# A Displacement Solution to Transverse Shear Loading of Composite Beams by BEM 

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#### Abstract

In this paper the boundary element method is employed to develop a displacement solution for the general transverse shear loading problem of composite beams of arbitrary constant cross section. The composite beam (thin or thick walled) consists of materials in contact, each of which can surround a finite number of inclusions. The materials have different elasticity and shear moduli and are firmly bonded together. The analysis of the beam is accomplished with respect to a coordinate system that has its origin at the centroid of the cross section, while its axes are not necessarily the principal bending ones. The transverse shear loading is applied at the shear center of the cross section, avoiding in this way the induction of a twisting moment. The evaluation of the transverse shear stresses at any interior point is accomplished by direct differentiation of a warping function. The shear deformation coefficients are obtained from the solution of two boundary value problems with respect to warping functions appropriately arising from the aforementioned one using only boundary integration, while the coordinates of the shear center are obtained from these functions using again only boundary integration. Three boundary value problems are formulated with respect to corresponding warping functions and solved employing a pure BEM approach. Numerical examples are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method. The accuracy of the obtained values of the resultant transverse shear stresses compared with those obtained from an exact solution is remarkable.


Keywords: Warping function; Transverse shear stresses; Shear center; Shear deformation coefficients; Principal shear axes; Composite; Beam; Boundary element method

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## 1 Introduction

One of the problems often encountered in engineering practice is the analysis of rectilinear members of composite structures subjected to transverse shear loads. Also, in recent years composite structural elements consisting of a relatively weak matrix reinforced by stronger inclusions or of different materials in contact are of increasing technological importance in engineering. Composite structures can produce very elegant solutions to complex structural engineering challenges. The use of composite beams is now a real trend in many engineering applications. Steel beams or columns totally encased in concrete are most common examples, while construction using steel beams as stiffeners of concrete plates is a quick, familiar and economical method for long bridge decks or for long span slabs. Moreover, composite beams or columns offer many significant advantages, such as high load capacity with small cross-section and economic material use, simple connection to other members as for steel construction, good fire resistance, excellent earthquake resistance, high resistance to compressive stresses reducing the risk of local buckling of the steel section and advantages in fabrication. The extensive use of the aforementioned structural elements necessitates a rigorous analysis.
The problem of a homogeneous prismatic beam subjected in shear torsionless loading has been widely studied from both the analytical and numerical point of view. Theoretical discussions concerning flexural shear stresses (Weber, 1924; Trefftz, 1935; Goodier, 1944), or the problem of the center of shear (Goodier, 1944; Osgood, 1943; Weinstein, 1947; Reissner and Tsai, 1972; Andreaus and Ruta, 1998) and text books giving detailed representations of these topics (Timoshenko and Goodier, 1984; Sokolnikoff, 1956; Love, 1952) are mentioned among the extended analytical studies.
Numerical methods have also been used for the analysis of the aforementioned problem. Among these methods the majority of researchers have employed the finite element method (FEM) based on assumptions for the displacement field (Mason and Herrmann, 1968; Koczyk, 1994; Gruttmann, Wagner und Sauer, 1998; Kraus, 2005) or introducing a stress function that fulfils the equilibrium equations for the evaluation of the shear stresses (Gruttmann, Sauer and Wagner, 1999) and the shear deformation coefficients (Gruttmann and Wagner, 2001). However, the FEM despite the generality of its application in engineering problems, is not free of drawbacks. The FEM require the whole cross section to be discretized into area (triangular or quadrilateral) elements. Hence, generation and inspection of the finite element mesh exhibit difficulty and are both laborious and time consuming, especially when the geometry of the cross section is not simple. For example, when there are holes, notches or corners, mesh refinement and high element density is required at these critical regions. Although the FEM computes accurately the filed
function, which is the unknown of the problem, it is ineffective in determining its derivatives, especially in areas of large gradients.
The boundary element method (BEM) (Amaziane et al., 2005; Botta and Venturini, 2005; Divo and Kassab, 2005; Dziatkiewicz and Fedelinski, 2007; Hatzigeorgiou and Beskos, 2002; Katsikadelis, 2002; Lie, et al., 2001; Liu, 2007; Mandolini, et al., 2001; Mansur, et al., 2004; Miers and Telles, 2004; Ochiai, 2001; Providakis, 2000; Rashed, 2004; Sanz, et al., 2006; Sapountzakis and Tsiatas, 2007; Shiah and Tan, 2000; Shiah, et al., 2005; Sun et al., 2004; Tan, et al., 2009; Tsai, et al., 2002; Wang et al., 2006; Wang and Yao, 2008; Zhang and Savaidis, 2003; Zhou, et al., 2006), on the other hand, seems to be an alternative powerful tool for the solution of the aforementioned problem. BEM solutions require only boundary discretization resulting in line or parabolic elements instead of area elements of the FEM solutions, making the numerical modeling easy and reducing the number of unknowns by one order. The method is particularly effective in computing the derivatives of the field function (e.g. stresses). The BEM allows evaluation of the solution and its derivatives at any point of the cross section, using the integral representation of the solution as a continuous mathematical expression. This is impossible with the FEM, since the solution is obtained only at the nodal points. The boundary element procedure was first employed by Sauer (1980) for the calculation of shear stresses based on Weber (1924) analysis. BEM was also used for the calculation of the shear center location in an arbitrary homogeneous cross section by Chou (1993) and for the presentation of a solution to the general flexure problem in an isotropic only simply connected arbitrary cross section beam by Friedman and Kosmatka (2000). More recently, the boundary element method has been applied to homogeneous orthotropic beams (Gaspari and Aristodemo 2005). In this research effort the analysis is accomplished with respect only to the principal bending system of axes of the cross section restricting in this way its generality. Moreover, Sapountzakis and Mokos (2005) presented a stress function solution employing the BEM for the general transverse shear loading problem of homogeneous beams of arbitrary constant cross section.
Contrary to these many efforts, to the authors' knowledge very little work has been done on the corresponding problem of composite beams of arbitrary constant cross section. In the pioneer work of Muskhelishvili (1963) the governing equations of the problem are formulated and an analytic solution of a composite cross section of simple geometry is presented. Nouri and Gay (1994) presented a numerical solution for the shear problem of composite beams of simply connected materials in contact, of arbitrary cross section, employing the 2-D FEM and taking into account the boundary conditions at the interfaces. In this reference the shear problem is formulated with respect to the principal bending system of axes, which as it is stated
below is different from the principal shear axes one, while the evaluation of the non-diagonal shear deformation coefficient is missing. Moreover, Fatmi and Zenzri (2004) based on the "exact" elastic beam theory presented a numerical solution of the shear problem of composite beams of arbitrary cross section employing the 3-D FEM. The last two references take into account the boundary conditions at the interfaces in contrast with all other research efforts in composite beams of arbitrary cross section that ignore them (Pilkey 2002), resulting in an analysis that is not completely rigorous. In the case of composite beams of thin-walled or laminated cross sections the aforementioned problem can also be solved using the "refined models" (Reddy, 1989; Touratier, 1992; Wagner and Gruttmann, 2002; Karama, Afaq and Mistou, 2003). However, these models do not satisfy the continuity conditions of transverse shear stress at layer interfaces and assume that the transverse shear stress along the thickness coordinate remains constant, leading to the fact that kinematic or static assumptions cannot be always be valid. It is also worth here noting that most of the commercial finite element packages can only handle the shear problem of homogeneous beams (MSC/NASTRAN 1999), while the corresponding ones handling composite beams usually ignore the boundary conditions at the interfaces (SectionBuilder 2002), with very few exceptions of scientific programs (Debard/RDM5. 01 1997). Furthermore, Mokos and Sapountzakis (2005) presented a stress function solution employing the BEM for the general transverse shear loading problem of composite beams of arbitrary constant cross section. Finally, the BEM has not yet been used for the solution of the aforementioned problem for composite beams based on assumptions for the displacement field.
In this paper the boundary element method is employed to develop a displacement solution for the general transverse shear loading problem of composite beams of arbitrary constant cross section. The composite beam (thin or thick walled) consists of materials in contact, each of which can surround a finite number of inclusions. The materials have different elasticity and shear moduli and are firmly bonded together. The analysis of the beam is accomplished with respect to a coordinate system that has its origin at the centroid of the cross section, while its axes are not necessarily the principal bending ones. The formulation of the problem follows the displacement field adopted in the FEM solutions presented in Gruttmann, Wagner und Sauer (1998) and Wagner and Gruttmann (2002). The transverse shear loading is applied at the shear center of the cross section, avoiding in this way the induction of a twisting moment. The evaluation of the transverse shear stresses at any interior point is accomplished by direct differentiation of a warping function. The shear deformation coefficients are obtained from the solution of two boundary value problems with respect to warping functions appropriately arising from the aforementioned one using only boundary integration, while the coordinates of the
shear center are obtained from these functions using again only boundary integration. Three boundary value problems are formulated with respect to corresponding warping functions and solved employing a pure BEM approach. In very thin-walled cross sections, special care is taken during the numerical evaluation of the line integrals in order to avoid their "near singular integral behaviour". According to this, boundary elements that are very close to each other (distance smaller than their length) are divided in sub elements, in each of which Gauss integration is applied (Katsikadelis, 2002). The essential features and novel aspects of the present formulation are summarized as follows:
i. The proposed displacement solution constitutes the first step to the solution of the nonuniform shear problem avoiding the use of stress functions.
ii. All basic equations are formulated with respect to an arbitrary coordinate system, which is not restricted to the principal axes.
iii. The boundary conditions at the interfaces between different material regions have been considered.
iv. The shear deformation coefficients are evaluated using an energy approach instead of Timoshenko's (Timoshenko and Goodier, 1984) and Cowper's (Cowper, 1966) definitions, for which several authors (Schramm, Kitis, Kang and Pilkey, 1994; Schramm, Rubenchik and Pilkey, 1997) have pointed out that one obtains unsatisfactory results or definitions given by other researchers (Stephen, 1980; Hutchinson, 2001), for which these factors take negative values.
v. The proposed method can be efficiently applied to homogeneous and composite beams of thin walled cross section and to laminated composite beams, without the restrictions of the "refined models".
vi. The developed procedure retains the advantages of a BEM solution over a pure domain discretization method since it requires only boundary discretization.

