

Elastic Instability of Pseudo-Elastic Rubber Balloons

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Abstract: Elastic instability for the inflation and deflation of a thin-walled spherical rubber balloon is examined within the framework of finite pseudo-elasticity. When a spherical rubber balloon is inflated, it is subject to a complex deformation after a pressure maximum has been obtained. One part of the balloon is lightly stretched while the remainder becomes highly stretched. So an aspherical deformation is observed after the initial spherical inflation. A pseudo-elastic strain energy function including a damage variable which may model the loading, unloading and reloading of rubber is used. The balloon is idealized as an elastic membrane and the inflation, deflation and re-inflation of the balloon is described in detail. Instability of solutions is discussed through energy comparison. Furthermore, the effect of temperature is discussed with a thermo-hyperelastic model and the residual strain is analyzed with a pseudo-elastic strain energy function including a residual strain variable.

Keyword: rubber balloon, pseudo-elasticity, thermo-hyperelasticity, aspherical deformation, instability

1 Introduction

Rubber-like materials such as rubber and polymeric materials are using in a broader and broader range of engineering field in recent years. So such nonlinear problems as the instability [Fu and Ogden R.W. (2001); Beatty (1987); Gent (2005)], cavity formation [[Yuan and Zhang (2005); Yuan et al (2006)]] and the damage or rupture [Gao et al (2006); Sharifi and Gahwaya (2006); Le Phong et al (2007); Timmel et al (2007)] of

hyper-elastic materials have attracted much attention as they play a fundamental role on the failure of materials. Problems with multiple solutions are an important class of material instability. A well-known example is the inflation of spherical rubber balloons [Gent (1999); Needleman (1977)]. When a spherical rubber balloon is inflated, it keeps spherical inflation at the initial stage until the pressure maximum is attained. After the pressure maximum, the pressure begins to fall with the increasing inflation. At the same time, one part of the balloon is lightly stretched while the remainder becomes highly stretched. So the balloon becomes noticeably aspherical. For a certain pressure, there exist one or more solutions corresponding to different inflations, so that instability is encountered.

But hyper-elastic theory is unable to describe such inelastic effects in the unloading and reloading of balloons as the notable retracing of a similar curve in the deflation path [Beatty (1987)]. This occurs because of the Mullins effect in rubber-like solids, which is a stress-softening effect induced by damage [Ogden (2003)]. The Mullins effect has been modeled using the notion of pseudo-elasticity in papers by Ogden [Ogden (2003)]. The effect was modeled by incorporating a discontinuous internal or damage variable into the strain-energy function of the material. As the damage variable may be active or inactive, an elastic stress-strain law is used in the loading path and a different elastic stress-strain law is used in the unloading path. So the hysteresis effects exhibited by rubber-like materials may be prescribed.

The importance of gaining a theoretical understanding of the thermo-mechanical behavior of rubber was illustrated by the role of the O-ring seals in the Challenger shuttle disaster. Not only the elasticity theory but also extensions of the the-

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ory to account for inelastic effects are involved for rubber-like materials. Such thermo-mechanical behavior of rubber-like materials is described as the thermo-hyperelastic model [Nicholson and Nelson (1990)].

The purpose of the present paper is to further investigate the instability of inflation and deflation of a thin-walled pseudo-elastic spherical rubber balloon. A pseudo-elastic strain energy function [Dorfmann and Ogden (2003)] including a damage variable which can model the loading, unloading and reloading of rubber is used here. The balloon is idealized as an elastic membrane and a simple energy method is used to achieve the general formula for the inflation pressure within the context of membrane theory. The inflation, deflation and re-inflation curves of the balloon are given. Instability of solutions in the inflation-deflation circle is discussed through energy comparison and the material response in the circle is described in details. Furthermore, the effect of temperature on the inflation of the balloon is discussed with a thermo-hyperelastic model. The residual strain is analyzed with a pseudo-elastic strain energy function including a residual strain variable and the basic law for the distribution of residual strain is given.

