An LGEM to Identify Time-Dependent Heat Conductivity Function by an Extra Measurement of Temperature Gradient

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Abstract: We consider an inverse problem for estimating an unknown heat conductivity parameter $\alpha(t)$ in a heat conduction equation $T_t(x,t) = \alpha(t)T_{xx}(x,t)$ with the aid of an extra measurement of temperature gradient on boundary. Basing on an establishment of the one-step Lie-group elements $\mathbf{G}(r)$ and $\mathbf{G}(\ell)$ for the semi-discretization of heat conduction equation in time domain, we can derive algebraic equations from $\mathbf{G}(r) = \mathbf{G}(\ell)$. The new method, namely the Lie-group estimation method (LGEM), is examined through numerical examples to convince that it is highly accurate and efficient; the maximum estimation error is smaller than 10^{-5} for smooth parameter and for discontinuous and oscillatory parameter the accuracy is still in the order of 10^{-2}. Although the estimation is carried out under a large measurement noise, the LGEM is also stable.

Keyword: Inverse problem, Parameter identification, Lie-group estimation method, Time-dependent heat conductivity

1 Introduction

Present study aims to estimate as accurately as possible the time-varying heat conductivity parameter by solving an inverse heat conduction problem under an overspecified boundary data. The estimation is based on a transient temperature gradient measurement undertaken by a thermocouple on the boundary of a heat conducting rod as schematically shown in Fig. 1.

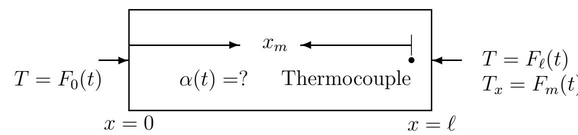

Figure 1: A schematic diagram of the inverse problem to identify parameter through an internal measurement of temperature or boundary temperature gradient.

Applications of inverse methods span over many heat transfer related topics. Sometimes the temperature and heat flux data on the boundary are known and one wants to determine the material properties of the material investigated. Those problems are often referred to as parameter identification problems in the literature [Beck and Arnold (1997); Luce and Perez (1995)]. An application to the determination of thermal heat conductivity of thermo plastics under moulding conditions was studied by Jurkowski, Jarny and Delaunay (1997), and a parameter identification problem for the determination of temperature dependent heat capacity under a convection process was carried out by Zueco, Alhama and González-Fernández (2003). If temperature and heat flux data are known then heat transfer coefficients under the specified boundary conditions may be determined. Some applications of these techniques are given by Kim and Lee (1997) and Su and Hewitt (2004).

Most inverse problems belong to a family of problems that have inherited the property of being ill-posed in the sense of Hadamard [Hadamard (1953); Maz'ya and Shaposhnikova (1998)]. Since the interest in these methods begun with one of the first published paper by Stolz (1960) in the 60's, the applications nowadays range over many scientific fields. Those fields include solid me-

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chanics, fluid dynamics and heat transfer to name only a few.

The parameter determination in partial differential equations from overspecified boundary data plays a crucial role in applied mathematics and physics. These problems are widely encountered in the modeling of physical phenomena [Chao, Chen and Lin (2001); Dehghan (2005); Shamsi and Dehghan (2007); Dehghan and Tatari (2007); Huang and Shih (2007); Liu (2008a); Marin, Power, Bowtell, Sanchez, Becker, Glover and Jones (2008)]. Therefore, we consider an inverse problem of finding an unknown parameter $\alpha(t)$ in a one-dimensional heat conduction equation, of which one needs to find the temperature distribution $T(x,t)$ as well as the heat conductivity function $\alpha(t)$ that simultaneously satisfy

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha(t)}\frac{\partial T(x,t)}{\partial t},\ 0<x<\ell,\ 0<t\leq t_f, \tag{1}$$

$$T(0,t)=F_0(t),\ \ T(\ell,t)=F_\ell(t), \tag{2}$$

$$T(x,0)=f(x). \tag{3}$$

Because the above problem has an unknown function $\alpha(t)$, it cannot be solved directly. This point is drastically different from the direct problem, where $\alpha(t)$ is given. Here, ℓ is a length of the heat conducting rod, and t_f is a terminal time.

A new method will be developed to estimate the unknown heat conductivity $\alpha(t)$ of the above inverse problem, which is subjected to the above boundary conditions and initial condition, as well as an overspecified temperature gradient boundary condition at $x=\ell$:

$$\frac{\partial T(\ell,t)}{\partial x} = F_m(t). \tag{4}$$

Sometimes the measurement of temperature gradient may be a difficult task. Then we may replace it by an internal measurement of temperature at x_m:

$$T(x_m,t)=T_m(t). \tag{5}$$

When the thermocouple is mounted as close as possible to the right-boundary as shown in Fig. 1, we can approximate the quantity in Eq. (4) by

$$\frac{\partial T(\ell,t)}{\partial x} \approx \frac{T(\ell,t)-T(x_m,t)}{\ell - x_m}. \tag{6}$$

For this inverse problem, $\alpha(t)$ can be estimated, provided that an extra measurement as that given by Eq. (4) or Eq. (5) is available. For the problem governed by Eqs. (1)-(3) and (5) there are many studies as can be seen from the papers by Dehghan (2005) and Shamsi and Dehghan (2007) and references therein.

Attempting here is to develop a new Lie-group estimation method (LGEM) for the inverse problem of parameter identification governed by Eqs. (1)-(4). If we consider the extra condition to be Eq. (5), the measurement position x_m is required to be near the right-boundary, such that the approximation in Eq. (6) does not deviate the true boundary value too much.

The parameter identification of $\alpha(t)$ is one of the inverse problems for the applications in heat conduction engineering by considering thermal aging of materials. The inverse problems are those in which one would like to determine the causes for an observed effect. One of the characterizing properties of many of the inverse problems is that they are usually ill-posed, in the sense that a solution that depends continuously on the data does not exist. For the present inverse problem of parameter identification the observed effect is the temperature gradient measurement $\partial T(\ell,t)/\partial x$ at the boundary point $x=\ell$ on the rod. We are interesting to search the cause of the unknown coefficient $\alpha(t)$ in Eq. (1), which induces the effect we observe through measurement. For the inverse problems the measurement error may often lead to a large discrepancy of the true cause.

Our approach of the above inverse problem is based on the numerical method of line, which is a well-developed numerical method that transforms the partial differential equations (PDEs) into a system of ordinary differential equations (ODEs), together with the group preserving scheme (GPS) developed previously by Liu (2001) for ODEs. Recently, Liu (2006a, 2006b, 2006c) has extended the GPS technique to solve the boundary value problems (BVPs), and the numerical results reveal that the GPS is a rather promising method to effectively calculate the two-point BVPs. In the construction of the Lie group method for the calculations of BVPs, Liu (2006a) has introduced the idea of one-step GPS by utilizing the closure property of Lie group, and hence, the new shooting method has been named the Lie-group shooting method (LGSM). The LGSM is also shown effective on the second order general boundary value problems [Liu (2006b)], the singularly perturbed BVPs [Liu (2006c)], and the backward heat conduction problems [Chang, Liu and Chang (2007a, 2007b)]. Recently, Liu (2008a) has employed the LGSM technique to solve accurately the inverse heat conduction problems of identifying the nonhomogeneous heat conductivity functions.