Numerical examples are worked out to illustrate the efficiency, the accuracy and the range of applications of the developed method. The accuracy of the obtained values of the resultant transverse shear stresses compared with those obtained from an exact solution is remarkable.

## 2 Statement of the problem

Consider a prismatic beam of length $L$ with an arbitrarily shaped composite cross section consisting of materials in contact, each of which can surround a finite num-
ber of inclusions, with modulus of elasticity $E_{j}$, shear modulus $G_{j}$, occupying the regions $\Omega_{j}(j=1,2, \ldots, K)$ of the $y, z$ plane (Fig.1). The materials of these regions are firmly bonded together and are assumed homogeneous, isotropic and linearly elastic. Let also the boundaries of the nonintersecting regions $\Omega_{j}$ be denoted by $\Gamma_{j}$ ( $j=1,2, \ldots, K$.). These boundary curves are piecewise smooth, i.e. they may have a finite number of corners. Without loss of generality, it may be assumed that the beam end with centroid at point $C$ is fixed, while the $x$-axis of the coordinate system is the line joining the centroids of the cross sections. The system $y, z$ is not necessarily the principal bending one.
When the beam is subjected to torsionless bending arising from a concentrated load

(a)


Figure 1: Prismatic beam subjected to torsionless bending (a) and two dimensional region $\Omega$ occupied by the composite cross-section (b).
$Q$ having as $Q_{y}, Q_{z}$ its components along yand $z$ axes, respectively, applied at the shear center $S$ of its free end cross section, the displacement components in the $x, y$ and $z$ directions are approximated as
$u(x, y, z)=z \theta_{y}(x)-y \theta_{z}(x)+\widetilde{\varphi}_{C}(y, z)$
$v(x, y, z)=v(x)$
$w(x, y, z)=w(x)$
where $\theta_{y}(x), \theta_{z}(x)$ are the angles of rotation about the centroidal $y$ and $z$ axes, while $v(x), w(x)$ describe the deflections of the reference point in $y$ and $z$ directions, respectively. The customary beam kinematic with inextensibility in transverse directions and plane cross sections is extended by a main warping function $\widetilde{\varphi}_{C}(y, z)$ due to shear with respect to the centroid $C$ of the cross section (Fig.2). From the above definition, it follows that this function is a parameter of the cross section assuming it independent of its $x$ coordinate. However, in a more refined model the influence of this coordinate may also be considered taking into account restrained warping due to shear of the cross section (nonuniform shear problem). It is worth here noting that the angles of rotation, also referred to as angles of slope, are the angles between the $x$ axis and the tangents to the deflection curve. Furthermore, the derivatives of these angles can be written as
$\frac{\partial \theta_{y}}{\partial x}=\kappa_{y}$
$\frac{\partial \theta_{z}}{\partial x}=\kappa_{z}$
where for small rotations $\kappa_{y}$ and $\kappa_{z}$ are the curvatures of the transverse displacement curve (Pilkey, 2002).
Employing the linearized strain-displacement equations of the three-dimensional elasticity (Love, 1952), which form the Cauchy strain tensor, the following strain components are easily obtained
$\varepsilon_{x x}=z \kappa_{y}-y \kappa_{z}$
$\gamma_{x y}=\frac{\partial v}{\partial x}-\theta_{z}+\frac{\partial \widetilde{\varphi}_{C}(y, z)}{\partial y}$
$\gamma_{x z}=\frac{\partial w}{\partial x}+\theta_{y}+\frac{\partial \widetilde{\varphi}_{C}(y, z)}{\partial z}$
$\varepsilon_{y y}=\varepsilon_{z z}=\gamma_{y z}=0$


Figure 2: Warping due to shear of a rectangular and a hollow square cross section.
while the resulting from three-dimensional elasticity stress components in the regions $\Omega_{j}(j=0,1,2, \ldots, K)$ are given as

$$
\begin{equation*}
\left(\sigma_{x x}\right)_{j}=E_{j}\left(z \kappa_{y}-y \kappa_{z}\right) \tag{4a}
\end{equation*}
$$

$\left(\tau_{x y}\right)_{j}=G_{j}\left(\frac{\partial v}{\partial x}-\theta_{z}+\frac{\partial \widetilde{\varphi}_{C}(y, z)}{\partial y}\right)$
$\left(\tau_{x z}\right)_{j}=G_{j}\left(\frac{\partial w}{\partial x}+\theta_{y}+\frac{\partial \widetilde{\varphi}_{C}(y, z)}{\partial z}\right)$
$\left(\sigma_{y y}\right)_{j}=\left(\sigma_{z z}\right)_{j}=\left(\tau_{y z}\right)_{j}=0$
It is worth noting that in the above relations the materials' Poisson ration is set equal to zero (engineering beam theory), since the kinematic assumption of eqns. (1) leads to inextensibility in the transverse directions.

Introducing the unit warping function $\varphi_{C}(y, z)$ due to shear as

$$
\begin{equation*}
\varphi_{C}(y, z)=\left(\frac{\partial v}{\partial x}-\theta_{z}\right) y+\left(\frac{\partial w}{\partial x}+\theta_{y}\right) z+\widetilde{\varphi}_{C}(y, z) \tag{5}
\end{equation*}
$$

Its derivatives with respect to $y$ and $z$ axes are given as

$$
\begin{align*}
& \frac{\partial \varphi_{C}(y, z)}{\partial y}=\frac{\partial v}{\partial x}-\theta_{z}+\frac{\partial \widetilde{\varphi}_{C}(y, z)}{\partial y}  \tag{6a}\\
& \frac{\partial \varphi_{C}(y, z)}{\partial z}=\frac{\partial w}{\partial x}+\theta_{y}+\frac{\partial \widetilde{\varphi}_{C}(y, z)}{\partial z} \tag{6b}
\end{align*}
$$

and the shear stress components $\left(\tau_{x y}\right)_{j},\left(\tau_{x z}\right)_{j}$ of eqns. (4b,c) and the resultant shear stress $\left(\tau_{\Omega}\right)_{j}$ in the regions $\Omega_{j}(j=1,2, \ldots, K)$ can be written as

$$
\begin{equation*}
\left(\tau_{x y}\right)_{j}=G_{j} \frac{\partial \varphi_{C}(y, z)}{\partial y} \tag{7a}
\end{equation*}
$$

$\left(\tau_{x z}\right)_{j}=G_{j} \frac{\partial \varphi_{C}(y, z)}{\partial z}$
$\left(\tau_{\Omega}\right)_{j}=\left[\left(\tau_{x y}\right)_{j}^{2}+\left(\tau_{x z}\right)_{j}^{2}\right]^{1 / 2}$
Applying the shear stress components (7a,b) in the first equation of equilibrium of the three-dimensional elasticity neglecting the body forces
$\frac{\partial\left(\sigma_{x}\right)_{j}}{\partial x}+\frac{\partial\left(\tau_{x y}\right)_{j}}{\partial y}+\frac{\partial\left(\tau_{x z}\right)_{j}}{\partial z}=0$
we obtain the following relation
$G_{j}\left(\frac{\partial^{2} \varphi_{C}}{\partial y^{2}}+\frac{\partial^{2} \varphi_{C}}{\partial z^{2}}\right)=-\frac{\partial\left(\sigma_{x}\right)_{j}}{\partial x}$
while the last two elasticity equations of equilibrium are identically satisfied. Differentiating eqn.(4a) with respect to $x$ the derivative of the normal stress component may be written as
$\frac{\partial\left(\sigma_{x}\right)_{j}}{\partial x}=E_{j}\left(z \frac{\partial \kappa_{y}}{\partial x}-y \frac{\partial \kappa_{z}}{\partial x}\right)$
Substituting equation (4a) in the relations of the bending moments
$M_{y}=\sum_{j=1}^{K}\left(M_{y}\right)_{j}=\sum_{j=1}^{K} \int_{\Omega_{j}}\left(\sigma_{x x}\right)_{j} z d \Omega_{j}$
$M_{z}=\sum_{j=1}^{K}\left(M_{z}\right)_{j}=-\sum_{j=1}^{K} \int_{\Omega_{j}}\left(\sigma_{x x}\right)_{j} y d \Omega_{j}$
the following system of equations is obtained

$$
\left[\begin{array}{cc}
I_{y y} & -I_{y z}  \tag{12}\\
-I_{y z} & I_{z z}
\end{array}\right]\left\{\begin{array}{l}
\kappa_{y} \\
\kappa_{z}
\end{array}\right\}=\frac{1}{E_{r e f}}\left\{\begin{array}{l}
M_{y} \\
M_{z}
\end{array}\right\}
$$

