacone, Filippou and Taucer (1996b)]. According to this model, the cross-section of a beam element is divided into a matrix of fibres; each of these fibres may have different material, thermal and mechanical properties. Normally, to simplify the simulation, the temperature in a fibre is assumed to be uniform along the longitudinal direction of members.

Heat transfer is a classical issue in physics. It is also a normal problem in Engineering. Up to today, many approaches have been developed to calculate the temperature distribution in structural members. To obtain accurate result, analytical approach can be used in some typical problems [Wang and Tan (2008); Chao,

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Chen and Chen (2009)]. On the other hand, approximate solution deduced from engineering assumption may be adopted for simplicity [Tan, Ting and Huang (2002)]. Numerical analysis is another alternative choice. For instance, MFS algorithm and residual correction method have been applied to the Cauchy problem in anisotropic heat conduction [Marin (2009)] and non-linear heat transfer [Cheng, Chen and Yang (2009)], respectively. Moreover, the hybrid numerical method has been used to analyze the heat conduction in a plate with functionally graded material. Besides, other numerical approaches, such as finite element analysis (FEA) [Huang, Tan and Phng (2007)], boundary element method (BEM) [Divo and Kassab (2006)] and Meshless [Sladek, Sladek, Tan and Atluri (2008); Sladek, Sladek, Solek, Wen and Atluri (2008); Sladek, Sladek, Solek and Wen (2008)] are also very useful methods to conduct heat transfer analysis. However, among these methods, FEA is the most popular tool used by engineers. On the other hand, due to the meshing of the cross-section, it is tedious to incorporate heat transfer analysis into fibre model formulations. Compared with FEA, BEM seems more convenient to be used since it only requires discretization on the boundaries of the cross-sections of beams and columns. But, BEM will become inefficient if the fibres have many different types of material properties. Although analytical approach is the basis of numerical validations for FEA and BEM, its use is rather limited as it is confined to well-posed problems. Engineers resorting to analytical approach will encounter great difficulties when dealing with a cross-section of complex shape consisting of anisotropic material. The fourth approach engineering assumption pre-supposes that the temperature distribution on a cross-section follows a linear or quadratic function in spatial domain. This method is easy to use but lacks rigor.

To overcome the difficulties mentioned above, the authors propose a mesh-free method based on cellular automaton (CA). The history of CA can be traced back to 1940s when Von Neumann et al. studied biological reproduction and crystal growth [Von and Burks (1966)]. Since then, CA has been used to model complex phenomena in various areas such as fluid dynamics [Wolfram (1986)], biology [Ermentrout and Edlestein (1993)] and emergency evacuation [Yuan and Tan (2007a); Yuan and Tan (2007b)]. Basically, CA is a discrete model which consists of a regular grid of cells, each cell in one of a finite number of states. In a CA model, time is also discretized into finite number of steps, and the current state of a specific cell is influenced and determined by the states of its neighboring cells at the last time step. When CA is applied to heat conduction problems, the rules which dominate the simulation are deduced directly from the physical phenomenon instead of solving differential equations. To implement the CA approach, random nodes are generated within a domain and along its boundary as well. Since meshing is not required, this approach can be used to model domains with arbitrary boundaries. Moreover, it

can also be used to calculate the temperature distribution in non-homogeneous materials. Therefore, the proposed approach provides a useful and practical numerical tool to simulate heat conduction in composite structural members, such as composite concrete columns with embedded steel I-section.

2 Formulation

Generally, a CA model requires a spatial domain to be discretized into regular cells. Besides, the time domain is divided into a series of intervals. As a main feature, a CA model is established based on several local rules which relate the current state of a particular cell to the states of its neighbouring cells at the last time step. To analyze heat conduction in solids, this basic concept of CA is extended in the present study. Firstly, irregular grid of random nodes in the spatial domain is used to replace regular cells in traditional CA. Secondly, a virtual time domain is generated and divided into constant intervals to match the features of CA. In this paper, only steady-state heat conduction is discussed, although with some modifications, the method can be extended to transient-state heat conduction.

