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The Boundary Contour Method for Piezoelectric Media with Quadratic Boundary Elements

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Abstract: This paper presents a development of the boundary contour method (BCM) for piezoelectric media. Firstly, the divergence-free of the integrand of the piezoelectric boundary element method is proved. Secondly, the boundary contour method formulations are obtained by introducing quadratic shape functions and Green's functions (*Computer Methods in Applied Mechanics and Engineering* 1998; **158**: 65-80) for piezoelectric media and using the rigid body motion solution to regularize the BCM and avoid computation of the corner tensor. The BCM is applied to the problem of piezoelectric media. Finally, numerical solutions for illustrative examples are compared with exact ones. The numerical results of the BCM coincide very well with the exact solution, and the feasibility and efficiency of the method are verified.

Keywords: boundary element method (BCM); piezoelectric media; quadratic shape function; divergence

1 Introduction

The conventional boundary element method (BEM) usually requires numerical evaluation of line integrals for two-dimensional problems and surface integrals for three-dimensional ones. So, more and more attention has been paid to those methods that do not require the use of internal cells. Atluri (2004) gave a detailed account of problems relating to application of the meshless method(MLPG) for domain & BIE discretizations. Yoshihiro and Vladimir (2004) gave a method using arbitrary internal points instead of internal cells, based on a three-dimensional interpolation method by using a poly-harmonic function with volume distribution in a three-dimensional BIEM. Han and Atluri (2004) developed three different truly Meshless Local Petrov-Galerkin (MLPG) methods for solving 3D elasto-static problems. Using the general MLPG concept, those methods were derived through

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the local weak forms of the equilibrium equations, by using different test functions, namely, the Heaviside function, the Dirac delta function, and the fundamental solutions. Reutskiy (2005) reduced the solution of an eigenvalue problem to a sequence of inhomogeneous problems with the differential operator studied using the method of fundamental solutions.

For piezoelectric media, many research works have been done for the numerical computations. Carrera and Boscolo (2007) gave classical amd mixed elements which are able to give a quasi-three-dimensional description of stress/strain mechanical and electrical fields in multilayered thick and thin piezoelectric plates for static and dynamic analysis. Carrera and Brischetto (2007) extended the Reissner Mixed Theorem to Static Analysis of Piezoelectric Shells. Carrera and Fagiano (2007) applied the Reissner Mixed Variation Theorem and the Unified Formulation to the analysis of multilayered anisotropic plates with embedded piezolectric layers with continuos transverse electric displacements. At the same time, the BEM have been derived [see, for example, Ding and Liang (1999); Ding, et al. (1998)]. But the boundary contour method (BCM) can achieve a further reduction in dimension by using the divergence-free property of the integrand of the conventional boundary element method. Using this method, three-dimensional problems can be reduced to numerical evaluation of line integrals over closed contours and two-dimensional problems to merely evaluation of functions at nodes on the boundary of the plane. This is true even for boundary elements of arbitrary shape with curved boundary lines (for two-dimensional problems) or curved surface (for three-dimensional problems).

Nagarajan, et al. (1994) have proposed this novel approach, called the BCM for linear elasticity problems. Nagarajan, et al. (1996) used the Stokes' theorem to transform surface integrals in the conventional boundary elements into line integrals in the bounding contours of these elements. Phan, et al. (1997) derived a BCM formulation and implemented the method for two- dimensional problems of linear elasticity with quadratic boundary elements. Zhou, et al. (2002) developed the BCM based on equivalent boundary integral equations and applied the traction BCM to crack problems and the bending problems of elastic thin plate.

However, to the authors' knowledge, no attempts in the literature have been made to solve problems of piezoelectric media by the BCM with quadratic boundary elements. This paper presents a development of the BCM for piezoelectric problems. Firstly, the divergence-free of the integrand of the piezoelectric boundary element method is proved, then, the BCM formulation is derived and potential functions are obtained by introducing quadratic shape functions and Green's functions for piezoelectric media and using the rigid body motion solution to regularize the BCM and avoid computation of the corner tensor. The BCM is applied to the problem of piezoelectric media. Finally, numerical solutions for illustrative examples are compared with exact ones. The numerical results of the BCM coincide very well with the exact solution, and the feasibility and efficiency of the method are verified.

2 General piezoelectric boundary integral equation and theory

For two-dimensional piezoelectric media, we define the general displacement **u**, general surface traction **t**, general stress **T** and general strain **S** as follows [Ding, et al. (1998)]:

$$\mathbf{u}=\begin{Bmatrix} u \\ w \\ \Phi \end{Bmatrix}, \quad \mathbf{t}=\begin{Bmatrix} t_x \\ t_z \\ -\omega \end{Bmatrix}, \quad \mathbf{T}=\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \\ D_x \\ D_z \end{Bmatrix}, \quad \mathbf{S}=\begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \\ -E_x \\ -E_z \end{Bmatrix} \tag{1}$$

where Φ is the electric potential, $D_{x(z)}$ is the components of electric displacement and $E_{x(z)} = -\frac{\partial \Phi}{\partial x(z)}$.
So the relation between general stress and general strain can be written as

$$\mathbf{T}=\mathbf{D}\mathbf{S} \tag{2}$$

where

$$\mathbf{D}=\begin{bmatrix} c_{11} & c_{13} & 0 & 0 & e_{31} \\ c_{13} & c_{33} & 0 & 0 & e_{33} \\ 0 & 0 & c_{44} & e_{15} & 0 \\ 0 & 0 & e_{15} & -\varepsilon_{11} & 0 \\ e_{31} & e_{33} & 0 & 0 & -\varepsilon_{33} \end{bmatrix}$$

Moreover, we define

$$\mathbf{U}^*=\begin{bmatrix} u^*_{11} & u^*_{12} & \Phi^*_1 \\ u^*_{21} & u^*_{22} & \Phi^*_2 \\ u^*_{31} & u^*_{32} & \Phi^*_3 \end{bmatrix}, \quad \mathbf{T}^*=\begin{bmatrix} t^*_{11} & t^*_{12} & -\omega^*_1 \\ t^*_{21} & t^*_{22} & -\omega^*_2 \\ t^*_{31} & t^*_{32} & -\omega^*_3 \end{bmatrix} \tag{3}$$

where u^*_{ij} and t^*_{ij} $(i,j=1,2)$ are, respectively, displacements and surface tractions at a field point Q in the X_j ($X_1 = x$, $X_2 = z$) coordinate directions due to a unit load acting in one of the X_i directions at a source point P on the boundary, u^*_{3j} and t^*_{3j} $(j=1,2)$ are, respectively, displacement components and surface tractions in the X_j coordinate directions at Q due to a unit electric charge at P, Φ^*_i and ω^*_i $(i=1,2)$ are, respectively, electric potential and surface charge at Q due to a unit load acting in one of the X_i directions at P, Φ_3^* and ω_3^* are, respectively, electric potential and surface charge at Q due to a unit electric charge at P. The full statement of $\mathbf{U}^*$ and $\mathbf{T}^*$ can be seen in Appendix A. It is assumed that there is neither body force nor electric charge. Based on the extended Somigliana equation, the boundary integral formulation is obtained

$$\mathbf{C}(P)\mathbf{u}(P) = \int_S \mathbf{U}^*(P,Q)\mathbf{t}(Q)d\mathbf{s} - \int_S \mathbf{T}^*(P,Q)\mathbf{u}(Q)d\mathbf{s} \tag{4}$$