where
$I_{y y}=\sum_{j=1}^{K} \lambda_{j} \int_{\Omega_{j}} z^{2} d \Omega_{j}$
$I_{z z}=\sum_{j=1}^{K} \lambda_{j} \int_{\Omega_{j}} y^{2} d \Omega_{j}$
$I_{y z}=\sum_{j=1}^{K} \lambda_{j} \int_{\Omega_{j}} y z d \Omega_{j}$
are the moments of inertia with respect to $y$ and $z$ axes and the product of inertia of the composite cross section, respectively and $\lambda_{j}$ is given from
$\lambda_{j}=\frac{E_{j}}{E_{\text {ref }}}=\frac{G_{j}}{G_{r e f}}$
with $E_{r e f}, G_{r e f}$ the modulus of elasticity and shear modulus of a reference material, respectively. It is worth here noting that any material of the composite cross section can be used as reference material for the reduction of eqns (12), (13). Moreover, the weighted elastic and shear moduli of the $j$-th material have the same value $\lambda_{j}$ since as it was mentioned before its Poisson ration has been set equal to zero. Solving the system of eqns. (12) the curvatures of the transverse displacements are obtained as
$\kappa_{y}=\frac{1}{E_{\text {ref }}} \frac{I_{z z} M_{y}+I_{y z} M_{z}}{I_{y y} I_{z z}-I_{y z}^{2}}$
$\kappa_{z}=\frac{1}{E_{r e f}} \frac{I_{y z} M_{y}+I_{y y} M_{z}}{I_{y y} I_{z z}-I_{y z}^{2}}$
Differentiating eqns. (15) with respect to $x$ and taking into account that
$\frac{\partial M_{y}}{\partial x}=Q_{z}$
$\frac{\partial M_{z}}{\partial x}=-Q_{y}$
the following relations are obtained
$\frac{\partial \kappa_{y}}{\partial x}=\frac{1}{E_{\text {ref }}} \frac{I_{z z} Q_{z}-I_{y z} Q_{y}}{\Delta}$
$\frac{\partial \kappa_{z}}{\partial x}=\frac{1}{E_{\text {ref }}} \frac{I_{y z} Q_{z}-I_{y y} Q_{y}}{\Delta}$
where $\Delta$ is defined as
$\Delta=\left(I_{y y} I_{z z}-I_{y z}^{2}\right)$
Substituting eqns. (17) in eqn.(10) the derivative of the normal stress component is written as

$$
\begin{equation*}
\frac{\partial\left(\sigma_{x}\right)_{j}}{\partial x}=\frac{\lambda_{j}}{\Delta}\left[\left(I_{y y} Q_{y}-I_{y z} Q_{z}\right) y+\left(I_{z z} Q_{z}-I_{y z} Q_{y}\right) z\right] \tag{19}
\end{equation*}
$$

and employing eqn.(9) the partial Poisson type differential equation governing the unit warping function $\varphi_{C}(y, z)$ is obtained as
$\nabla^{2} \varphi_{C}(y, z)=-\frac{1}{G_{r e f} \Delta}\left[\left(I_{y y} Q_{y}-I_{y z} Q_{z}\right) y+\left(I_{z z} Q_{z}-I_{y z} Q_{y}\right) z\right] \quad$ in $\Omega_{j} j=1,2, \ldots, K$
where $\left(\nabla^{2}\right)_{j} \equiv\left(\partial^{2} / \partial y^{2}\right)_{j}+\left(\partial^{2} / \partial z^{2}\right)_{j}$ is the Laplace operator and $\Omega=\cup_{j=1}^{K} \Omega_{j}$ denotes the whole region of the composite cross section.
The boundary condition of the aforementioned warping function will be derived from the following physical considerations:

- The traction vector in the direction of the normal vector $n$ vanishes on the free surface of the beam
- The traction vectors in the direction of the normal vector $n$ on the interfaces separating the $j$-th and $i$-th different materials are equal in magnitude and opposite in direction
- The displacement components $u, v, w$ remain continuous across the interfaces since it is assumed that the materials are firmly bonded together

The third condition ensures the continuity of the warping function across the boundaries $\Gamma_{j}(j=1,2, \ldots, K)$ separating different materials $\left(\left(\varphi_{C}\right)_{j}=\left(\varphi_{C}\right)_{i}=\varphi_{C}\right)$, while the first two lead to
$\left(\tau_{x n}\right)_{j}=\left(\tau_{x y}\right)_{j} n_{y}+\left(\tau_{x z}\right)_{j} n_{z}=0$ on the free surface of the beam
$\left(\tau_{x n}\right)_{j}=\left(\tau_{x n}\right)_{i}$ or $\left(\tau_{x y}\right)_{j} n_{y}+\left(\tau_{x z}\right)_{j} n_{z}=\left(\tau_{x y}\right)_{i} n_{y}+\left(\tau_{x z}\right)_{i} n_{z}$ on the interfaces
where $n_{y}=\cos \beta, n_{z}=\sin \beta$ (with $\beta=\widehat{y, n}$ as shown in Fig.1b) are the direction cosines of the normal vector $\mathbf{n}$ to the boundary $\Gamma_{j}(j=1,2, \ldots, K)$, while on both sides of the equality (21b) the normal vector $\mathbf{n}$ points in one and the same direction.

Substituting eqns. (7a,b) in eqns. (21a,b), the Neumann type boundary condition of the unit warping function can be written as
$G_{j}\left(\frac{\partial \varphi_{C}}{\partial n}\right)_{j}-G_{i}\left(\frac{\partial \varphi_{C}}{\partial n}\right)_{i}=0$ on $\Gamma_{j} j=1,2, \ldots, K$
where $G_{i}$ is the modulus of elasticity of the $\Omega_{i}$ region at the common part of the boundaries of $\Omega_{j}$ and $\Omega_{i}$ regions, or $G_{i}=0$ at the free part of the boundary of $\Omega_{j}$ region, while $(\partial / \partial n)_{j} \equiv n_{y}(\partial / \partial y)_{j}+n_{z}(\partial / \partial z)_{j}$ denotes the directional derivative normal to the boundary $\Gamma_{j}$. The vector $\mathbf{n}$ normal to the boundary $\Gamma_{j}$ is positive if it points to the exterior of the $\Omega_{j}$ region. It is worth here noting that the normal derivatives across the interior boundaries vary discontinuously.
Referring to the Neumann boundary value problem described by eqns. $(20,22)$ the following should be taken into account arising from the theory of partial differential equations of elliptic type:

- The aforementioned problem will have a solution if the following condition is satisfied (Muskhelishvili, 1963)

$$
\begin{equation*}
\int_{\Gamma_{j}}\left[G_{j}\left(\frac{\partial \varphi_{C}}{\partial n}\right)_{j}-G_{i}\left(\frac{\partial \varphi_{C}}{\partial n}\right)_{i}\right] d s=0 \tag{23}
\end{equation*}
$$

- The solution $\varphi_{C}$ is not determined uniquely, but to the approximation of an arbitrary constant term. That is, the exact solution (unit warping function) $\varphi_{C}$ can be written in the form
$\varphi_{C}(y, z)=\bar{\varphi}_{C}(y, z)+c$
where $\bar{\varphi}_{C}(y, z)$ is the function obtained form the solution of the aforedescribed Neumann problem and is called basic warping function and $c$ is a integration constant.

The first remark is identically satisfied taking into account the boundary condition of the Neumann problem (eqn. 22). According to the second remark, the stress components are not influenced by the integration constant $c$, since following eqns. $(6 a, b)$ only the derivatives of $\varphi_{C}$ are required for the evaluation of these quantities. With respect to the displacement $u$, the arbitrary constant introduces a rigid motion in the direction of the beam axis, which, however, does not influence the deformation of the composite cross section. The constant term $c$ can be determined from the condition

$$
\begin{equation*}
\sum_{j=1}^{K} E_{j} \int_{\Omega_{j}} \varphi_{C}(y, z) d \Omega_{j}=0 \tag{25}
\end{equation*}
$$

which after substituting eqn. (24) leads to
$c=-\frac{1}{A} \sum_{j=1}^{K} \lambda_{j} \int_{\Omega_{j}} \bar{\varphi}_{C}(y, z) d \Omega_{j}$
and the unit warping function is given from the relation
$\varphi_{C}(y, z)=\bar{\varphi}_{C}(y, z)-\frac{1}{A} \sum_{j=1}^{K} \lambda_{j} \int_{\Omega_{j}} \bar{\varphi}_{C}(y, z) d \Omega_{j}$
where
$A=\sum_{j=1}^{K} \lambda_{j} \int_{\Omega_{j}} d \Omega_{j}$
is the area of the composite cross section.
It can be proven that the resultants of the shear stress components $\left(\tau_{x y}\right)_{j},\left(\tau_{x z}\right)_{j}$ of eqns. (7) at the beam ends ( $x=0, x=L$ ) are the shear loads $Q_{y}, Q_{z}$, respectively, that is
$Q_{y}=\sum_{j=1}^{K} \int_{\Omega_{j}}\left(\tau_{x y}\right)_{j} d \Omega_{j}$
$Q_{z}=\sum_{j=1}^{K} \int_{\Omega_{j}}\left(\tau_{x z}\right)_{j} d \Omega_{j}$
Indeed, using the equilibrium equation (8) the integral of the shear stresses $\left(\tau_{x y}\right)_{j}$ is reformulated as follows

$$
\begin{array}{r}
\sum_{j=1}^{K} \int_{\Omega_{j}}\left(\tau_{x y}\right)_{j} d \Omega_{j}=\sum_{j=1}^{K} \int_{\Omega_{j}}\left[\left(\tau_{x y}\right)_{j}+y\left(\frac{\partial\left(\tau_{x y}\right)_{j}}{\partial y}+\frac{\partial\left(\tau_{x z}\right)_{j}}{\partial z}+\frac{\partial\left(\sigma_{x}\right)_{j}}{\partial x}\right)\right] d \Omega_{j} \\
=\sum_{j=1}^{K} \int_{\Omega_{j}}\left[\frac{\partial\left(y \tau_{x y}\right)_{j}}{\partial y}+\frac{\partial\left(y \tau_{x z}\right)_{j}}{\partial z}\right] d \Omega_{j}+\sum_{j=1}^{K} \int_{\Omega_{j}}\left[y \frac{\partial\left(\sigma_{x}\right)_{j}}{\partial x}\right] d \Omega_{j} \tag{30}
\end{array}
$$

