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Flexural-Torsional Buckling and Vibration Analysis of Composite Beams

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Abstract: In this paper the general flexural-torsional buckling and vibration problems of composite Euler-Bernoulli beams of arbitrarily shaped cross section are solved using a boundary element method. The general character of the proposed method is verified from the formulation of all basic equations with respect to an arbitrary coordinate system, which is not restricted to the principal one. The composite beam consists of materials in contact each of which can surround a finite number of inclusions. It is subjected to a compressive centrally applied load together with arbitrarily transverse and/or torsional distributed or concentrated loading, while its edges are restrained by the most general linear boundary conditions. The resulting problems are (i) the flexural-torsional buckling problem, which is described by three coupled ordinary differential equations and (ii) the flexural-torsional vibration problem, which is described by three coupled partial differential equations. Both problems are solved employing a boundary integral equation approach. Besides the effectiveness and accuracy of the developed method, a significant advantage is that the method can treat composite beams of both thin and thick walled cross sections taking into account the warping along the thickness of the walls. The proposed method overcomes the shortcoming of possible thin tube theory (TTT) solution, which its utilization has been proven to be prohibitive even in thin walled homogeneous sections. Example problems of composite beams are analysed, subjected to compressive or vibratory loading, to illustrate the method and demonstrate its efficiency and wherever possible its accuracy. Moreover, useful conclusions are drawn from the buckling and dynamic response of the beam.

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1 Introduction

Composite beams have been increasingly used in recent years as structural members due to their high strength/stiffness properties for light weight materials. The design of such structures subjected to compressive or vibratory loading, presents a serious challenge and necessitates a reliable and accurate analysis. This becomes much more complicated in the case the beam's cross section centroid does not coincide with its shear center (asymmetric beams), leading to the formulation of the flexural-torsional buckling or vibration problem of composite beams, respectively.

The flexural-torsional buckling or vibration problem of thin-walled composite beams, based on the assumptions of the thin tube theory, has been studied by many researchers. Among them, Kollar (2001a, b) and Sapkas and Kollar (2002) used a simple closed form solution showing the effect of shear deformations, Chen (2003) developed the differential quadrature element method, Bhaskar and Librescu (1995) based on a geometrically non-linear thin-walled beam theory, Lee and Kim (2001, 2002), Lee and Lee (2004) employed a displacement-based one-dimensional finite element model based on the classical lamination theory, and Shan and Qiao (2005) and Qiao, Zou and Davalos (2003) presented a combined analytical and experimental study for the buckling of pultruded fiber-reinforced plastic composite open channel or cantilever I- beams using the Rayleigh-Ritz method. Moreover, the flexural-torsional buckling and vibration problems have been studied by many researchers in laminated composite beams, such as Matsunaga (2001) employing Hamilton's principle, Vinogradov and Derrick (2000) based on analytic solutions and Song and Waas (1997), Lellep and Sakkov (2006) in stepped rectangular beams of simple boundary conditions. However, the aforementioned formulations concerning composite beams of thin walled cross sections or laminated cross-sections are analyzing these beams with respect to cross section mid lines ignoring the warping along the thickness of the walls. Moreover, they do not satisfy the continuity conditions of transverse shear stress at layer interfaces and assume that the transverse shear stress along the thickness coordinate remains constant, leading to the fact that kinematic or static assumptions cannot be always valid (Karama, Afaq and Mistou, 2003; Reddy, 1989; Touratier, 1992). To the authors' knowledge publications on the solution to the general flexural-torsional buckling or vibration problem of arbitrarily shaped composite cross sections do not exist.

In this investigation a boundary element method is developed for the general flexural-torsional buckling and vibration analysis of composite Euler-Bernoulli beams of arbitrarily shaped cross section. The composite beam consists of materials in contact each of which can surround a finite number of inclusions. The general character of the proposed method is verified from the fact that all basic equations are formulated with respect to an arbitrary coordinate system, which is not restricted to the principal one. The beam is subjected to a compressive centrally applied load together with arbitrarily transverse and/or torsional distributed or concentrated loading, while its edges are restrained by the most general linear boundary conditions. The resulting problems are (i) the flexural-torsional buckling problem, which is described by three coupled ordinary differential equations and (ii) the flexural-torsional vibration problem, which is described by three coupled partial differential equations. Among many respective Boundary Element and Meshless Methods such as the Meshless Regularized Integral Equation Method (MRIEM) (Liu, 2007), the Meshless Local Petrov-Galerkin (MLPG) Method (Andreaus, Batra and Porfiri, 2005) and the Dual Boundary Element Method (Purbolaksono and Aliabadi, 2005) the authors applied the Analog Equation Method (AEM) (Katsikadelis, 2002a) for the solution of the aforementioned problems. According to this method, the three coupled fourth order differential equations are replaced by three uncoupled ones subjected to fictitious load distributions under the same boundary conditions. Besides the effectiveness and accuracy of the developed method, a significant advantage is that the method can treat composite beams of both thin and thick walled cross sections taking into account the warping along the thickness of the walls. The method overcomes the shortcoming of possible thin tube theory (TTT) solution, which its utilization has been proven to be prohibitive even in thin walled homogeneous sections (Sapountzakis and Tsiatas, 2007a). Moreover, the method permits the inclusion of the torsion-bending and flexural coupling stiffnesses which play an important role in the response of the beam and have to be taken always into account (Sapountzakis and Tsiatas, 2007a). Example problems of composite beams are analysed to illustrate the method and demonstrate its efficiency and wherever possible its accuracy. Moreover, useful conclusions are drawn from the buckling and dynamic response of the beam.

