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The Lie-Group Shooting Method for Solving Classical Blasius Flat-Plate Problem

Chih-Wen Chang[1], Jiang-Ren Chang[1] and Chein-Shan Liu[2]

Abstract: In this paper, we propose a Lie-group shooting method to deal with the classical Blasius flat-plate problem and to find unknown initial conditions. The pivotal point is based on the erection of a one-step Lie group element $\mathbf{G}(T)$ and the formation of a generalized mid-point Lie group element $\mathbf{G}(r)$. Then, by imposing $\mathbf{G}(T) = \mathbf{G}(r)$ we can derive some algebraic equations to recover the missing initial conditions. It is the first time that we can apply the Lie-group shooting method to solve the classical Blasius flat-plate problem. Numerical examples are worked out to persuade that the novel approach has better efficiency and accuracy with a fast convergence speed by searching a suitable $r \in (0,\ 1)$ with the minimum norm to fit the targets.

Keyword: One-step group preserving scheme, Blasius equation, Boundary value problem, Shooting method, Estimation of missing initial condition

1 Introduction

When a two-dimensional (2D) steady flow of an incompressible constant property fluid with very low viscosity and high Reynolds number moves promptly over a semi-infinite flat plate, the friction between the fluid and the flat plate will induce the fluid to be obstructed within a thin region immediately adjacent to the boundary layer. The governing equation describing the boundary layer with such fluid characteristics and boundary conditions is called the Blasius equation [Schlichting (1979); Özisik (1979)]. Blasius (1908) gave a solution in the form of a power series for the Blasius equation and since then, it has led to much attention on solving this equation with emphasis on its solving techniques. After that, Töpfer (1912) began to adopt the Runge-Kutta algorithms to solve this equation, and until the era of Howarth (1938), the numerical solution by the Runge-Kutta method is still not as accurate and reliable as presently tabulated result [Schlichting (1979); Özisik (1979)]. Apart from this, Lock (1951, 1954) investigated two cases, where the lower stream was at rest as well as when it was in motion. Later, Potter (1957) extended the research to two fluids of different viscosities and densities, where both fluids were moving concurrently with different velocities. Moreover, Abussita (1994) took a differential equation of mixing layer into account that arises in the Blasius solutions for flow passing a flat plate, and manifested the existence of a solution for this model by using the Weyl techniques. Thereafter, Liao (1997, 1999) proposed a systematic depiction of a new kind of analytic technique for nonlinear problems, namely, the homotopy analysis method. He applied it to give an explicit and analytic solution of the 2D laminar viscous flow over a semi-infinite flat plate. This method may have higher accuracy but it is very complex in expression.

Yu and Chen (1998) converted the Blasius equation (a boundary value problem) to a pair of initial value problems, and then solved those by a differential transformation method. To speed up the convergent rate and the accuracy of calculation, the entire domain needs to be divided into subdomains. Besides, Khabibrakhmanov and Summers (1998) have employed the generalized La-

[1] Department of Systems Engineering and Naval Architecture, National Taiwan Ocean University, Keelung 20224, Taiwan. Corresponding author, Tel.: +886-2-24622192x6031. E-mail address: cjr@mail.ntou.edu.tw

[2] Department of Mechanical and Mechatronic Engineering, National Taiwan Ocean University, Keelung 20224, Taiwan; Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung 20224, Taiwan

guerre polynomials to compute a spectral solution of the Blasius equation on a semi-infinite interval; however, this method involves many calculations for nonlinear algebraic equations. For approximating the solution of the Blasius equation, He (1999) proposed the variational iteration method and Lin (1999) employed the parameter iteration method to deal with this problem. Recently, He (2003) has coupled the iteration method with the perturbation method to solve the Blasius equation. Later, Wang (2004) even proposed the Adomian decomposition method to the transformation of the Blasius equation. Abbasbandy (2007) also employed the same approach and compared it with the homotopy perturbation method. Furthermore, Hashim (2006) commented on "a new algorithm for solving classical Blasius equation" by Wang (2004). As for solutions and error estimates of the Blasius equation, Lee and Hung (2002) proposed the modified group preserving scheme with the shooting method; however, their method shows complicated in algorithms and seems indirect to solve the Blasius equation. After that, Lee (2006) employed the particle swarm optimization method to solve the Blasius equation. The merit of this approach is that the computer storage cell requirements are less than the one with the finite element method. In addition, Ahmad and Al-Barakati (2008) derived a short analytical expression ([4/3] Pade approximant) for the derivative of the solution. The approximant for $f'(\xi)$ is slightly modified to the influence that the resulting expression stands for the function on the entire domain $[0, \infty)$ with a remarkable accuracy; nevertheless, the computational procedures of this approach are still complicated.

In this paper, we propose a Lie-group shooting method to tackle the classical Blasius flat-plate problem. Our method of boundary value problems (BVPs) is based on the group preserving scheme (GPS) developed by Liu (2001) for the integration of initial value problems. The GPS is very effective to cope with the ordinary differential equations (ODEs) with special structures as shown by Liu (2005) for stiff equations and Liu (2006a) for ODEs with constraints. Moreover, a past cone structure and a backward GPS have been successfully developed, such that the one-step backward GPS has been employed to solve the backward in time Burgers equation by Liu (2006b), and the backward in time heat conduction equation by Liu, Chang and Chang (2006). In Liu (2006c, 2006d, 2006e) a Lie-group shooting method (LGSM) is first developed to solve the BVPs of the second order ODEs. Thereafter, Chang, Liu and Chang (2007) have used the LGSM to solve the backward heat conduction problem, and then Liu (2008) has extended the LGSM to estimate the thermophysical property of nonhomogeneous heat conductivity with an accurate result. It will be evident that our approach can be applied to the classical Blasius flat-plate problem, since we are able to search for the missing initial condition through a minimum solution of r in a compact space of $r \in (0, 1)$, where the factor r is used in a generalized mid-point rule for the Lie group of one-step GPS. Especially, the proposed scheme is easy to implement and time saving. Through this study, we may have an easy-implementation LGSM used in the calculations of the classical Blasius flat-plate problem and the accuracy of the proposed scheme will be seen better than other numerical methods.