Applying integration by parts for the first integral of the right hand side of eqn. (30) and using eqns. (7a,b), (23) the first integral of the right hand side vanishes, while substituting eqn. (19) in eqn. (30) the domain integral of the shear stresses $\left(\tau_{x y}\right)_{j}$ may be written as

$$
\begin{align*}
\sum_{j=1}^{K} \int_{\Omega_{j}}\left(\tau_{x y}\right)_{j} d \Omega_{j}=\frac{\left(I_{y y} Q_{y}-I_{y z} Q_{z}\right)}{\Delta} \sum_{j=1}^{K} & \lambda_{j} \int_{\Omega_{j}} y^{2} d \Omega_{j} \\
& +\frac{\left(I_{z z} Q_{z}-I_{y z} Q_{y}\right)}{\Delta} \sum_{j=1}^{K} \lambda_{j} \int_{\Omega_{j}} y z d \Omega_{j} \tag{31}
\end{align*}
$$

and considering eqns. (13b,c), (18) we obtain Eqn. (29a). An analogous derivation can be applied to the integral of the shear stresses $\left(\tau_{x z}\right)_{j}$, which yields the shear load $Q_{z}$.
Having in mind that the shear center $S$ is defined as the point of the cross section at which the torsional moment arising from the transverse shear stresses vanishes, the coordinates $\left\{y_{S}, z_{S}\right\}$ of this point with respect to the system of axes with origin the cross section centroid can be derived from the condition
$y_{s} Q_{z}-z_{S} Q_{y}=M_{x} \rightarrow y_{S} Q_{z}-z_{S} Q_{y}=\sum_{j=1}^{K} \int_{\Omega_{j}}\left[\left(\tau_{x z}\right)_{j} y-\left(\tau_{x y}\right)_{j} z\right] d \Omega_{j}$
For $\left\{Q_{y}=0, Q_{z}=1\right\}$, after substituting eqns. (6a,b) in eqn. (32), the $y_{S}$ coordinate of the shear center $S$ can be obtained from
$y_{S}=\sum_{j=1}^{K} G_{j} \int_{\Omega_{j}}\left(\frac{\partial \varphi_{C y}}{\partial z} y-\frac{\partial \varphi_{C y}}{\partial y} z\right) d \Omega_{j}$
while for $\left\{Q_{y}=1, Q_{z}=0\right\}$ the $z_{S}$ coordinate is given as
$z s=\sum_{j=1}^{K} G_{j} \int_{\Omega_{j}}\left(\frac{\partial \varphi_{C z}}{\partial y} z-\frac{\partial \varphi_{C z}}{\partial z} y\right) d \Omega_{j}$

Equations (33), (34) declare that the $\left\{y_{S}, z_{S}\right\}$ coordinates of the shear center $S$ are independent from shear loading. Moreover, it can be shown that the coordinates of the shear center $S$, given from the aforementioned eqns. (33), (34), coincide with the coordinates of the center of twist $M$, that is

$$
\begin{align*}
& y_{S}=y_{M}  \tag{35a}\\
& z_{S}=z_{M} \tag{35b}
\end{align*}
$$

where the equations for the coordinates $\left\{y_{M}, z_{M}\right\}$ are given in Sapountzakis (2000). This coincidence of these centers was first recognized by Weber (1926) applying the Betty-Maxwell reciprocal relations and Trefftz (1935) using an energy approach.
Furthermore, the shear deformation coefficients $a_{y}, a_{z}$ and $a_{y z}=a_{z y}$, which are introduced from the approximate formula for the evaluation of the shear strain energy per unit length (Schramm, Rubenchik and Pilkey, 1997)
$U_{\text {appr. }}=\frac{a_{y} Q_{y}^{2}}{2 A G_{\text {ref }}}+\frac{a_{z} Q_{z}^{2}}{2 A G_{\text {ref }}}+\frac{a_{y z} Q_{y} Q_{z}}{A G_{\text {ref }}}$
are evaluated equating this approximate energy with the exact one given from
$U_{\text {exact }}=\sum_{j=1}^{K} \frac{1}{2 G_{j}} \int_{\Omega_{j}}\left[\left(\tau_{x y}\right)_{j}^{2}+\left(\tau_{x z}\right)_{j}^{2}\right] d \Omega_{j}$
and are obtained for the cases
$\left\{Q_{y} \neq 0, Q_{z}=0\right\},\left\{Q_{y}=0, Q_{z} \neq 0\right\}$
and
$\left\{Q_{y} \neq 0, Q_{z} \neq 0\right\}$,
respectively, as
$a_{y}=\frac{1}{\kappa_{y}}=A G_{r e f} \sum_{j=1}^{K} G_{j} \int_{\Omega_{j}}\left[\left(\frac{\partial \varphi_{C z}}{\partial y}\right)_{j}^{2}+\left(\frac{\partial \varphi_{C z}}{\partial z}\right)_{j}^{2}\right] d \Omega_{j}$
$a_{z}=\frac{1}{\kappa_{z}}=A G_{r e f} \sum_{j=1}^{K} G_{j} \int_{\Omega_{j}}\left[\left(\frac{\partial \varphi_{C y}}{\partial y}\right)_{j}^{2}+\left(\frac{\partial \varphi_{C y}}{\partial z}\right)_{j}^{2}\right] d \Omega_{j}$
$a_{y z}=\frac{1}{\kappa_{y z}}=A G_{r e f} \sum_{j=1}^{K} G_{j} \int_{\Omega_{j}}\left[\left(\frac{\partial \varphi_{C y}}{\partial y}\right)_{j}\left(\frac{\partial \varphi_{C z}}{\partial y}\right)_{j}+\left(\frac{\partial \varphi_{C y}}{\partial z}\right)_{j}\left(\frac{\partial \varphi_{C z}}{\partial z}\right)_{j}\right] d \Omega_{j}$
where the factors $\kappa_{y}, \kappa_{z}, \kappa_{y z}$ are called shear correction factors or shear form factors or shear stiffness factors (Pilkey, 2002). It is worth here noting that the unit warping function $\varphi_{C y}$ of Eqns. (33), (38b,c) results from the solution of the Neumann boundary value problem
$\nabla^{2} \varphi_{C y}(y, z)=\frac{1}{G_{r e f} \Delta}\left(I_{y z} y-I_{z z} z\right) \quad$ in $\Omega_{j} j=1,2, \ldots, K$
$G_{j}\left(\frac{\partial \varphi_{C y}}{\partial n}\right)_{j}-G_{i}\left(\frac{\partial \varphi_{C y}}{\partial n}\right)_{i}=0 \quad$ on $\Gamma_{j} j=1,2, \ldots, K$
and the unit warping function $\varphi_{C z}$ of Eqns. (34), (38a,c) from the Neumann boundary value problem
$\nabla^{2} \varphi_{C z}(y, z)=\frac{1}{G_{r e f} \Delta}\left(I_{y z} z-I_{y y} y\right) \quad$ in $\Omega_{j} j=1,2, \ldots, K$
$G_{j}\left(\frac{\partial \varphi_{C z}}{\partial n}\right)_{j}-G_{i}\left(\frac{\partial \varphi_{C z}}{\partial n}\right)_{i}=0 \quad$ on $\Gamma_{j} j=1,2, \ldots, K$
Notice that the unit warping functions $\varphi_{C y}$ and $\varphi_{C z}$ are determined exactly apart from an arbitrary constant term (Neumann problem). However, the coordinates of the shear center and the shear deformation coefficients are not influenced by this arbitrary constant, since according to Eqns. (33), (34), (38a,b,c), only the derivatives of $\varphi_{C y}$ and $\varphi_{C z}$ are required for the evaluation of these quantities. Employing the shear deformation coefficients $a_{y}, a_{z}, a_{y z}$ using eqns. (38a,b,c) we can define the cross section shear rigidities of the Timoshenko's beam theory as
$G_{r e f} A_{s y}=G_{r e f}\left(A / a_{y}\right)=G_{r e f}\left(\kappa_{y} A\right)$
$G_{r e f} A_{s z}=G_{r e f}\left(A / a_{z}\right)=G_{r e f}\left(\kappa_{z} A\right)$
$G_{r e f} A_{s y z}=G_{r e f}\left(A / a_{y z}\right)=G_{r e f}\left(\kappa_{y z} A\right)$
The shear deformation coefficients $a_{y}, a_{z}, a_{y z}$ can be expressed in matrix form as

$$
\left[A_{s}\right]=\left[\begin{array}{cc}
a_{y} & a_{y z}  \tag{42}\\
a_{z y} & a_{z}
\end{array}\right]=A G_{r e f} \sum_{j=1}^{K} G_{j} \int_{\Omega_{j}}\left(\left\{B_{y}\right\}^{T}\left\{B_{y}\right\}+\left\{B_{z}\right\}^{T}\left\{B_{z}\right\}\right) d \Omega_{j}
$$

where
$\left\{B_{y}\right\}=\left\{\begin{array}{ll}\frac{\partial \varphi_{C_{z}}}{\partial y} & \frac{\partial \varphi_{C_{y}}}{\partial y}\end{array}\right\}$
$\left\{B_{z}\right\}=\left\{\begin{array}{ll}\frac{\partial \varphi_{C_{z}}}{\partial z} & \frac{\partial \varphi_{C_{Y}}}{\partial z}\end{array}\right\}$
The matrix $\left[A_{s}\right]$ is a symmetric tensor. The principal values of this tensor can be obtained form the solution of the eigenvalue problem

$$
\begin{equation*}
\left[\left[A_{s}\right]-a[I]\right]\{x\}=\{0\} \tag{44}
\end{equation*}
$$

where $[I]$ is the unity matrix. Solution of this problem yields the eigenpairs $\left(a_{i},\{x\}_{i}\right)$, $i=1,2$. The eigenvectors $\{x\}_{i}, i=1,2$ are the basis vectors of the principal coordinates, which is called principal shear system of axes (Pilkey, 2002), while the eigenvalues $a_{i}, i=1,2$ are the principal shear deformation coefficients which are always greater than or equal to 1 (Schramm, Rubenchik and Pilkey, 1997). The angle $\phi^{S}$ of the principal shear system with respect to the coordinate system $y, z$ (Fig.1) is obtained form

$$
\begin{equation*}
\tan 2 \varphi^{S}=\frac{2 a_{y z}}{a_{y}-a_{z}} \tag{45}
\end{equation*}
$$