2 Statement of the problem

Let us consider an initially straight Euler-Bernoulli beam of length l (Fig. 1), of constant arbitrary cross-section of area A. The cross section consists of materials in contact, each of which can surround a finite number of inclusions, with modulus of elasticity E_j and shear modulus G_j, occupying the regions Ω_j $(j = 1,2,\ldots,K)$ of the y, z plane (Fig. 1). The materials of these regions are assumed homogeneous, isotropic and linearly elastic. Let also the boundaries of the nonintersecting regions Ω_j be denoted

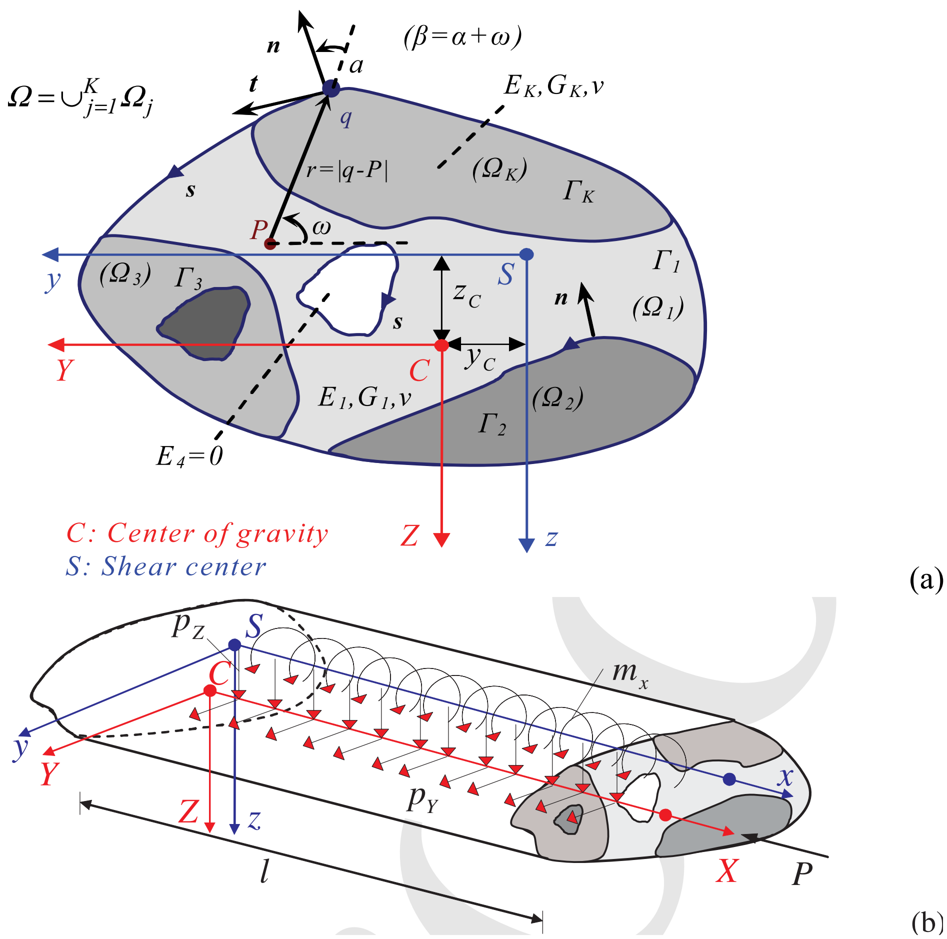

Figure 1: Prismatic element of an arbitrarily shaped constant cross section occupying region Ω (a) subjected in bending, torsional and/or buckling loading (b).

by $\Gamma_j\,(j=1,2,\ldots,K)$. These boundary curves are piecewise smooth, i.e. they may have a finite number of corners. In Fig. 1a CYZ and Syz are coordinate systems (not necessarily principal) through the cross section's centroid C and shear center S, respectively. Moreover, y_C, z_C are the coordinates of the centroid C with respect to Syz system of axes. The beam is subjected to a compressive centrally applied load together with arbitrarily transverse and/or torsional distributed or concentrated loading, while its edges are restrained by the most general linear boundary conditions. The two considered problems are:

(i) The flexural-torsional buckling problem, which is described by the following three coupled ordinary differential equations

$$E_1 I_Z \frac{d^4 v}{dx^4} + E_1 I_{YZ} \frac{d^4 w}{dx^4} + P\left(\frac{d^2 v}{dx^2} - z_C \frac{d^2\theta}{dx^2}\right) = p_Y \quad (1)$$

$$E_1 I_Y \frac{d^4 w}{dx^4} + E_1 I_{YZ} \frac{d^4 v}{dx^4} + P\left(\frac{d^2 w}{dx^2} + y_C \frac{d^2\theta}{dx^2}\right) = p_Z \quad (2)$$

$$E_1 C_S \frac{d^4\theta}{dx^4} - G_1 I_t \frac{d^2\theta}{dx^2} + P\left(i_\omega^2 \frac{d^2\theta}{dx^2} - z_C \frac{d^2 v}{dx^2} + y_C \frac{d^2 w}{dx^2}\right) = m_x + p_Z y_C - p_Y z_C \quad (3)$$

