

# A Fictitious Time Integration Method for a Quasilinear Elliptic Boundary Value Problem, Defined in an Arbitrary Plane Domain

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**Abstract:** Motivated by the evolutionary and dissipative properties of parabolic type partial differential equation (PDE), Liu (2008a) has proposed a natural and mathematically equivalent approach by transforming the quasilinear elliptic PDE into a parabolic one. However, the above paper only considered a rectangular domain in the plane, and did not treat the difficulty arisen from the quasilinear PDE defined in an arbitrary plane domain. In this paper we propose a new technique of internal and boundary residuals in a fictitious rectangular domain, which are driving forces for the ordinary differential equations based on the Fictitious Time Integration Method (FTIM). Several numerical examples validate the performance of the FTIM, which can easily handle the nonlinear boundary value problem, defined in an arbitrary plane domain.

**Keywords:** Quasilinear elliptic equation, Fictitious Time Integration Method (FTIM), Arbitrary plane domain, Internal and boundary residuals, Fictitious rectangular domain

## 1 Introduction

The Dirichlet problem of elliptic type PDE has appeared in many engineering applications. For a given quasilinear PDE, when the domain is complex, the finding of analytical solution is not easy. If one encounters a nonlinear problem in an arbitrary domain, both the geometric complexity and the nonlinearity encroach, and typically one needs to resort on a numerical technique for solving this problem.

For the quasilinear elliptic boundary value problems (BVPs), Chen and Zhou (2000) have presented some iteration methods, which include the mountain iteration algorithm, the scaling iterative algorithm, the monotone iterative algorithm, as well as a

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direct iterative algorithm. In general, a sequence of iterations is generated by different methods, but they are not guaranteed to converge to the true solution. Generally speaking, a stronger condition of the differential equation is needed to guarantee the convergence.

On the other hand, there are many papers to concern with the numerical solutions of linear elliptic type BVPs, like as, Liu (2007a,2007b,2007c,2008b). Meshless and meshfree methods are nowadays the main stream in numerical computations, as strongly advocated by many researchers, to name a few, Zhu, Zhang and Atluri (1998, 1999), Atluri and Zhu (1998a, 1998b), Atluri, Kim and Cho (1999), Atluri and Shen (2002), Cho, Golberg, Muleshkov and Li (2004), Jin (2004), Li, Lu, Huang and Cheng (2007), Liu (2007d,2007e), Tsai, Lin, Young and Atluri (2006), and Young, Chen, Chen and Kao (2007). However, it cannot be overemphasized the effectiveness of those methods, when nonlinear problems are encountered.

Many collocation techniques together with the expansions by functional bases were employed to solve the elliptic type BVPs; see, for example, Cheng, Golberg, Kansas and Zammito (2003), Hu, Li and Cheng (2005), Algahtani (2006), Tian, Reutskiy and Chen (2008), Hu and Chen (2008), and Libre, Emdadi, Kansa, Rahimian and Shekarchi (2008). Recently, Li, Lu, Hu and Cheng (2008) gave a very detailed description of the collocation Trefftz method. Basically, the above bases expansion methods are effective for linear problems. For nonlinear problems some iterations of those methods are necessary.

The purpose of this paper is to develop a non-iterative algorithm, having the advantage of easy to numerical implementation, and having a great flexibility applied to the most elliptic type BVPs. We begin with the following quasilinear elliptic equation:

$$\Delta u(x,y) = F(x,y,u,u_x,u_y,\dots), \quad (x,y) \in \Omega, \quad (1)$$

$$u(x,y) = H(x,y), \quad (x,y) \in \Gamma, \quad (2)$$

where  $\Delta$  is the Laplacian operator,  $\Gamma$  is the boundary of the problem domain  $\Omega$ , and  $F$  and  $H$  are given functions. Here, the set of  $\Omega$  is assumed to be a star convex one with respect to the geometric center located at the origin; hence, the boundary  $\Gamma$  can be described in the polar coordinates by a radius function with  $\Gamma = \{(r,\theta)|r = \rho(\theta), \theta \in [0, 2\pi]\}$ . In Fig. 1(a) we give an example of the elliptic boundary. Under a given Dirichlet boundary condition (2), we have to solve Eq. (1) in order to find the solution of  $u(x,y)$ .