In general, for an asymmetric cross section the principal shear axes do not coincide with the principal bending ones, defined by the engineering beam theory as
$\tan 2 \varphi^{B}=\frac{2 I_{y z}}{I_{y y}-I_{z z}}$
Due to this difference $\left(\varphi^{S} \neq \varphi^{B}\right)$, the deflection components in the $y$ and $z$ directions are in general coupled, even if the system of axes of the cross section coincides with the principal bending one (Pilkey, 2002). If the cross section is symmetric about an axis, the principal shear axes system coincides with the principal bending one. In this case, the deflection components with respect to the principal directions are not coupled any more ( $a_{y z}=a_{z y}=0$ and $I_{y z}=I_{z y}=0$ ).
Finally, the shear stress components at points on the boundary $\Gamma_{j}(j=1,2, \ldots, K)$ are evaluated from the established values of $\left(\varphi_{C}\right)_{j}$ and $\left(\partial v \operatorname{arphi} i_{C} / \partial n\right)_{j}$ using the following relations

$$
\begin{align*}
& \left(\tau_{x n}\right)_{j}=G_{j}\left(\frac{\partial \varphi_{C}}{\partial n}\right)_{j}  \tag{47a}\\
& \left(\tau_{x t}\right)_{j}=G_{j}\left(\frac{\partial \varphi_{C}}{\partial t}\right)_{j}  \tag{47b}\\
& \left(\tau_{\Gamma}\right)_{j}=\left[\left(\tau_{x n}\right)_{j}^{2}+\left(\tau_{x t}\right)_{j}^{2}\right]^{1 / 2} \tag{47c}
\end{align*}
$$

where the tangential derivative $\left(\partial \varphi_{C} / \partial t\right)_{j}=\left(\partial \varphi_{C} / \partial s\right)_{j}$ is computed numerically using appropriately central, backward or forward differences. It is worth noting that $\left(\tau_{x n}\right)_{j}$ is the bond stress at the interface part of the boundary $\Gamma_{j}$, while $\left(\tau_{\Gamma}\right)_{j}$ is the resultant boundary shear stress.

## 3 Integral representations - Numerical solution

According to the precedent analysis, the shear problem of a composite beam reduces in establishing the warping functions $\varphi_{C}(y, z), \varphi_{C y}(y, z)$ and $\varphi_{C z}(y, z)$ having continuous partial derivatives up to the second order and satisfying the boundary value problems described by eqns. [(20), (22)], [(39a), (39b)] and [(40a), (40b)]. The numerical solution of these problems is similar. For this reason, in the following we will analyze the solution of the problem of eqns. [(20), (22)].
The evaluation of the warping function $\varphi_{C}$ is accomplished using BEM (Katsikadelis, 2002) as this is presented in Mokos and Sapountzakis (2005). According to this method the Green identity

$$
\begin{equation*}
\int_{\Omega_{j}}\left(\Psi\left(\nabla^{2} \varphi_{C}\right)_{j}-\left(\varphi_{C}\right)_{j} \nabla^{2} \Psi\right) d \Omega_{j}=\int_{\Gamma_{j}}\left(\Psi\left(\frac{\partial \varphi_{C}}{\partial n}\right)_{j}-\left(\varphi_{C}\right)_{j} \frac{\partial \Psi}{\partial n}\right) d s \tag{48}
\end{equation*}
$$

when applied to the warping function $\varphi_{C}$ and to the fundamental solution
$\Psi=\frac{1}{2 \pi} \ln r(P, Q) \quad P, Q \in \Omega_{j}$
which is a particular singular solution of the Poisson equation
$\nabla^{2} \Psi=\delta(P, Q)$
where $\delta(P, Q)$ is the Dirac delta function in two dimensions, yields

$$
\begin{align*}
& \varepsilon\left(\varphi_{C}(P)\right)_{j}=\int_{\Omega_{j}} \ln r\left(\nabla^{2} \varphi_{C}(Q)\right)_{j} d \Omega_{Q} \\
&+\int_{\Gamma_{j}}\left[\left(\varphi_{C}(q)\right)_{j} \frac{\cos a}{r}-\left(\frac{\partial \varphi_{C}(q)}{\partial n}\right)_{j} \ln r\right] d s_{q} \tag{51}
\end{align*}
$$

with $\alpha=\widehat{r, n} ; r=|P-q|, P, Q \in \Omega_{j}, q \in \Gamma_{j}(j=1,2, \ldots, K)$ and $\varepsilon=2 \pi, \pi$ or 0depending on whether the point $P$ is inside the region $\Omega_{j}, P \equiv p$ on the boundary $\Gamma_{j}$ or $P$ outside $\Omega_{j}$, respectively. Note that the boundary has been assumed to be smooth at point $p \in \Gamma_{j}$. Using eqn. (20) the integral representation (51) is written as

$$
\begin{equation*}
\varepsilon\left(\varphi_{C}(P)\right)_{j}=\int_{\Omega_{j}} f(Q) \ln r d \Omega_{Q}+\int_{\Gamma_{j}}\left[\left(\varphi_{C}(q)\right)_{j} \frac{\cos a}{r}-\left(\frac{\partial \varphi_{C}(q)}{\partial n}\right)_{j} \ln r\right] d s_{q} \tag{52}
\end{equation*}
$$

where the function $f$ is defined as
$f=-\frac{1}{G_{r e f} \Delta}\left[\left(I_{y y} Q_{y}-I_{y z} Q_{z}\right) y+\left(I_{z z} Q_{z}-I_{y z} Q_{y}\right) z\right]$
Applying once more the Green identity given by eqn. (48) for the function $f$ satisfying the Laplace equation
$\nabla^{2} f=0$
and for the function $U$ defined as
$U=\frac{1}{8 \pi} r^{2}(\ln r-1)$
satisfying the Poisson equation
$\nabla^{2} U=\Psi$
the domain integral of eqn. (52) can be converted into a line integral along the boundaries of the cross section and the integral representation (52) is written as

$$
\begin{align*}
& \varepsilon\left(\varphi_{C}(P)\right)_{j}=\frac{1}{4} \int_{\Gamma_{j}}\left[f(q)(2 \ln r-1) r \cos a-\frac{\partial f(q)}{\partial n}(\ln r-1) r^{2}\right] d s_{q} \\
&+\int_{\Gamma_{j}}\left[\left(\varphi_{C}(q)\right)_{j} \frac{\cos a}{r}-\left(\frac{\partial \varphi_{C}(q)}{\partial n}\right) \ln r\right] d s_{q} \tag{57}
\end{align*}
$$

In eqn. (57) the subscript $q$ in the arc element $d s_{q}$ indicates that point $q$ varies along the boundaries of the cross section during integration and differentiation, while the point $P$ (or $p$ ) is retained constant. The values of the function $\left(\varphi_{C}(P)\right)_{j}$ inside the region $\Omega_{j}$ can be established from the integral representation (57) if $\left(\varphi_{C}\right)_{j}$ and its derivative $\left(\partial \varphi_{C} / \partial n\right)_{j}$ were known on the boundaries $\Gamma_{j}$. Thus,

$$
\begin{align*}
\left(\varphi_{C}(P)\right)_{j} & =\frac{1}{8 \pi} \int_{\Gamma_{j}}\left[f(q)(2 \ln r-1) r \cos a-\frac{\partial f(q)}{\partial n}(\ln r-1) r^{2}\right] d s_{q} \\
& +\frac{1}{2 \pi} \int_{\Gamma_{j}}\left[\left(\varphi_{C}(q)\right)_{j} \frac{\cos a}{r}-\left(\frac{\partial \varphi_{C}(q)}{\partial n}\right)_{j} \ln r\right] d s_{q}, P \in \Omega_{j}, q \in \Gamma_{j} \tag{58}
\end{align*}
$$

The unknown boundary quantities $\left(\varphi_{C}\right)_{j}$ and $\left(\partial \varphi_{C} / \partial n\right)_{j}$ can be evaluated from the solution of a boundary integral equation on the boundary $\Gamma_{j}$, which is derived working as follows.
Consider a point $p$ lying on the boundary $\Gamma_{j}(j=1,2, \ldots, K)$. For a point $q$ lying on the boundary $\Gamma_{j}$ of the region $\Omega_{j}$ eqn. (57) may be written as

$$
\begin{align*}
\pi\left(\varphi_{C}(p)\right)_{j}=\frac{1}{4} \int_{\Gamma_{j}} & {\left[f(q)(2 \ln r-1) r \cos a-\frac{\partial f(q)}{\partial n}(\ln r-1) r^{2}\right] d s_{q} } \\
& +\int_{\Gamma_{j}}\left[\left(\varphi_{C}(q)\right)_{j} \frac{\cos a}{r}-\left(\frac{\partial \varphi_{C}(q)}{\partial n}\right)_{j} \ln r\right] d s_{q}, q \in \Gamma_{j} \tag{59}
\end{align*}
$$

Similarly, for a point $q$ lying on the part of the boundary $\Gamma_{k}$ of the region $\Omega_{k}$, which is an interface between regions $\Omega_{j}$ and $\Omega_{k}$, eqn. (57) may be written as

$$
\begin{align*}
\pi\left(\varphi_{C}(p)\right)_{j}=\frac{1}{4} \int_{\Gamma_{k}} & {\left[-f(q)(2 \ln r-1) r \cos a+\frac{\partial f(q)}{\partial n}(\ln r-1) r^{2}\right] d s_{q} } \\
& +\int_{\Gamma_{k}}\left[-\left(\varphi_{C}(q)\right)_{k} \frac{\cos a}{r}+\left(\frac{\partial \varphi_{C}(q)}{\partial n}\right)_{k} \ln r\right] d s_{q}, q \in \Gamma_{k} \tag{60}
\end{align*}
$$

Moreover, for a point $q$ lying on the boundaries $\Gamma_{i}(i=1,2, \ldots, K, i \neq k)$ eqn. (57) yields

$$
\begin{align*}
& 0=\frac{1}{4} \int_{\Gamma_{i}}\left[-f(q)(2 \ln r-1) r \cos a+\frac{\partial f(q)}{\partial n}(\ln r-1) r^{2}\right] d s_{q} \\
&+\int_{\Gamma_{i}}\left[-\left(\varphi_{C}(q)\right)_{i} \frac{\cos a}{r}+\left(\frac{\partial \varphi_{C}(q)}{\partial n}\right)_{i} \ln r\right] d s_{q}, q \in \Gamma_{i} \tag{61}
\end{align*}
$$