2 Formulations

Consider the rubber balloon as an isotropic spherical membrane with undeformed radius r_0 and thickness d_0 ($d_0 \ll r_0$) and it is in a stress-free state at initial time t_0 . Suppose the spherical shape is preserved as the inflation pressure p deforms the balloon uniformly to a radius r and thickness d at time t . The undeformed and deformed configurations are described by the spherical coordinate systems (R, Θ, Φ) and (r, θ, φ) , respectively. Then the deformation function of the balloon is given as

$$r = r(R) > 0, \theta = \Theta, \varphi = \Phi \quad (1)$$

here $r(R)$ is an undetermined function. The associated deformation gradient tensor is

$$\mathbf{F} = \text{diag}(\dot{r}(R), r(R)/R, r(R)/R) \quad (2)$$

The principal stretches are

$$\lambda_r = \dot{r}(R), \lambda_\theta = \lambda_\varphi = r(R)/R \quad (3)$$

From the incompressibility condition of the material, let $\lambda_\theta = \lambda_\varphi = \lambda$, we have $\lambda_r = \lambda^{-2}$.

The strain energy function of the rubber material is given as the pseudo-elastic constitutive law used by Dorfmann and Ogden [Dorfmann and Ogden (2003)] to model the idealized Mullins effect with no residual strain in the form

$$W(\lambda, \eta) = \eta W_0(\lambda) + \phi(\eta) \quad (4)$$

Where, $W_0(\lambda)$ is the classical strain energy function for the rubber material. Here, we consider a particular constitutive law, namely that of the Gent material [Gent (1999)],

$$W_0 = -\frac{EJ_m}{6} \ln\left(1 - \frac{J_1}{J_m}\right) \quad (5)$$

Where, $J_1 = \lambda_r^2 + \lambda_\theta^2 + \lambda_\varphi^2 - 3$, J_m is the maximum permitted value of J_1 corresponding to a maximum extension. $\phi(\eta)$ is the damage function depends only on the damage variable η and it is given in the form

$$\phi(\eta) = -m(\eta - 1) \tanh^{-1}[r(\eta - 1)] - W_m(\eta - 1) - \frac{m}{2r} \log\left[1 - r^2(\eta - 1)^2\right] \quad (6)$$

Where material constant $r = 3.3$, $m = 0.3Nmm$, $W_m = W_0(\lambda_m)$, λ_m is the value of the principal stretch attained on the loading path. The damage variable η is inactive on the loading path, that is to say $\eta = 1$. When $\eta = 1$, $\phi(\eta) = 0$, we may have the classical strain energy function from the pseudo-energy function (4). The damage variable η is active on the unloading path, which is determined as

$$1 - \eta = \frac{1}{r} \tanh\left[\frac{W_m - W_0(\lambda)}{m}\right] \quad (7)$$

3 Inflation of the balloon

Following the mechanical energy principle that the time rate of change of the total mechanical energy for any part of a body is balanced by the total

mechanical power [Beatty (1987)], we have

$$\begin{aligned} \int_V \Delta W dV &= 4\pi r_0^2 t_0 W(\lambda) \\ &= 4\pi \int_{r_0}^r r^2 p(r) dr \\ &= 4\pi r_0^3 \int_1^\lambda \lambda^2 p(\lambda) d\lambda \end{aligned} \quad (8)$$

Differentiating it with respect to λ yields

$$p(\lambda) = \frac{t_0}{r_0 \lambda^2} \frac{dW(\lambda)}{d\lambda} \quad (9)$$

This is the general formula for the inflation pressure for an isotropic hyper-elastic spherical membrane inflated from the initial stress-free state. When the balloon is primarily inflated, the damage variable η is inactive and we have

$$p(\lambda) = \frac{d_0}{r_0 \lambda} \frac{2E}{3} \frac{1 - \lambda^{-6}}{1 - J_1/J_m} \quad (10)$$

Numerical result of (10) for the inflation curve is shown in Fig.1 with material constant $J_m = 97.2, E = 3.5 \times 10^6 \text{MPa}$. It is shown that there exists a maximum inflation pressure $p_1 = p(\lambda = \lambda_1 = 1.4)$. When the pressure is less than this maximum pressure, it increases rapidly with the increasing of stretch. But when the pressure is larger than this maximum pressure, it decreases with the increasing of stretch. Finally, when the pressure is larger than a minimum pressure $p_2 = p(\lambda = \lambda_2 = 4.0)$, it increases with the increasing of stretch in place of continuously decreasing.