The general surface $\mathbf{t}$ and the matrix $\mathbf{T}^*$ can be written as

$$\begin{Bmatrix} t_x \\ t_z \\ -\omega \end{Bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \\ D_x & D_z \end{bmatrix} \begin{Bmatrix} n_x \\ n_z \end{Bmatrix} \tag{5}$$

$$\begin{bmatrix} t_{11}^* & t_{21}^* & t_{31}^* \\ t_{12}^* & t_{22}^* & t_{32}^* \\ -\omega_1^* & -\omega_2^* & -\omega_3^* \end{bmatrix} = \begin{bmatrix} \sigma_{1x} & \tau_{1xz} & \sigma_{2x} & \tau_{2xz} & \sigma_{3x} & \tau_{3xz} \\ \tau_{1xz} & \sigma_{1z} & \tau_{2xz} & \sigma_{2z} & \tau_{3xz} & \sigma_{3z} \\ D_{1x} & D_{1z} & D_{2x} & D_{2z} & D_{3x} & D_{3z} \end{bmatrix} \begin{bmatrix} n_x & 0 & 0 \\ n_z & 0 & 0 \\ 0 & n_x & 0 \\ 0 & n_z & 0 \\ 0 & 0 & n_x \\ 0 & 0 & n_z \end{bmatrix} \tag{6}$$

It is more convenient to use the index notation rather than the matrix representation

$$t_i = T_{ij} n_j, \quad T_{ki}^* = \Sigma_{kij} n_j \tag{7}$$

where $\mathbf{T}$ is the general stress tensor and Σ is the Green's function stress tensor. Then, Eq.(4) can be rewritten as

$$c_{ki}(P)u_i(P) = \int_S \{u_{ki}^*(P,Q)T_{ij}(Q) - \Sigma_{kij}(P,Q)u_i(Q)\}\mathbf{e}_j \cdot d\mathbf{s} \tag{8}$$

where $\mathbf{e}_j$ are global Cartesian unit vectors.

Consider an arbitrary rigid body translation where $u_i(Q) = u_i(P)$=constant. Thus, $T_{ij}(Q) = 0$. Use of this rigid body motion solution in Eq.(8) gives

$$c_{ki}(P)u_i(P) = -\int_S \Sigma_{kij}(P,Q)u_i(P)\mathbf{e}_j \cdot d\mathbf{s} \tag{9}$$

Substituting Eq.(9) into Eq.(8) yields a new BEM equation

$$\int_S \{u_{ki}^*(P,Q)T_{ij}(Q) - \Sigma_{kij}(P,Q)[u_i(Q) - u_i(P)]\}\mathbf{e}_j \cdot d\mathbf{s} = 0 \tag{10}$$

Thus, the corner tensor c_{ki} is now eliminated from the BEM equation. Its evaluation is avoided and this is the first advantage of using the rigid body motion technique.
Now let

$$\mathbf{F}_k = \{u^*_{ki}(P,Q)T_{ij}(Q) - \Sigma_{kij}(P,Q)[u_i(Q) - u_i(P)]\}\mathbf{e}_j \tag{11}$$

It is easy to show that when we take the divergence of $\mathbf{F}_k$ at a field point Q, this vector is divergence free everywhere except at the source point P, i.e.

$$\begin{aligned}\nabla_Q \cdot \mathbf{F}_k &= \{u^*_{ki}(P,Q)T_{ij}(Q) - \Sigma_{kij}(P,Q)[u_i(Q) - u_i(P)]\}_{,j} \\ &= [S^*_{kij}(P,Q)T_{ij}(Q) - \Sigma_{kij}(P,Q)S_{ij}(Q)] \\ &\quad + u^*_{ki}(P,Q)T_{ij,j}(Q) - \Sigma_{kij,j}(P,Q)[u_i(Q) - u_i(P)] = 0\end{aligned} \tag{12}$$

Where $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, for $i = 1,2$; $S_{ij} = u_{i,j}$, for $i = 3$, and similarly for S^*_{kij}. Eq.(12) shows the existence of a function Φ_k such that

$$\mathbf{F}_k = \frac{\partial \Phi_k}{\partial z}\mathbf{e}_1 - \frac{\partial \Phi_k}{\partial x}\mathbf{e}_2 \tag{13}$$

The boundary is now discretized into n elements, and Eq.(10) becomes

$$\int\limits_S \mathbf{F}_k \cdot d\mathbf{s} = \sum_{e=1}^{n} \int_{Ee_1}^{Ee_2} \mathbf{F}_k \cdot d\mathbf{s} = \sum_{e=1}^{n} \int_{Ee_1}^{Ee_2} d\Phi_k = \sum_{e=1}^{n} [\Phi^e_k(Ee_2) - \Phi^e_k(Ee_1)] \tag{14}$$

which means that there is no need for any numerical integration for two dimensional piezoelectric problems.
It is important to observe that the above integrand contains unknown functions $\mathbf{u}$ and $\mathbf{t}$ on $d\mathbf{s}$ which must satisfy the basic equations of piezoelectric media. Thus, local shape functions for $\mathbf{u}$ must be chosen such that they satisfy the general Navier-Cauchy equations and the shape functions for $\mathbf{t}$ must be derived from those of $\mathbf{u}$.