subjected to the following boundary conditions

$$\alpha_1 v(x) + \alpha_2 R_Y(x) = \alpha_3 \quad (4a)$$

$$\overline{\alpha}_1 \frac{dv(x)}{dx} + \overline{\alpha}_2 M_Z(x) = \overline{\alpha}_3 \quad (4b)$$

$$\beta_1 w(x) + \beta_2 R_Z(x) = \beta_3 \quad (5a)$$

$$\overline{\beta}_1 \frac{dw(x)}{dx} + \overline{\beta}_2 M_Y(x) = \overline{\beta}_3 \quad (5b)$$

$$\gamma_1 \theta(x) + \gamma_2 M_t(x) = \gamma_3 \quad (6a)$$

$$\overline{\gamma}_1 \frac{d\theta(x)}{dx} + \overline{\gamma}_2 M_b(x) = \overline{\gamma}_3 \quad (6b)$$

at the beam ends $x = 0, l$, where $v = v(x)$, $w = w(x)$ are the deflections of the shear center along y, z axes, respectively; $\theta(x)$ is the angle of twist of the cross-section about the shear center S; I_Y, I_Z, I_{YZ} are the moments and the product of inertia with respect to the centroid C given as

$$I_Z = \sum_{j=1}^{K} \frac{E_j}{E_1} I_{Zj}, \quad (7a)$$

$$I_Y = \sum_{j=1}^{K} \frac{E_j}{E_1} I_{Yj}, \quad (7b)$$

$$I_{YZ} = \sum_{j=1}^{K} \frac{E_j}{E_1} I_{YZj}, \quad (7c)$$

with I_{Yj}, I_{Zj}, I_{YZj} $(j = 1, 2, \ldots, K)$ are the moments and the product of inertia of the materials, while, i_ω is the polar radius of inertia with respect to the shear center S (Kollar, 2001a)

$$i_\omega^2 = z_C^2 + y_C^2 + \frac{E_1 I_Z + E_1 I_Y}{\sum_{j=1}^{K} \frac{E_j}{E_1} A_j} \quad (8)$$

Moreover,

$$C_S = \sum_{j=1}^{K} \frac{E_j}{E_1} C_{Sj}, \quad (9a)$$

$$I_t = \sum_{j=1}^{K} \frac{G_j}{G_1} I_{tj} \quad (9b)$$

are the warping and torsion rigidities of the composite cross section, with C_{Sj} and I_{tj}, $(j = 1, 2, \ldots, K)$, the corresponding constants of the Ω_j region, respectively, (Sauer, 1980) which are established using a BEM procedure (Katsikadelis, 2002b).

In the boundary conditions (4), (5) R_Y, M_Y and R_Z, M_Z are the reactions and bending moments with respect to Y and Z axes, respectively, given as

$$R_Y = -P\frac{dv(x)}{dx} - E_1 I_Z \frac{d^3 v(x)}{dx^3} - E_1 I_{YZ} \frac{d^3 w(x)}{dx^3} \quad (10)$$

$$M_Y = -E_1 I_Y \frac{d^2 w(x)}{dx^2} - E_1 I_{YZ} \frac{d^2 v(x)}{dx^2} \quad (11)$$

$$R_Z = -P\frac{dw(x)}{dx} - E_1 I_Y \frac{d^3 w(x)}{dx^3} - E_1 I_{YZ} \frac{d^3 v(x)}{dx^3} \quad (12)$$

$$M_Z = E_1 I_Z \frac{d^2 v(x)}{dx^2} + E_1 I_{YZ} \frac{d^2 w(x)}{dx^2} \quad (13)$$

(ii) The flexural-torsional vibration problem, which is described by the following three coupled partial differential equations

$$E_1 I_Z \frac{\partial^4 v}{\partial x^4} + E_1 I_{YZ} \frac{\partial^4 w}{\partial x^4} + \rho_1 A \left(\frac{\partial^2 v}{\partial t^2} - z_C \frac{\partial^2 \theta}{\partial t^2} \right) = p_Y \quad (14)$$

$$E_1 I_Y \frac{\partial^4 w}{\partial x^4} + E_1 I_{YZ} \frac{\partial^4 v}{\partial x^4} + \rho_1 A \left(\frac{\partial^2 w}{\partial t^2} + y_C \frac{\partial^2 \theta}{\partial t^2} \right) = p_Z \quad (15)$$

$$E_1 C_S \frac{\partial^4 \theta}{\partial x^4} - G_1 I_t \frac{\partial^2 \theta}{\partial x^2} + \rho_1 I_S \frac{\partial^2 \theta}{\partial t^2} + \rho_1 A \left(-z_C \frac{\partial^2 v}{\partial t^2} + y_C \frac{\partial^2 w}{\partial t^2} \right) = m_x + p_Z y_C - p_Y z_C \quad (16)$$

subjected to the following boundary conditions

$$\alpha_1 v(x,t) + \alpha_2 R_Y(x,t) = \alpha_3 \quad (17a)$$

$$\overline{\alpha}_1 \frac{\partial v(x,t)}{\partial x} + \overline{\alpha}_2 M_Z(x,t) = \overline{\alpha}_3 \quad (17b)$$