For a certain pressure, there exist one or more solutions corresponding to different inflations, so that instability is encountered and it is necessary to compare the total potential energy for the solutions. The total potential energy of the inflated balloon with internal pressure from the initial stress-free state is

$$\begin{aligned} E &= \int_V W dV - \int_A p(r - r_0) dr \\ &= 4\pi r_0^2 d_0 \left(W(\lambda) - \frac{1}{3} \frac{r_0}{d_0} p(\lambda^3 - 1) \right) \end{aligned} \quad (11)$$

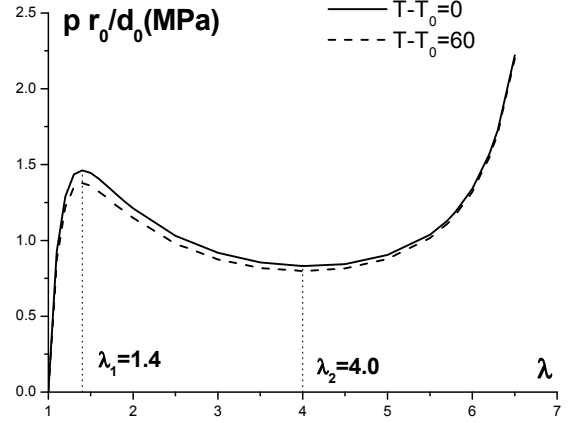


Figure 1: Inflation curve of the balloon

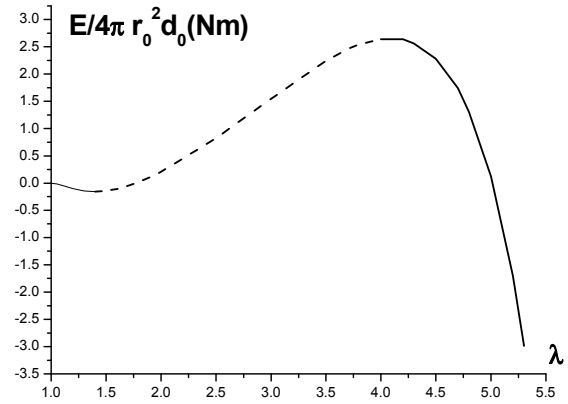


Figure 2: Energy curve of the inflated balloon

Numerical result of (11) for the inflated balloon is shown in Fig.2. It is shown that the total potential energy of the inflated balloon decreases with the increasing of stretch when $1 \leq \lambda \leq \lambda_1 = 1.4$ or $\lambda_2 = 4.0 \leq \lambda$. But it increases with the increasing of stretch when $\lambda_1 = 1.4 \leq \lambda \leq \lambda_2 = 4.0$. That is to say, the balloon attained a stable deformation state when the pressure is less than the maximum pressure and it takes up a roughly spherical shape. But when the pressure is larger than the maximum pressure, the deformation state is unstable. After a small inflation, the balloon is subject to a complex deformation. One part of the balloon is lightly stretched and the other is highly stretched. So the balloon becomes noticeably aspherical. When the pressure is larger than the minimum pressure, the balloon attains a second stable deformation and regains its spherical shape. This is accord with

observation from a typical balloon inflation experiment [Beatty (1987)].

4 Deflation of the balloon

When the primarily inflated balloon is unloaded from a certain maximum stretch λ_m , the damage variable η becomes active and its value may be given explicitly in terms of λ from (7). Then the pseudo-energy function during the unloading path may be given by (4) and (6). Substituting it into the general formula (9), the inflation pressure during the unloading path may be given as

$$p(\lambda) = \eta \frac{d_0}{r_0 \lambda} \frac{2E}{3} \frac{1 - \lambda^{-6}}{1 - J_1 J_m} \quad (12)$$

Numerical result of (12) for the unloading curve which is unloaded from the maximum stretch $\lambda_m = 6$ is shown in Fig.3. The pressure for deflation isn't monotonically decreasing with the stretch, too. It is shown that there is a notable retracing of a similar curve in the deflation path as that in the inflation path and it includes a maximum pressure at the same stretch just as that in the inflation path.