On the other hand, in order to effectively solve the backward in time problems of parabolic PDEs, a past cone structure and a backward group preserving scheme have been successfully developed, such that the new one-step Lie-group numerical methods have been used to solve the backward in time Burgers equation by Liu (2006d), and the backward in time heat conduction equation by Liu, Chang and Chang (2006a).

Liu (2006e, 2006f, 2007) has used the concept of one-step GPS to develop the numerical estimation method for the unknown temperature-dependent heat conductivity and heat capacity of one-dimensional heat conduction equation. Because the Lie-group method possesses a certain advantage than other numerical methods due to its group structure, the Lie-group estimation method (LGEM) is shown to be a powerful technique to solve the inverse problems of parameters identification. However, the methodology of LGEM is not yet applied to the identification of parabolic type linear PDEs with time-dependent coefficient in the open literature. It thus deserves our attention to develop an effective, accurate and stable numerical method for this specific inverse problem and to investige the numerical behavior based on the Lie-group properties.

The Lie-group method is originally used for the boundary value problems as designed by Liu (2006a, 2006b, 2006c) for direct problems. However, these methods are restricted only for the two-dimensional ODEs, and here we will extend them to the multi-dimensional problems. In a series of papers by the author and his coworkers, the Lie-group method reveals its excellent behavior on the numerical solutions of different boundary value problems, for example, Chang, Chang and Liu (2006) to treat the boundary layer equation in fluid mechanics, and Liu (2004), Liu, Chang and Chang (2006a), and Chang, Liu and Chang (2007a, 2007b) to treat the backward heat conduction equation, and Liu, Chang and Chang (2006b) to treat the Burgers equation. Under the advantage of Lie-group method, Liu (2008b) has extended it to solve the inverse Sturm-Liouville problems, and also used the LGEM technique by Liu (2008c) to solve the inverse vibrational problems.

It is interesting to note that the new method of LGEM does not require any a priori regularization when applying it to the inverse problem of parameter identification, and also exhibits several advantages than other methods. It would be clear that the new method can greatly reduce the computational time and is very easy to implement on the calculations of inverse problem of parameter identification. Especially, the present method of LGEM would provide much better computational results than others, which in turns greatly suggest us to use the LGEM in these calculations of inverse problems of parameter identification.

2 The numerical procedures

We are going to solve the present inverse problem of parameter identification through two steps. First, we solve the heat conduction problem in the spatial interval of $0 < x < \ell$ by subjecting it to the initial condition, and the Dirichlet boundary conditions. For this purpose, as that done by Chang, Liu and Chang (2005), Eq. (1) is transformed into the following equations:

$$\frac{\partial T(x,t)}{\partial x} = S(x,t), \tag{7}$$

$$\frac{\partial S(x,t)}{\partial x} = \frac{1}{\alpha(t)}\frac{\partial T(x,t)}{\partial t}. \tag{8}$$

Then, by using a semi-discretization method to discretize the quantities of $T(x,t)$ and $S(x,t)$ in the time domain, we can obtain a system of ODEs for T and S with x as an independent variable. Second, the Lie-group estimation method as first developed by Liu (2006e) is thus extended and applied to the following discretized equations:

$$\frac{\partial T^i(x)}{\partial x} = S^i(x), \;\; i=1,\ldots,n, \tag{9}$$

$$\frac{\partial S^i(x)}{\partial x} = \frac{T^{i+1}(x)-T^{i-1}(x)}{2\alpha_i \Delta t}, \;\; i=1,\ldots,n-1, \tag{10}$$

$$\frac{\partial S^n(x)}{\partial x} = \frac{3T^n(x)-4T^{n-1}(x)+T^{n-2}(x)}{2\alpha_n \Delta t}, \tag{11}$$

where $\Delta t = t_f/n$ is a uniform time increment, and $t_i = i\Delta t$ are the discretized times of which the measurement is sampling by a rate Δt. $T^i(x) = T(x,t_i)$, $S^i(x) = S(x,t_i)$ and $\alpha_i = \alpha(t_i)$ are the discretized quantities at the nodal points of time.

When $i=1$, the term $T^0(x)$ appeared in Eq. (10) is determined by the initial condition (3). While the central difference is used in Eq. (10), we may use the backward difference in Eq. (11) at the last time point in order to maintain the same second-order accuracy.

The two known boundary conditions are given by

$$T^i(0) = F_0(t_i), \;\; i=1,\ldots,n, \tag{12}$$

$$T^i(\ell) = F_\ell(t_i), \;\; i=1,\ldots,n, \tag{13}$$

which are obtained from Eq. (2) by discretizations.

3 Mathematical backgrounds

We will develop a numerical method to estimate the coefficients α_i based on the numerical method of line, which leads to a set of ODEs as already shown above. In order to explore our new method self-content, let us first briefly sketch the group-preserving scheme (GPS) for ODEs and one-step GPS in this section. The readers may refer the author's papers listed in the References for a detailed treatment.

3.1 The GPS

Let us write Eqs. (9)-(11) as in a vector form:

$$\mathbf{y}' = \mathbf{f}(x,\mathbf{y}), \tag{14}$$

where the prime denotes the differential with respect to x, and

$$\mathbf{y} := \begin{bmatrix} \mathbf{T} \\ \mathbf{S} \end{bmatrix}, \;\; \mathbf{f} := \begin{bmatrix} \mathbf{S} \\ \mathbf{h}(x,\mathbf{T}) \end{bmatrix}, \tag{15}$$

in which $\mathbf{T} = (T^1,\ldots,T^n)^t$ and $\mathbf{S} = (S^1,\ldots,S^n)^t$. The components of $\mathbf{h}$ represent the right-hand sides of Eqs. (10) and (11). The dependence of $\mathbf{h}$ on x is due to the dependence of initial condition (3) on x.

When both the vector $\mathbf{y}$ and its magnitude $\|\mathbf{y}\| := \sqrt{\mathbf{y}^t\mathbf{y}} = \sqrt{\mathbf{y}\cdot\mathbf{y}}$ are combined into a single augmented vector

$$\mathbf{X} = \begin{bmatrix} \mathbf{y} \\ \|\mathbf{y}\| \end{bmatrix}, \tag{16}$$

Liu (2001) has transformed Eq. (14) into an augmented system:

$$\mathbf{X}' = \mathbf{A}\mathbf{X} := \begin{bmatrix} \mathbf{0}_{2n\times 2n} & \frac{\mathbf{f}(x,\mathbf{y})}{\|\mathbf{y}\|} \\ \frac{\mathbf{f}^t(x,\mathbf{y})}{\|\mathbf{y}\|} & 0 \end{bmatrix} \mathbf{X}, \tag{17}$$

where $\mathbf{A}$ is an element of the Lie algebra $so(2n,1)$ satisfying

$$\mathbf{A}^t\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}, \tag{18}$$

and

$$\mathbf{g} = \begin{bmatrix} \mathbf{I}_{2n} & \mathbf{0}_{2n\times 1} \\ \mathbf{0}_{1\times 2n} & -1 \end{bmatrix} \tag{19}$$

is a Minkowski metric. Here, $\mathbf{I}_{2n}$ is the identity matrix, and the superscript t stands for the transpose.

