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An Alternating Iterative MFS Algorithm for the Cauchy Problem in Two-Dimensional Anisotropic Heat Conduction

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Abstract: In this paper, the alternating iterative algorithm originally proposed by Kozlov, Maz′ya and Fomin (1991) is numerically implemented for the Cauchy problem in anisotropic heat conduction using a meshless method. Every iteration of the numerical procedure consists of two mixed, well-posed and direct problems which are solved using the method of fundamental solutions (MFS), in conjunction with the Tikhonov regularization method. For each direct problem considered, the optimal value of the regularization parameter is chosen according to the generalized cross-validation (GCV) criterion. An efficient regularizing stopping criterion which ceases the iterative procedure at the point where the accumulation of noise becomes dominant and the errors in predicting the exact solutions increase, is also presented. The iterative MFS algorithm is tested for Cauchy problems related to heat conduction in two-dimensional anisotropic solids to confirm the numerical convergence, stability and accuracy of the method.

Keywords: Anisotropic Heat Conduction; Inverse Problem; Cauchy Problem; Iterative Method of Fundamental Solutions (MFS); Regularization.

1 Introduction

In the case of direct problems in heat transfer, the thermal equilibrium equation has to be solved, in a known geometry, subject to known thermal conductivities and heat sources, and appropriate initial and boundary conditions for the temperature and/or normal heat flux. If at least one of the aforementioned conditions is unknown or incomplete then one has to solve an inverse problem. A classical example of an inverse problem in heat transfer is the *Cauchy problem*. In this case, the boundary of the solution domain, the thermal conductivities and/or the heat sources are all

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known, while both Dirichlet and Neumann conditions are prescribed on a part of the boundary, while on the remaining portion of the boundary no boundary conditions are given. It is well known that Cauchy problems are generally ill-posed [Hadamard (1923)], in the sense that the existence, uniqueness and stability of their solutions are not always guaranteed. Consequently, a special numerical treatment of these problems is required.

There are numerous important contributions in the literature, as well as various approaches, to the theoretical and numerical solutions of the Cauchy problem associated with the steady state heat conduction. However, most of these are related to steady state heat conduction in isotropic solids, i.e. the Laplace equation, while just a few studies refer to steady state heat conduction in anisotropic media. Kozlov, Maz′ya and Fomin (1991) proposed an alternating iterative algorithm for the stable solution of the Cauchy problem for elliptic partial differential operators which was implemented using the boundary element method (BEM) for steady state heat conduction in isotropic and anisotropic media by Lesnic, Elliott and Ingham (1997) and Mera, Elliott, Ingham and Lesnic (2000), respectively. Ang, Nghia and Tam (1998) reformulated the Cauchy problem as an integral equation problem and solved the latter by using the Fourier transform, together with the Tikhonov regularization method. As a result of a variational approach to the Cauchy problem, the conjugate gradient method, in conjunction with the BEM, was proposed by Hào and Lesnic (2000) in order to obtain a stable solution. Cheng, Hon, Wei and Yamamoto (2001) transformed the original problem into a moment problem by using Green's formula and also provided an error estimate for the numerical solution. Hon and Wei (2001) converted the Cauchy problem into a classical moment problem whose numerical approximation can be achieved and also provided a convergence proof based on Backus-Gilbert algorithm. Cimetière, Delvare, Jaoua and Pons (2001) reduced the Cauchy problem for the Laplace equation to solving a sequence of optimization problems under equality constraints using the finite element method (FEM). The minimization functional consists of two terms that measure the gap between the optimal element and the over-specified data and the gap between the optimal element and the previous optimal element (regularization term), respectively. This method was later implemented using the BEM by Delvare, Cimetière and Pons (2002). Cimetière, Delvare, Jaoua and Pons (2002) reduced the solution of harmonic Cauchy problems to the resolution of a fixed point process, while the authors implemented numerically the proposed method by employing both the BEM and the FEM. Bourgeois (2005) approached the Cauchy problem for the Laplace equation by the mixed formulation of the method of quasi-reversibility, which finally led to a $\mathscr{C}^0$ FEM. In order to improve the reconstruction of the normal derivatives, Delvare and Cimetière (2008) extended the method originally proposed by Cimetière, Delvare, Jaoua and Pons (2001) to a higher-order one, which was implemented using the BEM. On assuming the available data to have a Fourier expansion, Liu (2008g) applied a modified indirect Trefftz method to solve the Cauchy problem for the Laplace equation.

The method of fundamental solutions (MFS) is a simple but powerful technique that has been used to obtain highly accurate numerical approximations of solutions to linear partial differential equations. Like the BEM, the MFS is applicable when a fundamental solution of the governing PDE is explicitly known. Since its introduction as a numerical method by Mathon and Johnston (1977), it has been successfully applied to a large variety of physical problems, an account of which may be found in the survey papers [Fairweather and Karageorghis (1998); Golberg and Chen (1999); Fairweather, Karageorghis and Martin (2003); Cho, Golberg, Muleshkov and Li (2004)].

The ease of implementation of the MFS and its low computational cost make it an ideal candidate for inverse problems as well. For these reasons, the MFS, in conjunction with various regularization methods (e.g. the Tikhonov regularization method, Morozov's discrepancy principle, singular value decomposition), have been used increasingly over the last decade for the numerical solution of inverse problems. For example, the Cauchy problem associated with the heat conduction equation [Hon and Wei (2002); Hon and Wei (2003); Hon and Wei (2004); Hon and Wei (2005); Mera (2005); Dong, Sun and Meng (2007); Wei, Hon and Ling (2007); Ling and Takeuchi (2008); Marin (2008); Young, Tsai, Chen and Fan (2008); Shigeta and Young (2009); Wei and Li (2009); Wei and Zhou (2009); Marin (2009)], linear elasticity [Marin and Lesnic (2004); Marin (2005a)], steady-state heat conduction in functionally graded materials [Marin (2005b)], Helmholtz-type equations [Marin (2005c); Marin and Lesnic (2005a); Jin and Zheng (2006)], Stokes problems [Chen, Young, Tsai and Murugesan (2005)], the biharmonic equation [Marin and Lesnic (2005b)] etc. have been successfully addressed by employing the MFS.

To our knowledge, the MFS has not been applied iteratively to the numerical solution of the Cauchy problem for steady state heat conduction in anisotropic solids as yet. Due to this fact and also encouraged by the recent results obtained by Marin (2009), who implemented the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991), in conjunction with the MFS, for Cauchy problems associated with the steady state heat conduction in isotropic media (i.e. the Laplace equation), we decided to extend, in this paper, the work of Marin (2009) to Cauchy problems in two-dimensional steady state anisotropic heat conduction. At every iteration, two mixed, well-posed and direct problems are solved using the MFS, in conjunction with the Tikhonov regularization method. For each of the aforementioned direct problems, the optimal value of the regularization parameter is chosen according to the generalized cross-validation (GCV) criterion. An efficient regularizing stopping criterion which ceases the iterative procedure at the point where the accumulation of noise becomes dominant and the errors in predicting the exact solutions increase, is also presented. The iterative MFS algorithm is then tested for Cauchy problems for steady state anisotropic heat conduction in two-dimensional simply and doubly connected domains with smooth boundaries.

2 Mathematical formulation

Consider an open bounded domain $\Omega \subset \mathbb{R}^2$ occupied by an anisotropic solid characterised by the homogeneous, symmetric and positive-definite thermal conductivity tensor $\mathbb{K} = \left[\mathrm{K}_{ij}\right]_{1\leq i,j\leq 2}$. We also assume that Ω is bounded by a smooth or piecewise smooth curve $\partial\Omega$, such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 \neq \emptyset$, $\Gamma_2 \neq \emptyset$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. In the absence of heat sources, the temperature distribution, u, in the domain Ω satisfies the following elliptic partial differential equation, also referred to as the heat balance equation

$$-\nabla \cdot \left(\mathbb{K}\nabla \mathrm{u}(\mathbf{x})\right) \equiv -\sum_{i,j=1}^{2} \mathrm{K}_{ij}\,\partial_i \partial_j \mathrm{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{1}$$

where $\partial_i \equiv \partial / \partial \mathrm{x}_i$. We now let $\mathbf{n}(\mathbf{x})$ be the unit outward normal vector at $\mathbf{x} \in \partial\Omega$ and $\mathrm{q}(\mathbf{x})$ be the normal heat flux at a point $\mathbf{x} \in \partial\Omega$ defined by

$$\mathrm{q}(\mathbf{x}) = -\mathbf{n}(\mathbf{x}) \cdot \left(\mathbb{K}\nabla \mathrm{u}(\mathbf{x})\right) = -\sum_{i,j=1}^{2} \mathrm{n}_i(\mathbf{x})\,\mathrm{K}_{ij}\,\partial_j \mathrm{u}(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \tag{2}$$

In the direct problem formulation, the knowledge of the thermal conductivity matrix $\mathbb{K}$, the location, shape and size of the entire boundary $\partial\Omega$, the temperature and/or normal heat flux on the entire boundary $\partial\Omega$ gives the corresponding Dirichlet, Neumann, or Robin conditions which enable us to determine the unknown boundary conditions, as well as the temperature distribution in the solution domain. A different and more interesting situation occurs when both the temperature and normal heat flux are prescribed on a part of the boundary, say Γ_1, whilst no boundary conditions are supplied on the remaining part of the boundary $\Gamma_2 = \partial\Omega \setminus \Gamma_1$. More precisely, we consider the following *Cauchy problem* for steady heat conduction in an anisotropic homogeneous medium:

$$-\sum_{i,j=1}^{2} \mathrm{K}_{ij}\,\partial_i \partial_j \mathrm{u}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{3a}$$

$$u(\mathbf{x}) = \widetilde{u}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{3b}$$

$$q(\mathbf{x}) = \widetilde{q}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{3c}$$

where $\widetilde{u}$ and $\widetilde{q}$ are prescribed temperature and normal heat flux, respectively, on Γ_1.