It may have different strategies by adding a time-like variable  $t$  in Eq. (1), and change it to an initial boundary value problem. Even there are many better time integration techniques to solve the resulting time-dependent equations by using a

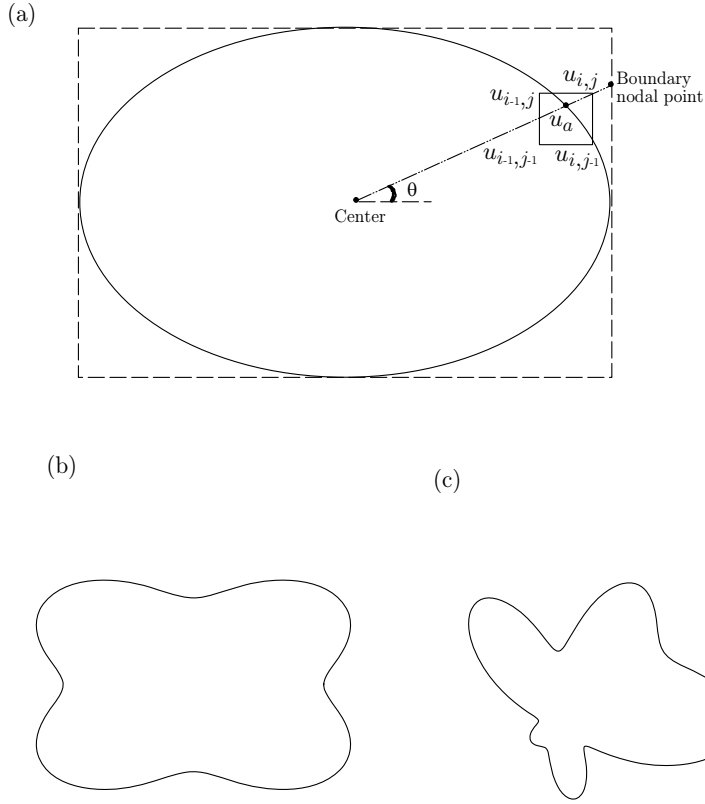


Figure 1: The problem domains used in the calculations: (a) an ellipse, and a fictitious rectangular boundary, (b) an epitrochoid, and (c) an amoeba-like domain.

semi-discretization technique, it has a big problem of the conventional approach by viewing it as a steady-state solution, because such an equation is much deviating from the original equation (1). In order to approach the steady state we should calculate this equation to a long time, such that it is very time consumption, and on the other hand, we do not know whether this equation has a steady state or not. Indeed, Sincovec and Madsen (1975) have described a general-purpose package for solving PDEs, where it is simply adding a differential term  $\partial u / \partial t$ , with respect to an artificial time  $t$ , on the right-hand side, by embedding a nonlinear elliptic PDE into a time-dependent one. Obviously, such a simple fashion makes no improvement of the numerical solution, because we can construct examples where this embedding technique inspires a very long and expensive time integration to achieve a descent approximation for the steady-state solution [Liu (2008a)].

In order to overcome these difficulties, Liu (2008a) was the first, who treated Eq. (1) by introducing a fictitious time coordinate, and mathematically equivalently transformed Eq. (1) into a parabolic type PDE, without destroying the structure of the original PDE. However, Liu (2008a) only considered a rectangular domain in the plane, and did not deal with the difficult problem of quasilinear equation defined in an arbitrary plane domain. In this paper we propose a new technique to extend the novel numerical solution to an arbitrary plane domain.

## 2 A fictitious time integration method

### 2.1 Transformation into a parabolic type PDE

First, we propose the following transformation:

$$v(x, y, t) := q(t)u(x, y), \quad (3)$$

where  $q(t)$  is a differentiable function of  $t$ , and  $q(0) = 1$ . Introducing a nonzero viscosity damping coefficient  $v$  in Eq. (1), we can obtain

$$0 = v\Delta u - vF(x, y, u, u_x, u_y, \dots). \quad (4)$$

By using Eq. (3) we have

$$0 = \frac{v}{q(t)}\Delta v - vF(x, y, u, u_x, u_y, \dots). \quad (5)$$

From Eq. (3) it follows that  $\partial v / \partial t = \dot{q}(t)u(x, y)$ , and by adding it on both the sides of the above equation we can obtain

$$\frac{\partial v}{\partial t} = \frac{v}{q(t)}\Delta v - vF(x, y, u, u_x, u_y, \dots) + \dot{q}(t)u. \quad (6)$$

Then, by using  $u = v/q(t)$ ,  $u_x = v_x/q(t)$  and  $u_y = v_y/q(t)$ , etc., we can recast Eqs. (1) and (2) into a parabolic type PDE:

$$\frac{\partial v}{\partial t} = \frac{v}{q(t)}\Delta v - vF\left(x, y, \frac{v}{q(t)}, \frac{v_x}{q(t)}, \frac{v_y}{q(t)}, \dots\right) + \frac{\dot{q}(t)v}{q(t)}, \quad (x, y) \in \Omega, \quad (7)$$

$$v(x, y, t) = q(t)H(x, y), \quad (x, y) \in \Gamma. \quad (8)$$

The above process was first developed by Liu (2008a) with  $q(t) = 1 + t$  to transform the original elliptic type Eqs. (1) and (2) into the parabolic type Eqs. (7) and (8). When  $v(x, y, t)$  is numerically solved to a certain time  $t_0$  according to a given

convergence criterion, the solution of  $u(x, y)$  is given by  $u(x, y) = v(x, y, t_0)/q(t_0)$  according to Eq. (3).