The augmented variable $\mathbf{X}$ can be viewed as a point in the Minkowski space $\mathbb{M}^{2n+1}$, satisfying the cone condition:

$$\mathbf{X}^t\mathbf{g}\mathbf{X} = \mathbf{y}\cdot\mathbf{y} - \|\mathbf{y}\|^2 = 0. \tag{20}$$

Accordingly, Liu (2001) has developed a group preserving scheme (GPS) to guarantee that each $\mathbf{X}_k$ locates on the cone:

$$\mathbf{X}_{k+1} = \mathbf{G}(k)\mathbf{X}_k, \tag{21}$$

where $\mathbf{X}_k$ denotes the numerical value of $\mathbf{X}$ at the discrete x_k, and $\mathbf{G}(k) \in SO_o(2n,1)$ satisfies

$$\mathbf{G}^t\mathbf{g}\mathbf{G} = \mathbf{g}, \tag{22}$$

$$\det \mathbf{G} = 1, \tag{23}$$

$$G_0^0 > 0, \tag{24}$$

where G_0^0 is the 00-th component of $\mathbf{G}$.

3.2 One-step Lie-group transformation

Throughout this paper we use the superscripted symbols $\mathbf{y}^0$ to denote the value of $\mathbf{y}$ at $x = 0$, and $\mathbf{y}^\ell$ the value of $\mathbf{y}$ at $x = \ell$. Notice that $\mathbf{y}^0$ cannot be a zero vector, i.e., $\|\mathbf{y}^0\| > 0$; otherwise, from Eq. (14) it follows that $\mathbf{y} \equiv \mathbf{0}$ for all $x \in (0,\ell]$.

Applying scheme (21) on Eq. (17) with a specified left-boundary condition $\mathbf{X}(0) = \mathbf{X}^0$ we can compute the solution $\mathbf{X}(x)$ by the GPS. Assuming that the spatial stepsize used in the GPS is $\Delta x = \ell/K$, and starting from an augmented left-boundary condition $\mathbf{X}^0 = ((\mathbf{y}^0)^t, \|\mathbf{y}^0\|)^t \neq \mathbf{0}$ we will calculate the value $\mathbf{X}^\ell = ((\mathbf{y}^\ell)^t, \|\mathbf{y}^\ell\|)^t$ at the right-boundary $x = \ell$.

By applying Eq. (21) step-by-step we can obtain

$$\mathbf{X}^\ell = \mathbf{G}_K(\Delta x)\cdots\mathbf{G}_1(\Delta x)\mathbf{X}^0. \tag{25}$$

However, let us recall that each $\mathbf{G}_i,\ i = 1,\dots,K$, is an element of the Lie group $SO_o(2n,1)$, and by the closure property of the Lie group, $\mathbf{G}_K(\Delta x)\cdots\mathbf{G}_1(\Delta x)$ is also a Lie group denoted by $\mathbf{G}$. Hence, from Eq. (25) it follows that

$$\mathbf{X}^\ell = \mathbf{G}\mathbf{X}^0. \tag{26}$$

This is a one-step transformation from $\mathbf{X}^0$ to $\mathbf{X}^\ell$.

It should be stressed that the one-step Lie-group transformation property is usually not shared by other numerical methods, because those methods do not belong to the Lie-group schemes. This important property has first pointed out by Liu (2006d) and used to solve the backward in time Burgers equation. After that Liu (2006e) has used this concept to establish a one-step estimation method to estimate the temperature-dependent heat conductivity, and then extended to estimate thermophysical properties of heat conductivity and heat capacity [Liu (2006f); Liu (2007); Liu, Liu and Hong (2007)].

The remaining problem is how to calculate $\mathbf{G}$. While an exact solution of $\mathbf{G}$ is not available, we can calculate $\mathbf{G}$ through a numerical method by a generalized mid-point rule, which is obtained from an exponential mapping of $\mathbf{A}$ by taking the values of the argument variables of $\mathbf{A}$ at a generalized mid-point. The Lie group generated from such an $\mathbf{A} \in so(2n,1)$ by an exponential mapping is

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_{2n} + \frac{(a-1)}{\|\hat{\mathbf{f}}\|^2}\hat{\mathbf{f}}\hat{\mathbf{f}}^t & \frac{b\hat{\mathbf{f}}}{\|\hat{\mathbf{f}}\|} \\ \frac{b\hat{\mathbf{f}}^t}{\|\hat{\mathbf{f}}\|} & a \end{bmatrix}, \tag{27}$$

where

$$\hat{\mathbf{y}} = r\mathbf{y}^0 + (1-r)\mathbf{y}^\ell, \tag{28}$$

$$\hat{\mathbf{f}} = \mathbf{f}(\hat{x},\hat{\mathbf{y}}), \tag{29}$$

$$a = \cosh\left(\frac{\ell\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{y}}\|}\right),\ \ b = \sinh\left(\frac{\ell\|\hat{\mathbf{f}}\|}{\|\hat{\mathbf{y}}\|}\right). \tag{30}$$

Here, we use the left-side $\mathbf{y}^0 = (\mathbf{T}(0),\mathbf{S}(0))$ and the right-side $\mathbf{y}^\ell = (\mathbf{T}(\ell),\mathbf{S}(\ell))$ through a suitable weighting factor r to calculate $\mathbf{G}$, where $r \in (0,1)$ is a parameter to be determined and $\hat{x} = r\ell$. To stress its dependence on r we denote this $\mathbf{G}$ by $\mathbf{G}(r)$.

3.3 A Lie-group mapping between two points on the cone

Let us define a new vector

$$\mathbf{F} := \frac{\hat{\mathbf{f}}}{\|\hat{\mathbf{y}}\|}, \tag{31}$$

such that Eqs. (27) and (30) can also be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_{2n} + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^t & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^t}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{32}$$

$$a = \cosh(\ell\|\mathbf{F}\|),\ \ b = \sinh(\ell\|\mathbf{F}\|). \tag{33}$$

From Eqs. (16), (26) and (32) it follows that

$$\mathbf{y}^\ell = \mathbf{y}^0 + \eta\mathbf{F}, \tag{34}$$

$$\|\mathbf{y}^\ell\| = a\|\mathbf{y}^0\| + b\frac{\mathbf{F}\cdot\mathbf{y}^0}{\|\mathbf{F}\|}, \tag{35}$$

where

$$\eta := \frac{(a-1)\mathbf{F}\cdot\mathbf{y}^0 + b\|\mathbf{y}^0\|\|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \tag{36}$$

Substituting $\mathbf{F}$ in Eq. (34) written as

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{y}^\ell - \mathbf{y}^0) \tag{37}$$

into Eq. (35) and dividing both the sides by $\|\mathbf{y}^0\|$ by noting $\|\mathbf{y}^0\| > 0$, we obtain

$$\frac{\|\mathbf{y}^\ell\|}{\|\mathbf{y}^0\|} = a + b\frac{(\mathbf{y}^\ell - \mathbf{y}^0)\cdot\mathbf{y}^0}{\|\mathbf{y}^\ell - \mathbf{y}^0\|\|\mathbf{y}^0\|}, \tag{38}$$

where, after inserting Eq. (37) for $\mathbf{F}$ into Eq. (33), a and b are now written as

$$a = \cosh\left(\frac{\ell\|\mathbf{y}^\ell - \mathbf{y}^0\|}{\eta}\right),\ \ b = \sinh\left(\frac{\ell\|\mathbf{y}^\ell - \mathbf{y}^0\|}{\eta}\right). \tag{39}$$