Notice that the sign in eqns. (60), (61) is reversed, since the unit vector normal to the boundary is negative. Multiplying eqn. (59) by $G_{j}$, eqn. (60) by $G_{k}$, eqn. (61) by $G_{i}(i=1,2, \ldots, K, i \neq k)$ and adding them yields

$$
\begin{align*}
& \pi\left(\varphi_{C}(p)\right)_{j}\left(G_{j}+G_{k}\right)= \\
& \frac{1}{4} \sum_{j=1}^{K} \int_{\Gamma_{j}}\left(G_{j}-G_{i}\right)\left[f(q)(2 \ln r-1) r \cos a-\frac{\partial f(q)}{\partial n}(\ln r-1) r^{2}\right] d s_{q} \\
& +\sum_{j=1}^{K} \int_{\Gamma_{j}}\left[\left(G_{j}-G_{i}\right)\left(\varphi_{C}(q)\right)_{j} \frac{\cos a}{r}-\left(G_{i}\left(\frac{\partial \varphi_{C}(q)}{\partial n}\right)_{j}-G_{j}\left(\frac{\partial \varphi_{C}(q)}{\partial n}\right)_{i}\right) \ln r\right] d s_{q} \tag{62}
\end{align*}
$$

which after taking into account the boundary condition of the unit warping function (eqn. 22) yields the following singular boundary integral equation

$$
\begin{align*}
& \pi\left(\varphi_{C}(p)\right)_{j}\left(G_{j}+G_{k}\right)=\sum_{j=1}^{K} \int_{\Gamma_{j}}\left(G_{j}-G_{i}\right)\left(\varphi_{C}(q)\right)_{j} \frac{\cos a}{r} d s_{q} \\
& \frac{1}{4} \sum_{j=1}^{K} \int_{\Gamma_{j}}\left(G_{j}-G_{i}\right)\left[f(q)(2 \ln r-1) r \cos a-\frac{\partial f(q)}{\partial n}(\ln r-1) r^{2}\right] d s_{q} \tag{63}
\end{align*}
$$

where the derivative of the function $f$ in the direction of the vector $\mathbf{n}$ normal to the boundary $\Gamma_{j}$ is defined as
$\frac{\partial f}{\partial n}=-\frac{1}{G_{r e f} \Delta}\left[\left(I_{y y} Q_{y}-I_{y z} Q_{z}\right) \cos \beta+\left(I_{z z} Q_{z}-I_{y z} Q_{y}\right) \sin \beta\right]$
It is worth here noting that in eqn. (63) the point $p$ lies on the boundary $\Gamma_{j}$ $(j=1,2, \ldots, K)$, which is an interface between regions $\Omega_{j}$ and $\Omega_{k}$, while the point $q$ varies along the boundary $\Gamma_{j}(j=1,2, \ldots, K)$, which is an interface between regions $\Omega_{j}$ and $\Omega_{i}$, while $G_{k}=G_{i}=0$ in the case $\Gamma_{j}$ is a free boundary. Moreover, in eqn. (63) the normal $\mathbf{n}$ to the boundary $\Gamma_{j}$ points to the exterior of the region $\Omega_{j}$ and $\Gamma_{j}$ is traveled only once.

For any given geometry of the composite cross section the warping function $\left(\varphi_{C}(s)\right)_{j}$ on the boundary $\Gamma_{j}(j=1,2, \ldots, K)$ is obtained from the solution of the boundary integral equation (63). Thus, using constant boundary elements to approximate the line integrals along the boundaries and a collocation technique the following well conditioned linear system of simultaneous algebraic equations is established (Katsikadelis, 2002)
$[A]\left\{\Phi_{C}\right\}=\{C\}$
where

$$
\left\{\Phi_{C}\right\}^{T}=\left\{\begin{array}{llll}
\left(\varphi_{C}\right)_{1} & \left(\varphi_{C}\right)_{2} & \ldots & \left(\varphi_{C}\right)_{N} \tag{66}
\end{array}\right\}
$$

are the values of the boundary quantities $\varphi_{C}$ at the nodal points of the $N$ boundary elements. Moreover, in eqn. (65) $[A]$ and $\{C\}$ are square $N \times N$ and column $N \times 1$ known coefficient matrices, respectively. From the solution of this system of simultaneous algebraic equations the values of the warping function $\varphi_{C}$ for all boundary nodal points are established.
The derivatives of $\left(\varphi_{C}\right)_{j}$ with respect to $y$ and $z$ at any interior point of the region $\Omega_{j}$, for the calculation of the stress resultants (eqns. 7a,b) are computed differentiating the integral representation (58) of the warping function $\left(\varphi_{C}\right)_{j}$ as

$$
\begin{align*}
& \left(\frac{\partial \varphi_{C}(P)}{\partial y}\right)_{j}=\frac{1}{2 \pi} \int_{\Gamma_{j}}\left[\left(\varphi_{C}(q)\right)_{j} \frac{\cos (\omega-a)}{r^{2}}+\left(\frac{\partial \varphi_{C}(q)}{\partial n}\right)_{j} \frac{\cos \omega}{r}\right] d s_{q} \\
& -\frac{1}{8 \pi} \int_{\Gamma_{j}}\left[f(q)(2 \cos \omega \cos a+(2 \ln r-1) \cos \beta)-\frac{\partial f(q)}{\partial n}(2 \ln r-1) r \cos \omega\right] d s_{q} \tag{67a}
\end{align*}
$$

$$
\begin{align*}
& \left(\frac{\partial \varphi_{C}(P)}{\partial z}\right)_{j}=\frac{1}{2 \pi} \int_{\Gamma_{j}}\left[\left(\varphi_{C}(q)\right)_{j} \frac{\sin (\omega-a)}{r^{2}}+\left(\frac{\partial \varphi_{C}(q)}{\partial n}\right)_{j} \frac{\sin \omega}{r}\right] d s_{q} \\
& -\frac{1}{8 \pi} \int_{\Gamma_{j}}\left[f(q)(2 \sin \omega \cos a+(2 \ln r-1) \sin \beta)-\frac{\partial f(q)}{\partial n}(2 \ln r-1) r \sin \omega\right] d s_{q} \tag{67b}
\end{align*}
$$

with $r=|P-q|, P \in \Omega_{j}, q \in \Gamma_{j}$ and $\omega=\widehat{x, r}$. The derivative $\left(\partial \varphi_{C} / \partial n\right)_{j}$ for the evaluation of the shear stresses is known only on the free parts of the boundaries $\Gamma_{j}$. Its values on the interfaces can be established from eqn. (22) and the solution of the singular integral eqn. (59) using the boundary values of $\left(\varphi_{C}\right)_{j}$ obtained from the solution of eqn. (62).

Moreover, since the torsionless bending problem of composite beams is solved using BEM, the domain integrals in Eqns. (13a,b,c), (26), (33), (34), (38a,b,c) have to be converted to boundary line ones in order to maintain the pure boundary character of the method. This can be achieved using integration by parts and the Green identity. Thus, for the moments of inertia, the product of inertia and the cross section area we can write the following relations
$I_{y y}=\sum_{j=1}^{K} \int_{\Gamma_{j}}\left(\lambda_{j}-\lambda_{i}\right)\left(y z^{2} \cos \beta\right) d s$
$I_{z z}=\sum_{j=1}^{K} \int_{\Gamma_{j}}\left(\lambda_{j}-\lambda_{i}\right)\left(z y^{2} \sin \beta\right) d s$
$I_{y z}=\frac{1}{2} \sum_{j=1}^{K} \int_{\Gamma_{j}}\left(\lambda_{j}-\lambda_{i}\right)\left(z y^{2} \cos \beta\right) d s$
$A=\frac{1}{2} \sum_{j=1}^{K} \int_{\Gamma_{j}}\left(\lambda_{j}-\lambda_{i}\right)(y \cos \beta+z \sin \beta) d s$
while the $\left\{y_{S}, z_{S}\right\}$ coordinates of the shear center $S$ are obtained from the calculation of the following boundary line integrals
$y_{S}=\sum_{j=1}^{K} \int_{\Gamma_{j}}\left(\lambda_{j}-\lambda_{i}\right)(y \sin \beta-z \cos \beta) \varphi_{c y} d s$
$z_{S}=\sum_{j=1}^{K} \int_{\Gamma_{j}}\left(\lambda_{j}-\lambda_{i}\right)(z \cos \beta-y \sin \beta) \varphi_{c z} d s$
Furthermore, the integration constant $c$ (eqn. 26) may be written in boundary integral form as

$$
\begin{array}{r}
c=-\frac{1}{8 A G_{r e f} \Delta} \sum_{j=1}^{K} \int_{\Gamma_{j}}\left(\lambda_{j}-\lambda_{i}\right)\left[8 y \bar{\varphi}_{C} G_{r e f} \Delta \cos \beta-\left(I_{y y} Q_{y}-I_{y z} Q_{z}\right) y^{4} \cos \beta\right. \\
\left.-2\left(I_{z z} Q_{z}-I_{y z} Q_{y}\right) y^{2} z^{2} \sin \beta\right] d s \tag{70}
\end{array}
$$

and the shear deformation coefficients $a_{y}, a_{z}, a_{y z}=a_{z y}$ are obtained from the following boundary line integrals

$$
\begin{align*}
a_{y}=\frac{A}{6 G_{r e f} \Delta^{2}} \sum_{j=1}^{K}\left(G_{j}-G_{i}\right) \int_{\Gamma_{j}} & {\left[I_{y y} h_{y} y^{3} z \sin \beta-I_{y z} h_{z} z^{3} y \cos \beta+\right.} \\
& \left.+3 G_{r e f} \Delta\left(I_{y y} \varphi_{C z} y^{2} \cos \beta-I_{y z} \varphi_{C z} z^{2} \sin \beta\right)\right] d s \tag{71a}
\end{align*}
$$