After multiplying  $1/q(t)$  on both sides of Eq. (7) we can obtain

$$\frac{\partial}{\partial t} \left[ \frac{v}{q(t)} \right] = \frac{v}{q(t)} \Delta \left[ \frac{v}{q(t)} \right] - \frac{v}{q(t)} F \left( x, y, \frac{v}{q(t)}, \frac{v_x}{q(t)}, \frac{v_y}{q(t)}, \dots \right). \quad (9)$$

Here we will use Eq. (3) again to derive a parabolic type PDE for  $u$ , but at this moment we must understand that  $u$  is not only a function of  $(x, y)$  but also a function of  $t$ . Therefore, from Eqs. (9) and (8) it is legal to write

$$\frac{\partial u}{\partial t} = \frac{v}{q(t)} \Delta u - \frac{v}{q(t)} F(x, y, u, u_x, u_y, \dots), \quad (x, y) \in \Omega, \quad (10)$$

$$u(x, y, t) = H(x, y), \quad (x, y) \in \Gamma. \quad (11)$$

The present approach is slightly better than that developed by Liu (2008a), because we can directly solve  $u$  numerically, and do not need through  $v$ .

The above idea was first proposed by Liu (2008c) to treat an inverse Sturm-Liouville problem by transforming an ordinary differential equation (ODE) into a PDE. Then, Liu and his coworkers [Liu (2008d, 2008e); Liu, Chang, Chang and Chen (2008)] extended this idea to develop new methods for estimating parameters in the inverse vibration problems. Recently, Liu and Atluri (2008a) have employed the above technique of fictitious time integration method (FTIM) to solve large system of nonlinear algebraic equations, and showed that high performance can be achieved by using the FTIM. Liu and Atluri (2008b) also employed this technique of FTIM to solve mixed-complementarity problems and optimization problems. Then, Liu and Atluri (2008c) employed the FTIM to solve the inverse Sturm-Liouville problems under specified eigenvalues. Furthermore, Liu (2008f) has used the FTIM technique to solve the nonlinear complementarity problems, whose numerical results are very well.

Liu (2009a) employed the fictitious time integration method to solve  $m$ -point boundary value problems of ODEs, and a higher-dimensional first-order ODEs obtained from the Euler-Bernoulli beam, subjecting to three-point boundary values. Because in the FTIM we do not require to inverse the resulting nonlinear algebraic equations, this method is very effective to find the numerical solutions of  $m$ -point boundary value problems of ODEs. Liu (2009b) also employed the fictitious time integration method to solve the backward in time and forward in time Burgers equation. Because the FTIM is integrated in a new direction of fictitious time, which is independent to the real time, the ill-posedness and noised disturbance for the backward in time Burgers equation can be handled rather well. This method is very effective

to find the numerical solutions of backward in time problems in PDEs. Liu (2009c) has found that the delay ordinary differential equations are easily calculated by using the FTIM.

There are many different choices of the time-like function  $q(t)$ . Recently, Liu and Atluri (2009) used the FTIM-based method to study the filtering effect of the FTIM by using the different time-like functions  $q(t)$  to solve the ill-posed linear systems. They showed that when  $q(t) = 1/(1+t)^\gamma$ ,  $0 < \gamma \leq 1$  is used, the filtering effect of the FTIM is better than that of the Tikhonov filter. Similarly, Ku, Yeih, Liu and Chi (2009) have advocated by using a new time-like function  $q(t) = 1/(1+t)^\gamma$ ,  $0 < \gamma \leq 1$  to enhance the convergence of the numerical solutions of nonlinear algebraic equations. In this paper we will use  $q(t) = 1/(1+t)^\gamma$  and take different  $\gamma$  in the numerical calculations. Moreover, Atluri, Liu and Kuo (2009) can extend the FTIM to a combination with the continuous Newton method, which is convergent very fast in the solutions of nonlinear algebraic equations.

## 2.2 Transformation into the ODEs

In order to facilitate the numerical implementation of Eqs. (10) and (11) by an equidistant finite difference scheme, we introduce a minimal rectangle to enclose the problem domain  $\Omega$ . See Fig. 1(a) for an example of the elliptic domain. Suppose that the rectangle is given by  $\bar{\Omega} := [-a_0, a_0] \times [-b_0, b_0]$ , where  $a_0$  and  $b_0$  are the maximum length of the domain  $\Omega$  in the  $x$ -axis, and in the  $y$ -axis, respectively. Now, we divide the rectangle  $\bar{\Omega}$  by a uniform grid with  $\Delta x = 2a_0/(m-1)$  and  $\Delta y = 2b_0/(m-1)$  being the uniform spatial grid lengths in the  $x$ - and  $y$ -direction, and let  $u_{i,j}(t) := u(x_i, y_j, t)$  be a numerical value of  $u$  at the grid point  $(x_i, y_j) \in \bar{\Omega}$ , varying with time  $t$ , where  $x_i = (i-1)\Delta x$  and  $y_j = (j-1)\Delta y$ .