For a certain pressure, there exist one or more solutions corresponding to different inflations, too, so that instability is encountered in the deflation path. It is necessary to compare the total potential energy for the solutions, too. The total potential energy of the deflated balloon with internal pressure from the maximum stretch λ_m is

$$\begin{aligned} E &= \int_V W dV - \int_A p(r - r_m) dr \\ &= 4\pi r_0^2 d_0 \left(W(\lambda) - \frac{1}{3} \frac{r_0}{d_0} p (\lambda^3 - \lambda_m) \right) \end{aligned} \quad (13)$$

Numerical result of (13) for the deflation curve unloaded from the maximum stretch $\lambda_m = 6$ is shown in Fig.4. It is shown that the total potential energy of the deflation balloon decreases with the decreasing of stretch when $1 \leq \lambda \leq \lambda_1 = 1.4$ or $\lambda_2 = 4.0 \leq \lambda$. But it increases with the decreasing of stretch when $\lambda_1 = 1.4 \leq \lambda \leq \lambda_2 = 4.0$. That is to say, the deflation balloon attained a stable deformation state when the pressure is larger than the minimum pressure and it takes up a roughly

spherical shape. But when the pressure is less than the minimum pressure, the deformation state is unstable. The balloon is subject to a complex deflated deformation as that in the inflation path and it becomes aspherical, too. Finally, when the pressure is less than the maximum pressure, the deflation balloon attains a second stable deformation and regains its spherical shape.

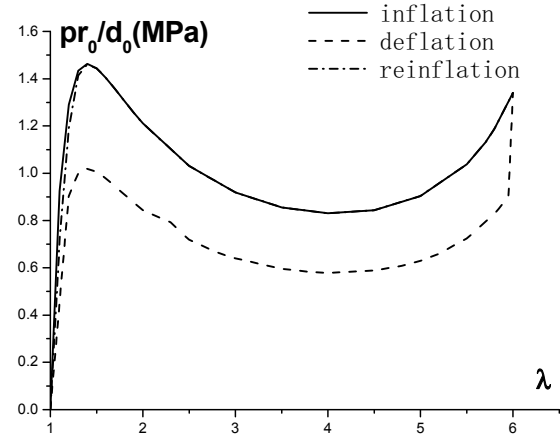


Figure 3: Inflation, deflation and re-inflation curves of the balloon

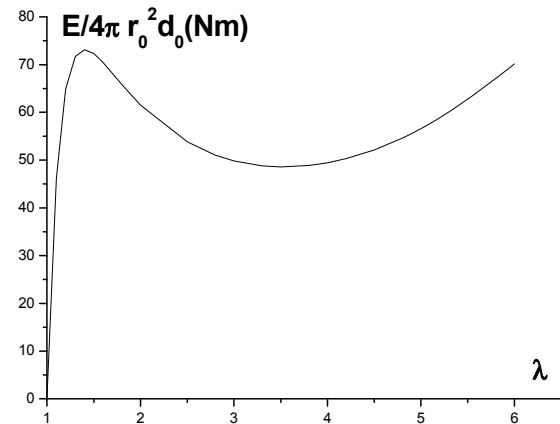


Figure 4: Energy curve of the deflated balloon

5 Re-inflation of the balloon

Re-inflation from a state $\lambda_u \in (1, \lambda_m)$ on the previous deflation curve is considered now. The pseudo-energy function during the reloading path

may be given as [Dorfmann and Ogden (2003)]

$$W_r(\lambda, \eta_r) = \eta_r W_0(\lambda) + \phi_r(\eta_r) \quad (14)$$

$\phi_r(\eta_r)$ is the damage function depends only on the damage variable η_r during reloading and it is given in the form

$$\begin{aligned} \phi_r(\eta_r) = & \phi(\eta_u) - (\eta_r - \eta_u) W_0(\lambda_u) \\ & - a_1 (\eta_r - \eta_u) \tanh^{-1} \left[\frac{(\eta_r - \eta_u)}{(1 - \eta_u)} \right] \\ & - \frac{a_1}{2} (1 - \eta_u) \log \left[1 - \frac{(\eta_r - \eta_u)^2}{(1 - \eta_u)^2} \right] \end{aligned} \quad (15)$$