3 Two dimensional piezoelectric plane strain with quadratic shape functions

It is easy to know that there are a total of fifteen linearly independent quadratic shape functions. The displacement components are written as arbitrary linear combinations of the fifteen functions as follows

$$\begin{aligned}\begin{Bmatrix} u \\ w \\ \Phi \end{Bmatrix} &= \delta_1 \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} + \delta_2 \begin{Bmatrix} x \\ 0 \\ 0 \end{Bmatrix} + \delta_3 \begin{Bmatrix} z \\ 0 \\ 0 \end{Bmatrix} + \delta_4 \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} + \delta_5 \begin{Bmatrix} 0 \\ x \\ 0 \end{Bmatrix} + \delta_6 \begin{Bmatrix} 0 \\ z \\ 0 \end{Bmatrix} + \delta_7 \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \\ &+ \delta_8 \begin{Bmatrix} 0 \\ 0 \\ x \end{Bmatrix} + \delta_9 \begin{Bmatrix} 0 \\ 0 \\ z \end{Bmatrix} + \delta_{10} \begin{Bmatrix} x^2 \\ k_1 xz \\ 0 \end{Bmatrix} + \delta_{11} \begin{Bmatrix} x^2 \\ 0 \\ k_2 xz \end{Bmatrix} + \delta_{12} \begin{Bmatrix} xz \\ k_3 x^2 \\ k_4 x^2 \end{Bmatrix} \\ &+ \delta_{13} \begin{Bmatrix} xz \\ k_5 z^2 \\ k_6 z^2 \end{Bmatrix} + \delta_{14} \begin{Bmatrix} z^2 \\ k_7 xz \\ 0 \end{Bmatrix} + \delta_{15} \begin{Bmatrix} z^2 \\ 0 \\ k_8 xz \end{Bmatrix} \end{aligned} \tag{15a}$$

where x and z are co-ordinate with respect to a global co-ordinate system, and

$$\begin{aligned} &k_1 = -\frac{2c_{11}}{c_{13}+c_{44}}, k_2 = -\frac{2c_{11}}{e_{15}+e_{31}}, k_7 = -\frac{2c_{44}}{c_{13}+c_{44}}, k_8 = -\frac{2c_{44}}{e_{15}+e_{31}}, \\ &\begin{Bmatrix} k_3 \\ k_4 \end{Bmatrix} = -\begin{bmatrix} 2c_{44} & 2e_{15} \\ 2e_{15} & -2\varepsilon_{11} \end{bmatrix}^{-1} \begin{Bmatrix} c_{13}+c_{44} \\ e_{15}+e_{31} \end{Bmatrix}, \\ &\begin{Bmatrix} k_5 \\ k_6 \end{Bmatrix} = -\begin{bmatrix} 2c_{33} & 2e_{33} \\ 2e_{33} & -2\varepsilon_{33} \end{bmatrix}^{-1} \begin{Bmatrix} c_{13}+c_{44} \\ e_{15}+e_{31} \end{Bmatrix}, \end{aligned} \tag{15b}$$

In matrix form, Eq.(15a) becomes

$$\{\mathbf{u}\} = [\mathbf{T}_u(x,z)]\{\delta\} \tag{16}$$

where $\delta = [\delta_1\ \delta_2 \cdots\ \delta_{15}]^T$.

These fifteen artificial variables require quadratic elements with fifteen physical variables. The configuration of a chosen quadratic boundary element is shown in Fig.1. Each element is divided into 4 equal segments by 2 traction and 3 displacement nodes. Thus, it has fifteen physical variables and the way they are numbered globally on the element (e) is also shown in the figure. It should be noted that the BCM equations are enforced at the displacement nodes only.

By using the Eqs.(2),(5) and (15), the fifteen physical variables on the element (e) can be described as follows

$$\left\{\mathbf{P}^{(e)}\right\} = [\mathbf{T}^{(e)}(x,z)]\left\{\delta^{(e)}\right\} \tag{17}$$

where

$$\begin{aligned}\mathbf{P}^{(e)} = \Big[\ & u^{(2e-1)}\quad w^{(2e-1)}\quad \Phi^{(2e-1)}\quad t_x^{(2e-1)}\quad t_z^{(2e-1)} - \omega^{(2e-1)}\quad u^{(2e)}\quad w^{(2e)} \\ & \Phi^{(2e)}\quad t_x^{(2e)}\quad t_z^{(2e)} - \omega^{(2e)}\quad u^{(2e+1)}\quad w^{(2e+1)}\quad \Phi^{(2e+1)}\ \Big]^T. \end{aligned}$$

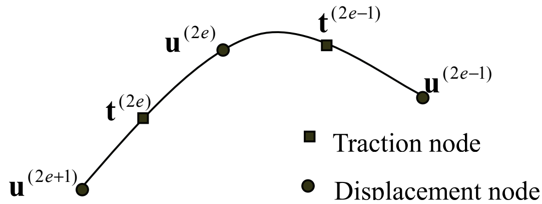

Figure 1: Quadratic boundary element(e)

The variables in $\mathbf{P}^{(e)}$ are chosen such that the matrix $\mathbf{T}^{(e)}$ is invertible.

A new coordinate system (ξ,η) centered at the source point j is introduced. This is done in order to make the shape function variables conform to those of u^*_{ki} and Σ_{kij}(which are functions of ξ and η only). The ξ and η axes are parallel to the global x and z axes, thus

$$\xi = x(Q) - x(P), \quad \eta = z(Q) - z(P) \tag{18}$$

By instituting Eq.(18) into Eq.(15), the displacements shape functions can be written as

$$\{\mathbf{u}\} = [\mathbf{T}_u(\xi,\eta)][\mathbf{B}_j]\{\boldsymbol{\delta}\} = [\mathbf{T}_u(\xi,\eta)]\left\{\hat{\delta}\right\} \tag{19}$$

in which $[\mathbf{B}_j]$ is a transformation matrix that depends only on the coordinates of the source point j.

If (h) is the element containing the source point at its first node, with this new coordinate system $u_1(P) = \hat{\delta}_1^{(h)}$, $u_2(P) = \hat{\delta}_4^{(h)}$ and $u_3(P) = \hat{\delta}_7^{(h)}$. So, for the element (e), we have

$$\{\mathbf{u}(Q) - \mathbf{u}(P)\} = [\mathbf{T}_u(\xi,\eta)]\left\{\tilde{\boldsymbol{\delta}}^{(e)}\right\} \tag{20}$$