Let

$$\cos\theta := \frac{(\mathbf{y}^\ell - \mathbf{y}^0)\cdot\mathbf{y}^0}{\|\mathbf{y}^\ell - \mathbf{y}^0\|\|\mathbf{y}^0\|}, \tag{40}$$

$$\ell_y := \ell\|\mathbf{y}^\ell - \mathbf{y}^0\|, \tag{41}$$

where $0 \le \theta \le \pi$ is the intersection angle between vectors $\mathbf{y}^\ell - \mathbf{y}^0$ and $\mathbf{y}^0$, and thus from Eqs. (38) and (39) it follows that

$$\frac{\|\mathbf{y}^\ell\|}{\|\mathbf{y}^0\|} = \cosh\left(\frac{\ell_y}{\eta}\right) + \cos\theta\sinh\left(\frac{\ell_y}{\eta}\right). \tag{42}$$

Upon defining

$$Z := \exp\left(\frac{\ell_y}{\eta}\right), \tag{43}$$

from Eq. (42) we obtain a quadratic equation for Z:

$$(1+\cos\theta)Z^2 - \frac{2\|\mathbf{y}^\ell\|}{\|\mathbf{y}^0\|}Z + 1 - \cos\theta = 0. \tag{44}$$

Because the following condition is satisfied (see Appendix A):

$$\left(\frac{\|\mathbf{y}^\ell\|}{\|\mathbf{y}^0\|}\right)^2 - 1 + \cos^2\theta \ge 0, \tag{45}$$

the solutions of Z are found to be

$$Z = \left(\frac{\|\mathbf{y}^\ell\|}{\|\mathbf{y}^0\|}\right)^{\pm 1},\ \text{ if }\ \cos\theta = \pm 1, \tag{46}$$

$$Z = \frac{\frac{\|\mathbf{y}^\ell\|}{\|\mathbf{y}^0\|} + \sqrt{\left(\frac{\|\mathbf{y}^\ell\|}{\|\mathbf{y}^0\|}\right)^2 - 1 + \cos^2\theta}}{1+\cos\theta}, \quad \text{if }\ -1 < \cos\theta < 1. \tag{47}$$

From Eqs. (43) and (41) it follows that

$$\eta = \frac{\ell\|\mathbf{y}^\ell - \mathbf{y}^0\|}{\ln Z}. \tag{48}$$

Therefore, we come to an important result that between any two points $(\mathbf{y}^0, \|\mathbf{y}^0\|)$ and $(\mathbf{y}^\ell, \|\mathbf{y}^\ell\|)$ on the cone, there exists a Lie group element $\mathbf{G} \in SO_o(2n,1)$ mapping $(\mathbf{y}^0, \|\mathbf{y}^0\|)$ onto $(\mathbf{y}^\ell, \|\mathbf{y}^\ell\|)$, which is given by

$$\begin{bmatrix} \mathbf{y}^\ell \\ \|\mathbf{y}^\ell\| \end{bmatrix} = \mathbf{G}\begin{bmatrix} \mathbf{y}^0 \\ \|\mathbf{y}^0\| \end{bmatrix}, \tag{49}$$

where $\mathbf{G}$ is uniquely determined by $\mathbf{y}^0$ and $\mathbf{y}^\ell$ through the following equations:

$$\mathbf{G}(\ell) = \begin{bmatrix} \mathbf{I}_{2n} + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^t & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \frac{b\mathbf{F}^t}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{50}$$

$$a = \cosh(\ell\|\mathbf{F}\|),\ \ b = \sinh(\ell\|\mathbf{F}\|), \tag{51}$$

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{y}^\ell - \mathbf{y}^0) = \frac{\ln Z}{\ell}\frac{\mathbf{y}^\ell - \mathbf{y}^0}{\|\mathbf{y}^\ell - \mathbf{y}^0\|}. \tag{52}$$

In view of Eqs. (46), (47) and (40), it can be seen that $\mathbf{G}$ is fully determined by $\mathbf{y}^0$ and $\mathbf{y}^\ell$.

It should be stressed that the above $\mathbf{G}$ is different from the one in Eq. (27). In order to feature its property as a Lie-group mapping between the quantities spanned a whole length ℓ we write it to be $\mathbf{G}(\ell)$. Conversely, $\mathbf{G}(r)$ is a function of r. However, these two Lie group elements $\mathbf{G}(r)$ and $\mathbf{G}(\ell)$ are both indispensable in our development of the Lie-group estimation method in the next section for the inverse problem of parameter identification.

4 The Lie-group estimation method

From Eqs. (9)-(13) it follows that

$$\mathbf{T}' = \mathbf{S}, \tag{53}$$

$$\mathbf{S}' = \mathbf{h}(x, \mathbf{T}), \tag{54}$$

$$\mathbf{T}(0) = \mathbf{T}^0, \;\; \mathbf{T}(\ell) = \mathbf{T}^\ell, \tag{55}$$

$$\mathbf{S}(0) = \mathbf{S}^0, \;\; \mathbf{S}(\ell) = \mathbf{S}^\ell, \tag{56}$$

where $\mathbf{T}^0$ and $\mathbf{T}^\ell$ are known from Eqs. (12) and (13), and $\mathbf{S}^\ell$ is obtained from the measurement in Eq. (4), but $\mathbf{S}^0$ is an unknown vector.

By using Eq. (15) for $\mathbf{y}$ we have

$$\mathbf{y}^0 = \begin{bmatrix} \mathbf{T}^0 \\ \mathbf{S}^0 \end{bmatrix}, \;\; \mathbf{y}^\ell = \begin{bmatrix} \mathbf{T}^\ell \\ \mathbf{S}^\ell \end{bmatrix}, \tag{57}$$

and further inserting them into Eq. (37) yields

$$\mathbf{F} := \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{bmatrix} = \frac{1}{\eta} \begin{bmatrix} \mathbf{T}^\ell - \mathbf{T}^0 \\ \mathbf{S}^\ell - \mathbf{S}^0 \end{bmatrix}. \tag{58}$$

From Eqs. (40), (46)-(48) by inserting Eq. (57) for $\mathbf{y}^0$ and $\mathbf{y}^\ell$ we obtain

$$\cos\theta := \frac{(\mathbf{T}^\ell - \mathbf{T}^0)\cdot \mathbf{T}^0 + (\mathbf{S}^\ell - \mathbf{S}^0)\cdot \mathbf{S}^0}{\sqrt{\|\mathbf{T}^\ell - \mathbf{T}^0\|^2 + \|\mathbf{S}^\ell - \mathbf{S}^0\|^2}\sqrt{\|\mathbf{T}^0\|^2 + \|\mathbf{S}^0\|^2}}, \tag{59}$$

$$Z = \left(\frac{\sqrt{\|\mathbf{T}^\ell\|^2 + \|\mathbf{S}^\ell\|^2}}{\sqrt{\|\mathbf{T}^0\|^2 + \|\mathbf{S}^0\|^2}} \right)^{\pm 1}, \;\text{ if }\; \cos\theta = \pm 1, \tag{60}$$

$$Z = \frac{\frac{\sqrt{\|\mathbf{T}^\ell\|^2 + \|\mathbf{S}^\ell\|^2}}{\sqrt{\|\mathbf{T}^0\|^2 + \|\mathbf{S}^0\|^2}} + \sqrt{\frac{\|\mathbf{T}^\ell\|^2 + \|\mathbf{S}^\ell\|^2}{\|\mathbf{T}^0\|^2 + \|\mathbf{S}^0\|^2} - 1 + \cos^2\theta}}{1 + \cos\theta}, \tag{61}$$

$$\text{if } \; -1 < \cos\theta < 1,$$

$$\eta = \frac{\ell\sqrt{\|\mathbf{T}^\ell - \mathbf{T}^0\|^2 + \|\mathbf{S}^\ell - \mathbf{S}^0\|^2}}{\ln Z}. \tag{62}$$