Applying a semi-discrete procedure on the PDE in Eq. (10), which being slightly extended to  $\bar{\Omega}$ , yields a coupled system of ODEs:

$$\dot{u}_{i,j} = \frac{v_1}{q(t)} R_{i,j} \quad (12)$$

where

$$R_{i,j} = \frac{1}{(\Delta x)^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] + \frac{1}{(\Delta y)^2} [u_{i,j+1} - 2u_{i,j} + u_{i,j-1}] - F \left( x_i, y_j, u_{i,j}, \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}, \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}, \dots \right), \quad i, j = 2, \dots, m-1, \quad (13)$$

are internal residuals.

Because Eq. (12) is an equidistant finite difference equation, it is hard to match the boundary condition (2) exactly. Usually, one uses an adaptive grid method to let the circumferential grid points locate on the boundary. However, this may complicate the numerical computation for arbitrary plane domain. Because the boundary condition is given on the contour  $\Gamma$ , not on the boundary of the rectangle  $\bar{\Omega}$ , we require to derive the governing equations of these  $u_{i,j}$  at the nodal points on the rectangular boundary as shown in Fig. 1(a), that is,  $u_{m,j}$ ,  $j = 1, \dots, m$ ,  $u_{i,m}$ ,  $i = 1, \dots, m-1$ ,  $u_{1,j}$ ,  $j = 1, \dots, m-1$ , and  $u_{i,1}$ ,  $i = 2, \dots, m-1$ .

For a typical boundary nodal point  $(x_i, y_j)$  as shown in Fig. 1(a) by a bullet, we can find its angle  $\theta$  with respect to the  $x$ -axis, and its corresponding point on the contour  $\Gamma$  by using  $(x_a, y_a) = (\rho(\theta) \cos \theta, \rho(\theta) \sin \theta)$ , which is marked by a bullet in the top of  $u_a$ . Usually, the point  $(x_a, y_a)$  does not just locate on a grid point, and it has an offset  $(\delta x, \delta y)$  with respect to the grid point  $(x_{i-1}, y_{j-1})$ . Denote the numerical values of  $u_{i,j}$  at the four neighboured grid points by  $u_{i-1,j-1}$ ,  $u_{i,j-1}$ ,  $u_{i,j}$  and  $u_{i-1,j}$ . Then the numerical value of  $u$  at the point  $(x_a, y_a)$ , denoted by  $u_a$ , can be approximated by the following linear interpolation equation:

$$u_a = u_{i-1,j-1} + \frac{\delta y}{\Delta y} [u_{i-1,j} - u_{i-1,j-1}] + \frac{\delta x}{\Delta x} \left\{ u_{i,j-1} + \frac{\delta y}{\Delta y} [u_{i,j} - u_{i,j-1}] - \left( u_{i-1,j-1} + \frac{\delta y}{\Delta y} [u_{i-1,j} - u_{i-1,j-1}] \right) \right\}. \quad (14)$$

However, the exact value at  $(x_a, y_a)$  is given by Eq. (2) with  $u_e = H(x_a, y_a)$ . Thus we have a difference given by

$$R_{i,j} = u_a - u_e, \quad (15)$$

where we denote this residual by  $R_{i,j}$ , since the corresponding boundary nodal point is  $(x_i, y_j)$ . Corresponding to Eq. (13), the residual defined in Eq. (15) is a boundary residual on a fictitious boundary of  $\bar{\Omega}$ .

Thus, on the rectangular boundary nodal points we can derive the FTIM-based governing equations as follows:

$$\begin{aligned} \dot{u}_{m,j} &= \frac{v_2}{q(t)} R_{m,j}, \quad j = 1, \dots, m, \\ \dot{u}_{i,m} &= \frac{v_2}{q(t)} R_{i,m}, \quad i = 1, \dots, m-1, \\ \dot{u}_{1,j} &= \frac{v_2}{q(t)} R_{1,j}, \quad j = 1, \dots, m-1, \\ \dot{u}_{i,1} &= \frac{v_2}{q(t)} R_{i,1}, \quad i = 2, \dots, m-1. \end{aligned} \quad (16)$$

In the above the coefficient is  $v_2$ , which is different from the damping coefficient  $v_1$  used in Eq. (12). These two sets of ODEs play different roles in the solution of  $u$ , and the weightings are also needed to be different. The damping coefficient  $v_1$  is always positive; conversely,  $v_2$  may be positive or negative.

In this section we have transformed the boundary value problem of the second-order elliptic PDE in Eqs. (1) and (2) in an arbitrary plane domain, to an evolutionary problem of a parabolic PDE in Eqs. (10) and (11), and finally arrived to an initial value problem in the  $n$ -dimensional ODEs system (12) and (16) with dimensions  $n = m^2$ , of which the driving force is the residual. The initial values of Eqs. (12) and (16) are given through a guess because the true initial condition of  $u(x, y, t)$  is not known a priori.