where the columns of $[\mathbf{T}_u(\xi,\eta)]$ are the fifteen shape functions

$$\begin{Bmatrix}1\\0\\0\end{Bmatrix},\begin{Bmatrix}\xi\\0\\0\end{Bmatrix},\begin{Bmatrix}\eta\\0\\0\end{Bmatrix},\begin{Bmatrix}0\\1\\0\end{Bmatrix},\begin{Bmatrix}0\\\xi\\0\end{Bmatrix},\begin{Bmatrix}0\\\eta\\0\end{Bmatrix},\begin{Bmatrix}0\\0\\1\end{Bmatrix},\begin{Bmatrix}0\\0\\\xi\end{Bmatrix},\begin{Bmatrix}0\\0\\\eta\end{Bmatrix},$$
$$\begin{Bmatrix}\xi^2\\k_1\xi\eta\\0\end{Bmatrix},\begin{Bmatrix}\xi^2\\0\\k_2\xi\eta\end{Bmatrix},\begin{Bmatrix}\xi\eta\\k_3\xi^2\\k_4\xi^2\end{Bmatrix},\begin{Bmatrix}\xi\eta\\k_5\eta^2\\k_6\eta^2\end{Bmatrix},\begin{Bmatrix}\eta^2\\k_7\xi\eta\\0\end{Bmatrix},\begin{Bmatrix}\eta^2\\0\\k_8\xi\eta\end{Bmatrix} \tag{21}$$

and

$$\{\tilde{\boldsymbol{\delta}}^{(e)}\} = \Big[\ (\hat{\delta}_1^{(e)} - \hat{\delta}_1^{(h)}) \quad \hat{\delta}_2^{(e)} \quad \hat{\delta}_3^{(e)} \quad (\hat{\delta}_4^{(e)} - \hat{\delta}_4^{(h)}) \quad \hat{\delta}_5^{(e)} \quad \hat{\delta}_6^{(e)} \quad (\hat{\delta}_7^{(e)} - \hat{\delta}_7^{(h)}) \quad \hat{\delta}_8^{(e)} \quad \hat{\delta}_9^{(e)} \quad \hat{\delta}_{10}^{(e)} \quad \hat{\delta}_{11}^{(e)} \quad \hat{\delta}_{12}^{(e)} \hat{\delta}_{13}^{(e)} \quad \hat{\delta}_{14}^{(e)} \quad \hat{\delta}_{15}^{(e)}\ \Big]^T \tag{22}$$

Potential functions must be obtained for each of fifteen states, for each of the directions and electric potentials corresponding to $k = 1 \sim 3$ in Eq.(14). Let $\varphi_{1,l}(l = 1 \sim 15)$ denote potentials for the above fifteen states for $k = 1$, corresponding to the unit force in the x direction. The function in Eq.(14) is a linear combination of $\varphi_{1,l}(l = 1 \sim 15)$. $\varphi_{1,l}$ can be calculated individually by using Eq.(13), with $\varphi_{1,l}$ replacing Φ_1, ξ and η replacing x and z, respectively.

Similarly, let $\varphi_{2,l}$ and $\varphi_{3,l}$ $(l = 1 \sim 15)$ denote potentials for the fifteen states, for $k = 2,3$ corresponding to the unit force in the $z$direction and the unit electric charge, respectively. The potentials $\varphi_{k,l}(k = 1 \sim 3, l = 1 \sim 15)$ are given in Appendix B. It should be noted that Φ_k in Eq.(14) is composed of $\varphi_{k,l}(l = 1 \sim 15)$, respectively.

Now, with the potential functions already derived, the BCM discretized equations are developed as follows.

For the source point j (source points are only placed at the ends and mid-point, i.e. displacement nodes, of each boundary element, see Fig.1.)

$$\sum_{e=1}^{n} [\Phi_k^e(Ee_2) - \Phi_k^e(Ee_1)] = \sum_{e=1}^{n} \sum_{l=1}^{15} [\varphi_{k,l}^j(Ee_2) - \varphi_{k,l}^j(Ee_1)] \tilde{\delta}_l^{(e)} = 0 \tag{23}$$

It should be noted that the potential functions $\varphi_{k,1}(\xi,\eta)$,$\varphi_{k,4}(\xi,\eta)$ and $\varphi_{k,7}(\xi,\eta)(k = 1 \sim 3)$corresponding to constant shape functions, are singular when a field point $Q \rightarrow$the source point P, i.e. when $(\xi,\eta) \rightarrow (0,0)$. But in this case $u_k(Q) - u_k(P) = 0(r)$, and Eq.(20) lead to

$$\begin{cases} \tilde{\delta}_1^{(e)} = (\hat{\delta}_1^{(e)} - \hat{\delta}_1^{(h)}) = 0 \\ \tilde{\delta}_4^{(e)} = (\hat{\delta}_4^{(e)} - \hat{\delta}_4^{(h)}) = 0 \\ \tilde{\delta}_7^{(e)} = (\hat{\delta}_7^{(e)} - \hat{\delta}_7^{(h)}) = 0 \end{cases} \tag{24}$$

so the evaluation of these potential functions can be avoided, i.e. expression(23) is now completely regular. This is the second advantage of the approach using the rigid body motion technique.

With $2n$ source pints corresponding to $2n$ displacement nodes on the boundary ds, one can get the final BCM linear system of equations

$$[\mathbf{K}]\{\mathbf{P}\} = \{\mathbf{0}\} \tag{25}$$

where $\{\mathbf{P}\}$ are degrees of freedom on the whole boundary ds. The global system of equations is condensed in accordance with continuity of displacements across element in the usual way.

Finally, the system of equation (25) needs to be reordered in accordance with the boundary conditions to form

$$[\mathbf{A}]\{\mathbf{X}\} = \{\mathbf{Z}\} \tag{26}$$

where $\{\mathbf{X}\}$ contains the unknown boundary quantities and $\{\mathbf{Z}\}$ is known in terms of prescribed boundary quantities and geometrical and material data of the problem.

After the solution of the global equation system (26) is obtained, one can easily derive the artificial variables $\{\boldsymbol{\delta}^{(e)}\}$ from Eq.(17). At this stage, the remaining physical variables at any node on the boundary can be easily calculated from Eq.(15a) and the corresponding relations for stresses and tractions in terms of their shape functions.

Evaluation of strain components at points inside a body requires transformation of equations of the strain BEM at an internal point to an integrated form analogous to Eq.(14). This can be done since the integrand is again divergence free. Stress calculations would then follow from the piezoelectric media fundamental equations.