A necessary condition for the Cauchy problem given by Eqs. (3a) – (3c) to be identifiable is that $\text{meas}(\Gamma_1) \geq \text{meas}(\Gamma_2)$. This inverse problem is much more difficult to solve both analytically and numerically than the direct problem since the solution does not satisfy the general conditions of well-posedness. Although the problem may have a unique solution, it is well known that this solution is unstable with respect to small perturbations into the data on Γ_1, see e.g. Hadamard (1923). Thus the problem is ill-posed and we cannot use a direct approach, such as the least-squares method, in order to solve the system of linear equations which arises from the discretisation of the Cauchy problem (3a) – (3c). Therefore, regularization methods are required in order to solve accurately the inverse problem (3a) – (3c) for steady state heat conduction in an anisotropic medium.

3 Description of the algorithm

Kozlov, Maz′ya and Fomin (1991) proposed the following iterative algorithm for the simultaneous reconstruction of the unknown temperature $u\big|_{\Gamma_2}$ and normal heat flux $q\big|_{\Gamma_2}$ on the under-specified boundary:

Step 1. (i) If $k = 1$ then specify an initial boundary temperature guess on Γ_2, namely $u^{(2k-1)} \in H^{1/2}(\Gamma_2)$.

(ii) If $k > 1$ then solve the following mixed, well-posed, direct problem:

$$-\sum_{i,j=1}^{2} K_{ij}\,\partial_i\partial_j u^{(2k-1)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{4a}$$

$$u^{(2k-1)}(\mathbf{x}) = \widetilde{u}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{4b}$$

$$q^{(2k-1)}(\mathbf{x}) = q^{(2k-2)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2, \tag{4c}$$

to determine $u^{(2k-1)}(\mathbf{x})$ for $\mathbf{x} \in \Omega$ and $u^{(2k-1)}(\mathbf{x})$ for $\mathbf{x} \in \Gamma_2$.

Step 2. Having constructed the approximation $u^{(2k-1)}$, $k \geq 1$, the following mixed, well-posed, direct problem:

$$-\sum_{i,j=1}^{2} K_{ij}\,\partial_i\partial_j u^{(2k)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{5a}$$

$$q^{(2k)}(\mathbf{x}) = \widetilde{q}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_1, \tag{5b}$$

$$\mathrm{u}^{(2k)}(\mathbf{x}) = \mathrm{u}^{(2k-1)}(\mathbf{x}), \quad \mathbf{x} \in \Gamma_2, \tag{5c}$$

is solved in order to determine $\mathrm{u}^{(2k)}(\mathbf{x})$ for $\mathbf{x} \in \Omega$ and $\mathrm{q}^{(2k)}(\mathbf{x}) \equiv -\mathbf{n}(\mathbf{x}) \cdot (\mathbb{K}\nabla \mathrm{u}^{(2k)}(\mathbf{x}))$ for $\mathbf{x} \in \Gamma_2$.

Step 3. Repeat steps 1 and 2 until a prescribed stopping criterion is satisfied.

Let $\mathrm{H}^1(\Omega)$ be the Sobolev space and $\mathrm{H}^{1/2}(\partial\Omega)$ be the space of traces on $\partial\Omega$ corresponding to $\mathrm{H}^1(\Omega)$, see e.g. Lions and Magenes (1972). We denote by $\mathrm{H}^{1/2}(\Gamma_i)$ the space of functions from $\mathrm{H}^{1/2}(\partial\Omega)$ that are bounded on Γ_i and by $\left(\mathrm{H}^{1/2}(\Gamma_i)\right)^*$ the dual space of $\mathrm{H}^{1/2}(\Gamma_i)$, for $i = 1,2$. Kozlov, Maz′ya and Fomin (1991) showed that if $\partial\Omega$ is smooth, $\widetilde{\mathrm{u}} \in \mathrm{H}^{1/2}(\Gamma_1)$ and $\widetilde{\mathrm{q}} \in \left(\mathrm{H}^{1/2}(\Gamma_1)\right)^*$, then the alternating iterative algorithm based on steps $1-3$ produces two sequences of approximate solutions $\left\{\mathrm{u}^{(2k-1)}\right\}_{k\geq 1}$ and $\left\{\mathrm{u}^{(2k)}\right\}_{k\geq 1}$ which both converge in $\mathrm{H}^1(\Omega)$ to the solution u of the Cauchy problem $(3a)-(3c)$ for any initial guess $\mathrm{u}^{(1)} \in \mathrm{H}^{1/2}(\Gamma_2)$, provided that a solution to this Cauchy problem exists. Furthermore, the alternating iterative algorithm has a regularizing character. Also, the same conclusion holds if at the step 1 one specifies an initial guess for the unknown normal heat flux on Γ_2, i.e. $\mathrm{q}^{(1)} \in \left(\mathrm{H}^{1/2}(\Gamma_2)\right)^*$, instead of an initial guess for the temperature, $\mathrm{u}^{(1)} \in \mathrm{H}^{1/2}(\Gamma_2)$, and we modify steps 1 and 2 accordingly.

4 Method of fundamental solutions

4.1 MFS approximation

The fundamental solution G of the heat balance equation (1) or (3a) for two-dimensional steady heat conduction in anisotropic homogeneous media is given by, see e.g. Fairweather and Karageorghis (1998)

$$G(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{2\pi\sqrt{\det\mathbb{K}}} \log\left(\frac{1}{\mathrm{R}}\right), \quad \mathbf{x} \in \overline{\Omega}, \quad \boldsymbol{\xi} \in \mathbb{R}^2 \setminus \overline{\Omega}, \tag{6}$$

where $\mathrm{R} = \sqrt{(\mathbf{x}-\boldsymbol{\xi}) \cdot \mathbb{K}^{-1}(\mathbf{x}-\boldsymbol{\xi})}$ and $\boldsymbol{\xi}$ is a singularity or source point. The main idea of the MFS consists of the approximation of the temperature in the solution domain by a linear combination of fundamental solutions with respect to M singularities $\boldsymbol{\xi}^{(j)}, j = 1,\ldots,M$, in the form

$$\mathrm{u}(\mathbf{x}) \approx \mathrm{u}_M(\mathbf{c}, \boldsymbol{\xi}; \mathbf{x}) = \sum_{j=1}^{M} \mathrm{c}_j\, G(\mathbf{x}, \boldsymbol{\xi}^{(j)}), \quad \mathbf{x} \in \overline{\Omega}, \tag{7}$$

where $\mathbf{c} = [\mathrm{c}_1,\ldots,\mathrm{c}_M]^\mathsf{T}$ and $\boldsymbol{\xi} \in \mathbb{R}^{2M}$ is a vector containing the coordinates of the singularities $\boldsymbol{\xi}^{(j)}$, $j = 1,\ldots,M$. On taking into account the definitions of the normal heat flux (2) and the fundamental solution (6) then the normal heat flux, through a curve defined by the outward unit normal vector $\mathbf{n}(\mathbf{x})$, can be approximated on the boundary $\partial\Omega$ by

$$\mathrm{q}(\mathbf{x}) \approx \mathrm{q}_M(\mathbf{c},\boldsymbol{\xi};\mathbf{x}) = \sum_{j=1}^{M} \mathrm{c}_j H(\mathbf{x},\boldsymbol{\xi}^{(j)}), \quad \mathbf{x} \in \partial\Omega, \tag{8}$$

where

$$H(\mathbf{x},\boldsymbol{\xi}) = -\mathbf{n}(\mathbf{x}) \cdot \left(\mathbb{K}\nabla_{\mathbf{x}} G(\mathbf{x},\boldsymbol{\xi})\right) = \frac{1}{2\pi \mathrm{R}\sqrt{\det\mathbb{K}}} \left[\frac{\mathbf{x}-\boldsymbol{\xi}}{\mathrm{R}} \cdot \mathbf{n}(\mathbf{x})\right], \quad \mathbf{x} \in \partial\Omega, \quad \boldsymbol{\xi} \in \mathbb{R}^2 \setminus \overline{\Omega}. \tag{9}$$

Next, we select the N_1 MFS collocation points $\left\{\mathbf{x}^{(i)}\right\}_{i=1}^{N_1}$ on the boundary Γ_1 and the N_2 MFS collocation points $\left\{\mathbf{x}^{(i)}\right\}_{i=N_1+1}^{N_1+N_2}$ on the boundary Γ_2, such that the total number of MFS collocation points used to discretise the boundary $\partial\Omega$ of the solution domain Ω is given by $N = N_1 + N_2$.