Comparing Eq. (58) with Eq. (31) and by Eqs. (15) and (57), we can obtain

$$\mathbf{T}^\ell = \mathbf{T}^0 + \frac{\eta}{\|\hat{\mathbf{y}}\|}\hat{\mathbf{S}}, \tag{63}$$

$$\mathbf{S}^\ell = \mathbf{S}^0 + \frac{\eta}{\|\hat{\mathbf{y}}\|}\hat{\mathbf{h}}, \tag{64}$$

where

$$\|\hat{\mathbf{y}}\| = \sqrt{\|\hat{\mathbf{T}}\|^2 + \|\hat{\mathbf{S}}\|^2} = \sqrt{\|r\mathbf{T}^0 + (1-r)\mathbf{T}^\ell\|^2 + \|r\mathbf{S}^0 + (1-r)\mathbf{S}^\ell\|^2}, \tag{65}$$

$$\hat{\mathbf{h}} = \mathbf{h}(\hat{x}, \hat{\mathbf{T}}). \tag{66}$$

For the later use $\hat{\mathbf{h}}$ is written explicitly as

$$\hat{\mathbf{h}} = \begin{bmatrix} \frac{\hat{T}^2 - \hat{T}^0}{2\alpha_1 \Delta t} \\ \vdots \\ \frac{\hat{T}^n - \hat{T}^{n-2}}{2\alpha_{n-1}\Delta t} \\ \frac{3\hat{T}^n - 4\hat{T}^{n-1} + \hat{T}^{n-2}}{2\alpha_n \Delta t} \end{bmatrix}, \tag{67}$$

where $\hat{T}^i = rT^i(0) + (1-r)T^i(\ell) = rF_0(t_i) + (1-r)F_\ell(t_i)$, $i = 1, \ldots, n$, and $\hat{T}^0 = f(\hat{x})$. We shoud stress that $\hat{\mathbf{h}}$ is an unknown vector due to the appearence of the unknown coefficients α_i.

The above governing equations (63) and (64) are obtained by letting the two $\mathbf{F}$'s in Eqs. (31) and (52) be equal, which in terms of the Lie group elements $\mathbf{G}(r)$ and $\mathbf{G}(\ell)$ is essentially identical to the specification of $\mathbf{G}(r) = \mathbf{G}(\ell)$. In this sense we call our method the Lie-group estimation method (LGEM).

As mentioned in the above $\mathbf{S}^0$ and $\hat{\mathbf{h}}$ are two unknown vectors but the three vectors $\mathbf{T}^0$, $\mathbf{T}^\ell$, and $\mathbf{S}^\ell$ are known, given by

$$\mathbf{T}^0 = \begin{bmatrix} F_0(t_1) \\ \vdots \\ F_0(t_n) \end{bmatrix}, \;\; \mathbf{T}^\ell := \begin{bmatrix} F_\ell(t_1) \\ \vdots \\ F_\ell(t_n) \end{bmatrix}, \;\; \mathbf{S}^\ell := \begin{bmatrix} F_m(t_1) \\ \vdots \\ F_m(t_n) \end{bmatrix}. \tag{68}$$

Although $\mathbf{S}^0$ and $\hat{\mathbf{h}}$ are unknowns we can evaluate them as follows. By using

$$\hat{\mathbf{S}} = r\mathbf{S}^0 + (1-r)\mathbf{S}^\ell, \tag{69}$$

from Eqs. (63) and (64) we can solve $\mathbf{S}^0$ and $\hat{\mathbf{h}}$ as follows:

$$\mathbf{S}^0 = \frac{\|\hat{\mathbf{y}}\|}{\eta}(\mathbf{T}^\ell - \mathbf{T}^0) - \frac{(1-r)\eta}{\|\hat{\mathbf{y}}\|}\hat{\mathbf{h}}, \quad (70)$$

$$\hat{\mathbf{h}} = \frac{\|\hat{\mathbf{y}}\|}{\eta}(\mathbf{S}^\ell - \mathbf{S}^0). \quad (71)$$

Because $\mathbf{T}^0$, $\mathbf{T}^\ell$ and $\mathbf{S}^\ell$ are all available, for a specified r, we can use Eqs. (70) and (71), starting from an initial guess of $(\mathbf{S}^0, \hat{\mathbf{h}})$, for example, $(\mathbf{S}^0, \hat{\mathbf{h}}) = (\mathbf{0}, \mathbf{0})$, to generate the new $(\mathbf{S}^0, \hat{\mathbf{h}})$, until they converge according to a given stopping criterion:

$$\sqrt{\|\mathbf{S}^0_{i+1} - \mathbf{S}^0_i\|^2 + \|\hat{\mathbf{h}}_{i+1} - \hat{\mathbf{h}}_i\|^2} \le \varepsilon, \quad (72)$$

which means that the norm of the difference between the $i+1$-th and the i-th iterations of $(\mathbf{S}^0, \hat{\mathbf{h}})$ is smaller than ε.

If the new $\hat{\mathbf{h}}$ is available, then by Eq. (67) we can calculate α_i by

$$\alpha_i = \frac{\hat{F}(t_{i+1}) - \hat{F}(t_{i-1})}{2\Delta t \hat{h}_i}, \quad i = 1, \ldots, n-1, \quad (73)$$

$$\alpha_n = \frac{3\hat{F}(t_n) - 4\hat{F}(t_{n-1}) + \hat{F}(t_{n-2})}{2\Delta t \hat{h}_n}, \quad (74)$$

where $\hat{h}_i$ denotes the i-th component of $\hat{\mathbf{h}}$ and $\hat{F}(t_i) = rF_0(t_i) + (1-r)F_\ell(t_i)$ is known for the specified r.

Under the above new left-boundary condition $\mathbf{S}^0$ together with the known boundary condition of $\mathbf{T}^0$ and the new coefficients α_i, we can return to Eqs. (9)-(11) and integrate them to obtain $\mathbf{T}(\ell)$ and $\mathbf{S}(\ell)$. The above process can be done for all r in the interval of $r \in (0,1)$. Among these solutions we can pick up the best r, which leads to the smallest error of

$$\min_{r \in (0,1)} \sqrt{\|\mathbf{T}(\ell) - \mathbf{T}^\ell\|^2}, \quad (75)$$

such that the right-boundary condition specified by Eq. (2) can be fulfilled as best as possible.

When the process terminates, by inserting the best r and $\hat{h}_i$ into Eqs. (73) and (74) we can estimate the time-dependent coefficient $\alpha(t)$ at the discretized time t_i. As a byproduct we can also obtain the unknown left-boundary condition of $\mathbf{S}^0$, i.e., the left-boundary value of temperature gradient $\partial T(0,t)/\partial x$, and the temperature distribution in the whole rod can be calculated by our method.