In the previous paper by Liu (2008a), only the rectangular plane domains were considered, and the difficult problem of quasilinear equation defined in an arbitrary plane domain was not treated. Here, by using the interpolation equation (14) and the ODEs in Eq. (16), which are forced by the residuals defined in Eq. (15), we can extend the novel FTIM-based numerical solution to an arbitrary plane domain.

### 2.3 Numerical procedures

Starting from an initial value of  $u_{i,j}$ , which can be guessed in a rather free way, we may employ a simple Euler method to integrate Eqs. (12) and (16) from  $t = 0$  to a selected final time  $t_f$ . In the numerical integration process we can check the convergence of  $u_{i,j}$  at the  $\ell$ - and  $\ell + 1$ -steps by

$$\sqrt{\sum_{i,j=1}^m [u_{i,j}^{\ell+1} - u_{i,j}^{\ell}]^2} \leq \varepsilon, \quad (17)$$

where  $\varepsilon$  is a selected convergence criterion. If at a time  $t_0 \leq t_f$  the above criterion is satisfied, then the solution of  $u$  is obtained. In practice, if a suitable  $t_f$  is selected we find that the numerical solution also approaches very well to the true solution, even the above convergence criterion is not satisfied. The viscosity coefficients  $v_1$  and  $v_2$  introduced in Eqs. (12) and (16) can be used to enhance the stability of numerical integration.

In particular, we should emphasize that the present method is a new fictitious time integration method (FTIM). Because it does not need to face the nonlinearity in the spatial domain, this new FTIM can calculate the boundary value problem of quasilinear elliptic equation very stably and effectively without needing of any iteration. Below we give numerical examples to display some advantages of the present FTIM.



### 3 Numerical examples

In this section we will apply the FTIM to the semilinear and quasilinear BVPs. In order to focus on the observation of the effect of the complex domain on the new method, we suppose that some exact solutions are known and can be compared with the numerical solutions, and the domain, in addition to the ellipse plotted in Fig. 1(a), we also consider the other two domains in Figs. 1(b) and 1(c). The contours of these domains are given, respectively, by

$$\rho(\theta) = \frac{ab}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}, \quad (18)$$

$$\rho(\theta) = \sqrt{(a+b)^2 + 1 - 2(a+b) \cos(a\theta/b)}, \quad (19)$$

$$\rho = \exp(\sin \theta) \sin^2(2\theta) + \exp(\cos \theta) \cos^2(2\theta). \quad (20)$$

For Eq. (18) we use  $a = 1.5$  and  $b = 1$ , while for Eq. (19) we use  $a = 4$  and  $b = 1$ .

#### 3.1 Example 1

We consider an analytical solution

$$u(x, y) = x^3 + 2xy \quad (21)$$

of a nonlinear Poisson equation:

$$\Delta u = u^2 + 6x - x^6 - 4x^4y - 4x^2y^2. \quad (22)$$

For simplicity we apply the Euler method to integrate the resulting ODEs from the FTIM, where the time stepsize is fixed to be  $\Delta t = 0.001$ . By using the following parameters:  $m = 41$ ,  $v_1 = 0.8$ ,  $v_2 = -5$ ,  $\varepsilon = 10^{-4}$  and  $\gamma = 0.5$  we calculate this problem inside an elliptic domain with semiaxes lengths  $a = 1.5$  and  $b = 1$ . Through 1891 steps the numerical solution is obtained, whose absolute error is plotted in Fig. 2 inside the elliptic contour, and its maximum error is  $8.2 \times 10^{-3}$ .

Similarly, we use the following parameters:  $m = 101$ ,  $v_1 = 0.01$ ,  $v_2 = -5$ ,  $\varepsilon = 10^{-4}$  and  $\gamma = 1$  to calculate this problem inside an epitrochoid described by Eq. (19). Through a few steps the numerical solution is obtained, whose absolute error is plotted in Fig. 3 inside the contour, and its maximum error is smaller than 0.018. We also use the following parameters:  $m = 61$ ,  $v_1 = 0.1$ ,  $v_2 = -0.5$ ,  $\varepsilon = 10^{-4}$  and  $\gamma = 1$  to calculate this problem inside an amoeba-like irregular shape described by Eq. (20). Through a few steps the numerical solution is obtained, whose absolute error is plotted in Fig. 4 inside the contour, and its maximum error is smaller than 0.025.