2 One-step GPS

2.1 The GPS

Although we do not know previously the symmetry group of nonlinear differential equations system, Liu (2001) has embedded it into an augmented system and found an internal symmetry of the new system. That is, for an ODEs system with dimensions n:

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}, t), \quad \mathbf{u} \in R^n,\ t \in R, \tag{1}$$

we can embed it into the following n+1-dimensional augmented system:

$$\frac{d}{dt}\mathbf{X} := \frac{d}{dt}\begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{u},t)}{\|\mathbf{u}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{u},t)}{\|\mathbf{u}\|} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \|\mathbf{u}\| \end{bmatrix}. \tag{2}$$

It is obvious that the first row in Eq. (2) is the same as the original equation (1), but the inclusion of the second row in Eq. (2) gives us a

Minkowskian structure of the augmented system for $\mathbf{X}$ satisfying the cone condition:

$$\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = \mathbf{u}\cdot\mathbf{u} - \|\mathbf{u}\|^2 = 0, \tag{3}$$

where

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & -1 \end{bmatrix} \tag{4}$$

is a Minkowski metric. $\mathbf{I}$ is the identity matrix of order n, and the superscript T denotes the transpose. The cone condition is a natural constraint imposed on the system (2).

Consequently, we have an n+1-dimensional augmented system:

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} \tag{5}$$

with a constraint (3), where

$$\mathbf{A} := \begin{bmatrix} \mathbf{0}_{n\times n} & \frac{\mathbf{f}(\mathbf{u},t)}{\|\mathbf{u}\|} \\ \frac{\mathbf{f}^{\mathrm{T}}(\mathbf{u},t)}{\|\mathbf{u}\|} & 0 \end{bmatrix} \tag{6}$$

is an element of the Lie algebra $so(n,1)$ satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}. \tag{7}$$

Therefore, Liu (2001) has developed a group-preserving numerical scheme as follows:

$$\mathbf{X}_{\mathbf{l}+1} = \mathbf{G}(\mathbf{l})\mathbf{X}_{\mathbf{l}}, \tag{8}$$

where $\mathbf{X}_{\mathbf{l}}$ denotes the numerical value of $\mathbf{X}$ at the discrete time $\mathbf{t}_{\mathbf{l}}$, and $\mathbf{G}(\mathbf{l}) \in SO_o(n,1)$ satisfies

$$\mathbf{G}^{\mathrm{T}}\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{9}$$

$$\det \mathbf{G} = 1, \tag{10}$$

$$G_0^0 > 0, \tag{11}$$

where G_0^0 is the 00th component of $\mathbf{G}$.

2.2 Generalized mid-point rule

Applying scheme (8) to Eq. (5) with a specified initial condition $\mathbf{u}(0) = \mathbf{u}_0$, we can compute the solution $\mathbf{u}(t)$ by the GPS. Assuming that the total time T is divided by K steps, that is, the time stepsize we use in the GPS is $\Delta t = T/K$. Starting from an initial augmented condition $\mathbf{X}_0 = \mathbf{X}(0) = (\mathbf{u}_0^{\mathrm{T}},\ \|\mathbf{u}_0\|)^{\mathrm{T}}$, we want to calculate the value $\mathbf{X}(T) = (\mathbf{u}^{\mathrm{T}}(T),\ \|\mathbf{u}(T)\|)^{\mathrm{T}}$ at a desired time $t = T$.

By applying Eq. (8) step-by-step, we can obtain

$$\mathbf{X}_T = \mathbf{G}_K(\Delta t)\ldots\mathbf{G}_1(\Delta t)\mathbf{X}_0, \tag{12}$$

where $\mathbf{X}_T$ approximates the exact $\mathbf{X}(T)$ with a certain accuracy depending on Δt. However, let us recall that each $\mathbf{G}_i$, $i = 1,\ldots,K$, is an element of the Lie group $SO_o(n,1)$, and by the closure property of Lie group, $\mathbf{G}_K(\Delta t)\ldots\mathbf{G}_1(\Delta t)$ is also a Lie group denoted by $\mathbf{G}$. Hence, we have

$$\mathbf{X}_T = \mathbf{G}\mathbf{X}_0. \tag{13}$$

This is a one-step transformation from $\mathbf{X}_0$ to $\mathbf{X}_T$; see, e.g., Liu (2006f, 2006g).

We can simply calculate $\mathbf{G}$ by a generalized mid-point rule, which is obtained from an exponential mapping of $\mathbf{A}$ by taking the values of the argument variables of $\mathbf{A}$ at a generalized mid-point. The Lie group generated from $\mathbf{A} \in so(n,1)$ is known as a proper orthochronous Lorentz group, which admits a closed-form representation as follows:

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{(a-1)}{\|\hat{\mathbf{f}}\|^2}\hat{\mathbf{f}}\hat{\mathbf{f}}^{\mathrm{T}} & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^{\mathrm{T}}}{\|\hat{\mathbf{f}}\|} & a \end{bmatrix}, \tag{14}$$

where

$$\hat{\mathbf{u}} = r\mathbf{u}_0 + (1-r)\mathbf{u}_T, \tag{15}$$

$$\hat{\mathbf{f}} = \mathbf{f}(\hat{\mathbf{u}}, \hat{t}), \tag{16}$$

$$a = \cosh\left(\frac{T\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{u}}\|}\right), \tag{17}$$

$$b = \sinh\left(\frac{T\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{u}}\|}\right). \tag{18}$$

Here, we employ the initial $\mathbf{u}_0$ and the final $\mathbf{u}_T$ through a suitable weighting factor r to calculate $\mathbf{G}$, where $0 < r < 1$ is a parameter and $\hat{t} = rT$. The above method is applied a generalized mid-point rule on the calculation of $\mathbf{G}$, and the result is a single-parameter Lie group element $\mathbf{G}(r)$.