$$\beta_1 w(x,t) + \beta_2 R_Z(x,t) = \beta_3 \quad (18a)$$

$$\overline{\beta}_1 \frac{\partial w(x,t)}{\partial x} + \overline{\beta}_2 M_Y(x,t) = \overline{\beta}_3 \quad (18b)$$

$$\gamma_1\theta(x,t)+\gamma_2 M_t(x,t)=\gamma_3 \tag{19a}$$

$$\bar{\gamma}_1\frac{\partial\theta(x,t)}{\partial x}+\bar{\gamma}_2 M_b(x,t)=\bar{\gamma}_3 \tag{19b}$$

at the beam ends $x=0,l$, where $v=v(x,t)$, $w=w(x,t)$ are the deflections of the shear center along y, z axes, respectively; $\theta(x,t)$ is the angle of twist of the cross-section about the shear center S; A is the cross-sectional area and I_S the polar moment of inertia with respect to the shear center S given as

$$A=\sum_{j=1}^{K}\frac{\rho_j}{\rho_1}A_j, \tag{20a}$$

$$I_S=\sum_{j=1}^{K}\frac{\rho_j}{\rho_1}I_{Sj} \tag{20b}$$

with ρ_j, A_j $(j=1,2,\ldots,K)$ being the mass densities and the areas of the materials, respectively.

The initial conditions are

$$v(x,0)=\bar{v}_0(x) \tag{21a}$$

$$\dot{v}(x,0)=\dot{\bar{v}}_0(x) \tag{21b}$$

$$w(x,0)=\bar{w}_0(x) \tag{22a}$$

$$\dot{w}(x,0)=\dot{\bar{w}}_0(x) \tag{22b}$$

$$\theta(x,0)=\bar{\theta}_0(x) \tag{23a}$$

$$\dot{\theta}(x,0)=\dot{\bar{\theta}}_0(x) \tag{23b}$$

In the boundary conditions (17), (18) R_Y, M_Y and R_Z, M_Z are the reactions and bending moments with respect to Y and Z axes, respectively, given as

$$R_Y(x,t)=-E_1I_Z\frac{\partial^3 v(x,t)}{\partial x^3}-E_1I_{YZ}\frac{\partial^3 w(x,t)}{\partial x^3} \tag{24}$$

$$M_Y(x,t)=-E_1I_Y\frac{\partial^2 w(x,t)}{\partial x^2}-E_1I_{YZ}\frac{\partial^2 v(x,t)}{\partial x^2} \tag{25}$$

$$R_Z(x,t)=-E_1I_Y\frac{\partial^3 w(x,t)}{\partial x^3}-E_1I_{YZ}\frac{\partial^3 v(x,t)}{\partial x^3} \tag{26}$$

$$M_Z(x,t)=E_1I_Z\frac{\partial^2 v(x,t)}{\partial x^2}+E_1I_{YZ}\frac{\partial^2 w(x,t)}{\partial x^2} \tag{27}$$

while in Eqns. (6) and (19) M_t and M_b are the torsional and warping moments, respectively, given as (Sapountzakis and Tsiatas, 2007a) (in the buckling case the time is excluded)

$$M_t(x,t)=-E_1C_S\frac{\partial^3\theta(x,t)}{\partial x^3}+G_1I_t\frac{\partial\theta(x,t)}{\partial x} \tag{28}$$

$$M_b(x,t)=-E_1C_S\frac{\partial^2\theta(x,t)}{\partial x^2} \tag{29}$$

Finally, α_k, $\bar{\alpha}_k$, β_k, $\bar{\beta}_k$, γ_k, $\bar{\gamma}_k$ $(k=1,2,3)$ are functions specified at the beam ends $x=0,l$. Eqns. (4-6) and (17-19) describe the most general linear boundary conditions associated with the problem at hand and can include elastic support or restrain. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived form these equations by specifying appropriately these functions (e.g. for a clamped edge it is $\alpha_1=\beta_1=\gamma_1=1$, $\bar{\alpha}_1=\bar{\beta}_1=\bar{\gamma}_1=1$, $\alpha_2=\alpha_3=\beta_2=\beta_3=\gamma_2=\gamma_3=\bar{\alpha}_2=\bar{\alpha}_3=\bar{\beta}_2=\bar{\beta}_3=\bar{\gamma}_2=\bar{\gamma}_3=0$).

3 Integral representations – numerical solution

3.1 The flexural-torsional buckling problem

The flexural-torsional buckling problem of a composite beam reduces in establishing the displacement components $v(x)$, $w(x)$ and $\theta(x)$ having continuous derivatives up to the fourth order satisfying the coupled governing Eqns. (1)-(3) inside the beam and the boundary conditions (4)-(6) at the beam ends $x=0,l$.

Eqns. (1)-(3) are solved using the Analog Equation Method as it was developed for ordinary differential equations in (Sapountzakis and Katsikadelis, 2000; Katsikadelis and Tsiatas, 2005; Sapountzakis and Tsiatas, 2007b). This method is applied for the problem at hand as follows. Let $v(x)$, $w(x)$ and $\theta(x)$ be the sought solution of the boundary value problem described by Eqns. (1)-(3) and (4)-(6). Setting as $u_1(x)=v(x)$, $u_2(x)=w(x)$, $u_3(x)=\theta(x)$ and differentiating these functions four times yields

$$\frac{d^4u_i}{dx^4}=b_i(x)\quad (i=1,\,2,\,3) \tag{30}$$

Eqns. (30) indicate that the solution of Eqns. (1)-(3) can be established by solving Eqns. (30) under the same boundary conditions (4)-(6), provided that the fictitious load distributions $b_i(x)$ $(i = 1, 2, 3)$ are first established. These distributions can be determined using BEM.