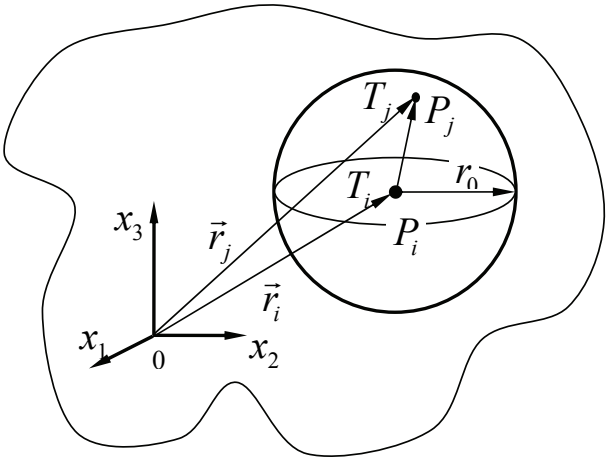

Figure 1: Illustration of an arbitrary three-dimensional domain

Consider a three-dimensional solid domain Ω as shown in Figure 1. The term T_i is the temperature at an arbitrary point P_i within Ω. The point P_i is located at the centre of its neighbouring volume which is illustrated by a sphere with a radius r_0. Within the sphere, P_j is a typical node among N_i monitoring points. Similar toP_i, the temperature at point P_j is denoted by T_j. To construct a CA model, the relationship between P_i and other points within its neighbouring sphere is defined

by Eq. (1).

$$T_i = \sum_{j=1}^{N_i} \omega_{ij} T_j \tag{1}$$

where ω_{ij} is a weighting factor where subscript $i$denotes point P_i and subscript j relates to the congregate effects of temperature at neighbouring points P_j. Thus, clearly, from Eq. (1), the temperature at P_i is determined by the neighbouring temperatures. This rule is based on the assumption that the variable T_i will not vary too much within a very small sphere. It is implicitly assumed that there is continuous variation of temperature over the spatial distance within the sphere. Eq. (1) also indicates that the influence of point P_j to T_i is dependant on the value of weighting factor ω_{ij}. To allow for anisotropic heat transfer, it is assumed that the material in the domain is non-homogeneous, where ω_{ij} is given by Eq. (2):

$$\omega_{ij} = (k_i + k_j) \Big/ \sum_{l=1}^{N_i} (k_i + k_l) \tag{2}$$

where k_i, k_j and k_l are the respective thermal conductivities at points P_i, P_j and P_l. One observes that $\sum_{j=1}^{N_i} \omega_{ij} = 1$.

From Eqs. (1) and (2), one finds that the temperature at each point within the neighbouring area of P_i affects the value of T_i. The definition in Eq. (2) accords with the physical law in heat transfer in that, the larger the average thermal conductivity between points P_i and P_j, the closer are the temperatures at these two points.

To complete the construction of the CA model, Eq. (1) can be extended to Eq. (3) by incorporating virtual time steps t:

$$T_i(t_m) = \sum_{j=1}^{N_i} \omega_{ij} T_j(t_{m-1}) \tag{3}$$

The interpretation for Eq. (3) is that the temperature at P_i at t_m is determined by the temperatures at neighbouring points of P_j at the previous time step t_{m-1}.

To implement the CA model described above, analogous to points P_i and P_j, one can define many random points within Ω and on its boundary. During an analysis, the temperature at each point is updated according to Eq. (3) until it converges to a stable value. The final converged value of the temperature at each point forms a state which represents the solution of the particular domain. One may realize that Eqs. (1) $\sim$ (3) are entirely based on physical intuition of the heat conduction

phenomenon. It should be noted that differential equations are not involved in the derivation. Like traditional CA model, the procedure to calculate the distribution of temperature forms a loop which contains the following steps:

(1) Generate random nodes within the domain and on its boundary.

(2) Assign an initial value to each node where temperature is unknown.

(3) Based on the state at time t_{m-1}, calculate the temperature at each node at t_m using Eq. (3).

(4) Repeat step (3) till the final time step t_M. At t_M, the temperature at each node has converged to a stable value. It cannot be further updated compared with the previous time t_{M-1}.