According to the MFS approximations (7) and (8), the discretised versions of the the boundary value problems (4a) – (4c) and (5a) – (5c) recast as

$$\mathbf{A}^{(1)}\,\mathbf{c}^{(2k-1)} = \mathbf{b}^{(2k-1)}, \quad k > 1, \tag{10}$$

and

$$\mathbf{A}^{(2)}\,\mathbf{c}^{(2k)} = \mathbf{b}^{(2k)}, \quad k \geq 1, \tag{11}$$

respectively. Here the components of the MFS matrices and right-hand side vectors corresponding to Eqs. (10) and (11) are given by

$$\mathrm{A}_{ij}^{(1)} = \begin{cases} G(\mathbf{x}^{(i)},\boldsymbol{\xi}^{(j)}), & i = 1,\ldots,N_1, \quad j = 1,\ldots,M, \\ H(\mathbf{x}^{(i)},\boldsymbol{\xi}^{(j)}), & i = N_1+1,\ldots,N_1+N_2, \quad j = 1,\ldots,M, \end{cases} \tag{12a}$$

$$\mathrm{b}_i^{(2k-1)} = \begin{cases} \widetilde{\mathrm{u}}(\mathbf{x}^{(i)}), & i = 1,\ldots,N_1, \\ \mathrm{q}^{(2k-2)}(\mathbf{x}^{(i)}), & i = N_1+1,\ldots,N_1+N_2, \end{cases} \tag{12b}$$

and

$$\mathrm{A}_{ij}^{(2)} = \begin{cases} H(\mathbf{x}^{(i)},\boldsymbol{\xi}^{(j)}), & i = 1,\ldots,N_1, \quad j = 1,\ldots,M, \\ G(\mathbf{x}^{(i)},\boldsymbol{\xi}^{(j)}), & i = N_1+1,\ldots,N_1+N_2, \quad j = 1,\ldots,M, \end{cases} \tag{13a}$$

$$\mathrm{b}_i^{(2k)} = \begin{cases} \widetilde{\mathrm{q}}(\mathbf{x}^{(i)}), & i = 1, \ldots, N_1, \\ \mathrm{u}^{(2k-1)}(\mathbf{x}^{(i)}), & i = N_1+1, \ldots, N_1+N_2, \end{cases} \tag{13b}$$

respectively.

Each of Eqs. (10) and (11) represents a system of N linear algebraic equations with M unknowns, namely the MFS coefficients $\mathbf{c}^{(2k-1)} = \left[\mathrm{c}_1^{(2k-1)}, \ldots, \mathrm{c}_M^{(2k-1)}\right]^{\mathsf{T}}$ and $\mathbf{c}^{(2k)} = \left[\mathrm{c}_1^{(2k)}, \ldots, \mathrm{c}_M^{(2k)}\right]^{\mathsf{T}}$, respectively. It should be noted that in order to uniquely determine the solutions $\mathbf{c}^{(2k-1)} \in \mathbb{R}^M$ and $\mathbf{c}^{(2k)} \in \mathbb{R}^M$ to the systems of linear algebraic equations (10) and (11), respectively, the number N of MFS boundary collocation points on the boundary $\partial\Omega$ and the number M of singularities must satisfy the inequality $M \leq N$. However, the systems of linear algebraic equations (10) and (11) cannot be solved by direct methods, such as the least-squares method, since such an approach would produce a highly unstable solution in the case of noisy Cauchy data on Γ_1.

4.2 MFS boundary collocation points and singularities

In order to implement the MFS, the location of the singularities has to be determined and this is usually achieved by considering either the static or the dynamic approach. In the static approach, the singularities are pre-assigned and kept fixed throughout the solution process, whilst in the dynamic approach, the singularities and the unknown coefficients are determined simultaneously during the solution process, see Fairweather and Karageorghis (1998). Thus the dynamic approach transforms the inverse problem into a more difficult nonlinear ill-posed problem which is also computationally much more expensive. The advantages and disadvantages of the MFS with respect to the location of the fictitious sources are described at length in Heise (1978) and Burgess and Maharejin (1984).

Recently, Gorzelańczyk and Kołodziej (2008) thoroughly investigated the performance of the MFS with respect to the shape of the pseudo-boundary on which the source points are situated, proving that, for the same number of boundary collocation points and sources, more accurate results are obtained if the shape of the pseudo-boundary is similar to that of the boundary of the solution domain. Therefore, we have decided to employ the static approach in our computations, at the same time accounting for the findings of Gorzelańczyk and Kołodziej (2008).

5 Regularization

It is well-known that the MFS discretisation matrices $\mathbf{A}^{(i)}$, $i = 1, 2$, are severely ill-conditioned. The accurate and stable solutions of Eqs. (10) and (11) are very important for obtaining physically meaningful numerical results. It is the purpose of this section to present a classical regularization procedure for obtaining stable solutions to the systems of linear algebraic equations (10) and (11), as well as details regarding the optimal choice of the regularization parameter.

5.1 Tikhonov regularization method

Several regularization techniques used for the stable solution of systems of linear and nonlinear algebraic equations are available in the literature, such as the singular value decomposition [Hansen (1998)], the Tikhonov regularization method [Tikhonov and Arsenin (1986)] and various iterative methods [Kunisch and Zou (1998)]. Recently, Liu (2008a) proposed a new and robust numerical technique for the stable solution of ill-posed systems of linear algebraic equations, namely the Fictitious Time Integration Method (FTIM). This method consists of introducing a fictitious time variable that plays the role of a regularization parameter, while its filtering effect is better than that of the Tikhonov and exponential filters. The FTIM was successfully applied to solving inverse vibration problems [Liu (2008b); Liu (2008c); Liu, Chang, Chang and Chen (2008)], thermal stress reconstruction [Liu (2008d)], nonlinear complementarity problems [Liu (2008e)], large systems of nonlinear algebraic equations [Liu and Atluri (2008a)], boundary value problems for elliptic partial differential equations [Liu (2008f)], inverse Sturm-Liouville problems [Liu and Atluri (2008b)], Burgers equation [Liu (2009a)] and delay ordinary differential equations [Liu (2009b)]. Liu and Atluri (2009) have recently shown that, when applied to solving an ill-posed system of linear equations, the general FTIM may be viewed as a special case of the Tiknonov regularization method.

Consider the following system of linear algebraic equations

$$\mathbf{A}\,\mathbf{c} = \mathbf{b}, \tag{14}$$

where $N \geq M$, $\mathbf{A} \in \mathbb{R}^{N\times M}$, $\mathbf{c} \in \mathbb{R}^{M}$ and $\mathbf{b} \in \mathbb{R}^{N}$. Note that Eq. (14) may describe each of the MFS systems of linear equations (10) and (11), provided that

$$\mathbf{A} = \mathbf{A}^{(1)}, \qquad \mathbf{c} = \mathbf{c}^{(2k-1)}, \qquad \mathbf{b} = \mathbf{b}^{(2k-1)}, \qquad k > 1, \tag{15}$$

and

$$\mathbf{A} = \mathbf{A}^{(2)}, \qquad \mathbf{c} = \mathbf{c}^{(2k)}, \qquad \mathbf{b} = \mathbf{b}^{(2k)}, \qquad k \geq 1, \tag{16}$$

respectively. The Tikhonov zeroth-order regularized solution to the generically written system of linear algebraic equations (14) is sought as, see Tikhonov and Arsenin (1986)

$$\mathbf{c}_\lambda: \quad \mathscr{F}_\lambda(\mathbf{c}_\lambda) = \min_{\mathbf{c}\in\mathbb{R}^M} \mathscr{F}_\lambda(\mathbf{c}), \tag{17}$$

where $\mathscr{F}_\lambda$ represents the Tikhonov zeroth-order regularization functional given by, see Tikhonov and Arsenin (1986)

$$\mathscr{F}_\lambda(\cdot): \mathbb{R}^M \longrightarrow [0,\infty), \quad \mathscr{F}_\lambda(\mathbf{c}) = \left\|\mathbf{A}\,\mathbf{c}-\mathbf{b}\right\|^2 + \lambda^2\left\|\mathbf{c}\right\|^2, \tag{18}$$

and $\lambda > 0$ is the regularization parameter to be prescribed. Formally, the Tikhonov regularized solution $\mathbf{c}_\lambda$ of the system of linear algebraic equations (14) is given as the solution of the normal equation

$$\left(\mathbf{A}^\mathsf{T}\mathbf{A} + \lambda^2\mathbf{I}_M\right)\mathbf{c} = \mathbf{A}^\mathsf{T}\,\mathbf{b}, \tag{19}$$

where $\mathbf{I}_M \in \mathbb{R}^{M\times M}$ is the identity matrix. More precisely, the Tikhonov regularized solution of the system of linear algebraic equations (14) is obtained as

$$\mathbf{c}_\lambda = \mathbf{A}^\dagger\,\mathbf{b}, \quad \mathbf{A}^\dagger \equiv \left(\mathbf{A}^\mathsf{T}\mathbf{A} + \lambda^2\mathbf{I}_M\right)^{-1}\mathbf{A}^\mathsf{T}. \tag{20}$$

To summarize, the Tikhonov regularization method solves a constrained minimization problem using a smoothness norm in order to provide a stable solution which fits the data and also has a minimum structure.