Where material constant $a(W_0(\lambda_u)) = c_0 + c_1 W_0(\lambda_u)$, $c_0 = 0.2227 Nmm$, $c_1 = 0.3723$. The damage variable η_r increases from η_u to the final value 1 during reloading and when this final value is reached the material response switches from the reloading path to the loading path. It is determined by

$$\frac{\eta_r}{\eta_u} - 1 = \frac{1 - \eta_u}{\eta_u} \tanh \left[\frac{W_0(\lambda) - W_0(\lambda_u)}{a_1} \right] \quad (16)$$

Substituting (14) into the general formula (9), the inflation pressure during the reloading path may be given as

$$p(\lambda) = \eta_r \frac{d_0}{r_0 \lambda} \frac{2E}{3} \frac{1 - \lambda^{-6}}{1 - J_1 J_m} \quad (17)$$

Numerical result of (17) for the reloading curve is shown in Fig.3 with $\lambda_u = 1.0$. It is shown that when the pressure is less than the maximum pressure, it increases rapidly with the increasing of stretch but it is always lightly smaller than that in the primary inflation. When the maximum inflation pressure $p_1 = p(\lambda = \lambda_1 = 1.4)$ is attained, η_r reaches the final value 1 and the re-inflation curve is consonant with the primary inflation curve.

6 Effect of temperature

Now, we consider the effect of temperature on the inflation of the balloon. Assuming there is a uniform temperature field $T = \text{const}$ for the balloon and consider the generalized Gent

thermo-hyperelastic material [Nicholson and Nelson (1990)],

$$W = -\frac{EJ_m}{6} \ln \left(1 - \frac{J_1}{J_m} \right) + \rho C_3 T \ln \left(\frac{T}{T_0} \right) + 2C_4 (T - T_0) J_1 \quad (18)$$

where, $C_3 = C_e$, $C_4 = -\alpha\lambda$ are material constants [Nicholson and Lin (1996)], C_e , α , λ are the specific heat at constant strain, the volumetric thermal expansion coefficient and the second Lamé coefficient, ρ is the mass density, T_0 is a reference temperature.

Substituting (18) into the general formula (9), the inflation pressure during the loading path may be given as

$$p(\lambda) = \frac{d_0}{r_0 \lambda} \left(\frac{2E}{3} \frac{1 - \lambda^{-6}}{1 - J_1 J_m} + 8C_4 (T - T_0) (1 - \lambda^{-6}) \right) \quad (19)$$

Numerical result of (19) for the inflation balloon is shown in Fig.1 with constant [Nicholson and Lin (1996)] $C_3 = 1506 Jkg^{-1}K^{-1}$, $\rho = 950 kgm^{-3}$, $C_4 = -6.36 \times 0.4245 \times 10^{-4} MPa \cdot K^{-1}$, $T_0 = 300^0K$, $T_0 = 360^0K$. It is shown that when the temperature is raised, the inflation curve is as similar as that under the reference temperature but the pressure is always lightly smaller than that under the reference temperature at the same stretch.

7 Residual strain

To analyze the residual strain of the balloon as shown in the typical balloon inflation experiment [Beatty (1987)], a pseudo-elastic constitutive law used by Dorfmann and Ogden [Dorfmann and Ogden (2004)] to model the idealized Mullins effect with residual strain in the following form is used

$$W(\lambda, \eta) = \eta_1 W_0(\lambda) + (1 - \eta_2) N(\lambda) + \phi_1(\eta_1) + \phi_2(\eta_2) \quad (20)$$

Where,

$$N(\lambda) = \frac{1}{2} [v_1 (\lambda_r^2 - 1) + v_2 (\lambda_\theta^2 - 1) + v_3 (\lambda_\phi^2 - 1)] \quad (21)$$

Material constants v_1, v_2, v_3 only depend on the value of the principal stretch λ_m attained on the loading path. η_1 is the damage variable determined as

$$\eta_1 = 1 - \frac{1}{r} \tanh \left[\frac{W_m - W_0(\lambda)}{\mu m} \right] \quad (22)$$

$\phi_1(\eta_1)$ is the damage function determined as

$$\phi_1(\eta_1) = -\mu m (\eta_1 - 1) \tanh^{-1} [r(\eta_1 - 1)] - W_m (\eta_1 - 1) - \frac{\mu m}{2r} \log [1 - r^2 (\eta_1 - 1)^2] \quad (23)$$