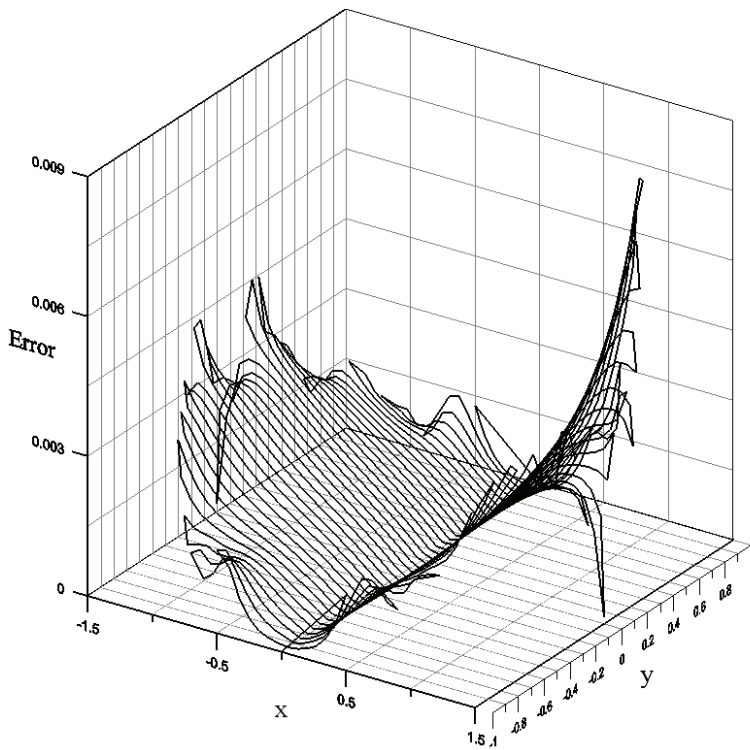


Figure 2: The error distribution of Example 1 over an ellipse.

### 3.2 Example 2

The following nonlinear diffusion reaction equation is considered:

$$\Delta u = 4u^3(x^2 + y^2 + a^2). \quad (23)$$

The domains are the same as those used in Example 1. The analytic solution

$$u(x,y) = \frac{-1}{x^2 + y^2 - a^2} \quad (24)$$

is singular on the circle with a radius  $a$ .

Algahtani (2005) has solved this problem by using a radial basis method, whose results as shown there in Fig. 8 are not matched well to the exact solution.

By using the following parameters:  $m = 41$ ,  $v_1 = 0.8$ ,  $v_2 = -5$ ,  $\varepsilon = 10^{-4}$  and  $\gamma = 0.1$  we calculate this problem inside an elliptic domain with semiaxes lengths

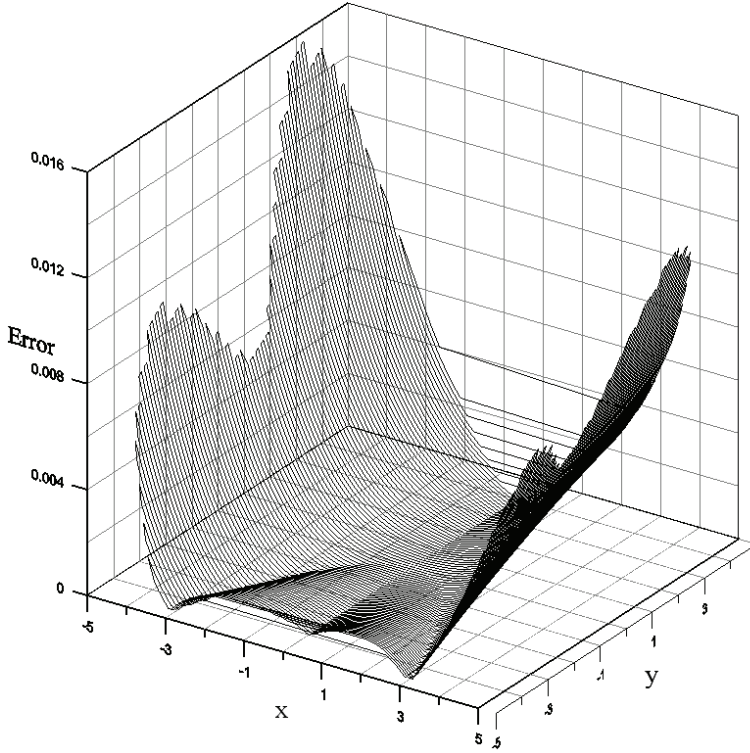


Figure 3: The error distribution of Example 1 over an epitrochoid.

1.5 and 1. Through 953 steps the numerical solution is obtained, whose absolute error is plotted in Fig. 5 inside the contour, and its maximum error is  $2.46 \times 10^{-3}$ . It can be seen that even the singularity is near to the problem domain, the numerical solution is also good with the third-order accuracy.

Similarly, we use the following parameters:  $m = 101$ ,  $v_1 = 0.01$ ,  $v_2 = -5$ ,  $\varepsilon = 10^{-4}$  and  $\gamma = 1$  to calculate this problem inside an epitrochoid described by Eq. (19). Through a few steps the numerical solution is obtained, whose absolute error is plotted in Fig. 6 inside the contour, and its maximum error is smaller than 0.00036. We also use the following parameters:  $m = 61$ ,  $v_1 = 0.1$ ,  $v_2 = -0.5$ ,  $\varepsilon = 10^{-4}$  and  $\gamma = 1$  to calculate this problem inside an amoeba-like irregular shape described by Eq. (20). Through a few steps the numerical solution is obtained, whose absolute error is plotted in Fig. 7 inside the contour, and its maximum error is smaller than 0.00012.