5 Numerical examples

Now, we are ready to apply the LGEM on the estimations of $\alpha(t)$ through the tests of numerical examples. When the input measured temperature gradient data $\partial T(\ell,t)/\partial x$ are contaminated by random noise, we are very concerned with the stability of LGEM, which is investigated by adding different levels of random noise on the measured data:

$$\hat{F}_m(t_i) = F_m(t_i) + sR(i), \quad (76)$$

where $F_m(t_i)$ is the exact data, and s specifies the level of noise. We use the function RANDOM_NUMBER given in Fortran to generate the $R(i)$, which are random numbers in $[-1,1]$. Then, the noisy data $\hat{F}_m(t_i)$ is used as input in the calculations.

5.1 Example 1

Let us first consider a simple inverse problem with an exact solution of $\alpha = 1+t$, where $T(x,t)$ is given by

$$T(x,t) = x^2 + (1+t)^2, \quad (77)$$

with boundary conditions

$$T(0,t) = (1+t)^2, \quad T(\ell,t) = \ell^2 + (1+t)^2, \quad (78)$$

and initial condition

$$T(x,0) = 1 + x^2. \quad (79)$$

The data $F_m(t_i)$ is supposed to be the exact value given by $F_m(t_i) = 2\ell$.

We consider $\ell = 0.1$ and $t_f = 1$. The other parameters used in this calculation were $\Delta t = 0.02$ and $\Delta x = 0.002$.

Before employing the numerical method of LGEM to calculate this example we use it to demonstrate how to pick up the best r as specified by Eq. (75). In the calculation we were fixed the stopping criterion used in Eq. (72) to be $\varepsilon = 10^{-10}$. We plot the error of mis-matching the target with respect to r in Fig. 2(a) in a fine range of $0.4 < r < 0.6$. It can be seen that there is a minimum point. Under this r the left-boundary condition and the calculated α_i derived from the LGEM provide the best match to the right-boundary condition. Then we can use the given $\mathbf{T}^0$ and the estimated $\mathbf{S}^0$ to calculate the whole temperature in the rod. In Fig. 2(b) we compare the exact α with the numerical one, of which the numerical error as shown in Fig. 2(c) is very small in the orders from 10^{-8} to 10^{-7}. One may appreciate the high accuracy of the present method of LGEM.

When the length of the rod is increased to $\ell = 1$, the numerical error increases to the order of 10^{-4} as shown in Fig. 2(c) by the dashed line. The accuracy is limited by $(\Delta t)^2$ of the same order. In the case when $F_m(t_i)$ is contaminated by a random noise with level $s = 0.005$ the numerical result as shown in Fig. 2(b) by the dashed line fitted with solid points is slightly deviating from the exact result.

5.2 *Example 2*

In order to compare our numerical results with that calculated by Dehghan (2005), let us consider the following example with $T(x,t)$ given by

$$T(x,t) = \exp\left(\frac{x}{2}\right)\left[\frac{1+2t^3}{1+t^3} + \sin\frac{t}{2}\right]. \tag{80}$$

Simple calculations give

$$T(0,t) = \left[\frac{1+2t^3}{1+t^3} + \sin\frac{t}{2}\right],$$

$$T(\ell,t) = \exp\left(\frac{\ell}{2}\right)\left[\frac{1+2t^3}{1+t^3} + \sin\frac{t}{2}\right], \tag{81}$$

$$T(x,0) = \exp\left(\frac{x}{2}\right), \tag{82}$$

$$\frac{\partial T(\ell,t)}{\partial x} = \frac{1}{2}\exp\left(\frac{\ell}{2}\right)\left[\frac{1+2t^3}{1+t^3} + \sin\frac{t}{2}\right], \tag{83}$$

$$\alpha(t) = \frac{2\left[6t^2 + (1+t^3)^2\cos\frac{t}{2}\right]}{(1+t^3)\left[1+2t^3+(1+t^3)\sin\frac{t}{2}\right]}. \tag{84}$$

Dehghan (2005) had made a mistake to write the term $(1+t^3)$ in the denominator of Eq. (84) as $(1+t)^3$. However, this mistake has been later corrected by Shamsi and Dehghan (2007), but not mentioned that mistake.

In the calculation of this example the length is taken to be $\ell = 1$ and $\Delta x = 0.02$ as that used by Dehghan (2005); however, the time stepsize $\Delta t = 0.001$ is ten times large than $\Delta t = 0.0001$ used by Dehghan (2005). The data $F_m(t_i)$ is supposed to be the exact value given by Eq. (83). Under these parameters values, in Fig. 3(a) we compare the numerical solution of α with the exact one given by Eq. (84) in the time interval of $t \in [0,1]$. These two curves are almost coincident, and the error is plotted in Fig. 3(b), which is smaller than 4×10^{-6}.

The exact values and the so-called exact values by Dehghan (2005) are compared in Table 1, from which it can be seen that the errors due to the above mentioned mistake are about in the order of 10^{-1}. Therefore, as compared our numerical errors with the corrected numerical errors of Dehghan (2005) for α at different times, we can say that the present method can produce about

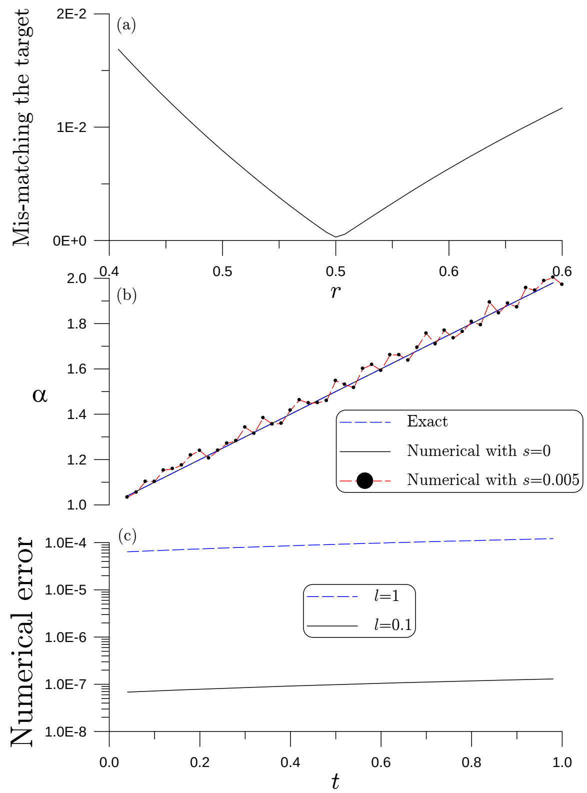

Figure 2: For Example 1: (a) plotting the error of mis-matching the target with respect to r in a fine interval, (b) comparing the numerical results under $s = 0$ and $s = 0.005$ with exact result, and (c) displaying the numerical errors for different lengths.