5.2 Selection of the optimal regularization parameter

The performance of regularization methods depends crucially on the suitable choice of the regularization parameter. One extensively studied criterion is the discrepancy principle, see e.g. Morozov (1966). Although this criterion is mathematically rigorous, it requires a reliable estimation of the amount of noise added into the data which may not be available in practical problems. Heuristical approaches are preferable in the case when no *a priori* information about the noise is available. For the Tikhonov zeroth-order regularization method, several heuristical approaches have been proposed, including the L-curve criterion, see Hansen (1998), and the generalized cross-validation (GCV), see Wahba (1977). In this paper, we employ the GCV criterion to determine the optimal regularization parameter, λ_{opt}, for the Tikhonov zeroth-order regularization method, namely

$$\lambda_{\text{opt}}: \quad \mathscr{G}\left(\lambda_{\text{opt}}\right) = \min_{\lambda>0} \mathscr{G}(\lambda). \tag{21}$$

Here

$$\mathscr{G}(\cdot):(0,\infty)\longrightarrow[0,\infty),\quad \mathscr{G}(\lambda)=\frac{\left\|\mathbf{A}\,\mathbf{c}_\lambda-\mathbf{b}^\epsilon\right\|^2}{\left[\mathrm{trace}\left(\mathbf{I}_N-\mathbf{A}\,\mathbf{A}^\dagger\right)\right]^2},\tag{22}$$

where $\mathbf{c}_\lambda$ is given by Eq. (20) with the perturbed right-hand side $\mathbf{b}^\epsilon$ such that $\left\|\mathbf{b}^\epsilon-\mathbf{b}\right\|\leq\epsilon$.

6 Numerical results and discussion

In this section, we present the performance of the proposed numerical method, namely the alternating iterative MFS described in Sections 3 and 4. To do so, we solve numerically the Cauchy geometric problem (3a) – (3c) for the steady state anisotropic heat conduction in the two-dimensional geometries described below.

6.1 Examples

Example 1. (Smooth simply connected geometry) We consider the following analytical solutions for the temperature and normal heat flux

$$\mathrm{u}^{(\mathrm{an})}(\mathbf{x})=\mathrm{x}_1^2-4\mathrm{x}_1\mathrm{x}_2+\mathrm{x}_2^2,\quad \mathbf{x}=(\mathrm{x}_1,\mathrm{x}_2)\in\overline{\Omega},\tag{23a}$$

$$\mathrm{q}^{(\mathrm{an})}(\mathbf{x})=3\mathrm{x}_2\mathrm{n}_1(\mathbf{x})+3\mathrm{x}_1\mathrm{n}_2(\mathbf{x}),\quad \mathbf{x}=(\mathrm{x}_1,\mathrm{x}_2)\in\partial\Omega,\tag{23b}$$

respectively, in the unit disk $\Omega=\left\{\mathbf{x}=(\mathrm{x}_1,\mathrm{x}_2)\,\middle|\,\rho(\mathbf{x})<\mathrm{r}\right\}$, where $\rho(\mathbf{x})=\sqrt{\mathrm{x}_1^2+\mathrm{x}_2^2}$ is the radial polar coordinate of $\mathbf{x}$ and $\mathrm{r}=1.0$. Here $\mathrm{K}_{11}=1.0$, $\mathrm{K}_{12}=\mathrm{K}_{21}=0.5$, $\mathrm{K}_2=1.0$, $\Gamma_1=\left\{\mathbf{x}\in\partial\Omega\,\middle|\,0\leq\theta(\mathbf{x})\leq3\pi/2\right\}$ and $\Gamma_2=\left\{\mathbf{x}\in\partial\Omega\,\middle|\,3\pi/2<\theta(\mathbf{x})<2\pi\right\}$, where $\theta(\mathbf{x})$ is the angular polar coordinate of $\mathbf{x}$.

Example 2. (Smooth doubly connected geometry) We consider the following analytical solutions for the temperature and normal heat flux

$$\mathrm{u}^{(\mathrm{an})}(\mathbf{x})=\frac{1}{5}\mathrm{x}_1^3-\mathrm{x}_1^2\mathrm{x}_2+\mathrm{x}_1\mathrm{x}_2^2+\frac{1}{3}\mathrm{x}_2^3,\quad \mathbf{x}=(\mathrm{x}_1,\mathrm{x}_2)\in\overline{\Omega},\tag{24a}$$

$$\begin{aligned}&\mathrm{q}^{(\mathrm{an})}(\mathbf{x})=-\left(\mathrm{x}_1^2-6\mathrm{x}_1\mathrm{x}_2+6\mathrm{x}_2^2\right)\mathrm{n}_1(\mathbf{x})-\left(\frac{1}{5}\mathrm{x}_1^2-2\mathrm{x}_1\mathrm{x}_2+3\mathrm{x}_2^2\right)\mathrm{n}_2(\mathbf{x}),\\&\mathbf{x}=(\mathrm{x}_1,\mathrm{x}_2)\in\partial\Omega,\end{aligned}\tag{24b}$$

respectively, in the annular domain $\Omega=\left\{\mathbf{x}=(\mathrm{x}_1,\mathrm{x}_2)\,\middle|\,\mathrm{r}_{\mathrm{int}}<\rho(\mathbf{x})<\mathrm{r}_{\mathrm{out}}\right\}$, where $\mathrm{r}_{\mathrm{int}}=2.0$ and $\mathrm{r}_{\mathrm{out}}=3.0$. Here $\mathrm{K}_{11}=5.0$, $\mathrm{K}_{12}=\mathrm{K}_{21}=2.0$, $\mathrm{K}_2=1.0$, $\Gamma_1=\left\{\mathbf{x}\in\partial\Omega\,\middle|\,\rho(\mathbf{x})=\mathrm{r}_{\mathrm{out}}\right\}$ and $\Gamma_2=\left\{\mathbf{x}\in\partial\Omega\,\middle|\,\rho(\mathbf{x})=\mathrm{r}_{\mathrm{int}}\right\}$.

The Cauchy problems investigated in this paper have been solved using the uniform distribution of both the MFS boundary collocation points $\mathbf{x}^{(i)}$, $i=1,\ldots,N$, and the singularities $\xi^{(j)}$, $j = 1, \ldots, M$. Furthermore, the numbers of MFS boundary collocation points N_1 and N_2 corresponding to the over- and under-specified boundaries Γ_1 and Γ_2, respectively, the number of singularities M and the distance d_S between the physical boundary $\partial\Omega$ and the pseudo-boundary $\partial\Omega_S$ on which the singularities are situated, were set to:

(i) $N_1/3 = N_2 = N/4 \in \{10, 20, 30\}$, $M = N/2$ and $d_S = 3.0$ for Example 1;

(ii) $N_1/3 = N_2/2 = N/5 \in \{10, 20, 30\}$, $M = N_1 + N_2/2 = 4N/5$, while $d_S = 0.5$ and $d_S = 2.0$ for the inner and outer boundaries, respectively, for Example 2.

6.2 Initial guess

An arbitrary real valued function $\mathrm{u}^{(1)} \in \mathrm{H}^{1/2}(\Gamma_2)$ may be specified as an initial guess for the unknown temperature on the under-specified boundary Γ_2. In order to improve the rate of convergence of the iterative algorithm, one may choose a real valued function which ensures the continuity of the boundary temperature at the common endpoints of the over- and under-specified boundaries Γ_1 and Γ_2, respectively, and which is also linear with respect to the angular polar coordinate θ, see e.g. Mera, Elliott, Ingham and Lesnic (2000) and Marin (2009). More precisely, for Example 1 the following initial guess for the unknown temperature on Γ_2 may be chosen:

$$\mathrm{u}^{(1)}(\mathbf{x}) = \frac{\theta\left(\mathbf{x}^{(2)}\right) - \theta(\mathbf{x})}{\theta\left(\mathbf{x}^{(2)}\right) - \theta\left(\mathbf{x}^{(1)}\right)} \mathrm{u}^{(\mathrm{an})}\left(\mathbf{x}^{(1)}\right) + \frac{\theta(\mathbf{x}) - \theta\left(\mathbf{x}^{(1)}\right)}{\theta\left(\mathbf{x}^{(2)}\right) - \theta\left(\mathbf{x}^{(1)}\right)} \mathrm{u}^{(\mathrm{an})}\left(\mathbf{x}^{(2)}\right), \quad \mathbf{x} \in \Gamma_2, \tag{25}$$

where $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are the common endpoints of the over- and under-specified boundaries, i.e. $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\}$.