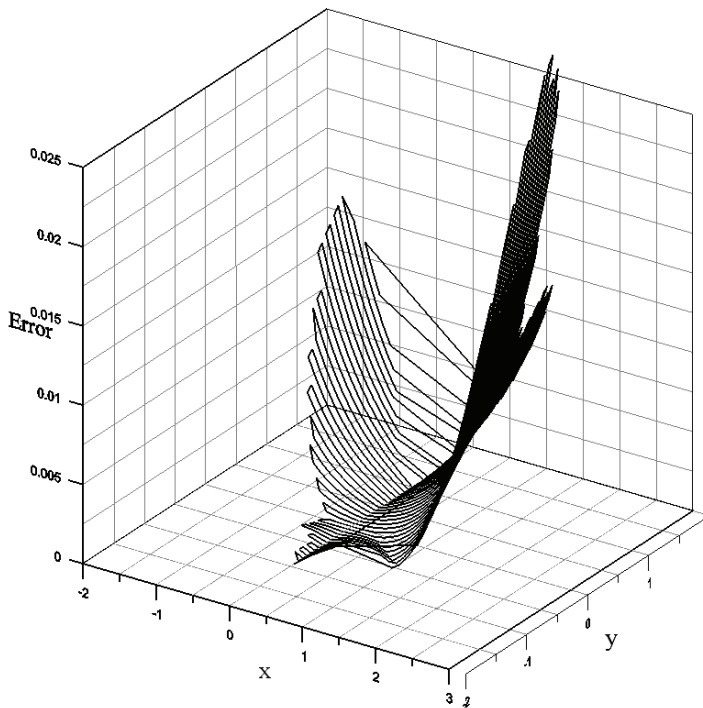


Figure 4: The error distribution of Example 1 over an amoeba-like domain.

### 3.3 Example 3

The following quasilinear equation is calculated:

$$\Delta u + u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy} = 0, \quad (25)$$

which is known as a minimal surface equation. The domain  $\Omega$  is an ellipse with  $a = 1$  and  $b = 0.8$ , and the boundary value is zero.

Under the following parameters of  $\Delta x = 2/50$ ,  $\Delta y = 1.6/50$ ,  $\Delta t = 0.0003$ ,  $u_{i,j}(0) = 0.15$ ,  $v_1 = 0.03$ ,  $v_2 = -9$ ,  $\varepsilon = 10^{-4}$  and  $\gamma = 0.1$  we find that the numerical solution is convergent at the 2000 time steps. We plot a minimal surface above the ellipse in Fig. 8.

## 4 Conclusions

The present paper has extended a previous result obtained by the author for the FTIM-based method of the quasilinear elliptic PDE in a rectangular domain to the

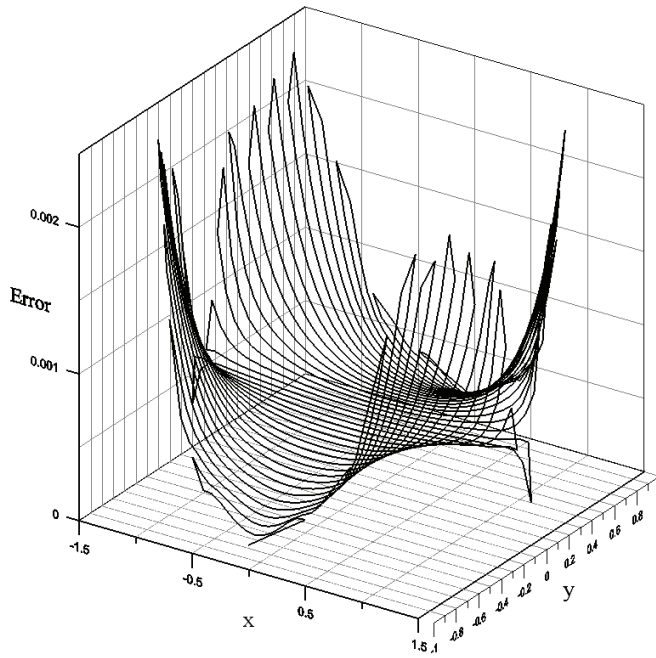


Figure 5: The error distribution of Example 2 over an ellipse.