Following the numerical procedure analytically described in (Sapountzakis and Katsikadelis, 2000; Katsikadelis and Tsiatas, 2005) and employing the constant element assumption, the discretized integral form of the solution of Eqns. (30) and their derivatives at the N collocation points is

$$\mathbf{u}_i = \mathbf{C}_4\mathbf{b}_i - (\mathbf{E}_1\hat{\mathbf{u}}_i + \mathbf{E}_2\hat{\mathbf{u}}_{i,x} + \mathbf{E}_3\hat{\mathbf{u}}_{i,xx} + \mathbf{E}_4\hat{\mathbf{u}}_{i,xxx}) \tag{31a}$$

$$\mathbf{u}_{i,x} = \mathbf{C}_3\mathbf{b}_i - (\mathbf{E}_1\hat{\mathbf{u}}_{i,x} + \mathbf{E}_2\hat{\mathbf{u}}_{i,xx} + \mathbf{E}_3\hat{\mathbf{u}}_{i,xxx}) \tag{31b}$$

$$\mathbf{u}_{i,xx} = \mathbf{C}_2\mathbf{b}_i - (\mathbf{E}_1\hat{\mathbf{u}}_{i,xx} + \mathbf{E}_2\hat{\mathbf{u}}_{i,xxx}) \tag{31c}$$

$$\mathbf{u}_{i,xxx} = \mathbf{C}_1\mathbf{b}_i - \mathbf{E}_1\hat{\mathbf{u}}_{i,xxx} \tag{31d}$$

$$\mathbf{u}_{i,xxxx} = \mathbf{b}_i \tag{31e}$$

where $\mathbf{C}_j$ $(j = 1,2,3,4)$ are $N \times N$ known matrices; $\mathbf{E}_j$ $(j = 1,2,3,4)$ are $N \times 2$ also known matrices and $\mathbf{u}_i$, $\mathbf{u}_{i,x}$, $\mathbf{u}_{i,xx}$, $\mathbf{u}_{i,xxx}$, $\mathbf{u}_{i,xxxx}$ are vectors including the values of $u_i(x)$ and their derivatives at the N nodal points. Moreover,

$$\hat{\mathbf{u}}_i = \{u_i(0)\, u_i(l)\}^{\mathrm{T}}, \tag{32a}$$

$$\hat{\mathbf{u}}_{i,x} = \left\{\frac{du_i(0)}{dx}\frac{du_i(l)}{dx}\right\}^{\mathrm{T}} \tag{32b}$$

$$\hat{\mathbf{u}}_{i,xx} = \left\{\frac{d^2u_i(0)}{dx^2}\frac{d^2u_i(l)}{dx^2}\right\}^{\mathrm{T}}, \tag{32c}$$

$$\hat{\mathbf{u}}_{i,xxx} = \left\{\frac{d^3u_i(0)}{dx^3}\frac{d^3u_i(l)}{dx^3}\right\}^{\mathrm{T}} \tag{32d}$$

are vectors including the two unknown boundary values of the respective boundary quantities and $\mathbf{b}_i = \left\{b_1^i\, b_2^i \ldots b_N^i\right\}^{\mathrm{T}}$ $(i = 1, 2, 3)$ is the vector including the N unknown nodal values of the fictitious load.

Employing the aforementioned numerical procedure for the coupled boundary conditions (4), (5) the following set of linear equations is obtained

$$\begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{0} & \mathbf{D}_{14} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{18} \\ \mathbf{0} & \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{27} & \mathbf{0} \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_{42} & \mathbf{E}_{43} & \mathbf{E}_{44} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{53} & \mathbf{0} & \mathbf{D}_{55} & \mathbf{D}_{56} & \mathbf{0} & \mathbf{D}_{58} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_{63} & \mathbf{0} & \mathbf{0} & \mathbf{D}_{66} & \mathbf{D}_{67} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{E}_{42} & \mathbf{E}_{43} & \mathbf{E}_{44} \end{bmatrix} \cdot \begin{Bmatrix} \hat{\mathbf{u}}_1 \\ \hat{\mathbf{u}}_{1,x} \\ \hat{\mathbf{u}}_{1,xx} \\ \hat{\mathbf{u}}_{1,xxx} \\ \hat{\mathbf{u}}_2 \\ \hat{\mathbf{u}}_{2,x} \\ \hat{\mathbf{u}}_{2,xx} \\ \hat{\mathbf{u}}_{2,xxx} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\alpha}_3 \\ \overline{\boldsymbol{\alpha}}_3 \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{\beta}_3 \\ \overline{\boldsymbol{\beta}}_3 \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{F}_3 \\ \mathbf{F}_4 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{b}_1 + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{bmatrix} \mathbf{b}_2 \tag{33}$$