Table 1: For Example 2 the comparison of present results with the results by Dehghan (2005)

t	exact α	α [Dehghan (2005)]	error of Dehghan (2005)	present error
0.1	2.0145622	1.9796344	3.2×10^{-3}	3.6×10^{-6}
0.2	2.2228619	2.0145622	3.3×10^{-3}	3.0×10^{-6}
0.3	2.5528873	2.0982827	3.3×10^{-3}	1.7×10^{-6}
0.4	2.9043892	2.2228619	3.4×10^{-3}	2.3×10^{-8}
0.5	3.1712532	2.3783814	3.5×10^{-3}	1.6×10^{-6}
0.6	3.2802085	2.5528873	3.6×10^{-3}	2.5×10^{-6}
0.7	3.2151758	2.7328931	3.9×10^{-3}	2.6×10^{-6}
0.8	3.0100474	2.9043892	3.8×10^{-3}	2.1×10^{-6}
0.9	2.7212903	3.0541794	3.8×10^{-3}	1.3×10^{-6}
1.0	2.4022955	3.1712532	3.8×10^{-3}	1.3×10^{-6}

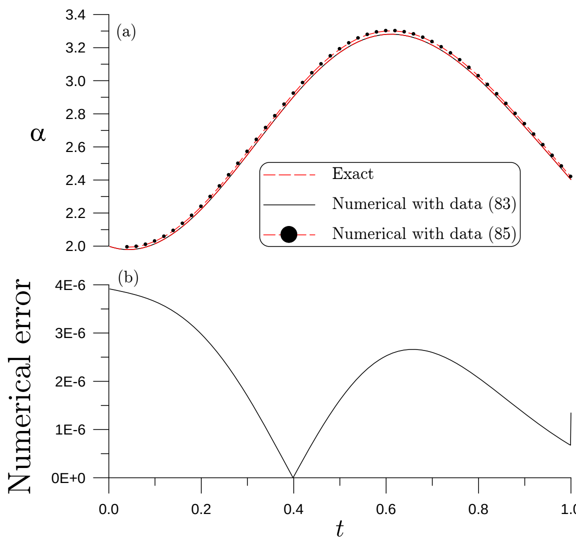

Figure 3: For Example 2: (a) comparing the numerical results using data (83) and (85) with exact result, and (b) displaying the numerical error.

10^5 times accuracy than that calculated by Dehghan (2005). Recently, Shamsi and Dehghan (2007) have employed the pseudospectral Legendre method to calculate the same example, whose accuracy is also in the order of 10^{-6}. However, the LGEM is much easy to implement than the above method.

In the case of replacing the exact boundary data (83) by an approximation with

$$\frac{\partial T(\ell,t)}{\partial x} \approx \frac{T(\ell,t)-T(x_m,t)}{\ell-x_m} = \frac{\exp\left(\frac{\ell}{2}\right)-\exp\left(\frac{x_m}{2}\right)}{\ell-x_m}\left[\frac{1+2t^3}{1+t^3}+\sin\frac{t}{2}\right], \quad (85)$$

we have computed the $\alpha(t)$ under the same parameters values as given in the above, but supplemented with an extra measurement of temperature at a position $x_m = 0.99$. In this case the numerical result as shown in Fig. 3(a) by the dashed line fitted with solid points is slightly deviating from the exact result. The numerical result is acceptable.

5.3 *Example 3*

Let us consider the following example with $T(x,t)$ given by

$$T(x,t) = \exp(x)\frac{2+3t^3}{1+t^3}. \quad (86)$$

Simple calculations give

$$T(0,t) = \frac{2+3t^3}{1+t^3}, \quad T(\ell,t) = \exp(\ell)\frac{2+3t^3}{1+t^3}, \quad (87)$$

$$T(x,0) = 2\exp(x), \quad (88)$$

$$\frac{\partial T(\ell,t)}{\partial x} = \exp(\ell)\frac{2+3t^3}{1+t^3}, \quad (89)$$

$$\alpha(t) = \frac{3t^2}{2+5t^3+3t^6}. \quad (90)$$

Again, in order to claim the accuracy of his numerical method being third-order, Dehghan (2005) intentionally made a mistake to write an

Table 2: For Example 3 the comparison of present results with the results by Dehghan (2005)

t	exact α	α [Dehghan (2005)]	error of Dehghan (2005)	present error
0.1	0.01496257	0.00374883	4.9×10^{-3}	4.9×10^{-5}
0.2	0.05881799	0.01496257	4.8×10^{-3}	4.2×10^{-5}
0.3	0.12633429	0.03346705	4.8×10^{-3}	2.6×10^{-5}
0.4	0.20580649	0.05881799	4.9×10^{-3}	6.4×10^{-7}
0.5	0.28070175	0.09019378	4.9×10^{-3}	2.5×10^{-5}
0.6	0.33540706	0.12633429	5.1×10^{-3}	4.2×10^{-5}
0.7	0.36136164	0.16554876	5.1×10^{-3}	4.5×10^{-5}
0.8	0.35911801	0.20580649	5.0×10^{-3}	3.7×10^{-5}
0.9	0.33566675	0.24490672	5.0×10^{-3}	2.5×10^{-5}
1.0	0.3	0.28070175	4.9×10^{-3}	2.7×10^{-5}

incorrect exact data as shown in Table 4 therein and in Table 2 of the present paper. For example at $t = 1$ the calculation by Eq. (90) is 0.3 obviously, but Dehghan (2005) wrote 0.2807017544, and claimed that the accuracy was 4.9×10^{-3} as shown in Table 2. However, the accuracy of the results by Dehghan (2005) is at most in the order of 10^{-1}.

In our calculation of this example the length is taken to be $\ell = 0.1$ and use $\Delta x = 0.02$ as that used by Dehghan (2005); however, the time stepsize $\Delta t = 0.01$ is one hundred times large than $\Delta t = 0.0001$ used by Dehghan (2005). The data $F_m(t_i)$ is supposed to be the exact value given by Eq. (89). Under these parameters values, in Fig. 4(a) we compare the numerical solution of α with the exact one given by Eq. (90) in the time interval of $t \in [0,1]$. These two curves are almost coincident, and the error is plotted in Fig. 4(b), which is smaller than 5×10^{-5}. Therefore, as compared our numerical errors with the corrected numerical errors of Dehghan (2005) for α at different times, we can say that the present method can produce about 10^4 times accuracy than the method by Dehghan (2005).

In the case by adding a noise with a level $s = 0.01$ on the input data, we plotted the numerical result in Fig. 4(b) by the dashed line fitted with solid points. It can be seen that the present method is robust against the noise. Even under a large noise the accuracy is still better than the order of 10^{-2}.

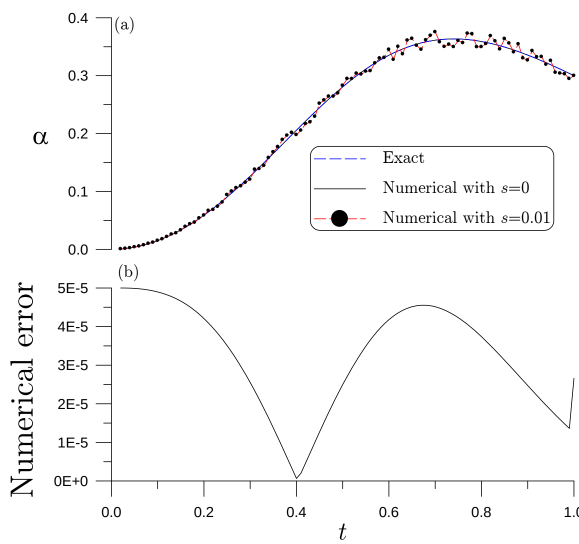

Figure 4: For Example 3: (a) comparing the numerical results under $s = 0$ and $s = 0.01$ with exact result, and (b) displaying the numerical error.