2.3 A Lie group mapping between two points on the cone

Let us define a new vector

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{u}}\|}, \tag{19}$$

such that Eqs. (14), (17) and (18) can also be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_n + \frac{(a-1)}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^{\mathrm{T}} & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^{\mathrm{T}}}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{20}$$

$$a = \cosh(T\|\mathbf{F}\|), \tag{21}$$

$$b = \sinh(T\|\mathbf{F}\|), \tag{22}$$

respectively. From Eqs. (13) and (20) it follows that

$$\mathbf{u}_T = \mathbf{u}_0 + \eta\mathbf{F}, \tag{23}$$

$$\|\mathbf{u}_T\| = a\|\mathbf{u}_0\| + b\frac{\mathbf{F}\cdot\mathbf{u}_0}{\|\mathbf{F}\|}, \tag{24}$$

where

$$\eta := \frac{(a-1)\mathbf{F}\cdot\mathbf{u}_0 + b\|\mathbf{u}_0\|\|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \tag{25}$$

Substituting

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{u}_T - \mathbf{u}_0) \tag{26}$$

into Eq. (24), we obtain

$$\frac{\|\mathbf{u}_T\|}{\|\mathbf{u}_0\|} = a + b\frac{(\mathbf{u}_T - \mathbf{u}_0)\cdot\mathbf{u}_0}{\|\mathbf{u}_T - \mathbf{u}_0\|\|\mathbf{u}_0\|}, \tag{27}$$

where

$$a = \cosh\left(\frac{T\|\mathbf{u}_T - \mathbf{u}_0\|}{\eta}\right), \tag{28}$$

$$b = \sinh\left(\frac{T\|\mathbf{u}_T - \mathbf{u}_0\|}{\eta}\right) \tag{29}$$

are obtained by inserting Eq. (26) for $\mathbf{F}$ into Eqs. (21) and (22).

Let

$$\cos\theta := \frac{(\mathbf{u}_T - \mathbf{u}_0)\cdot\mathbf{u}_0}{\|\mathbf{u}_T - \mathbf{u}_0\|\|u_0\|}, \tag{30}$$

$$S := T\|\mathbf{u}_T - \mathbf{u}_0\|, \tag{31}$$

and from Eqs. (27)-(29) it follows that

$$\frac{\|\mathbf{u}_T\|}{\|\mathbf{u}_0\|} = \cosh\left(\frac{S}{\eta}\right) + \cos\theta\sinh\left(\frac{S}{\eta}\right). \tag{32}$$

By defining

$$Z := \exp\left(\frac{S}{\eta}\right), \tag{33}$$

we obtain a quadratic equation for Z from Eq. (32):

$$(1+\cos\theta)Z^2 - \frac{2\|\mathbf{u}_T\|}{\|\mathbf{u}_0\|}Z + 1 - \cos\theta = 0. \tag{34}$$

The solution is found to be

$$Z = \frac{\frac{\|\mathbf{u}_T\|}{\|\mathbf{u}_0\|} + \sqrt{\left(\frac{\|\mathbf{u}_T\|}{\|\mathbf{u}_0\|}\right)^2 - 1 + \cos^2\theta}}{1+\cos\theta}, \tag{35}$$

and then from Eqs. (33) and (31) we obtain

$$\eta = \frac{T\|\mathbf{u}_T - \mathbf{u}_0\|}{\ln Z}. \tag{36}$$

Thus, between any two points $(\mathbf{u}_0, \|\mathbf{u}_0\|)$ and $(\mathbf{u}_T, \|\mathbf{u}_T\|)$ on the cone, there exists a single-parameter Lie group element $\mathbf{G}(T) \in SO_o(n,1)$ mapping $(\mathbf{u}_0, \|\mathbf{u}_0\|)$ onto $(\mathbf{u}_T, \|\mathbf{u}_T\|)$, which is given by

$$\begin{bmatrix} \mathbf{u}_T \\ \|\mathbf{u}_T\| \end{bmatrix} = G(T)\begin{bmatrix} \mathbf{u}_0 \\ \|\mathbf{u}_0\| \end{bmatrix}, \tag{37}$$

where $\mathbf{G}$ is uniquely determined by $\mathbf{u}_0$ and $\mathbf{u}_T$ through the following equations:

$$\mathbf{G}(T) = \begin{bmatrix} \mathbf{I}_n + \frac{(a-1)}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^{\mathrm{T}} & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^{\mathrm{T}}}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{38}$$

$$a = \cosh(T\|\mathbf{F}\|), \tag{39}$$

$$b = \sinh(T\|\mathbf{F}\|), \tag{40}$$

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{u}_T - \mathbf{u}_0), \tag{41}$$

in which η is still calculated by Eq. (36).

3 The Lie-group shooting method

To this point we have only considered the solutions of differential equations for which the initial conditions are known. However, many engineering applications of differential equations do not specify all initial conditions, but rather some given boundary conditions. Let us consider the following third order boundary value problem:

$$af''' + ff'' = 0, \quad f = f(\xi), \tag{42}$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1, \tag{43}$$

where the prime stands for the differential with respect to ξ. When the factor a is equal to 2, Eq. (42) is a form of the Blasius relation for the flat-plate. Cortell (2005) mentioned that the factor of 2 in Eq. (42) has been omitted in favor of a canonical form of this equation. The conditions for the constant a in the interval $1 \le a \le 2$, which guarantee an existed solution for Eqs. (42) and (43) should be checked before any numerical scheme is applied; otherwise, a list of meaningless output may be generated. Note that the problem in Eqs. (42) and (43) is depicted on a semi-infinite physical domain. Because this is not very convenient for computations, the condition of $\xi = \infty$ in Eq. (43) is usually replaced by a condition with a "sufficiently large" ξ; see, e.g., Asaithambi (1997, 1998, 2004, 2004, 2005).

Let $y_1 = f$, $y_2 = f'$ and $y_3 = f''$. We can rewrite the considered third order boundary value problem as

$$y_1' = y_2, \tag{44}$$

$$y_2' = y_3, \tag{45}$$

$$y_3' = \frac{-y_1 y_3}{a} = Y(y_1, y_3), \tag{46}$$

$$y_1(0) = \alpha = 0, \quad y_1(T) = A, \tag{47}$$

$$y_2(0) = \beta = 0, \quad y_2(T) = B = 1, \tag{48}$$

$$y_3(0) = \delta, \quad y_3(T) = C = 0, \tag{49}$$

where A, T and δ are three unknown constants. Here, we have left T as an unknown to replace ∞ and alternatively, we are imposed a physically reasonable condition $f''(T) = 0$.