while for the boundary conditions (6) we have

$$\begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} & \mathbf{0} & \mathbf{E}_{14} \\ \mathbf{0} & \mathbf{E}_{22} & \mathbf{E}_{23} & \mathbf{0} \\ \mathbf{E}_{31} & \mathbf{E}_{32} & \mathbf{E}_{33} & \mathbf{E}_{34} \\ \mathbf{0} & \mathbf{E}_{42} & \mathbf{E}_{43} & \mathbf{E}_{44} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}}_3 \\ \hat{\mathbf{u}}_{3,x} \\ \hat{\mathbf{u}}_{3,xx} \\ \hat{\mathbf{u}}_{3,xxx} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\gamma}_3 \\ \overline{\boldsymbol{\gamma}}_3 \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{F}_3 \\ \mathbf{F}_4 \end{bmatrix} \mathbf{b}_3 \tag{34}$$

where $\mathbf{D}_{11}$, $\mathbf{D}_{12}$, $\mathbf{D}_{14}$, $\mathbf{D}_{18}$, $\mathbf{D}_{21}$, $\mathbf{D}_{22}$, $\mathbf{D}_{27}$, $\mathbf{D}_{53}$, $\mathbf{D}_{55}$, $\mathbf{D}_{56}$, $\mathbf{D}_{58}$, $\mathbf{D}_{63}$, $\mathbf{D}_{66}$, $\mathbf{D}_{67}$, $\mathbf{E}_{22}$, $\mathbf{E}_{23}$, $\mathbf{E}_{1j}$, $(j = 1,2,4)$ are 2×2 known square matrices including the values of the functions a_j, $\overline{a}_j$, β_j, $\overline{\beta}_j$ $(j = 1,2)$ of Eqns. (4)-(6); $\boldsymbol{\alpha}_3$, $\overline{\boldsymbol{\alpha}}_3$, $\boldsymbol{\beta}_3$, $\overline{\boldsymbol{\beta}}_3$, $\boldsymbol{\gamma}_3$, $\overline{\boldsymbol{\gamma}}_3$ are 2×1 known column matrices including the boundary values of the functions a_3, $\overline{a}_3$, β_3, $\overline{\beta}_3$, γ_3, $\overline{\gamma}_3$ of Eqns. (4)-(6); $\mathbf{E}_{jk}$, $(j = 3,4,\, k = 1,2,3,4)$ are square 2×2 known coefficient matrices resulting from the values of the kernels at the beam ends and $\mathbf{F}_j$ $(j = 3,4)$ are $2 \times N$ rectangular known matrices originating from the integration of the kernels on the axis of the beam.

Eqns. (31) after eliminating the boundary quantities employing Eqns. (33) and (34), can be written as

$$\mathbf{u}_i = \mathbf{T}_i\mathbf{b}_i + \mathbf{t}_i, \tag{35a}$$

$$\mathbf{u}_{i,x} = \mathbf{T}_{ix}\mathbf{b}_i + \mathbf{t}_{ix} \tag{35b}$$

$$\mathbf{u}_{i,xx} = \mathbf{T}_{ixx}\mathbf{b}_i + \mathbf{t}_{ixx}, \tag{35c}$$

$$\mathbf{u}_{i,xxx} = \mathbf{T}_{ixxx}\mathbf{b}_i + \mathbf{t}_{ixxx}, \tag{35d}$$

$$\mathbf{u}_{i,xxxx} = \mathbf{b}_i \tag{35e}$$

where $\mathbf{T}_i$, $\mathbf{T}_{ix}$, $\mathbf{T}_{ixx}$, $\mathbf{T}_{ixxx}$ are known $N\times N$ matrices and $\mathbf{t}_i$, $\mathbf{t}_{ix}$, $\mathbf{t}_{ixx}$, $\mathbf{t}_{ixxx}$ are known $N\times 1$ matrices. It is worth here noting that for homogeneous boundary conditions ($\alpha_3 = \overline{\alpha}_3 = \beta_3 = \overline{\beta}_3 = \gamma_3 = \overline{\gamma}_3 = 0$) it is $\mathbf{t}_i = \mathbf{t}_{ix} = \mathbf{t}_{ixx} = \mathbf{t}_{ixxx} = \mathbf{0}$.

In the conventional BEM, the load vectors $\mathbf{b}_i$ are known and Eqns. (35) are used to evaluate u_i and their derivatives at the N nodal points. This, however, can not be done here since $\mathbf{b}_i$ are unknown. For this purpose, $3N$ additional equations are derived, which permit the establishment of $\mathbf{b}_i$. These equations result by applying Eqns. (1)-(3) to the N collocation points, leading to the formulation of the following set of $3N$ simultaneous equations

$$(\mathbf{A}+P\mathbf{B})\begin{Bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{Bmatrix} = \begin{Bmatrix}\mathbf{p}_Y\\ \mathbf{p}_Z\\ \mathbf{m}_x+\mathbf{p}_Z y_C-\mathbf{p}_Y z_C+\mathbf{GI}_t\mathbf{t}_{3xx}\end{Bmatrix} - P\begin{Bmatrix}\mathbf{t}_{1xx}-z_C\mathbf{t}_{3xx}\\ \mathbf{t}_{2xx}+y_C\mathbf{t}_{3xx}\\ -z_C\mathbf{t}_{1xx}+y_C\mathbf{t}_{2xx}+i_\omega^2\mathbf{t}_{3xx}\end{Bmatrix} \tag{36}$$