3 Verification and application

3.1 Example 1: A rectangular domain under thermal conditions

As shown in Figure 2, a rectangular two-dimentional domain a-b-d-c is subjected to thermal conditions. In the domain, the varying thermal conductivity of the material is described by the function $k(x,y) = e^{x^2-y^2}$ where x and y represent the physical domain. On the boundaries, it is assumed that the temperatures are given as $T_{ab} = 0$, $T_{ac} = 0$, $T_{bd} = 2y$ and $T_{cd} = x$.

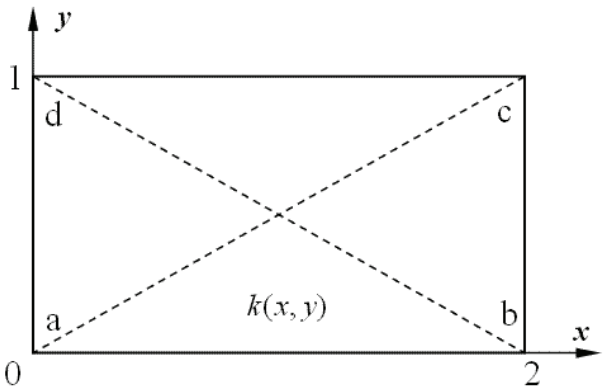

Figure 2: A rectangular domain under thermal conditions

For comparison purpose, the analytical solution can be derived for this example. From the heat transfer theory, Eq. (4) is the governing partial differential equation

for example 1.

$$\partial(k\partial T/\partial x)/\partial x+\partial(k\partial T/\partial y)/\partial y=0 \tag{4}$$

The solution of Eq. (4) can be obtained without difficulty. Its expression is:

$$T=xy \tag{5}$$

The diagonals a-c and d-b can be respectively described by $y=x/2$ and $y=1-x/2$. Hence, the temperature distributions on the diagonals of rectangle a-b-d-c can be derived:

$$T=\begin{cases} x^2/2, & (y=x/2) \\ x-x^2/2, & (y=1-x/2) \end{cases} \tag{6}$$

On the other hand, to evaluate the temperature distribution using CA method, 1200 points inside the domain and 120 points along the four edges are generated randomly, as shown in Figure 3. In the figure, there are 21 points (in red dots) distributed along the two diagonals of rectangle a-b-d-c. Temperatures at these points are used for comparison with analytical answers given by Eq.(6). The value of spatial distance of the sphere r_0 is set to 0.1 in the calculation.

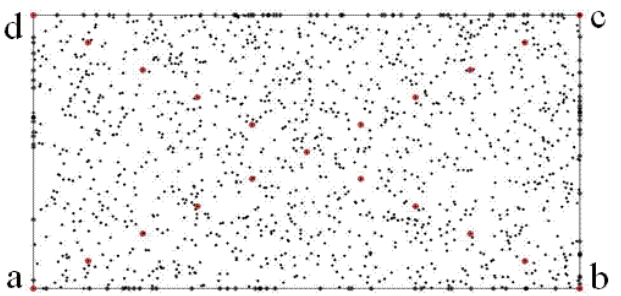

Figure 3: Distribution of points generated randomly in example 1

Both numerical and analytical results are depicted in Figure 4. It can be seen from the figure that the CA predictions agree very well with the analytical results. In fact, to minimize numerical errors, one can conduct several trial runs based on different distribution of random points. The average values of few trial run results will be more accurate.