However, in the general situation when the over- and under-specified boundaries have no common points, as is the case of Example 2, one cannot use the procedure described above. Therefore, in this case, the initial guess for the unknown temperature on the under-specified boundary Γ_2 is chosen as

$$\mathrm{u}^{(1)}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_2. \tag{26}$$

In this study, we have decided to use the initial guess (26). In this way, the most general situations regarding the geometry of the solution domain are accounted for and the robustness of the alternating iterative algorithm with respect to the initial guess for the unknown temperature on Γ_2 is also tested.

6.3 Convergence of the algorithm

If N_i MFS collocation points, $\{\mathbf{x}^{(\ell)}\}_{\ell=1}^{N_i}$, are considered on the boundary $\Gamma_i \subset \partial\Omega$ then the *root mean square error* (RMS error) associated with the real valued function $f(\cdot) : \Gamma_i \longrightarrow \mathbb{R}$ on Γ_i is defined by

$$\mathrm{RMS}_{\Gamma_i}(f) = \sqrt{\frac{1}{N_i}\sum_{\ell=1}^{N_i} f\big(\mathbf{x}^{(\ell)}\big)^2}, \tag{27}$$

In order to investigate the convergence of the algorithm, at every iteration, $k \geq 1$, we evaluate the following accuracy errors corresponding to the temperature and normal heat flux on the under-specified boundary, Γ_2, which are defined as *relative RMS errors*, i.e.

$$\mathrm{e_u}(k) = \frac{\mathrm{RMS}_{\Gamma_2}\big(\mathrm{u}^{(2k-1)} - \mathrm{u}^{(\mathrm{an})}\big)}{\mathrm{RMS}_{\Gamma_2}\big(\mathrm{u}^{(\mathrm{an})}\big)} = \frac{\big\|\mathrm{u}^{(2k-1)}\big|_{\Gamma_2} - \mathrm{u}^{(\mathrm{an})}\big|_{\Gamma_2}\big\|_2}{\big\|\mathrm{u}^{(\mathrm{an})}\big|_{\Gamma_2}\big\|_2}, \quad k \geq 1, \tag{28a}$$

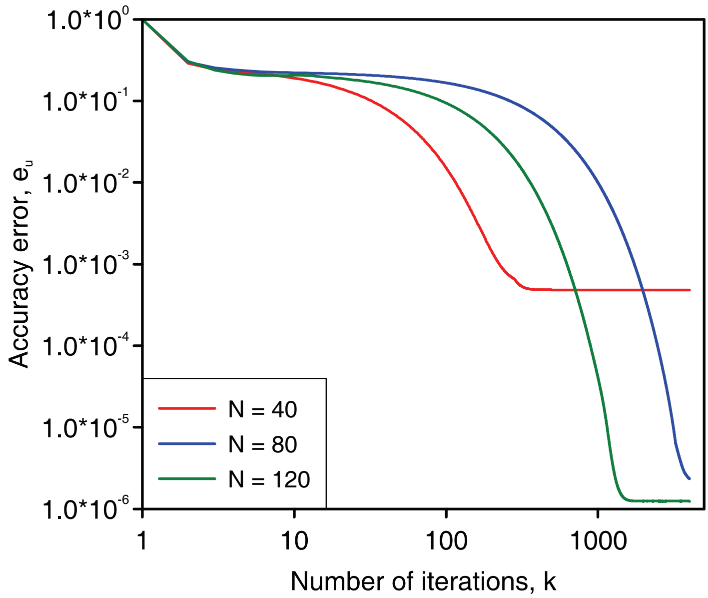
and

$$\mathrm{e_q}(k) = \frac{\mathrm{RMS}_{\Gamma_2}\big(\mathrm{q}^{(2k)} - \mathrm{q}^{(\mathrm{an})}\big)}{\mathrm{RMS}_{\Gamma_2}\big(\mathrm{q}^{(\mathrm{an})}\big)} = \frac{\big\|\mathrm{q}^{(2k)}\big|_{\Gamma_2} - \mathrm{q}^{(\mathrm{an})}\big|_{\Gamma_2}\big\|_2}{\big\|\mathrm{q}^{(\mathrm{an})}\big|_{\Gamma_2}\big\|_2}, \quad k \geq 1, \tag{28b}$$

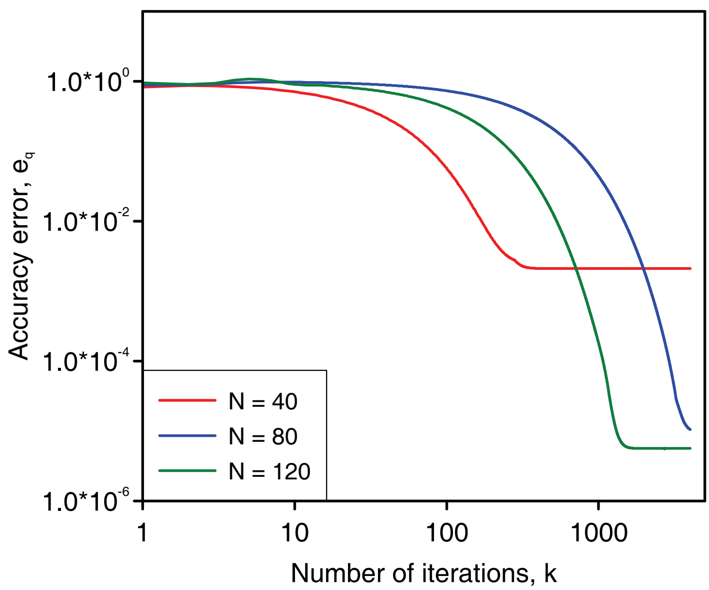
where $\mathrm{u}^{(2k-1)}$ and $\mathrm{q}^{(2k)}$ are the temperature and normal heat flux on the boundary Γ_2 retrieved after k iterations by solving the well-posed, mixed, direct, boundary value problems (4a) – (4c) and (5a) – (5c), respectively. The error in predicting the temperature inside the solution domain, Ω, may also be evaluated, but it has an evolution similar to that of the errors $\mathrm{e_u}$ and $\mathrm{e_q}$ given by Eqs. (28a) and (28b), respectively, and hence this is not pursued herein.

Figs. $1(a)$ and $1(b)$ display the accuracy errors $\mathrm{e_u}$ and $\mathrm{e_q}$, respectively, as functions of the number of iterations, k, obtained using exact Cauchy data on the over-specified boundary, Γ_1, and various numbers of MFS collocation points, for the inverse problems given by Example 1. It can be seen from these figures that both errors $\mathrm{e_u}$ and $\mathrm{e_q}$ decrease even after a large number of iterations, e.g. $k = 4000$, and as expected $\mathrm{e_u} < \mathrm{e_q}$ for all MFS discretisations employed, i.e. normal heat fluxes are more inaccurate than temperatures. Furthermore, as N increases, the errors $\mathrm{e_u}$ and $\mathrm{e_q}$ decrease showing that $N \geq 80$ ensures a sufficient discretisation for the accuracy to be achieved in the case of Example 1. Similar results have been obtained for Example 2 and, therefore, they are not illustrated.