Let

$$\mathbf{u} := \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \tag{50}$$

From Eqs. (41) and (47)-(49) it follows that

$$\mathbf{F} := \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \frac{1}{\eta} \begin{bmatrix} A-\alpha \\ B-\beta \\ C-\delta \end{bmatrix}. \tag{51}$$

Starting from an initial guess of (A, T, δ), we use the following equation to calculate η:

$$\eta = \frac{T\sqrt{(\alpha-A)^2 + (\beta-B)^2 + (\delta-C)^2}}{\ln Z}, \tag{52}$$

in which Z is calculated by

$$Z = \frac{\frac{\sqrt{A^2+B^2+C^2}}{\sqrt{\alpha^2+\beta^2+\delta^2}} + \sqrt{\frac{A^2+B^2+C^2}{\alpha^2+\beta^2+\delta^2} - (1-\cos^2\theta)}}{1+\cos\theta}, \tag{53}$$

where

$$\cos\theta = \frac{\alpha(A-\alpha) + \beta(B-\beta) + \delta(C-\delta)}{\sqrt{(\alpha-A)^2+(\beta-B)^2+(\delta-C)^2}\sqrt{\alpha^2+\beta^2+\delta^2}}. \tag{54}$$

The above three equations are obtained from Eqs. (36), (35) and (30) by inserting Eq. (50) for **u**.

By comparing Eq. (51) with Eq. (19), and with the aid of Eqs. (15), (16), and (44)-(49) we obtain

$$A = \alpha + \frac{\eta[r\beta + (1-r)B]}{\rho}, \tag{55}$$

$$T = \frac{2\rho \ln Z}{r[r\alpha + (1-r)A]\sqrt{(\alpha-A)^2+(\beta-B)^2+(\delta-C)^2}}, \tag{56}$$

$$\delta = \frac{\rho(B-\beta)}{r\eta}, \tag{57}$$

where

$$\hat{Y} := Y(r\alpha + (1-r)A, r\delta + (1-r)C), \tag{58}$$

$$\rho := \sqrt{\begin{array}{c}[r\alpha + (1-r)A]^2 + [r\beta + (1-r)B]^2 \\ +[r\delta + (1-r)C]^2\end{array}}. \tag{59}$$

The above derivation of the governing equations (52)-(59) is stemmed from by letting the two **F**'s in Eqs. (19) and (41) be equal, which is essentially identical to the specification of $\mathbf{G}(T) = \mathbf{G}(r)$ in terms of the Lie group elements $\mathbf{G}(T)$ and $\mathbf{G}(r)$.

For a specified r and the given vector field Y, Eqs. (55), (56) and (57) can be used to generate the new (A, T, δ) by repeating the above process in Eqs. (52)-(59) until (A, T, δ) converges according to a given stopping criterion:

$$\sqrt{(A_{i+1}-A_i)^2 + (T_{i+1}-T_i)^2 + (\delta_{i+1}-\delta_i)^2} \le \varepsilon_1, \tag{60}$$

which means that the norm of the difference between the $\imath$+1-th and the $\imath$-th iterations of (A, T, δ) is smaller than a given stopping criterion ε_1, say $\varepsilon_1 = 10^{-10}$. If δ is available, we can return to Eqs. (44)-(49) but with merely integrating the following equations by a forward integration scheme as the one given in Section 2:

$$y_1' = y_2, \tag{61}$$

$$y_2' = y_3, \tag{62}$$

$$y_3' = \frac{-y_1 y_3}{a}, \tag{63}$$

$$y_1(0) = \alpha, \tag{64}$$

$$y_2(0) = \beta, \tag{65}$$

$$y_3(0) = \delta. \tag{66}$$

So far, we have not yet said that how to determine r. Let $y_n^r(T)$ denote the above solution of y_n at T. We start from an r to determine δ by Eqs. (52)-(60) and then numerically integrate Eqs. (61)-(66) from t = 0 to τ = T, and compare the end values of $y_2^r(T)$ and $y_3^r(T)$ with the exact B and X. Then, we apply the minimum norm to fit two targets to select a suitable r, which requires us to calculate Eqs. (61)-(66) at each calculation of $\sqrt{(y_2^r - B)^2 + (y_3^r - C)^2}$, until $\sqrt{(y_2^r - B)^2 + (y_3^r - C)^2}$ is smaller enough to satisfy the criterion of $\sqrt{(y_2^r - B)^2 + (y_3^r - C)^2} < \varepsilon_{\min}$, where $\varepsilon_{\min}$is a given error tolerance, say $\varepsilon_{\min} = 0.02$. Because the numerical method is very stable, we can fast carry off the correct range of r through some trials and modifications.

4 Interpretative results

Following Section 3, when the factor a is equal to 2, we apply the LGSM to the classical Blasius flat-plate problem with an initial (A, δ) = (3, 2). Through some trials we took r = 0.627891. By using a stepsize $\Delta\xi = 0.0001$ the numerical results are shown in Fig. 1 and Table 1, respectively. From Table 1, it is apparent that our results are in great agreement with those given by Howarth (1938).

When the factor a is equal to 1, 1.2, 1.5 and 1.8, we apply the LGSM to the same problem with an initial (A, δ) = (3, 2) and through some trials we took r = 0.520748, r = 0.5502778, r = 0.5851937 and r = 0.612575, respectively. By using a stepsize $\Delta\xi = 0.0001$, the numerical results are shown in Table 2. From Table 2, it is obvious that functions f and f' decrease with the increasing a. Evidently, the function f'' at ξ= 0 decreases with the increasing a. The numerical results for a = 1.2, 1.5, 1.8 and 2 are plotted in Figs. 2 to 5, respectively.