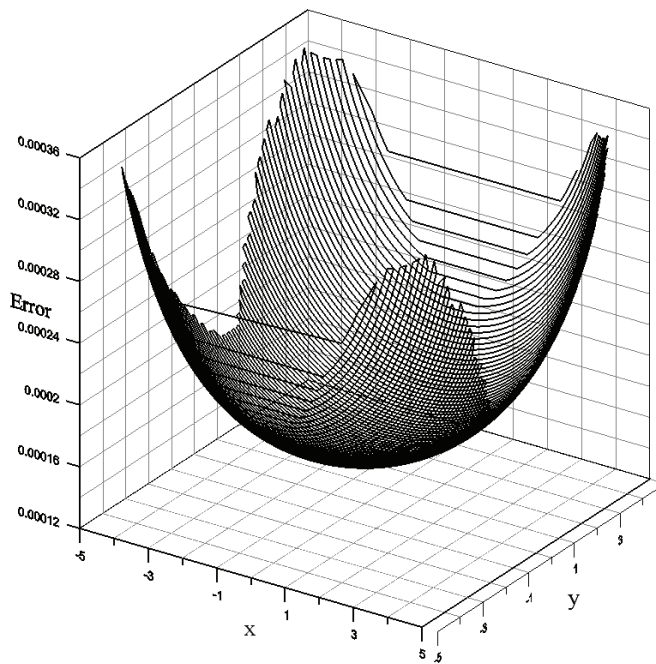


Figure 6: The error distribution of Example 2 over an epitrochoid.

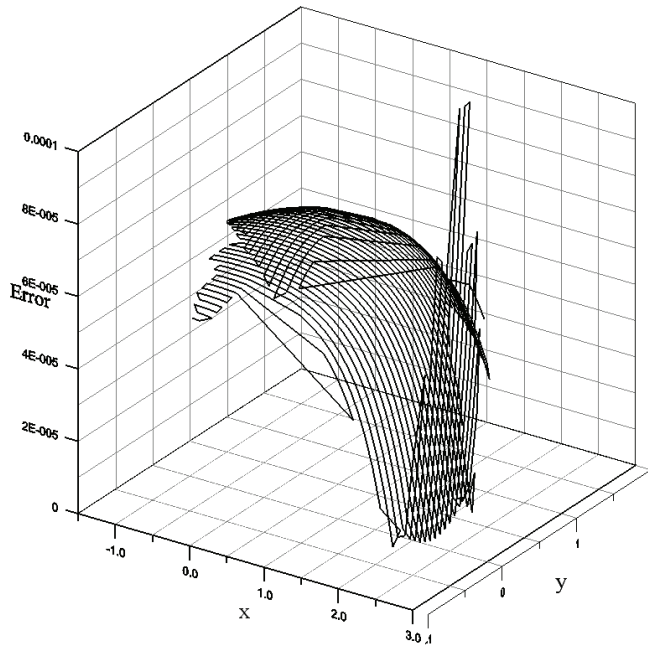


Figure 7: The error distribution of Example 2 over an amoeba-like domain.

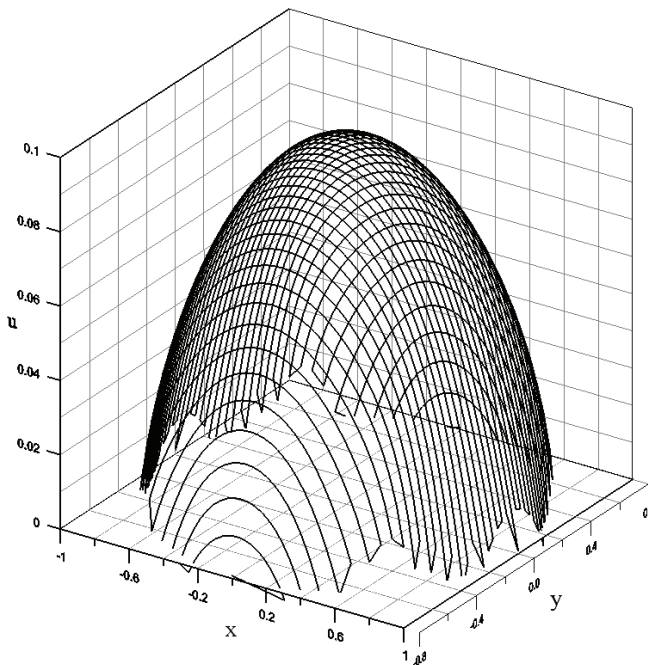


Figure 8: The minimal surface over an ellipse, calculated by the FTIM.

one in an arbitrary plane domain. In the past, the numerical methods to solve the elliptic boundary value problems are frustrated by nonlinearity and geometric complexity, and then require some iterations, because these methods were carried out in a spatial domain. We have introduced the concepts of internal residual and boundary residual in a fictitious rectangular domain, which just encloses the problem domain, such that the geometric complexity can be resolved. In the present paper, the nonlinearity and geometric complexity of quasilinear elliptic equation defined in an arbitrary plane domain can be detoured by adding a fictitious time coordinate, since we only required numerically integrating the ODEs, driving by the residuals, to a certain time to obtain numerical solutions. Hence, the present FTIM can work very effectively and accurately for the solution of boundary value problem of quasilinear elliptic equation defined in an arbitrary plane domain. No iteration is required, and thus the present method is very time saving.

**Acknowledgement:** Taiwan's National Science Council project NSC-97-2221-E-002-264-MY3 granted to the author is highly appreciated.

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