The analytical and numerical solutions for the temperature $\mathrm{u}\big|_{\Gamma_2}$ and the normal heat flux $\mathrm{q}\big|_{\Gamma_2}$ obtained with exact Cauchy data after $k = 4000$ iterations, for the

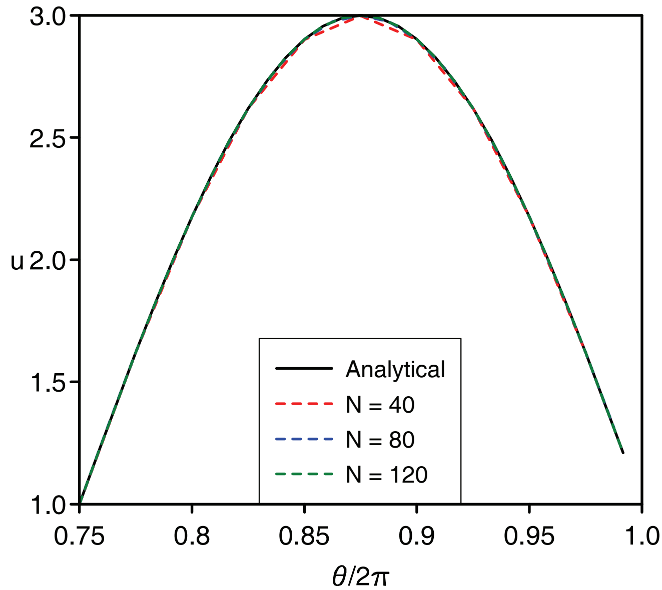


(a) Accuracy error, e_u

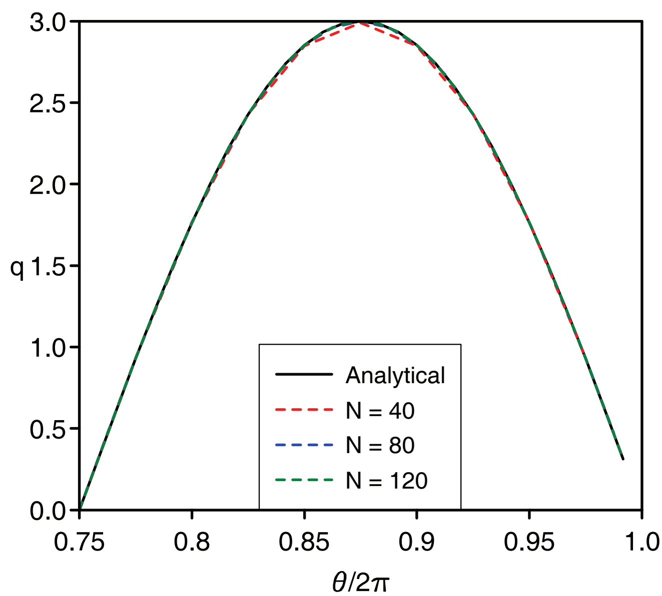


(b) Accuracy error, e_q

Figure 1: The accuracy errors (a) e_u, and (b) e_q, as functions of the number of iterations, k, obtained using exact Cauchy data on Γ_1 and various numbers of MFS boundary collocation points, for (a) Example 1.

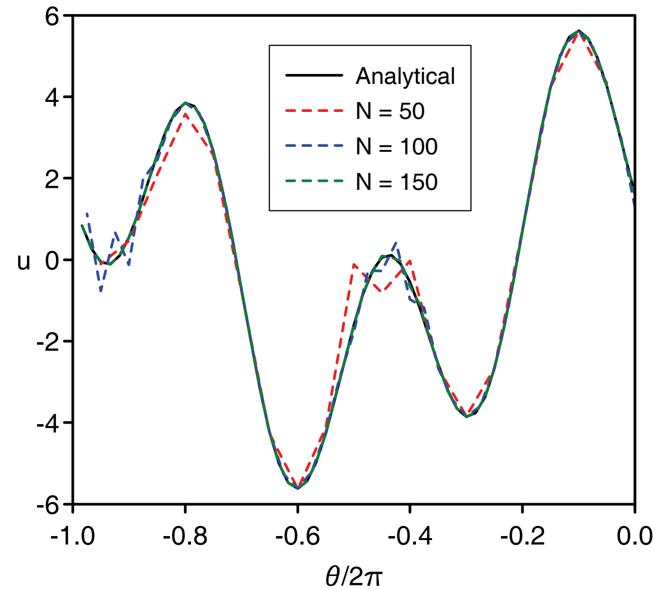


(a) Example 1: Temperatures on Γ_2

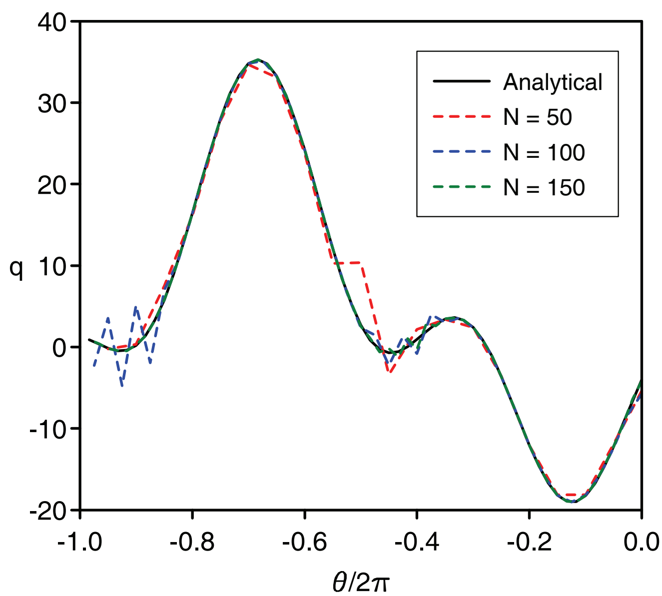


(b) Example 1: Normal heat fluxes on Γ_2

Figure 2: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using exact Cauchy data on Γ_1, $k = 4000$ iterations and various numbers of MFS boundary collocation points, namely $N \in \{40, 80, 120\}$, for Example 1.



(a) Example 2: Temperatures on Γ_2



(b) Example 2: Normal heat fluxes on Γ_2

Figure 3: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using exact Cauchy data on Γ_1, $k = 1000$ iterations and various numbers of MFS boundary collocation points, namely $N \in \{50, 100, 150\}$, for Example 2.

Cauchy problem given by Example 1 are presented in Figs. $2(a)$ and $2(b)$, respectively. Figs. $3(a)$ and $3(b)$ illustrate the analytical and numerical values for the temperature $\mathrm{u}\big|_{\Gamma_2}$ and the normal heat flux $\mathrm{q}\big|_{\Gamma_2}$, respectively, retrieved with exact Cauchy data after $k = 1000$ iterations, in the case of Example 2. From these figures, it can be seen that the accuracy in predicting both the temperature distribution and normal heat flux on the boundary Γ_2 is very good. As expected, the errors in predicting the normal heat flux $\mathrm{q}\big|_{\Gamma_2}$ are larger than the errors in predicting the temperature $\mathrm{u}\big|_{\Gamma_2}$ since the normal heat flux contains higher-order derivatives of the latter.

From Figs. $1-3$, it can be concluded that the MFS-based alternating iterative algorithm described in Sections 3 and 4 produces an accurate and convergent numerical solution for both the missing boundary temperature and normal heat flux with respect to increasing the number of iterations, k, and the number of MFS boundary collocation points, N, provided that exact input Cauchy data are used. However, exact data are seldom available in practice since measurement errors always include noise in the prescribed boundary conditions and this is investigated next.

6.4 Stopping criterion

Once the convergence, with respect to increasing the number of iterations, k, and the number of MFS boundary collocation points, N, of the numerical solution to the exact solution has been established, we fix $N = 80$ and $N = 100$ for Examples 1 and 2, respectively, and investigate the stability of the numerical solution. In what follows, the temperature, $\mathrm{u}\big|_{\Gamma_1} = \mathrm{u}^{(\mathrm{an})}\big|_{\Gamma_1}$, and/or the normal heat flux, $\mathrm{q}\big|_{\Gamma_1} = \mathrm{q}^{(\mathrm{an})}\big|_{\Gamma_1}$, on the over-specified boundary have been perturbed as

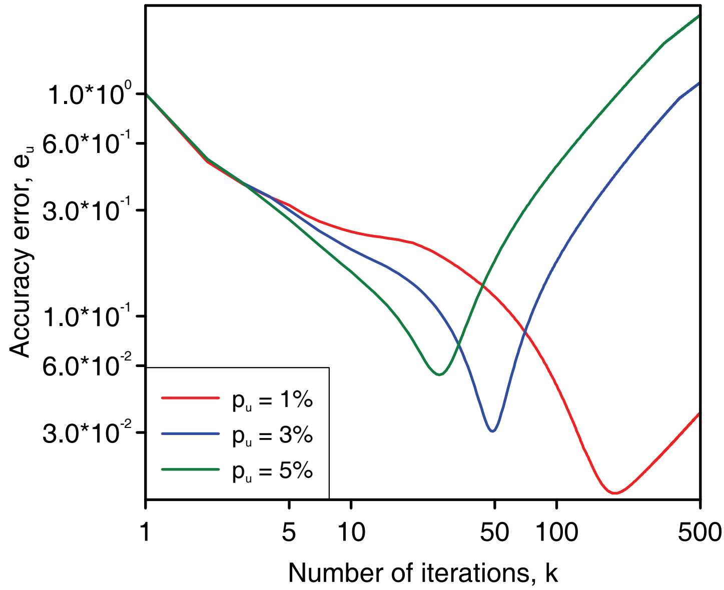
$$\widetilde{\mathrm{u}}^{\epsilon}\big|_{\Gamma_1} = \mathrm{u}\big|_{\Gamma_1} + \delta\mathrm{u}, \quad \delta\mathrm{u} = \mathtt{G05DDF}(0, \sigma_{\mathrm{u}}), \quad \sigma_{\mathrm{u}} = \max_{\Gamma_1} |\mathrm{u}| \times (p_{\mathrm{u}}/100), \tag{29}$$