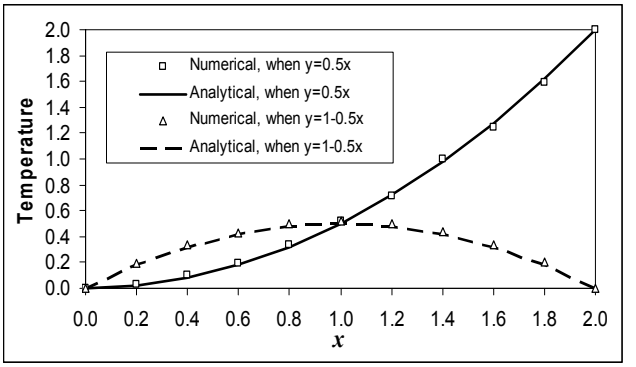

Figure 4: Numerical and analytical temperatures on diagonal lines

3.2 3.2 Example 2: A hollow ball under different inner and outer temperatures

Figure 5 illustrates a hollow ball with an inner radius $r_I = 0.25$ and an outer radius $r_O = 1.0$. The thermal conductivity is assumed to be constant throughout the whole domain. The temperatures on the two surfaces are $T_O = 200^0C$ and $T_I = 500^0C$, respectively.

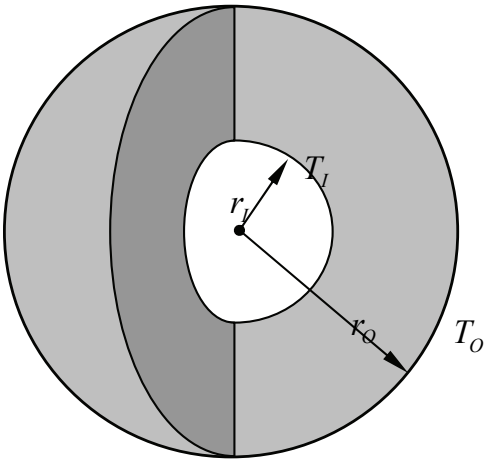

Figure 5: A hollow ball under different inner and outer temperatures

Since the thermal conductivity is constant, this example turns out to be a one-dimensional problem. However, to investigate the applicability of the proposed approach in three-dimensional domain, 14744 points including 4000 in the domain

and 10744 on the surfaces were generated randomly, as shown partly in Figure 6. In the modelling, the value of r_0 defined in Figure 1 is set to 0.08. To monitor the temperature in the domain, 11 nodes were distributed evenly along a straight line between the inner and outer surfaces. Through the comparison depicted in Figure 7, one finds that the numerical result agrees well with analytical result.

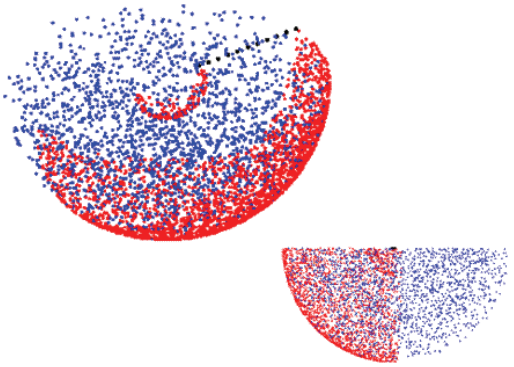

Figure 6: Random nodes generated for the calculation

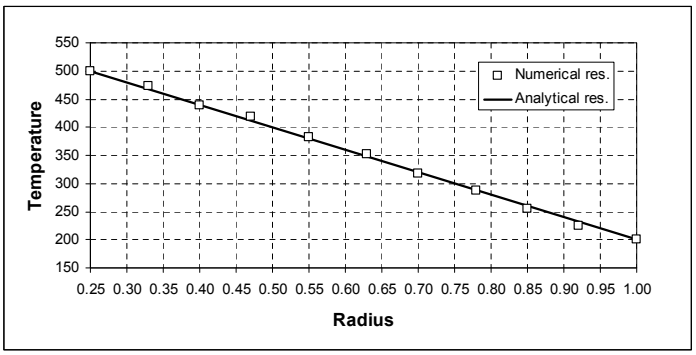

Figure 7: Temperatures at monitoring points

During the study in example 1 and 2, it has also been found that the accuracy of simulation is affected by the number of nodes and the value of r_0. In general, more nodes will result in more accurate results. Needless to say, a large number of random nodes will require a longer CPU time.

3.3 Example 3: A composite beam under thermal conditions

Figure 8 shows the cross-section of a rectangular composite concrete column modelling using fibre model. It can be seen that an I-section steel is embedded in the concrete interior. The dimensions and the thermal boundary conditions are illustrated in the figure. In the analysis, it is assumed that the temperatures on the top and bottom boundaries are $T_1 = 100^0C$ and $T_2 = 500^0C$, respectively. It is also defined that $k_C/k_S = 1/50$ where k_C and k_S denote the thermal conductivities of concrete and steel, respectively.