$$
\begin{align*}
a_{z}=\frac{A}{6 G_{r e f} \Delta^{2}} \sum_{j=1}^{K}\left(G_{j}-G_{i}\right) \int_{\Gamma_{j}} & {\left[I_{z z} g_{y} z^{3} y \cos \beta-I_{y z} g_{z} y^{3} z \sin \beta+\right.} \\
& \left.+3 G_{r e f} \Delta\left(I_{z z} \varphi_{C y} z^{2} \sin \beta-I_{y z} \varphi_{C y} y^{2} \cos \beta\right)\right] d s \tag{71b}
\end{align*}
$$

$$
\begin{align*}
& a_{y z}=\frac{A}{6 G_{r e f} \Delta^{2}} \sum_{j=1}^{K}\left(G_{j}-G_{i}\right) \int_{\Gamma_{j}}\left[I_{y y} g_{z} y^{3} z \sin \beta-I_{y z} g_{y} z^{3} y \cos \beta+\right. \\
&\left.+3 G_{r e f} \Delta\left(I_{y y} \varphi_{C y} y^{2} \cos \beta-I_{y z} \varphi_{C y} z^{2} \sin \beta\right)\right] d s \tag{71c}
\end{align*}
$$

where
$h_{y}=\frac{1}{2} I_{y z} z-I_{y y} y$
$h_{z}=I_{y z} z-\frac{1}{2} I_{y y} y$
$g_{y}=\frac{1}{2} I_{y z} y-I_{z z} z$
$g_{z}=I_{y z} y-\frac{1}{2} I_{z z} z$
Finally, the coordinates of the centroid $C$ with respect to the arbitrary coordinate system $O \widetilde{y z}$ are obtained from
$\widetilde{y}_{C}=\frac{1}{A} \sum_{j=1}^{K} \int_{\Gamma_{j}}\left(\lambda_{j}-\lambda_{i}\right)(\widetilde{y z} \sin \beta) d s$
$\widetilde{z}_{C}=\frac{1}{A} \sum_{j=1}^{K} \int_{\Gamma_{j}}\left(\lambda_{j}-\lambda_{i}\right)(\widetilde{y z} \cos \beta) d s$

## 4 Numerical Examples

On the basis of the analytical and numerical procedures presented in the previous sections, a FORTRAN program has been written and representative examples have been studied to demonstrate the efficiency, wherever possible the accuracy and the range of applications of the developed method. In all the examples treated each cross section has been analysed employing $N=300$ constant boundary elements along the boundary of the cross section, which are enough to ensure the convergence of the solution procedure. Moreover, the CPU time at a Personal Computer $\operatorname{Intel}(\mathrm{R}) 2.00 \mathrm{GHz}$ for the analysis of each example is less than 4 seconds.

Table 1: Resultant transverse shear stresses $\tau_{\Omega}(k P a)$ at points A and B for $Q_{y}=1 k N$ of the composite cross section of Example 1.

| $\frac{G_{1}}{G_{2}}$ | $\frac{G_{3}}{G_{2}}$ | $\left(\tau_{\Omega}^{A}\right)_{2}$ |  | $\left(\tau_{\Omega}^{B}\right)_{1}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Present study | Exact <br> (Muskhelishvili, 1963) | Present study | Exact (Muskhelishvili, 1963) |
| 1 | 0 | 8.9401 | 8.9524 | 5.5141 | 5.5147 |
|  | 2 | 17.7319 | - | 6.69481 | - |
| 2 | 0 | 6.5326 | 6.5425 | 5.1021 | 5.1084 |
|  | 4 | 15.6410 | - | 6.6843 | - |
| 3 | 0 | 5.1287 | 5.1389 | 4.7134 | 4.7229 |
|  | 6 | 13.8086 | - | 6.3768 | - |
| 4 | 0 | 4.2173 | 4.2281 | 4.4271 | 4.4385 |
|  | 8 | 12.3228 | - | 6.0610 | - |
| 5 | 0 | 3.5792 | 3.5906 | 4.2143 | 4.2273 |
|  | 10 | 11.1132 | - | 5.7794 | - |

Table 2: Shear correction factors $\kappa_{y}=\kappa_{z}$ and warping function $\varphi_{C z}(\mathrm{~mm})$ at points A and B of the composite cross section of Example 1.

| $\frac{G_{1}}{G_{2}}$ | $\frac{G_{3}}{G_{2}}$ | $\kappa_{y}=\kappa_{z}$ | $\left(\varphi_{C z}^{A}\right)_{2}$ | $\left(\varphi_{C z}^{B}\right)_{1}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 0 | 0.6819 | 0.7903 | 0.9301 |
|  |  | (0.6818, Cowper, 1966 \& Renton, 1997) <br> (0.7144, Nastran 4.0 Soft) |  |  |
|  | 2 | 0.8862 | 0.3974 | 0.6038 |
| 2 | 0 | 0.6412 | 0.5403 | 0.6259 |
|  | 4 | 0.7813 | 0.2581 | 0.4109 |
| 3 | 0 | 0.6163 | 0.4135 | 0.4759 |
|  | 6 | 0.7054 | 0.1993 | 0.3248 |
| 4 | 0 | 0.6004 | 0.3356 | 0.3848 |
|  | 8 | 0.6536 | 0.1649 | 0.2726 |
| 5 | 0 | 0.5896 | 0.2825 | 0.3233 |
|  | 10 | 0.6165 | 0.1417 | 0.23642 |

Table 3: Resultant transverse shear stresses $\tau_{\Omega}(\mathrm{kPa})$ at points A and B for $Q_{y}=1 \mathrm{kN}$ and shear correction factors $\kappa_{y}=\kappa_{z}$ for various boundary discretization schemes of the composite cross section of Example 1.

|  | Number of boundary elements $N$ |  |  |  |  | Exact |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 100 | 200 | 300 | 600 | 1000 |  |
|  | Resultant transverse shear stress $\left(\tau_{\Omega}^{A}\right)_{2}$ at point A for $G_{1} / G_{2}=2, G_{3} / G_{2}=0$ |  |  |  |  | Muskhelishvili (1963) |
| computed value | 6.4059 | 6.5166 | 6.5326 | 6.5401 | 6.5416 | 6.5425 |
| error (\%) | 2.0879 | 0.3959 | 0.1513 | 0.0367 | 0.0138 |  |
|  | Resultant transverse shear stress $\left(\tau_{\Omega}^{B}\right)_{1}$ at point B for $G_{1} / G_{2}=2, G_{3} / G_{2}=0$ |  |  |  |  | Muskhelishvili (1963) |
| computed value | 4.8743 | 5.0936 | 5.1021 | 5.1068 | 5.1075 | 5.1084 |
| error (\%) | 4.5826 | 0.2897 | 0.1233 | 0.0313 | 0.0176 |  |
|  | $\begin{aligned} & \text { Shear correction factors } \kappa_{y}=\kappa_{z} \\ & \text { for } G_{1} / G_{2}=1, G_{3} / G_{2}=0 \end{aligned}$ |  |  |  |  | $\begin{gathered} \hline \text { Cowper } \\ (1966) \end{gathered}$ |
| computed value | 0.6920 | 0.6823 | 0.6819 | 0.6818 | 0.6818 | 0.6818 |
| error (\%) | 1.4960 | 0.0733 | 0.0147 | 0.0000 | 0.0000 |  |



Figure 3: Composite circular tube cross section of the cantilever beam of Example 1.


Figure 4: Contours of the warping function $\varphi_{C y}(m)$ for $G_{1} / G_{2}=2, G_{3} / G_{2}=0$ (a) and $G_{1} / G_{2}=2, G_{3} / G_{2}=4(\mathrm{~b})$ of the composite cross section of Example 1.


Figure 5: Transverse shear stresses $\tau_{x y}(k P a)$ for $Q_{y}=1 k N$ along the $y$-axis of homogeneous circular cross section of Example 1.

Table 4: Shear correction factors $\kappa_{y}, \kappa_{z}$ of the composite cross section of Example 2.

| $\frac{G_{3}}{G_{1}}$ | $\frac{G_{2}}{G_{1}}$ | $\kappa_{y}$ | $\kappa_{z}$ |
| :---: | :---: | :---: | :---: |
| 10 | 0.0 | 0.6948 | 0.0479 |
|  |  | $(0.8271$, SectionBuilder, 2002) | $(0.0506$, SectionBuilder, 2002) |
|  | 0.2 | 0.8046 | 0.1292 |
| 20 | 0.0 | 0.7526 | 0.0249 |
|  | 0.4 | 0.8273 | 0.1091 |
| 30 | 0.0 | 0.7763 | 0.0168 |
|  | 0.6 | 0.8319 | 0.1013 |
| 40 | 0.0 | 0.7893 | 0.0127 |
|  | 0.8 | 0.8334 | 0.0969 |
| 50 | 0.0 | 0.7974 | 0.0102 |
|  | 1.0 | 0.8337 | 0.0939 |
|  |  | $(0.833$, Fatmi and Zenzri, 2004) | (0.094, Fatmi and Zenzri, 2004) |