and

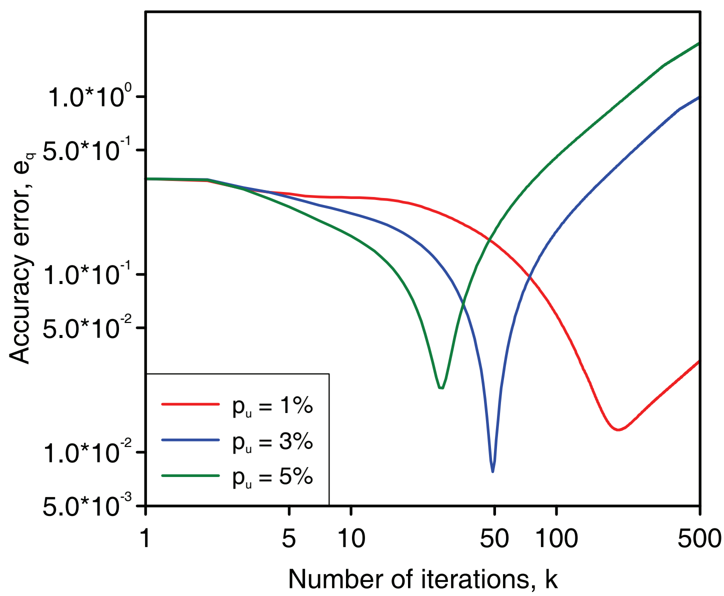
$$\widetilde{\mathrm{q}}^{\epsilon}\big|_{\Gamma_1} = \mathrm{q}\big|_{\Gamma_1} + \delta\mathrm{q}, \quad \delta\mathrm{q} = \mathtt{G05DDF}(0, \sigma_{\mathrm{q}}), \quad \sigma_{\mathrm{q}} = \max_{\Gamma_1} |\mathrm{q}| \times \big(p_{\mathrm{q}}/100\big), \tag{30}$$

respectively. Here $\delta\mathrm{u}$ and $\delta\mathrm{q}$ are Gaussian random variables with mean zero and standard deviations σ_{u} and σ_{q}, respectively, generated by the NAG subroutine `G05DDF` [NAG Library Mark 21 (2007)], while $p_{\mathrm{u}}\%$ and $p_{\mathrm{q}}\%$ are the percentages of additive noise included into the input boundary temperature, $\mathrm{u}\big|_{\Gamma_1}$, and normal heat flux, $\mathrm{q}\big|_{\Gamma_1}$, respectively, in order to simulate the inherent measurement errors.

Figs. $4(a)$ and $4(b)$ present the accuracy errors $\mathrm{e_u}$ and $\mathrm{e_q}$, respectively, for various levels of Gaussian random noise $p_{\mathrm{u}} \in \{1\%, 2\%, 3\%\}$ added into the temperature



(a) Example 1: Accuracy error e_u



(b) Example 1: Accuracy error e_q

Figure 4: The accuracy errors (a) e_u, and (b) e_q, as function of the number of iterations, k, obtained using $N = 100$ MFS boundary collocation points and various levels of noise added into the Dirichlet data on Γ_1, namely $p_u \in \{1\%, 3\%, 5\%\}$, for Example 2.

data $u|_{\Gamma_1}$, in the case of Example 2. From these figures it can be seen that as p_u decreases then e_u and e_q decrease. However, the errors in predicting the temperature and the normal heat flux on the under-specified boundary Γ_2 decrease up to a certain iteration number and after that they start increasing. If the iterative process is continued beyond this point then the numerical solutions lose their smoothness and become highly oscillatory and unbounded, i.e. unstable. Therefore, a regularizing stopping criterion must be used in order to terminate the iterative process at the point where the errors in the numerical solutions start increasing.

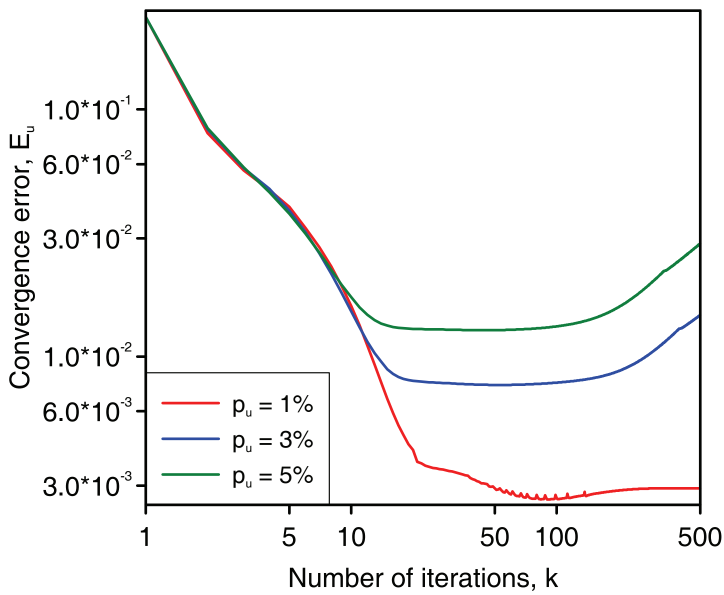


Figure 5: The convergence error, E_u, as a function of the number of iterations, k, obtained using $N = 100$ MFS boundary collocation points and various levels of noise added into the Dirichlet data on Γ_1, namely $p_u \in \{1\%, 3\%, 5\%\}$, for Example 2.

After each iteration, k, we evaluate the following convergence error which is associated with the temperature on the over-specified boundary, Γ_1, namely

$$E_u(k) = \frac{\mathrm{RMS}_{\Gamma_1}\left(u^{(2k)} - \widetilde{u}^{\epsilon}\right)}{\mathrm{RMS}_{\Gamma_1}\left(\widetilde{u}^{\epsilon}\right)} = \frac{\| u^{(2k)}|_{\Gamma_1} - \widetilde{u}^{\epsilon}|_{\Gamma_1} \|_2}{\| \widetilde{u}^{\epsilon}|_{\Gamma_1} \|_2}, \quad k \geq 1, \tag{31}$$

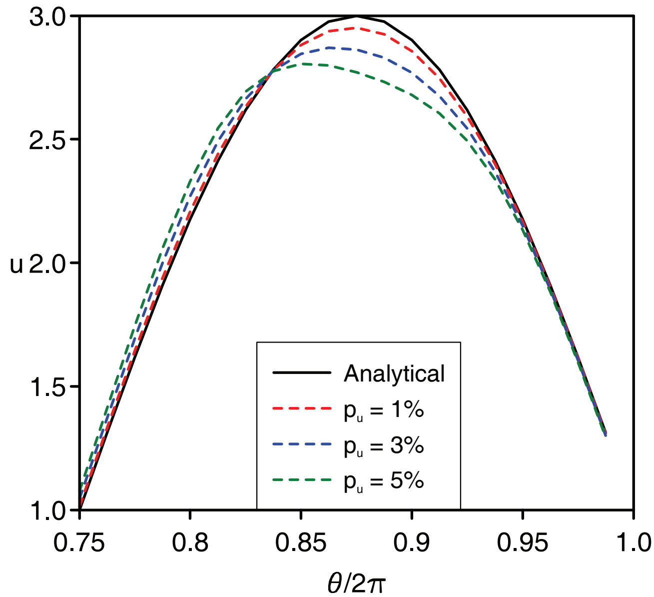
where $u^{(2k)}$ is the temperature on the over-specified boundary Γ_1 retrieved numerically after k iterations by solving the well-posed, mixed, direct, boundary value problem (5a) – (5c). This error E_u should tend to zero as the sequences $\{u^{(2k-1)}\}_{k\geq 1}$ and $\{u^{(2k)}\}_{k\geq 1}$ tend to the analytical solution, $u^{(an)}$, in the space $H^1(\Omega)$ and hence they are expected to provide an appropriate stopping criterion. Indeed, if we investigate the error E_u obtained at every iteration in the case of Example 2, for various levels of Gaussian random noise added into the input temperature data $u|_{\Gamma_1}$, we obtain the curves graphically represented in Fig. 5. By comparing Figs. 4 and 5, it can be noticed that the convergence error E_u, as well as the accuracy errors e_u and e_q, attain their corresponding minimum at around the same number iterations. Therefore, for noisy Cauchy data a natural stopping criterion ceases the MFS alternating iterative algorithm at the optimal number of iterations, k_{opt}, given by:

$$k_{opt}: \quad E_u(k_{opt}) = \min_{k\geq 1} E_u(k). \tag{32}$$

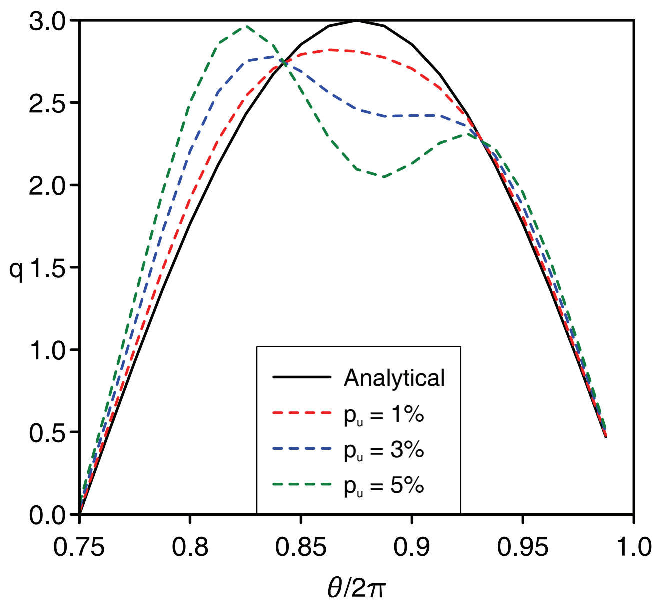
As mentioned in the previous section, for exact data the iterative process is convergent with respect to increasing the number of iterations, k, since the accuracy errors e_u and e_q keep decreasing even after a large number of iterations, see Figs. 1(a) and 1(b). It should be noted in this case that a stopping criterion is not necessary since the numerical solution is convergent with respect to increasing the number of iterations. Nonetheless, even in this case the errors E_u, e_u and e_q have a similar behaviour and the error E_u may be used to stop the iterative process at the point where the rate of convergence is very small and no substantial improvement in the numerical solution is obtained even if the iterative process is continued. Therefore, it can be concluded that the regularizing stopping criterion proposed is very efficient in locating the point where the errors start increasing and the iterative process should be ceased.

6.5 Stability of the algorithm

Based on the stopping criterion described in Section 6.4, the analytical and numerical values for the temperature, u, and normal heat flux , q, on the under-specified boundary Γ_2, obtained using $N = 80$, $M = 40$ and various levels of noise added into the temperature data on the over-specified boundary Γ_1 for Example 1, are illustrated in Figs. 6(a) and 6(b), respectively. From Fig. 6(a) it can be seen that the accuracy in predicting the missing boundary temperature, $u|_{\Gamma_2}$, is reasonable and the numerical solution converges to the exact solution as the level of noise, p_u, added into the input Dirichlet data decreases. However, the numerical solutions obtained for the unknown normal heat flux on the under-specified boundary Γ_2 are poor approximations for their exact values, as can be seen from Fig. 6(b), at the same time exhibiting an oscillatory behaviour. The reason for this is that

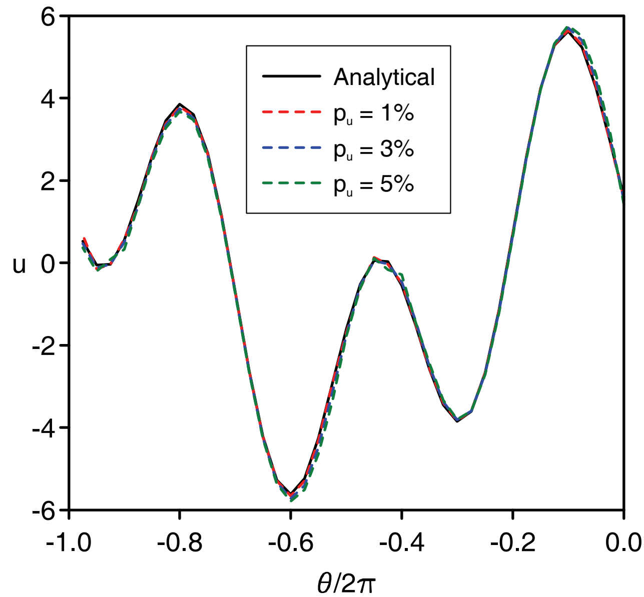


(a) Example 1: Temperatures on Γ_2

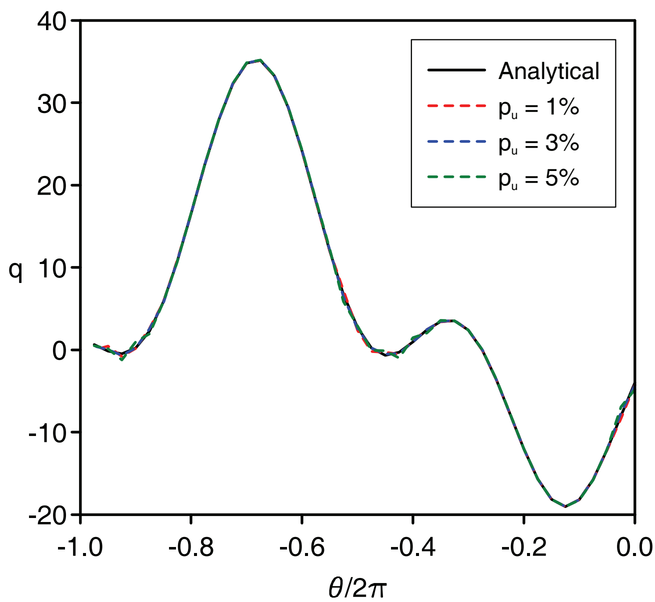


(b) Example 1: Normal heat fluxes on Γ_2

Figure 6: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using $N = 80$ MFS boundary collocation points and various levels of noise added into the Dirichlet data on Γ_1, namely $p_{\mathrm{u}} \in \{1\%, 3\%, 5\%\}$, for Example 1.



(a) Example 2: Temperatures on Γ_2



(b) Example 2: Normal heat fluxes on Γ_2

Figure 7: The analytical and numerical (a) temperatures u, and (b) normal heat fluxes q, on the under-specified boundary Γ_2, obtained using $N = 100$ MFS boundary collocation points and various levels of noise added into the Dirichlet data on Γ_1, namely $p_u \in \{1\%, 3\%, 5\%\}$, for Example 2.

$\overline{\Gamma}_1 \cap \overline{\Gamma}_2 \neq \emptyset$ and it is well known that the gradient of the temperature possesses singularities at the points where the data changes from temperature boundary conditions to normal heat flux boundary conditions, even if the temperature and the flux data are of class $\mathscr{C}^\infty$.

The proposed MFS-alternating iterative algorithm, in conjunction with the stopping criterion (32), works very well for the Cauchy problem associated with anisotropic heat conduction in a doubly connected domain with a smooth boundary, such as the annulus investigated in Example 2. Figs. $7(a)$ and $7(b)$ show the numerical results for the temperature and normal heat flux on the boundary Γ_2, obtained using the stopping criterion introduced in Section 6.4, $N = 100$, $M = 80$ and various amounts of noise added into the Dirichlet data, namely $p_{\rm u} \in \{1\%, 2\%, 3\%\}$, in comparison with their corresponding analytical values, in the case of Example 2. Although not illustrated, it is reported that for Example 2 very good results have also been retrieved for both the unknown temperature, $\mathrm{u}\big|_{\Gamma_2}$, and normal heat flux, $\mathrm{q}\big|_{\Gamma_2}$, for perturbed Neumann data on the over-specified boundary Γ_1. As expected, the numerical results obtained using the proposed MFS alternating iterative algorithm, in conjunction with the aforementioned stopping criterion, were found to be more sensitive to perturbations in the normal heat flux on the over-specified boundary than to noisy boundary temperature on Γ_1.

From the numerical results presented in this section, it can be concluded that the stopping criterion developed in Section 6.4 has a regularizing effect and the numerical solution obtained by the iterative MFS described in this paper is convergent and stable with respect to increasing the number of MFS boundary collocation points and decreasing the level of noise added into the Cauchy input data, respectively.

7 Conclusions

In this paper, the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991) was implemented, for the Cauchy problem associated with the two-dimensional steady state heat conduction in anisotropic solids, using a meshless method. The two mixed, well-posed and direct problems corresponding to every iteration of the numerical procedure were solved using the MFS, in conjunction with the Tikhonov regularization method. For each direct problem considered, the optimal value of the regularization parameter was selected according to the GCV criterion. An efficient regularizing stopping criterion which ceases the iterative procedure at the point where the accumulation of noise becomes dominant and the errors in predicting the exact solutions increase, was also presented. The MFS-based iterative algorithm was tested for Cauchy problems in steady state anisotropic heat conduction in two-dimensional simply and doubly connected domains with smooth boundaries.

From the numerical results presented in this study, it can be concluded that the proposed method is consistent, accurate, convergent with respect to increasing the number of MFS boundary collocation points and stable with respect to decreasing the amount of noise added into the Cauchy data. One possible disadvantage of the MFS-based iterative algorithm is related to the optimal choice of the regularization parameter associated with the Tikhonov regularization method which requires, at each step of the alternating iterative algorithm of Kozlov, Maz′ya and Fomin (1991), additional iterations with respect to the regularization parameter. However, this inconvenience can be overcome by introducing relaxation procedures in the MFS iterative algorithm and this is currently being under investigation.

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