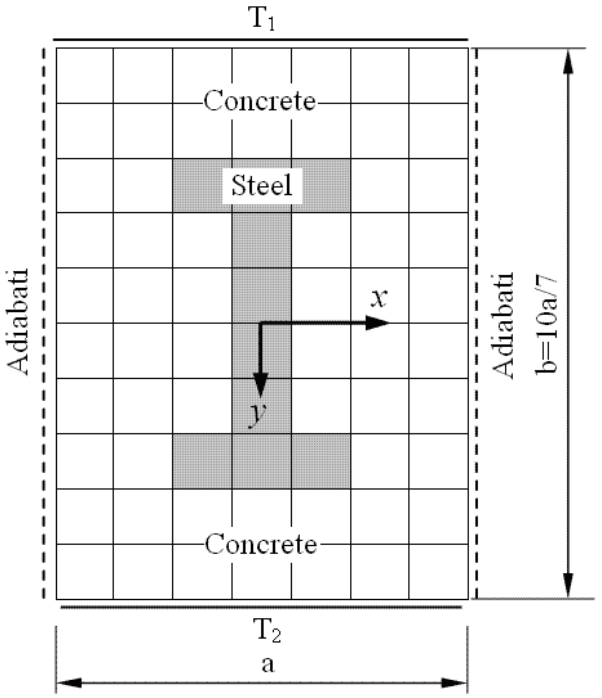

Figure 8: The cross-section of a composite concrete column

As shown by the temperature contour in Figure 9, the temperature distribution on the cross-section is symmetric about the column centroid $x = 0$. Along the y direction, between $y = b/2$ and $y = -b/2$, the temperature increases gradually from T_1 to T_2. More details can be seen in Figures 10 and 11. However, it is noteworthy that the temperature on the column cross-section is not linearly distributed.

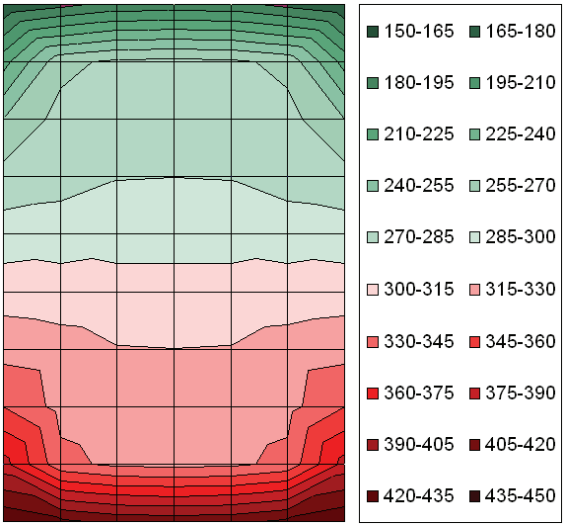

Figure 9: Temperature distribution on the cross-section of a composite concrete column

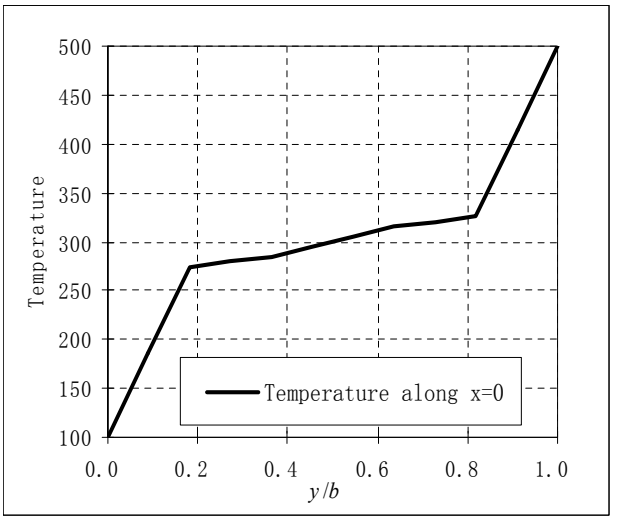

Figure 10: Temperatures along $x = 0$

4 The relationship between differential equation and cellular automaton

The three examples illustrate that the proposed CA model can be used as a numerical tool to analyze steady-state heat conduction, although the local rules in-