5.4 Example 4

In order to test the present method on the estimation of discontinuous and oscillatory heat conductivity, let us consider

$$\alpha(t) = \begin{cases} 2 & t \in [0,0.3], \\ 4 & t \in (0.3,0.6), \\ 2+\sin(10\pi t) & t \in [0.6,2]. \end{cases} \quad (91)$$

Here we let $t_f = 2$. Subjecting to the boundary conditions:

$$T(0,t) = 2t, \;\; T(\ell,t) = 2t+\ell, \quad (92)$$

and the initial condition

$$T(x,0) = x, \tag{93}$$

we can apply the LGEM to calculate this example. The data $F_m(t_i)$ is obtained by applying a numerical method, for example the fourth-order Runge-Kutta method, on the corresponding direct problem, supposing that $\alpha(t)$ is known from Eq. (91). In this identification of $\alpha(t)$ we have fixed $\Delta x = 0.01$ and $\Delta t = 0.02$. In Fig. 5(a) we compare the numerical solution of $\alpha(t)$ with exact solution. The errors are rather small in the order of 10^{-2}. From this example one may appreciate the accuracy of the LGEM provided here even for identifying a highly discontinuous and oscillatory parameter $\alpha(t)$ in the above. In this case the accuracy is not so good as the previous three examples, whose reason is attributed to that the present function of $\alpha(t)$ is more difficult to estimate, and the input data $F_m(t_i)$ is not exact.

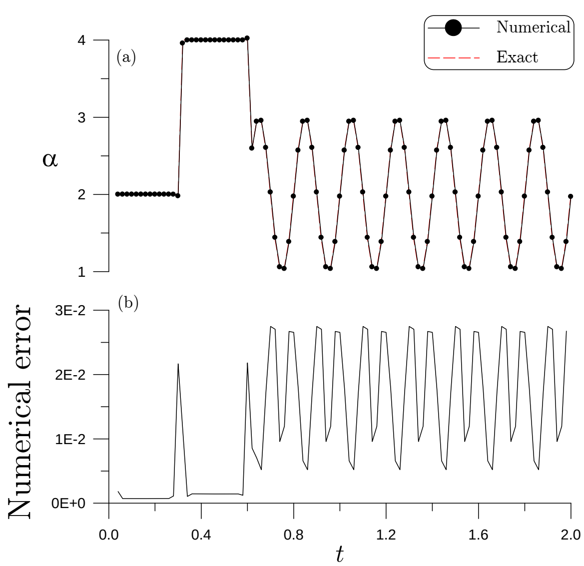

Figure 5: For Example 4: (a) comparing the numerical result with exact result, and (b) displaying the numerical error.

6 Conclusions

In order to estimate the time-dependent heat conductivity under an extra measured temperature gradient on boundary, we have employed the LGEM to derive algebraic equations and solved them by iteration process, where the construction of one-step Lie-groups $\mathbf{G}(r)$ and $\mathbf{G}(\ell)$, and the full use of the $n+1$ equations (34) and (35) are utmost important, the latter of which are the Lie-group transformation between left- and right-boundary temperature and temperature gradient in the augmented Minkowski space.

Numerical examples were worked out, which show that our LGEM is applicable even under a large noise on the measured data. Through this study, we can conclude that the new estimation method is accurate, effective and stable. Its numerical implementation is simple and the computational speed is fast. When an internal measurement of temperature is promoted, the LGEM is also workable by putting the measuring position near the boundary. According to these facts, the present LGEM can be used in practice as an accurate and effective mathematical tool to estimate the unknown time-dependent heat conductivity function.

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Appendix A

In this appendix we prove Eq. (45), which can be written as

$$\left(\frac{\|\mathbf{y}^\ell\|}{\|\mathbf{y}^0\|}\right)^2 - \sin^2\theta \geq 0. \tag{A1}$$

For this purpose we need to verify

$$\|\mathbf{y}^\ell\| \geq \|\mathbf{y}^0\| \sin\theta. \tag{A2}$$

When $\theta = 0$ and $\theta = \pi$, the above inequality holds obviously. Then, we consider $0 < \theta < \pi$, and divide the discussions into three cases: $\theta = \pi/2$, $0 < \theta < \pi/2$ and $\pi/2 < \theta < \pi$ as shown in Fig. 6.

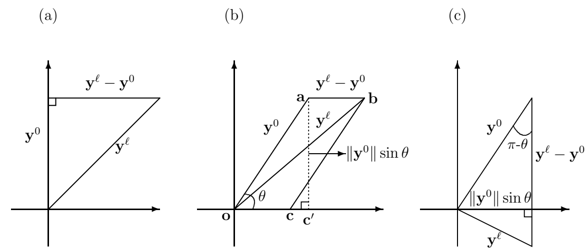

Figure 6: The proof of Eq. (45) by considering three cases: (a) $\theta = \pi/2$, (b) $0 < \theta < \pi/2$, and (c) $\pi/2 < \theta < \pi$.

Because the length and intersection angle are geometric invariants independent of coordinates rotation, we can choose some special simple coordinates to simplify our proof.

For the first case the vertical axis is parallel to $\mathbf{y}^0$ and we have

$$\|\mathbf{y}^\ell\|^2 = \|\mathbf{y}^0\|^2 + \|\mathbf{y}^\ell - \mathbf{y}^0\|^2. \tag{A3}$$

Because of $\|\mathbf{y}^\ell - \mathbf{y}^0\|^2 \geq 0$ and $\theta = \pi/2$, Eq. (A2) follows directly.

For the second case as shown in Fig. 6(b), we choose the horziontal axis parallel to $\mathbf{y}^\ell - \mathbf{y}^0$. Then parallel to the sides $\mathbf{y}^0$ and $\mathbf{y}^\ell - \mathbf{y}^0$ we make a parallelogram **oabc**. The dashed line **ac**′ is the projection of $\mathbf{y}^0$ in the vertical direction, such that the length of **ac**′ is $\|\mathbf{y}^0\| \sin\theta$ because the angle of $\angle$**aoc**′ is θ. There are three possible cases: the perpendicular foot **c**′ is on the left-side of **c**, **c**′ and **c** are coincident, and **c**′ is on the right-side of **c**. We only show the first case in Fig. 6(b). No matter which case is, it is easy to see that $\|\mathbf{ac}'\| < \|\mathbf{cb}\| < \|\mathbf{ob}\|$. Thus, $\|\mathbf{y}^0\| \sin\theta < \|\mathbf{y}^\ell\|$ is proved, and Eq. (A2) follows readily.

For the third case as shown in Fig. 6(c), we choose the vertical axis parallel to $\mathbf{y}^\ell - \mathbf{y}^0$. Because of

$\|\mathbf{y}^0\| \sin(\pi - \theta) = \|\mathbf{y}^0\| \sin\theta$, it is easy to see that the projection of $\mathbf{y}^0$ on the horziontal axis has a length $\|\mathbf{y}^0\| \sin\theta$ as marked in the figure. Therefore, it is easy to see that $\|\mathbf{y}^0\| \sin\theta < \|\mathbf{y}^\ell\|$, and Eq. (A2) follows.