In the above set of equations the matrices $\mathbf{A}$ and $\mathbf{B}$ are evaluated from the expressions

$$\mathbf{A} = \begin{bmatrix}\mathbf{E}_1\mathbf{I}_Z & \mathbf{E}_1\mathbf{I}_{YZ} & \mathbf{0}\\ \mathbf{E}_1\mathbf{I}_{YZ} & \mathbf{E}_1\mathbf{I}_Y & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{E}_1\mathbf{C}_S-\mathbf{G}_1\mathbf{I}_t\mathbf{T}_{3xx}\end{bmatrix} \tag{37}$$

$$\mathbf{B} = \begin{bmatrix}\mathbf{T}_{1xx} & \mathbf{0} & -z_C\mathbf{T}_{3xx}\\ \mathbf{0} & \mathbf{T}_{2xx} & y_C\mathbf{T}_{3xx}\\ -z_C\mathbf{T}_{1xx} & y_C\mathbf{T}_{2xx} & i_\omega^2\mathbf{T}_{3xx}\end{bmatrix} \tag{38}$$

where $\mathbf{E}_1\mathbf{I}_Y$, $\mathbf{E}_1\mathbf{I}_Z$, $\mathbf{E}_1\mathbf{I}_{YZ}$, $\mathbf{E}_1\mathbf{C}_S$, $\mathbf{G}_1\mathbf{I}_t$are $N\times N$ diagonal matrices including the values of the E_1I_Y, E_1I_Z, E_1I_{YZ}, E_1C_S, G_1I_t quantities, respectively, at the N nodal points. Moreover, $\mathbf{p}_Y$, $\mathbf{p}_Z$ and $\mathbf{m}_x$ are vectors containing the values of the external loading at these points.

Solving the linear system of Eqns. (36) for the fictitious load distributions $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ the displacements and their derivatives in the interior of the beam are computed using Eqns. (35).

Buckling equation

In this case it is $\alpha_3 = \overline{\alpha}_3 = \beta_3 = \overline{\beta}_3 = \gamma_3 = \overline{\gamma}_3 = 0$ (homogeneous boundary conditions) and $\mathbf{p}_X = \mathbf{p}_Y = \mathbf{m}_x = \mathbf{0}$. Thus, Eq. (36) becomes

$$(\mathbf{A}+P\mathbf{B})\begin{Bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{Bmatrix} = \mathbf{0} \tag{39}$$

The condition that Eq. (39) has a non-trivial solution yields the buckling equation

$$\det(\mathbf{A}+P\mathbf{B}) = 0 \tag{40}$$

3.2 The flexural-torsional vibration problem

The solution procedure to the vibration problem of homogeneous beams has been described in detail in (Sapountzakis and Tsiatas, 2007a). However, for the completeness of the paper we present the semidiscretized equation of motion, which for the case of the composite beam takes the form

$$\mathbf{M}\begin{Bmatrix}\ddot{\mathbf{b}}_1\\ \ddot{\mathbf{b}}_2\\ \ddot{\mathbf{b}}_3\end{Bmatrix} + \mathbf{K}\begin{Bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{Bmatrix} = \mathbf{f} \tag{41}$$

with

$$\mathbf{M} = \begin{bmatrix}\rho_1 A\mathbf{T}_1 & \mathbf{0} & -\rho_1 Az_C\mathbf{T}_3\\ \mathbf{0} & \rho_1 A\mathbf{T}_2 & \rho_1 Ay_C\mathbf{T}_3\\ -\rho_1 Az_C\mathbf{T}_1 & \rho_1 Ay_C\mathbf{T}_2 & \rho_1 I_S\mathbf{T}_3\end{bmatrix} \tag{42}$$

$$\mathbf{K} = \begin{bmatrix}\mathbf{E}_1\mathbf{I}_Z & \mathbf{E}_1\mathbf{I}_{YZ} & \mathbf{0}\\ \mathbf{E}_1\mathbf{I}_{YZ} & \mathbf{E}_1\mathbf{I}_Y & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{E}_1\mathbf{C}_S-\mathbf{G}_1\mathbf{I}_t\mathbf{T}_{3xx}\end{bmatrix} \tag{43}$$

$$\mathbf{f} = \begin{Bmatrix}\mathbf{p}_Y\\ \mathbf{p}_Z\\ \mathbf{m}_x+\mathbf{p}_Z y_C-\mathbf{p}_Y z_C+\mathbf{G}_1\mathbf{I}_t\mathbf{t}_{3xx}\end{Bmatrix} \tag{44}$$

playing the role of the generalized mass matrix, stiffness matrix and force vector, respectively.

Eq. (41) can be solved numerically, using any time step integration technique, to establish the time dependent vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$. Subsequently, the displacements as well as the stress resultants are computed at any cross-section of the beam.

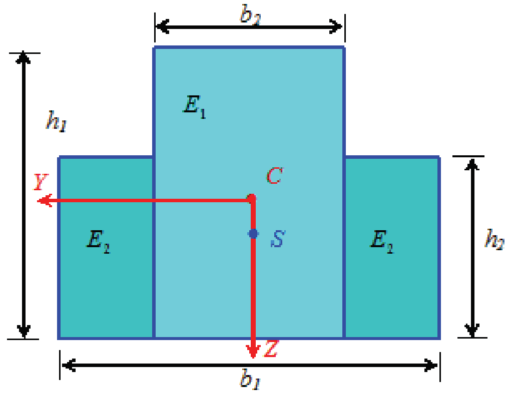

Figure 2: Composite cross section of the beam of Example 1.