5 Conclusions

There are two significant points deserved further inform in this paper. One is the erection of a one-step group $\mathbf{G}(T)$ and the full use of Eqs. (23) and (24), which are the Lie group transformation between initial conditions and final conditions in the augmented Minkowski space. The other is the use of a generalized mid-point rule to erect another Lie group element $\mathbf{G}(r)$. In order to evaluate the missing initial conditions for the boundary value problems of the Blasius equation, we have employed the equation $\mathbf{G}(T)$ = $\mathbf{G}(r)$ to derive

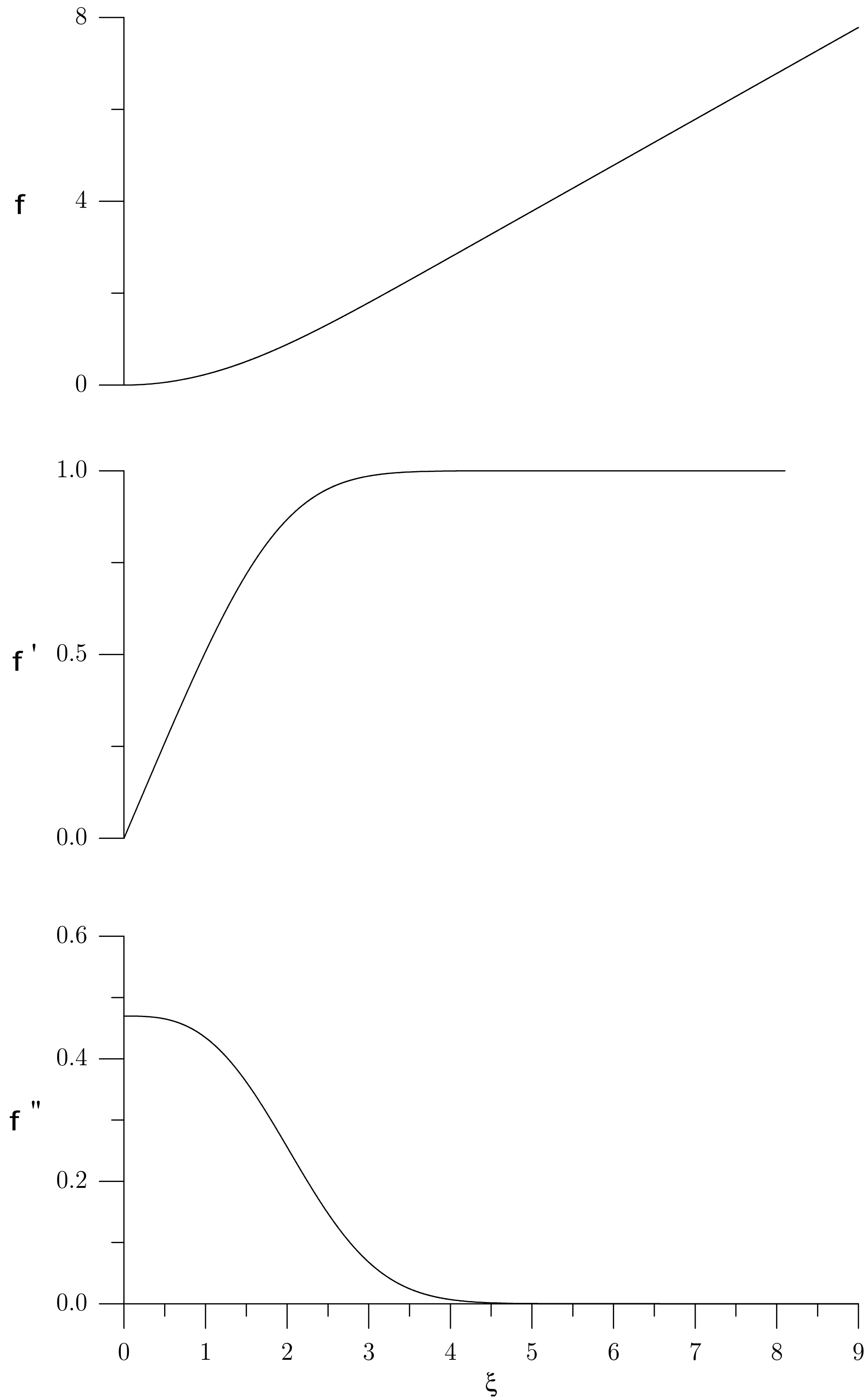

Figure 1: Variations of functions f, f' and f'' with respect to ξ at $a = 1$.