4 Numerical examples

4.1 Example 1

Consider a **PZT-4** piezoelectric column of size $2a \times 2b$ subjected to uniform axial tension. The problem is treated as a plane-strain one and the material used in the computations is a piezoelectric ceramic, poled in the $z$direction, with transversely isotropic properties. Its elastic and piezoelectric constants are

$$c_{11} = 12.6 \times 10^{10} Nm^{-2}, \quad c_{13} = 7.43 \times 10^{10} Nm^{-2}, \quad c_{33} = 11.5 \times 10^{10} Nm^{-2},$$

$$c_{44} = 2.56 \times 10^{10} Nm^{-2}, \quad e_{15} = 12.7 Cm^{-2}, \quad e_{31} = -5.2 Cm^{-2}, \quad e_{33} = 15.1 Cm^{-2},$$

$$\varepsilon_{11} = 6.463 \times 10^{-9} CV^{-1} m^{-1}, \quad \varepsilon_{33} = 5.622 \times 10^{-9} CV^{-1} m^{-1}$$

The corresponding boundary conditions are

$$z = \pm b: \quad \sigma_z = P, \quad \tau_{xz} = 0, \quad D_z = 0 \tag{27}$$

$$x = \pm a: \quad \sigma_x = 0, \quad \tau_{xz} = 0, \quad D_x = 0 \tag{28}$$

In the calculation, we set $a = 3m$, $b = 10m$, $P = 10Nm^{-2}$, and total 12 elements are used. The corresponding results are compared with the exact ones(in parentheses) only at four reference points shown in Table 1.

Table 1: Comparison of BCM results with the exact ones(*)

Location (x,z)	$u\times 10^{10}(m)$	$w\times 10^{9}(m)$	$\Phi(V)$	$\sigma_z(Nm^{-2})$
(2,0)	-0.9665 (-0.9672)	0.0001 (0.0000)	0.0001 (0.0000)	10.0000 (10.0000)
(3,0)	-1.4500 (-1.4508)	0.0001 (0.0000)	0.0000 (0.0000)	10.0000 (10.0000)
(0,5)	0.0001 (0.0000)	+0.4997 (+0.5006)	+0.6876 (+0.6888)	10.0000 (10.0000)
(0,10)	0.0000 (0.0000)	+0.9994 (+1.0011)	+1.3752 (+1.3775)	10.0000 (10.0000)

4.2 Example 2

Consider a simply-supported rectangular beam of length L, height h and width b subjected to uniformly distributed loads on the upper and bottom surfaces. For numerical calculation, we consider the beam the same material constants as Example 1 and use totally twelve quadratic boundary elements.

The problem is treated as a plane-stress one and the boundary conditions are

$$z=+h/2: \quad \sigma_z(x)=q_1(x), \quad \tau_{xz}=0, \quad D_z=0 \tag{29}$$

$$z=-h/2: \quad \sigma_z(x)=q_2(x), \quad \tau_{xz}=0, \quad D_z=0 \tag{30}$$

$$x=0,\, L: \quad \sigma_x=0, \quad w=0, \quad \Phi=0 \tag{31}$$

where $q_1(x)=q_0\sin\lambda x$, $\lambda=\pi/L$ and $q_2(x)=0$.

To obtain the exact solution to this problem, we constitute the displacement function as follows

$$\psi_j=\sin\lambda x(A_j\cosh\lambda z_j+B_j\sinh\lambda z_j), \quad (j=1,\,2,\,3), \tag{32}$$

where A_j and B_j are unknown constants to be determined. From Ding, et al. (1998), we get

$$u=\sum_{j=1}^{3}\lambda\cos\lambda x(A_j\cosh\lambda z_j+B_j\sinh\lambda z_j), \tag{33a}$$

$$w=\sum_{j=1}^{3}\alpha_{j1}\lambda\sin\lambda x(A_j\sinh\lambda z_j+B_j\cosh\lambda z_j), \tag{33b}$$

$$\Phi = \sum_{j=1}^{3} \alpha_{j2}\lambda \sin\lambda x (A_j \sinh\lambda z_j + B_j \cosh\lambda z_j), \tag{33c}$$

$$\sigma_z = \sum_{j=1}^{3} k_{12}^{j}\lambda^2 \sin\lambda x (A_j \cosh\lambda z_j + B_j \sinh\lambda z_j), \tag{33d}$$

$$D_z = \sum_{j=1}^{3} k_{13}^{j}\lambda^2 \sin\lambda x (A_j \cosh\lambda z_j + B_j \sinh\lambda z_j), \tag{33e}$$

$$\sigma_x = \sum_{j=1}^{3} k_{11}^{j}\lambda^2 \sin\lambda x (A_j \cosh\lambda z_j + B_j \sinh\lambda z_j), \tag{33f}$$

$$\tau_{xz} = \sum_{j=1}^{3} k_{14}^{j}\lambda^2 \cos\lambda x (A_j \sinh\lambda z_j + B_j \cosh\lambda z_j), \tag{33g}$$

$$D_x = \sum_{j=1}^{3} k_{15}^{j}\lambda^2 \cos\lambda x (A_j \sinh\lambda z_j + B_j \cosh\lambda z_j), \tag{33h}$$

It is seen that the boundary conditions in Eq.(31) is satisfied automatically. From Eqs. (29) and (30), we can get six linear inhomogeneous algebraic equations from which the unknown coefficients A_j and B_j $(j = 1,2,3)$ can be obtained. Substituting $A_j$and B_j back into Eq. (33a-h) yields the exact solution for a rectangular beam that is simply-supported on the left and right surfaces and uniformly pressured on the upper surface. Let $L = 0.20m$,$h = 0.02m$, $b = 0.001m$ and $q_0 = -10Nm^{-2}$. Table 2 compares the exact and numerical results at eight reference points, i.e. A(0.100, 0.000), B(0.125, 0.000), C(0.150, 0.000), D(0.175, 0.000), E(0.100, +0.010), F(0.125, +0.010), G(0.150, +0.010) and H(0.175, +0.010). The exact results are given in parentheses.

5 Conclusions

In this paper, the BCM is presented for two dimensional piezoelectric media based on the fundamental solution of an infinite piezoelectric plane. This approach does not require any numerical integration at all for 2D problem, even with curved boundary elements, and it requires only numerical evaluation of contour integrals for 3D problems. Numerical results for 2D problem show that the BCM performs very well.