Material constants $r = 1.25$, $m = 0.965$, $\mu = 1.24 \text{MPa}$. η_2 is the residual strain variable determined as

$$\eta_2 = \frac{1}{\tanh(1)} \tanh \left[\left(\frac{W_0(\lambda)}{W_m} \right)^{\alpha(W_m)} \right] \quad (24)$$

Where function

$$\alpha = 0.3 + 0.16 \frac{W_m}{\mu} \quad (25)$$

The residual strain function $\phi_2(\eta_2)$ is determined as

$$\phi_2'(\eta_2) = N(\lambda) \quad (26)$$

The two variables are inactive on the loading path, that is to say $\eta_1 = \eta_2 = 1$. When $\eta_1 = \eta_2 = 1$, $\phi_1(\eta_1) = \phi_2(\eta_2) = 0$, we may have the classical strain energy function from the pseudo-energy function (20).

For the given material constants $v_1 = 0.4\mu \left[1 - \frac{1}{3.5} \tanh \left(\frac{\lambda_m - 1}{0.1} \right) \right]$, $v_2 + v_3 = 0.8\mu$, numerical result for the inflation balloon and deflation balloon from different λ_m are shown in Fig.5. Residual strain of the deflated balloon with variable of λ_m is shown in Fig.6. It is shown that the residual strain is depend on the maximum value of the principal stretch attained on the loading path and there is an asymptotic value when λ_m is larger than 2.0. This is accord with observation from a typical balloon inflation experiment [Beatty (1987)], too.

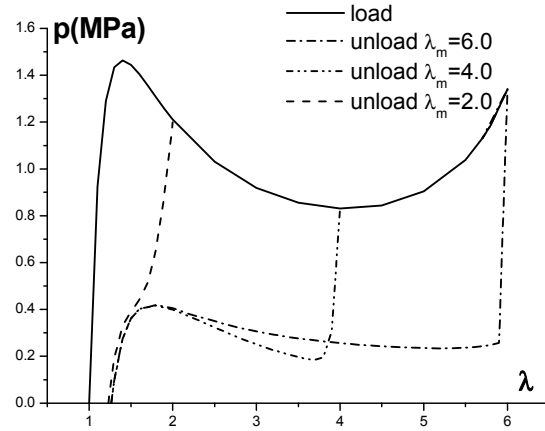


Figure 5: Deformation curves of the balloon

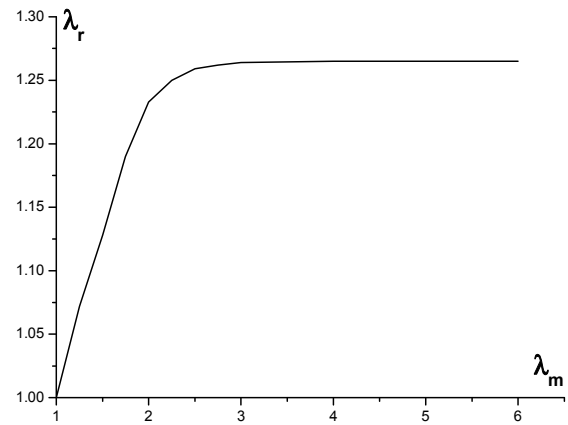


Figure 6: Residual strain of the deflated balloon

8 Conclusions

When a spherical pseudo-elastic rubber balloon is inflated, there exist one or more solutions corresponding to different inflations for a certain pressure. When the pressure is larger than the maximum pressure, the deformation state is unstable and it is subject to a complex deformation. One part of the balloon is lightly stretched while the remainder becomes highly stretched and an aspherical deformation is observed after the initial spherical inflation. When the primarily inflated balloon is unloaded from a certain maximum stretch, there is a notable retracing of a similar curve in the deflation path. When the deflation balloon is re-inflated from a state on the previous deflation curve, the pressure is always lightly smaller than

that in the primary inflation until the maximum inflation pressure is attained and finally the re-inflation curve is consonant with the primary inflation curve. Furthermore, the pressure is always lightly smaller than that under the reference temperature as the temperature is raised. The residual strain is analyzed with a pseudo-elastic strain energy function including a residual strain variable and it is depend on the maximum value of the principal stretch attained on the loading path and there is an asymptotic value.

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