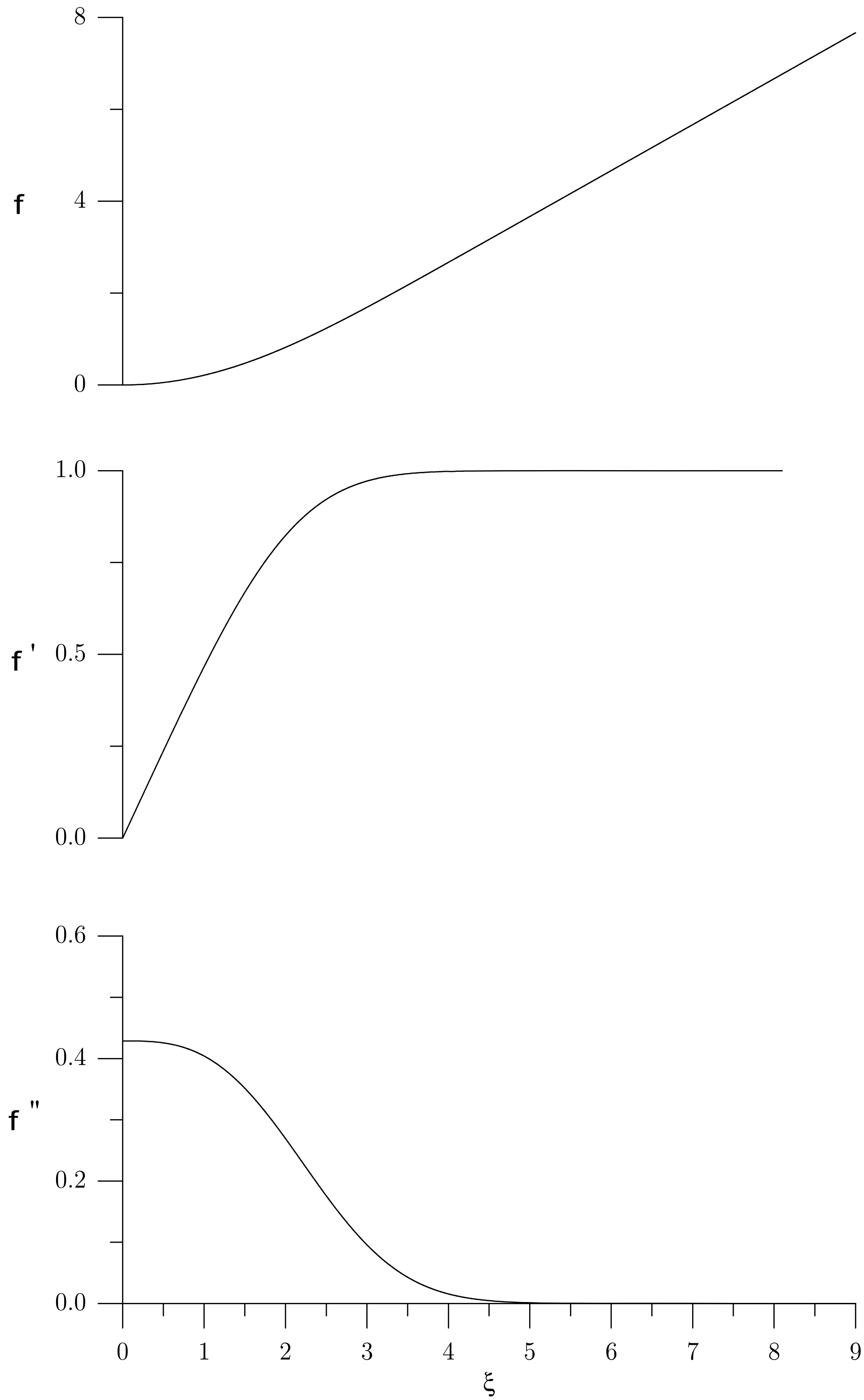

Figure 2: Variations of functions f, f' and f'' with respect to ξ at $a = 1.2$.

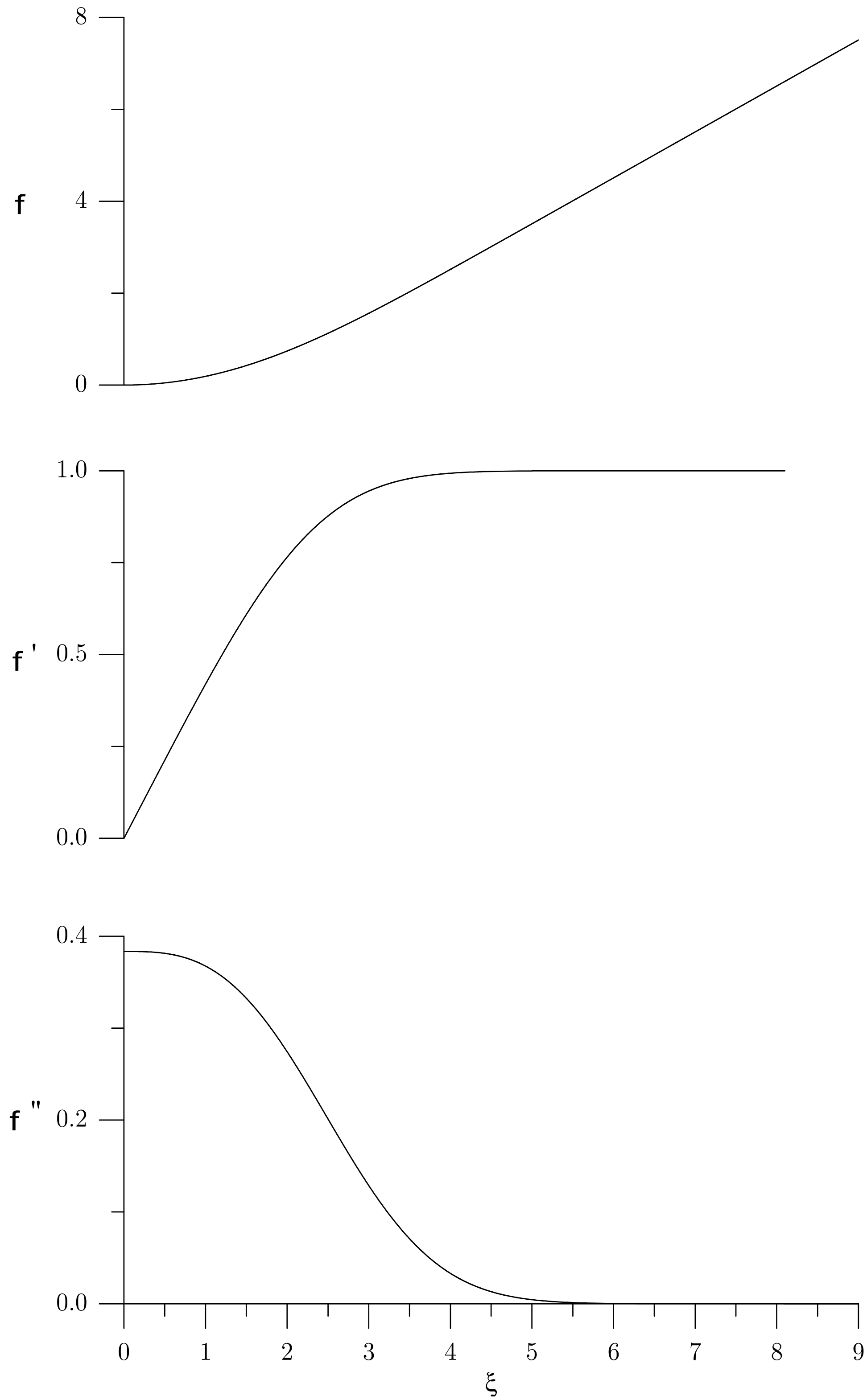

Figure 3: Variations of functions f, f' and f'' with respect to ξ at $a = 1.5$.