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Table 2: Simply-supported rectangular beam under uniformly distributed load

Point	w	Φ	σ_z	D_z	σ_x
A	-0.1987E-8 (-0.2000E-8)	-0.2310E-1 (-0.2314E-1)	-4.9979 (-5.0000)	-0.1110E-10 (-0.1114E-10)	+0.2798E-1 (+0.2818E-1)
B	-0.1796E-8 (-0.1847E-8)	-0.2123E-1 (-0.2138E-1)	-4.6168 (-4.6190)	-0.1032E-10 (-0.1029E-10)	+0.2583E-1 (+0.2604E-1)
C	-0.1378E-8 (-0.1414E-8)	-0.1625E-1 (-0.1637E-1)	-3.5336 (-3.5351)	-0.7896E-11 (-0.7875E-11)	+0.1976E-1 (+0.1993E-1)
D	-0.7692E-9 (-0.7652E-9)	-0.8798E-2 (-0.8857E-2)	-1.9124 (-1.9130)	-0.4275E-11 (-0.4262E-11)	+0.1062E-1 (+0.1078E-1)
E	-0.1972E-8 (-0.1993E-8)	-0.8807E-2 (-0.8728E-2)	-10.000 (-10.000)	0.0000 (0.0000)	-0.6055E+3 (-0.6104E+3)
F	-0.1864E-8 (-0.1842E-8)	-0.8132E-2 (-0.8063E-2)	-9.2388 (-9.2388)	0.0000 (0.0000)	-0.5585E+3 (-0.5640E+3)
G	-0.1374E-8 (-0.1409E-8)	-0.6097E-2 (-0.6172E-2)	-7.0711 (-7.0711)	0.0000 (0.0000)	-0.4389E+3 (-0.4316E+3)
H	-0.7698E-9 (-0.7628E-9)	-0.3281E-2 (-0.3340E-2)	-3.8268 (-3.8268)	0.0000 (0.0000)	-0.2323E+3 (-0.2336E+3)

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Appendix A: Fundamental solutions

$$u_{11}^{*}=\sum_{j=1}^{3}A_j\ln R_j,\quad u_{12}^{*}=\sum_{j=1}^{3}\alpha_{j1}A_j\arctan\frac{x}{z_j},\quad \Phi_1^{*}=\sum_{j=1}^{3}\alpha_{j2}A_j\arctan\frac{x}{z_j}$$

$$u_{21}^{*}=-\sum_{j=1}^{3}L_j\arctan\frac{x}{z_j},\quad u_{22}^{*}=\sum_{j=1}^{3}\alpha_{j1}L_j\ln R_j,\quad \Phi_2^{*}=\sum_{j=1}^{3}\alpha_{j2}L_j\ln R_j$$

$$u_{31}^{*}=-\sum_{j=1}^{3}M_j\arctan\frac{x}{z_j},\quad u_{32}^{*}=\sum_{j=1}^{3}\alpha_{j1}M_j\ln R_j,\quad \Phi_3^{*}=\sum_{j=1}^{3}\alpha_{j2}M_j\ln R_j$$

$$t_{11}^{*}=\sum_{j=1}^{3}k_{11}^{j}\frac{A_j}{R_j^2}(xn_x+zn_z),\quad t_{12}^{*}=\sum_{j=1}^{3}k_{12}^{j}\frac{A_j}{R_j^2}(-s_j^2zn_x+xn_z)$$

$$\omega_1^{*}=-\sum_{j=1}^{3}k_{13}^{j}\frac{A_j}{R_j^2}(-s_j^2zn_x+xn_z)$$

$$t_{21}^{*}=\sum_{j=1}^{3}k_{24}^{j}\frac{L_j}{R_j^2}(-s_j^2zn_x+xn_z),t_{22}^{*}=\sum_{j=1}^{3}k_{24}^{j}\frac{L_j}{R_j^2}(xn_x+zn_z)$$

$$\omega_2^{*}=-\sum_{j=1}^{3}k_{25}^{j}\frac{L_j}{R_j^2}(xn_x+zn_z)$$

$$t_{31}^{*}=\sum_{j=1}^{3}k_{24}^{j}\frac{M_j}{R_j^2}(-s_j^2zn_x+xn_z),t_{32}^{*}=\sum_{j=1}^{3}k_{24}^{j}\frac{M_j}{R_j^2}(xn_x+zn_z)$$

$$\omega_3^* = -\sum_{j=1}^{3} k_{25}^j \frac{M_j}{R_j^2}(xn_x + zn_z)$$

where $z_j = s_j z$, $R_j = \sqrt{x^2 + z_j^2}$, and α_{im}, k_{ij}^i, A_j, L_j and M_j are listed as in Ding, *et al* (1998).

Appendix B: Potential functions

$$\varphi_{1,1} = \sum_{j=1}^{3} \frac{k_{11}^j A_j}{s_j} \arctan\frac{\xi}{\eta_j}$$

$$\varphi_{1,2} = \sum_{j=1}^{3} [\frac{k_{11}^j - c_{11}}{s_j} A_j \xi \arctan\frac{\xi}{\eta_j} + c_{11} A_j \eta (\ln R_j - 1)]$$

$$\varphi_{1,3} = \sum_{j=1}^{3} [-c_{44} A_j \xi (\ln R_j - 1) + (k_{14}^j - c_{44} s_j) A_j \eta \arctan\frac{\xi}{\eta_j}]$$

$$\varphi_{1,4} = \sum_{j=1}^{3} k_{12}^j A_j \ln R_j$$

$$\varphi_{1,5} = \sum_{j=1}^{3} [-c_{44} A_j \xi (\ln R_j - 1) - (c_{44} + k_{12}^j) A_j s_j \eta \arctan\frac{\xi}{\eta_j} + k_{12}^j A_j \xi]$$

$$\varphi_{1,6} = \sum_{j=1}^{3} [\frac{k_{12}^j - c_{13}}{s_j} A_j \xi \arctan\frac{\xi}{\eta_j} + c_{13} A_j \eta (\ln R_j - 1) + k_{12}^j A_j \eta]$$

$$\varphi_{1,7} = \sum_{j=1}^{3} k_{13}^j A_j \ln R_j$$

$$\varphi_{1,8} = \sum_{j=1}^{3} [-e_{15} A_j \xi (\ln R_j - 1) - (e_{15} + k_{13}^j) A_j s_j \eta \arctan\frac{\xi}{\eta_j} + k_{13}^j A_j \xi]$$

$$\varphi_{1,9} = \sum_{j=1}^{3}[\frac{k_{13}^{j}-e_{31}}{s_j}A_j\xi\arctan\frac{\xi}{\eta_j}+e_{31}A_j\eta(\ln R_j-1)+k_{13}^{j}A_j\eta]$$

$$\varphi_{1,10} = \sum_{j=1}^{3}[-c_{44}k_1A_j\xi\eta\ln R_j+\frac{c_{44}k_1-2k_{12}^{j}s_j^2+k_{12}^{j}k_1}{2s_j}A_j\xi^2\arctan\frac{\xi}{\eta_j}$$
$$-\frac{c_{44}k_1+k_{12}^{j}k_1}{2}A_js_j\eta^2\arctan\frac{\xi}{\eta_j}+\frac{c_{44}k_1+k_{12}^{j}k_1}{2}A_j\xi\eta]$$