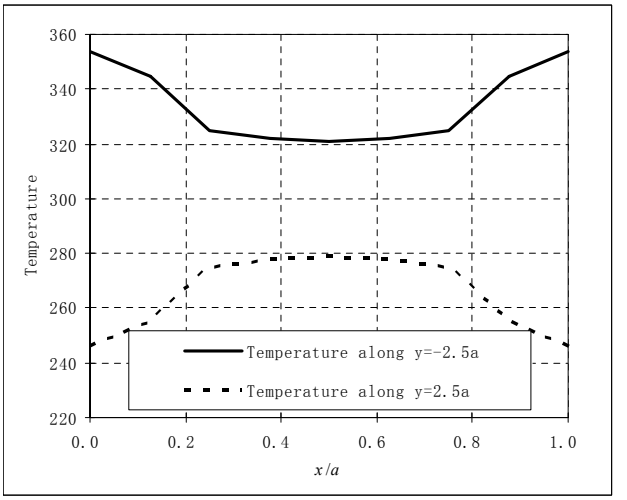

Figure 11: Temperatures along $y = \pm 2.5a$

corporated into the CA model are deduced without the consideration of differential equation. However, it should be realized that differential equation is an analytical tool applied to the same problem. Hence, it is believed that there exists a relationship between the present CA model and classical differential equation since they both describe the problem correctly. It is well-known that one-dimensional heat conduction is governed by:

$$d(kdT/dx)/dx = 0 \tag{7}$$

Based on this, one infers that $kdT/dx = C_1$ where C_1 is a constant. Applying classical limit theorem to Eq. (7), one obtains:

$$\lim_{x_j \to x_i} (k_j T_j - k_i T_i)/(x_j - x_i) = C_1 \tag{8}$$

Let spatial distance $\Delta_{ji} = x_j - x_i$ and thermal conductivity$k_j = k_i = (k_j + k_i)/2 = \eta_{ji}$ arbitrarily. By applying Eq. (8) to all points within the neighbouring domain of point P_i, one obtains Eq. (9).

$$\sum_{j=1}^{N_i} \eta_{ji} (T_j - T_i) = C_1 \sum_{j=1}^{N_i} \Delta_{ji} \tag{9}$$

Thus,

$$\sum_{j=1}^{N_i} (\eta_{ji} T_j) - \left(\sum_{j=1}^{N_i} \eta_{ji} \right) T_i = C_1 \sum_{j=1}^{N_i} \Delta_{ji} \tag{10}$$

It should be noted that P_j is a random point in the neighbourhood of P_i. Therefore, according to statistics, the term $C_1 \sum_{j=1}^{N_i} \Delta_{ji} \to 0$ when N_i is large enough. For this situation, one obtains:

$$T_i = \sum_{j=1}^{N_i} (\eta_{ji} T_j) / \sum_{l=1}^{N_i} \eta_{li} = \sum_{j=1}^{N_i} \omega_{ij} T_j \tag{11}$$

5 Conclusion

Based on the concept of cellular automata, a new numerical approach is developed without the consideration of differential equation for analyses of heat conduction in anisotropic domain. Numerical examples show that this approach can be applied to steady-state heat conduction. Since CA is a mesh-free method, the proposed approach is convenient to be used when it is incorporated into the fibre model in structural analyses for composite concrete beams or columns. Additionally, no complicated mathematical derivation is required during the development of the proposed approach. The numerical implementation of this model is simple because it is based on one local rule (Eq. (1)). After necessary improvement, such a methodology can be used in other fields, for instance, columns subjected to torsion since this kind of problems is governed by Poisson equation. In this study, only steady-state heat conduction is discussed. In the future, the authors would extend the CA model for transient problems.

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