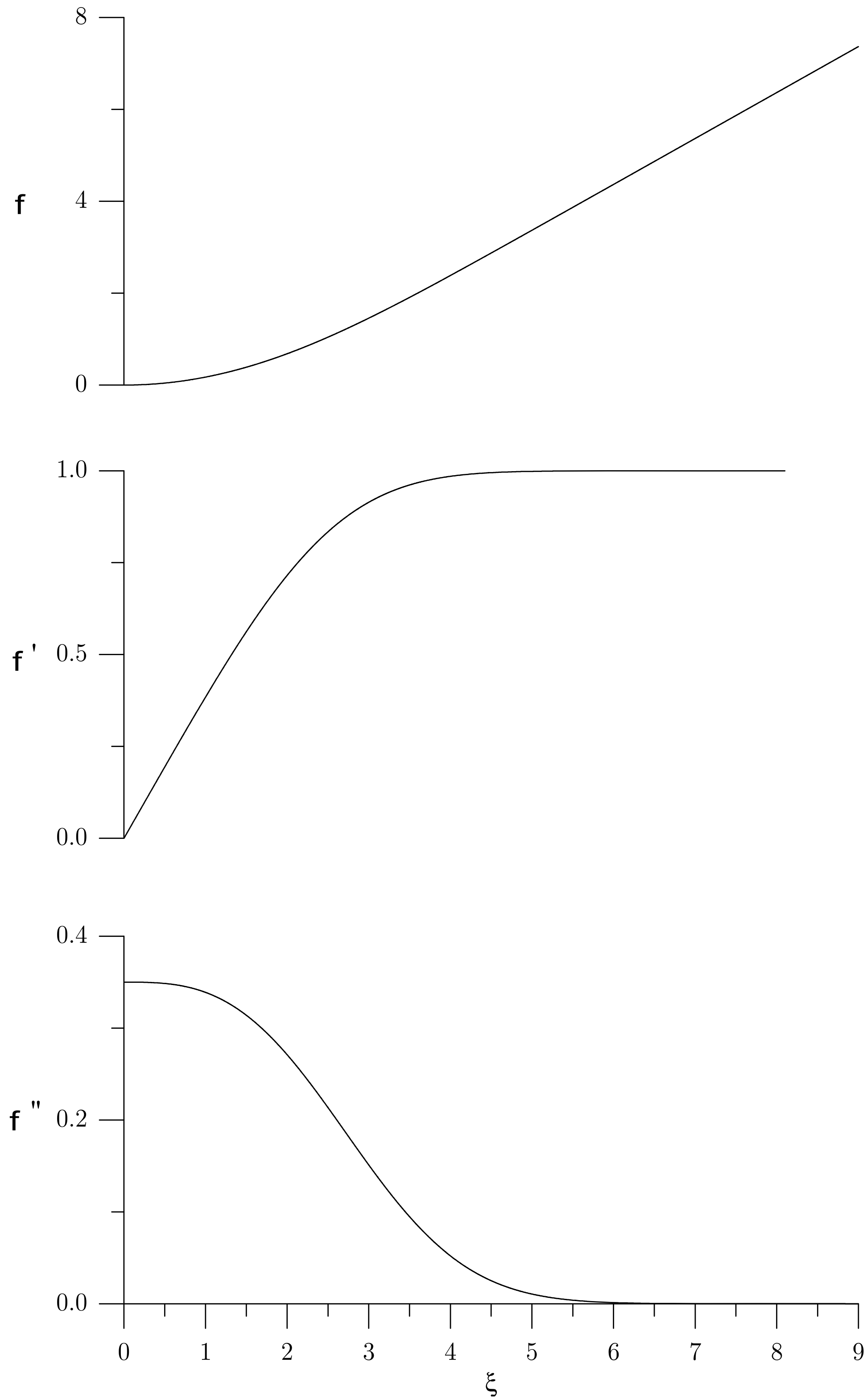

Figure 4: Variations of functions f, f' and f'' with respect to ξ at a = 1.8.

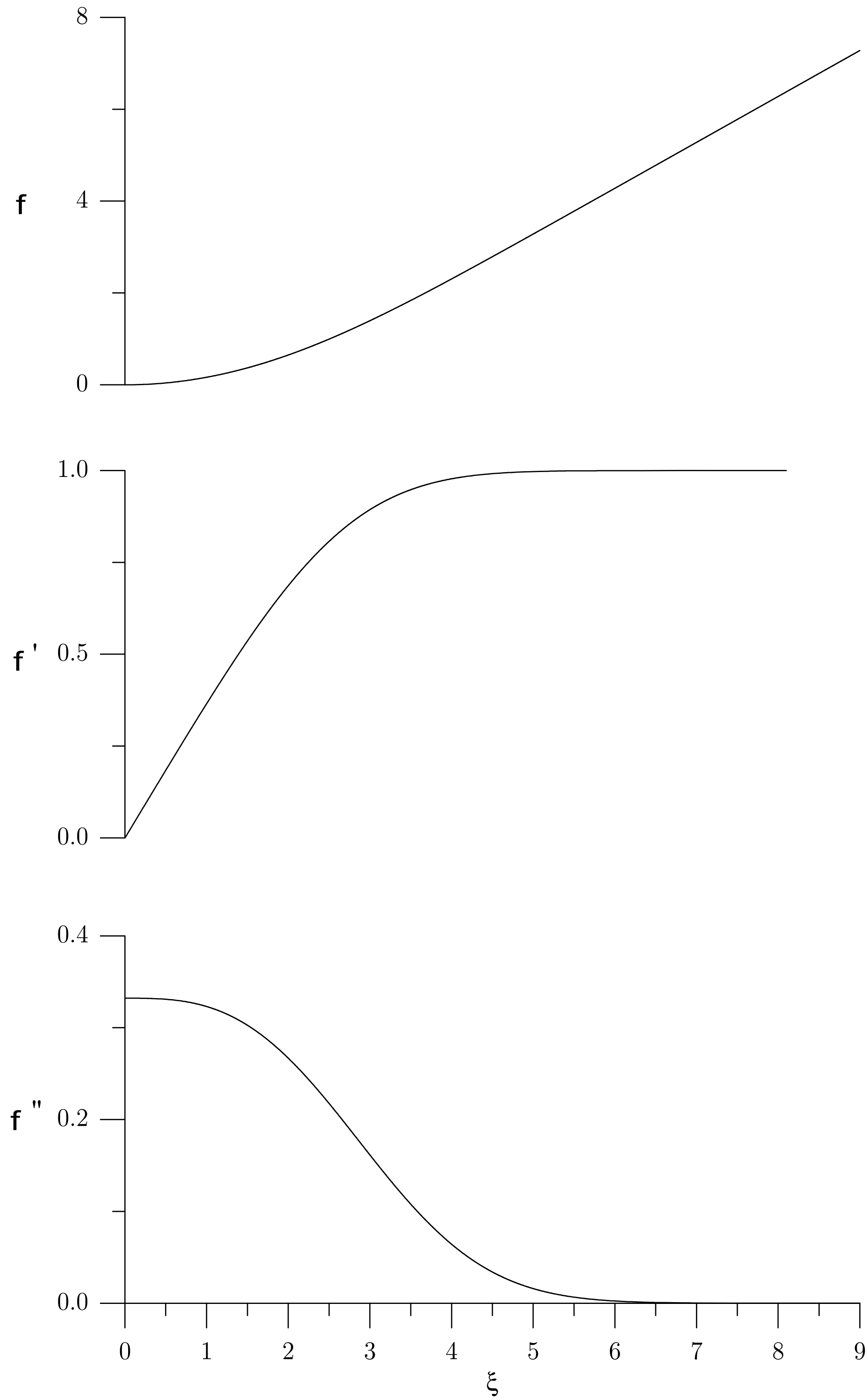

Figure 5: Variations of functions f, f' and f'' with respect to ξ at $a = 2$.