$$\varphi_{1,11} = \sum_{j=1}^{3}[-e_{15}k_2A_j\xi\eta\ln R_j+\frac{e_{15}k_2+2k_{11}^{j}+k_{13}^{j}k_2}{2s_j}A_j\xi^2\arctan\frac{\xi}{\eta_j}$$
$$-\frac{e_{15}k_2+k_{13}^{j}k_2}{2}A_js_j\eta^2\arctan\frac{\xi}{\eta_j}+\frac{e_{15}k_2+k_{13}^{j}k_2}{2}A_j\xi\eta]$$

$$\varphi_{1,12} = \sum_{j=1}^{3}[\frac{c_{13}A_j}{2}\xi^2(\ln R_j-\frac{1}{2})-\frac{c_{11}-k_{11}^{j}}{s_j}A_j\xi\eta\arctan\frac{\xi}{\eta_j}$$
$$+\frac{c_{11}A_j}{2}\eta^2(\ln R_j-\frac{1}{2})+(\frac{c_{13}}{4}+\frac{c_{11}}{4s_j^2})A_j\xi^2]$$

$$\varphi_{1,13} = \sum_{j=1}^{3}[-\frac{c_{44}A_j}{2}\xi^2(\ln R_j-\frac{1}{2})-(c_{44}+k_{12}^{j})A_js_j\xi\eta\arctan\frac{\xi}{\eta_j}$$
$$+\frac{c_{11}+2c_{13}k_5+2e_{31}k_6}{2}A_j\eta^2(\ln R_j-\frac{1}{2})+(\frac{c_{11}+2c_{13}k_5+2e_{31}k_6}{4}-\frac{c_{44}s_j^2}{4})A_j\eta^2]$$

$$\varphi_{1,14} = \sum_{j=1}^{3}[c_{13}k_7A_j\xi\eta\ln R_j-\frac{c_{13}-k_{12}^{j}}{2s_j}A_jk_7\xi^2\arctan\frac{\xi}{\eta_j}$$
$$-\frac{(2+k_7)(c_{44}+k_{12}^{j})}{2}A_js_j\eta^2\arctan\frac{\xi}{\eta_j}-\frac{c_{13}-k_{12}^{j}}{2}A_jk_7\xi\eta]$$

$$\varphi_{1,15} = \sum_{j=1}^{3}[e_{31}k_8A_j\xi\eta\ln R_j-\frac{e_{31}-k_{13}^{j}}{2s_j}A_jk_8\xi^2\arctan\frac{\xi}{\eta_j}$$
$$+\frac{-2k_{12}^{j}+(e_{31}-k_{13}^{j})k_8}{2}A_js_j\eta^2\arctan\frac{\xi}{\eta_j}-\frac{e_{31}-k_{13}^{j}}{2}A_jk_8\xi\eta]$$

$$\varphi_{2,1} = -\sum_{j=1}^{3} \frac{k_{21}^{j}}{s_j} L_j \ln R_j$$

$$\varphi_{2,2} = \sum_{j=1}^{3} [-\frac{k_{21}^{j} + c_{11}}{s_j} L_j \xi (\ln R_j - 1) - c_{11} L_j \eta \arctan \frac{\xi}{\eta_j} - \frac{k_{21}^{j}}{s_j} L_j \xi]$$

$$\varphi_{2,3} = \sum_{j=1}^{3} [c_{44} L_j \xi \arctan \frac{\xi}{\eta_j} + (k_{22}^{j} - c_{44}) L_j s_j \eta (\ln R_j - 1) + k_{22}^{j} L_j s_j \eta]$$

$$\varphi_{2,4} = \sum_{j=1}^{3} \frac{k_{24}^{j}}{s_j} L_j \arctan \frac{\xi}{\eta_j}$$

$$\varphi_{2,5} = \sum_{j=1}^{3} [c_{44} L_j \xi \arctan \frac{\xi}{\eta_j} + (k_{24}^{j} - c_{44} s_j) L_j \eta (\ln R_j - 1)]$$

$$\varphi_{2,6} = \sum_{j=1}^{3} [-\frac{k_{22}^{j} + c_{13}}{s_j} L_j \xi (\ln R_j - 1) - c_{13} L_j \eta \arctan \frac{\xi}{\eta_j}]$$

$$\varphi_{2,7} = \sum_{j=1}^{3} \frac{k_{25}^{j}}{s_j} L_j \arctan \frac{\xi}{\eta_j}$$

$$\varphi_{2,8} = \sum_{j=1}^{3} [e_{15} L_j \xi \arctan \frac{\xi}{\eta_j} + (k_{23}^{j} - e_{15}) L_j s_j \eta (\ln R_j - 1)]$$

$$\varphi_{2,9} = \sum_{j=1}^{3} [-\frac{k_{23}^{j} + e_{31}}{s_j} L_j \xi (\ln R_j - 1) - e_{31} L_j \eta \arctan \frac{\xi}{\eta_j}]$$

$$\varphi_{2,10} = \sum_{j=1}^{3} [c_{44} k_1 L_j \xi \eta \arctan \frac{\xi}{\eta_j} + \frac{c_{44} k_1 + 2k_{22}^{j} s_j^2 - k_{22}^{j} k_1}{2 s_j} L_j \xi^2 (\ln R_j - \frac{1}{2})$$
$$- \frac{c_{44} - k_{22}^{j}}{2} k_1 L_j s_j \eta^2 (\ln R_j - \frac{1}{2}) + \frac{k_{22}^{j} L_j s_j}{2} \xi^2]$$

$$\varphi_{2,11} = \sum_{j=1}^{3}[e_{15}k_2 L_j \xi\eta \arctan\frac{\xi}{\eta_j} + \frac{e_{15}k_2 - 2k_{21}^j - k_{23}^j k_2}{2s_j} L_j \xi^2 (\ln R_j - \frac{1}{2}) - \frac{e_{15} - k_{23}^j}{2} k_2 L_j s_j \eta^2 (\ln R_j - \frac{1}{2}) - \frac{k_{21}^j L_j}{2s_j}\xi^2]$$

$$\varphi_{2,12} = \sum_{j=1}^{3}(-\frac{c_{13}}{2} L_j \xi^2 \arctan\frac{\xi}{\eta_j} - \frac{c_{11}}{2} L_j \eta^2 \arctan\frac{\xi}{\eta_j} - \frac{c_{11} + k_{21}^j}{s_j} L_j \xi\eta \ln R_j + \frac{c_{11}}{2s_j} L_j \xi\eta)$$

$$\varphi_{2,13} = \sum_{j=1}^{3}[\frac{c_{44}}{2} L_j \xi^2 \arctan\frac{\xi}{\eta_j} - \frac{c_{11} + 2c_{13}k_5 + 2e_{31}k_6}{2} L_j \eta^2 \arctan\frac{\xi}{\eta_j} + (k_{22}^j - c_{44}) L_j s_j \xi\eta \ln R_j + \frac{c_{44}}{2} L_j s_j \xi\eta]$$