## Example 1

A cantilever beam having the composite circular tube cross section shown in Fig. 3 has been studied. In Table 1 the obtained values of the resultant transverse shear stresses $\left(\tau_{\Omega}^{A}\right)_{2},\left(\tau_{\Omega}^{B}\right)_{1}$ at points $A$ and $B$ of the cross section of the beam loaded at its free end by a concentrated force $Q_{y}=1 k N$ are presented, as compared wherever possible with those obtained from an exact solution (Muskhelishvili, 1963). The results are in excellent agreement. In Table 2 the shear correction factors $\kappa_{y}=\kappa_{z}$ (values in parentheses come from an analytical formula developed by Cowper (1966) and Renton (1997) as well as from a 2-D FEM solution using the Nastran code) and the warping function $\left(\varphi_{C z}^{A}\right)_{2},\left(\varphi_{C z}^{B}\right)_{1}$ at points $A$ and $B$ of the cross section are presented. Moreover, in Table 3 the computed values of the resultant transverse shear stresses $\left(\tau_{\Omega}^{A}\right)_{2},\left(\tau_{\Omega}^{B}\right)_{1}$ at points $A$ and $B$ for the case $\left\{G_{1} / G_{2}=2, \quad G_{3} / G_{2}=0\right\}$ and $Q_{y}=1 k N$ as well as the shear correction factor $\kappa_{y}=\kappa_{z}$ for the case $\left\{G_{1} / G_{2}=1, G_{3} / G_{2}=0\right\}$ (homogeneous circular tube cross section) are presented for various boundary discretization schemes. From this table the convergence and stability of the proposed method is concluded. It is apparent that 300 boundary elements are enough to ensure the convergence of the solution procedure. It is also worth here noting that a faster convergence can be achieved by employing another type of constant element, which approximates the geometry with a parabolic arc (Katsikadelis, 2002). Also, in Fig. 4 for the cases $\left\{G_{1} / G_{2}=2, \quad G_{3} / G_{2}=0\right\}$ and $\left\{G_{1} / G_{2}=2, G_{3} / G_{2}=4\right\}$ the contours of the warping function $\varphi_{C y}$ are presented, while in Fig. 5 for the case $G_{1} / G_{2}=$
$G_{3} / G_{2}=0$ (homogeneous circular cross section) the distribution of the transverse shear stresses $\tau_{x y}\left(=\tau_{\Omega}\right)$ for $Q_{y}=1 \mathrm{kN}$ along the $y$-axis of the cross section are presented ( $\max \tau_{x y}^{B E M}=21.2207 \mathrm{kPa}$ ) as compared with those obtained from the exact solution (Sokolnikoff(1956), max $\tau_{x y}^{\text {Exact }}=21.2206 \mathrm{kPa}$ ) and the engineering beam theory ( $\max \tau_{x y}^{E B T}=18.8628 \mathrm{kPa}$ ). From the aforementioned figure both the accuracy of the proposed method $(\operatorname{error}(\%)=0.0005)$ and the discrepancy of the engineering beam theory (max $\operatorname{error}(\%)=11.1109)$ are verified.

Table 5: Shear correction factors $\kappa_{y}, \kappa_{z}, \kappa_{y z}$ coordinates $\left\{y_{S}, z_{S}\right\}$ of the shear center S and angles of the principal shear $\phi^{S}$ and bending $\phi^{B}$ system of the composite cross section of Example 4.

| $\frac{G_{1}}{G_{2}}$ | $\kappa_{y}$ | $\kappa_{z}$ | $\kappa_{y z}$ | $y_{S}$ | $z_{S}$ | $\phi^{S}$ | $\phi^{B}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0.7616 | 0.8553 | -15.6709 | -0.0695 | -0.0718 | $-20.7934^{o}$ | $-4.8601^{o}$ |
| 4 | 0.6672 | 0.8308 | -12.8046 | -0.0808 | -0.0905 | $-13.9448^{o}$ | $-5.5983^{o}$ |
| 6 | 0.6064 | 0.8145 | -10.8031 | -0.0848 | -0.0983 | $-11.8593^{o}$ | $-5.8253^{o}$ |
| 8 | 0.5559 | 0.8007 | -9.2984 | -0.0877 | -0.1042 | $-10.6812^{o}$ | $-5.9714^{o}$ |
| 10 | 0.5145 | 0.7892 | -8.1765 | -0.0899 | -0.1089 | $-9.9370^{o}$ | $-6.0731^{o}$ |
| Schramm, Rubenchik and Pilkey $(1997)$, Pilkey $(2002)$ |  |  |  |  |  |  |  |
| 1 | 0.7614 | 0.8551 | -15.6843 | -0.0695 | -0.0720 | $-20.7787^{\circ}$ | $-4.8601^{o}$ |

## Example 2

As a second example the composite rectangular cross section shown in Fig. 6 has been analyzed. In Table 4 the shear correction factors $\kappa_{y}, \kappa_{z}$ of the cross section are presented, where the values in parentheses come from a $2-$ D FEM solution ignoring the boundary conditions at the interfaces (SectionBuilder, 2002) and from a 3-D FEM solution (Fatmi and Zenzri, 2004) of the 'exact' elastic beam theory (Ladevéze and Simmonds, 1998). The accuracy of the results between BEM (present study) and 3-D FEM solution is remarkable, while the discrepancy of the results between BEM (present study) and 2-D FEM arisen from the ignorance of the boundary conditions at the interfaces is easily verified. Moreover, for the case $\left\{G_{3} / G_{1}=10, \quad G_{2} / G_{1}=0.2\right\}$ in Fig.7a,b the warping surfaces $\varphi_{C y}$ and $\varphi_{C z}$ of the composite cross section are presented. Finally, for the case $\left\{G_{3} / G_{1}=10, \quad G_{2} / G_{1}=0.2\right\}$ in Fig. 8 a for $Q_{z}=1 \mathrm{kN}$ and in Fig. 8 b for $Q_{y}=1 \mathrm{kN}$ the distributions of the resultant transverse shear stresses $\tau_{\Omega}$ in the interior of the cross section are presented, respectively.

## Example 3

A cantilever beam of a steel IPB600-section (Eurocode No 3) in contact with a concrete rectangular one, as shown in Fig.9, has been studied. In Fig.10a,b the


Figure 6: Composite rectangular cross section of the cantilever beam of Example 2.


Figure 7: Warping surface $\varphi_{C y}(m)$ (a) and $\varphi_{C z}(m)$ (b) for $G_{3} / G_{1}=10, G_{2} / G_{1}=$ 0.2 of the composite cross section of Example 2.
warping functions $\varphi_{C y}$ and $\varphi_{C z}$ along the boundaries of the composite cross section are presented. Furthermore, in Fig.11a for $Q_{z}=1 k N$ and in Fig.11b for $Q_{y}=1 k N$ the distributions of the resultant transverse shear stress $\tau_{\Gamma}$ along the boundary of


Figure 8: Distributions of the resultant transverse shear stresses $\tau_{\Omega}(k P a)$ in the interior of the cross section of Example 2 for $G_{3} / G_{1}=10, G_{2} / G_{1}=0.2$ and for $Q_{z}=1 k N$ (a) and $Q_{y}=1 k N$ (b).
the concrete rectangular part of the cross section are presented, respectively. From Fig.11a it is concluded that the shear stresses in the upper surface of the concrete rectangular section are not zero as it is assumed in engineering beam theory and also that the shear stresses along the thickness of the concrete rectangular section are not constant as it is assumed in thin tube theory (Vlasov, 1961).

## Example 4

As a final example the composite trapezoidal cross section of Fig. 12 with no axis of symmetry has been analyzed. In Table 5 the shear correction factors $\kappa_{y}, \kappa_{z}, \kappa_{y z}$, the coordinates $\left\{y_{S}, z_{S}\right\}$ of the shear center $S$ and the angles of the principal shear $\phi^{S}$ and bending $\phi^{B}$ system with respect to the centroidal coordinate system $y, z$ are presented, as compared wherever possible (homogeneous cross section) with those obtained from a 2-D FEM solution (Schramm, Rubenchik and Pilkey, 1997,


Figure 9: Cantilever composite beam of Example 3, consisting of a steel IPB600section in contact with a concrete rectangular one.


Figure 10: Warping function $\varphi_{C y}$ (a) and $\varphi_{C z}$ (b) along the boundary of the composite cross section of Example 3.

Pilkey, 2002). From the last row of Table 5 the accuracy of the developed method is verified. Also, from this table it can be easily seen that the principal bending and shear axes are of different orientation. Finally, in Fig.13a for $Q_{z}=1 \mathrm{kN}$ and in Fig.13b for $Q_{y}=1 \mathrm{kN}$ the distributions of the resultant transverse shear stresses $\tau_{\Gamma}$ along the boundary of the composite cross section for $G_{1} / G_{2}=4$ are presented, respectively.


For $Q_{z}=1 k N: \tau_{\Gamma}{ }^{A}(k P a)=4.4127$
(a)


For $Q_{y}=1 k N: \tau_{\Gamma}{ }^{B}(k P a)=8.5171$
(b)

Figure 11: Distributions of the resultant transverse shear stresses $\tau_{\Gamma}$ along the boundary of the concrete rectangular part of the composite cross section of Example 3.


Figure 12: Composite trapezoidal cross section of the cantilever beam of Example 4.


Figure 13: Distributions of the resultant transverse shear stresses $\tau_{\Gamma}$ along the boundary of the composite cross section of Example 4, for $G_{1} / G_{2}=4$.

## 5 Concluding Remarks

In this paper the boundary element method is employed to develop a displacement solution for the general transverse shear loading problem of composite beams of arbitrary constant cross section. Three boundary value problems are formulated with respect to corresponding warping functions and solved employing a pure BEM approach. The evaluation of the transverse shear stresses at any interior point is accomplished by direct differentiation of these warping functions, while both the coordinates of the shear center and the shear deformation coefficients are obtained from these functions using only boundary integration. The main conclusions that can be drawn from this investigation are
a. The numerical technique presented in this investigation is well suited for computer aided analysis for composite beams of arbitrary cross section, while the analysis is performed with respect to an arbitrary system of axes and not necessarily the principal one.
b. The convergence of the obtained results employing the proposed numerical procedure with those obtained from an analytical solution and a 3-D FEM is easily verified.
c. Accurate results are obtained using a relatively small number of boundary elements.
d. The inaccuracy of the engineering beam theory for arbitrary cross section is verified.
e. The assumption that the transverse shear stress along the thickness coordinate remains constant is right only in thin-walled cross sections.
f. The developed procedure retains the advantages of a BEM solution over a pure domain discretization method since it requires only boundary discretization.
g. Further research is needed to investigate the influence of the restrained warping due to shear of the composite cross section (nonuniform shear problem).

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