Table 1: Values of functions f, f' and f'' with $a = 2$. LGSM–Lie-group shooting method; C–Cortell's solution (2005); H–Howarth's solution (1938); Yu–Yu and Chen's solution (1998)

a		ξ	LGSM	C	H	Yu
2	f	0	0.00000	0.00000	0.00000	0.00000
		1	0.16556	0.16557	0.16557	0.16556
		2	0.65003	0.65003	0.65003	0.65002
		3	1.39685	1.39682	1.39682	1.39681
		4	2.30584	2.30576	2.30576	2.30575
		5	3.28342	3.28330	3.28329	3.28327
		6	4.27982	4.27965	4.27964	4.27962
		7	5.27948	5.27927	5.27926	5.27924
		8	6.27950	6.27925	6.27923	6.27921
		9	7.27954	7.27925	7.27923	7.27921
	f'	0	0.00000	0.00000	0.00000	0.00000
		1	0.32979	0.32978	0.32979	0.32978
		2	0.62979	0.62977	0.62977	0.62977
		3	0.84608	0.84605	0.84605	0.84604
		4	0.95556	0.95552	0.95552	0.95552
		5	0.99158	0.99155	0.99155	0.99154
		6	0.99901	0.99898	0.99898	0.99897
		7	0.99996	0.99993	0.99992	0.99992
		8	1.00003	1.00000	1.00000	0.99999
		9	1.00003	1.00000	1.00000	0.99999
	f''	0	0.33206	0.33206	0.33206	0.33206
		1	0.32301	0.32301	0.32301	0.32301
		2	0.26675	0.26675	0.26675	0.26675
		3	0.16136	0.16136	0.16136	0.16136
		4	0.06423	0.06423	0.06424	0.06423
		5	0.01590	0.01591	0.01591	0.01591
		6	0.00240	0.00240	0.00240	0.00240
		7	0.00022	0.00022	0.00022	0.00022
		8	0.00001	0.00001	0.00001	0.00001
		9	0.00000	0.00000	0.00000	0.00000

algebraic equations. Hence, we can solve them through a minimum solution in a compact space of $r \in (0,\ 1)$. Various numerical examples in the interval $1 \le a \le 2$ are examined to ensure that the new algorithm has a fast convergence speed on the solution of r in a pre-selected range smaller than (0, 1) by using the minimum norm to fit two targets, which usually require only small number of iterations. Through this study, it can be concluded that the new shooting method is accurate, effective and stable. Its numerical implementation is very simple and the computation speed is very fast. Therefore, it is highly advocated to be used in the numerical computations of the classical Blasius flat-plate problem.

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Table 2: Values of functions f, f' and f'' for several values of a. LGSM–Lie-group shooting method; C–Cortell's solution (2005)

a	ξ	f of LGSM	f of C	f' of LGSM	f' of C	f'' of LGSM	f'' of C
1	0	0.00000	0.00000	0.00000	0.00000	0.46960	0.46960
	1	0.23298	0.23299	0.46065	0.46063	0.43439	0.43438
	2	0.88682	0.88681	0.81675	0.81670	0.25568	0.25567
	3	1.79567	1.79558	0.96912	0.96906	0.06770	0.06771
	4	2.78407	2.78390	0.99783	0.99777	0.00687	0.00687
	5	3.78349	3.78325	0.99999	0.99994	0.00026	0.00026
	6	4.78354	4.78324	1.0	1.0	0.0	0.0
1.2	0	0.00000	0.00000	0.00000	0.00000	0.42868	0.42868
	1	0.21307	0.21308	0.42243	0.42242	0.40398	0.40397
	2	0.82002	0.82001	0.76876	0.76872	0.26911	0.26910
	3	1.69251	1.69244	0.94679	0.94674	0.09563	0.09563
	4	2.66945	2.66931	0.99387	0.99382	0.01558	0.01558
	5	3.66738	3.66719	0.99971	0.99966	0.00111	0.00111
	6	4.66734	4.66710	1.0	0.99999	0.00003	0.00003
	7	5.66740	5.66710	1.0	1.0	0.0	0.0
1.5	0	0.00000	0.00000	0.00000	0.00000	0.38342	0.38342
	1	0.19089	0.19090	0.37941	0.37939	0.36747	0.36747
	2	0.74236	0.74235	0.70792	0.70789	0.27421	0.27420
	3	1.56322	1.56317	0.90915	0.90910	0.12858	0.12858
	4	2.51741	2.51730	0.98327	0.98323	0.03312	0.03313
	5	3.51053	3.51038	0.99832	0.99827	0.00444	0.00444
	6	4.50999	4.50979	0.99994	0.99990	0.00031	0.00030
	7	5.51001	5.50976	1.0	0.99999	0.00001	0.00001
	8	6.51006	6.50977	1.0	1.0	0.0	0.0
1.8	0	0.00000	0.00000	0.00000	0.00000	0.35002	0.35002
	1	0.17444	0.17444	0.34722	0.34721	0.33888	0.33887
	2	0.68277	0.68276	0.65813	0.65810	0.27096	0.27095
	3	1.45737	1.45732	0.87080	0.87075	0.15101	0.15101
	4	2.38498	2.38488	0.96782	0.96777	0.05217	0.05217
	5	3.36984	3.36969	0.99504	0.99500	0.01056	0.01056
	6	4.36790	4.46766	0.99957	0.99964	0.00123	0.00096
	7	5.36779	5.36754	1.0	0.99997	0.00008	0.00008
	8	6.36783	6.36754	1.0	1.0	0.0	0.0

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