$$\varphi_{2,14} = \sum_{j=1}^{3}[-c_{13}k_7 L_j \xi\eta \arctan\frac{\xi}{\eta_j} - \frac{c_{13} + k_{22}^j}{2s_j} k_7 L_j \xi^2 (\ln R_j - \frac{1}{2}) + \frac{(k_{22}^j - c_{44})(2 + k_7)}{2} L_j s_j \eta^2 (\ln R_j - \frac{1}{2}) + \frac{k_{22}^j s_j L_j}{2}\eta^2]$$

$$\varphi_{2,15} = \sum_{j=1}^{3}[-e_{31}k_8 L_j \xi\eta \arctan\frac{\xi}{\eta_j} - \frac{e_{31} + k_{23}^j}{2s_j} k_8 L_j \xi^2 (\ln R_j - \frac{1}{2}) + \frac{2k_{22}^j + (e_{31} + k_{23}^j)k_8}{2} L_j s_j \eta^2 (\ln R_j - \frac{1}{2}) + \frac{k_{22}^j s_j L_j}{2}\eta^2]$$

$$\varphi_{3,1} = -\sum_{j=1}^{3}\frac{k_{21}^j}{s_j} M_j \ln R_j$$

$$\varphi_{3,2} = \sum_{j=1}^{3}[-\frac{k_{21}^j + c_{11}}{s_j} M_j \xi (\ln R_j - 1) - c_{11} M_j \eta \arctan\frac{\xi}{\eta_j} - \frac{k_{21}^j}{s_j} M_j \xi]$$

$$\varphi_{3,3}=\sum_{j=1}^{3}[c_{44}M_j\xi\arctan\frac{\xi}{\eta_j}+(k_{22}^j-c_{44})M_js_j\eta(\ln R_j-1)+k_{22}^jM_js_j\eta]$$

$$\varphi_{3,4}=\sum_{j=1}^{3}\frac{k_{24}^j}{s_j}M_j\arctan\frac{\xi}{\eta_j}$$

$$\varphi_{3,5}=\sum_{j=1}^{3}[c_{44}M_j\xi\arctan\frac{\xi}{\eta_j}+(k_{24}^j-c_{44}s_j)M_j\eta(\ln R_j-1)]$$

$$\varphi_{3,6}=\sum_{j=1}^{3}[-\frac{k_{22}^j+c_{13}}{s_j}M_j\xi(\ln R_j-1)-c_{13}M_j\eta\arctan\frac{\xi}{\eta_j}]$$

$$\varphi_{3,7}=\sum_{j=1}^{3}\frac{k_{25}^j}{s_j}M_j\arctan\frac{\xi}{\eta_j}$$

$$\varphi_{3,8}=\sum_{j=1}^{3}[e_{15}M_j\xi\arctan\frac{\xi}{\eta_j}+(k_{23}^j-e_{15})M_js_j\eta(\ln R_j-1)]$$

$$\varphi_{3,9}=\sum_{j=1}^{3}[-\frac{k_{23}^j+e_{31}}{s_j}M_j\xi(\ln R_j-1)-e_{31}M_j\eta\arctan\frac{\xi}{\eta_j}]$$

$$\begin{aligned}\varphi_{3,10}=\sum_{j=1}^{3}[c_{44}k_1M_j\xi\eta\arctan\frac{\xi}{\eta_j}&+\frac{c_{44}k_1+2k_{22}^js_j^2-k_{22}^jk_1}{2s_j}M_j\xi^2(\ln R_j-\frac{1}{2})\\&-\frac{c_{44}-k_{22}^j}{2}k_1M_js_j\eta^2(\ln R_j-\frac{1}{2})+\frac{k_{22}^jM_js_j}{2}\xi^2]\end{aligned}$$

$$\begin{aligned}\varphi_{3,11}=\sum_{j=1}^{3}[e_{15}k_2M_j\xi\eta\arctan\frac{\xi}{\eta_j}&+\frac{e_{15}k_2-2k_{21}^j-k_{23}^jk_2}{2s_j}M_j\xi^2(\ln R_j-\frac{1}{2})\\&-\frac{e_{15}-k_{23}^j}{2}k_2M_js_j\eta^2(\ln R_j-\frac{1}{2})-\frac{k_{21}^jM_j}{2s_j}\xi^2]\end{aligned}$$

$$\varphi_{3,12} = \sum_{j=1}^{3} (-\frac{c_{13}}{2} M_j \xi^2 \arctan\frac{\xi}{\eta_j} - \frac{c_{11}}{2} M_j \eta^2 \arctan\frac{\xi}{\eta_j} - \frac{c_{11}+k_{21}^j}{s_j} M_j \xi\eta \ln R_j + \frac{c_{11}}{2s_j} M_j \xi\eta)$$

$$\varphi_{3,13} = \sum_{j=1}^{3} [\frac{c_{44}}{2} M_j \xi^2 \arctan\frac{\xi}{\eta_j} - \frac{c_{11}+2c_{13}k_5+2e_{31}k_6}{2} M_j \eta^2 \arctan\frac{\xi}{\eta_j} + (k_{22}^j - c_{44}) M_j s_j \xi\eta \ln R_j + \frac{c_{44}}{2} M_j s_j \xi\eta]$$

$$\varphi_{3,14} = \sum_{j=1}^{3} [-c_{13}k_7 M_j \xi\eta \arctan\frac{\xi}{\eta_j} - \frac{c_{13}+k_{22}^j}{2s_j} k_7 M_j \xi^2 (\ln R_j - \frac{1}{2}) + \frac{(k_{22}^j - c_{44})(2+k_7)}{2} M_j s_j \eta^2 (\ln R_j - \frac{1}{2}) + \frac{k_{22}^j s_j M_j}{2} \eta^2]$$

$$\varphi_{3,15} = \sum_{j=1}^{3} [-e_{31}k_8 M_j \xi\eta \arctan\frac{\xi}{\eta_j} - \frac{e_{31}+k_{23}^j}{2s_j} k_8 M_j \xi^2 (\ln R_j - \frac{1}{2}) + \frac{2k_{22}^j + (e_{31}+k_{23}^j)k_8}{2} M_j s_j \eta^2 (\ln R_j - \frac{1}{2}) + \frac{k_{22}^j s_j M_j}{2} \eta^2]$$

where $\eta_j = s_j \eta$, $R_j = \sqrt{\xi^2 + \eta_j^2}$.