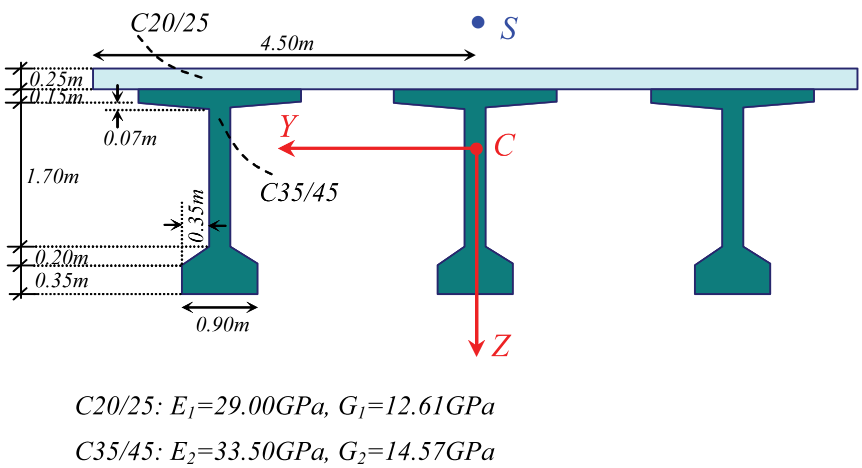

Figure 3: Composite slab-and-beams cross section of the beam of Example 2.

4 Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency, wherever possible the accuracy and the range of applications of the developed method. The program, based on the numerical implementation described in the previous, can be used with no modifications for both homogeneous (Sapountzakis and Tsiatas, 2007a) and composite beams of arbitrary cross section, subjected to any linear boundary conditions and to an arbitrarily dynamic loading.

Example 1

For comparison reasons, the composite beam of Fig. 2, with length $l = 1.0$ m, $\nu_1 = \nu_2 = 0.2$, $\rho_1 = \rho_2 = 2500\,\text{kg/m}^3$, $b_1 = h_1 = 0.4$ m has been studied. Three different types starting from a thin-walled and ending with a thick-walled cross-section are considered, that is (i) $b_2 = h_2 = 0.02$ m (ii) $b_2 = h_2 = 0.08$ m and (iii) $b_2 = h_2 = 0.20$ m. In Table 1 the computed values of the buckling load P for the cases of hinged-hinged (HH), fixed-hinged (FH) and fixed-fixed (FF) boundary conditions are presented. From the obtained results it can be concluded that the influence of the boundary conditions on the buckling load is significant, while the buckling load is increasing monotonically with the ratio E_2/E_1. Moreover, in Tables 2 through 4 the first seven eigenfrequencies of the aforementioned beam (hinged-hinged boundary conditions) are presented for various values of the ratio E_2/E_1 as compared with those presented in (Sapountzakis and Tsiatas, 2007a) for $E_2/E_1 = 1$ and found to be in excellent agreement.

Table 1: Buckling load P of the composite beam of Example 1.

$b_2 = h_2$	0.02m			0.08m			0.20m		
E_2/E_1	HH	FH	FF	HH	FH	FF	HH	FH	FF
0.5	648	776	1008	34710	46046	60766	213968	425148	710847
1	992	1157	1462	54931	67326	84864	349988	678619	917804
2	1747	1991	2441	92100	107058	130933	513241	1048796	1166912

Table 2: Eigenfrequencies of the composite HH beam of Example 1 ($b_2 = h_2 = 0.02$ m).

$b_2 = h_2$	0.02m			
E_2/E_1	0.5	1		2
	Present study	Present study	(Sapountzakis and Tsiatas, 2007a)	Present study
ω_1	2.204	2.500	2.500	3.029
ω_2	5.507	6.075	6.075	7.158
ω_3	10.407	11.284	11.284	13.026
ω_4	14.529	16.983	16.983	20.910
ω_5	17.095	18.349	18.349	21.633
ω_6	19.475	21.681	21.681	23.720
ω_7	25.639	27.354	27.354	30.922

Table 3: Eigenfrequencies of the composite HH beam of Example 1 ($b_2 = h_2 = 0.08$ m).

$b_2 = h_2$	0.08m			
E_2/E_1	0.5	1		2
	Present study	Present study	(Sapountzakis and Tsiatas, 2007a)	Present study
ω_1	7.891	9.768	9.768	12.027
ω_2	15.812	18.285	18.285	22.194
ω_3	18.526	20.336	20.336	22.709
ω_4	21.132	24.280	24.280	28.535
ω_5	39.552	44.475	44.475	51.527
ω_6	56.476	69.411	69.411	82.225
ω_7	64.292	71.529	71.529	88.798

Table 4: Eigenfrequencies of the composite HH beam of Example 1 ($b_2 = h_2 = 0.20$ m).

$b_2 = h_2$	0.20m			
E_2/E_1	0.5	1		2
	Present study	Present study	(Sapountzakis and Tsiatas, 2007a)	Present study
ω_1	13.351	17.080	17.080	20.682
ω_2	17.866	19.025	19.025	22.887
ω_3	24.446	27.132	27.132	30.063
ω_4	47.682	55.362	55.362	64.421
ω_5	58.005	70.978	70.978	82.748
ω_6	71.478	76.117	76.117	92.123
ω_7	81.240	91.556	91.556	107.400

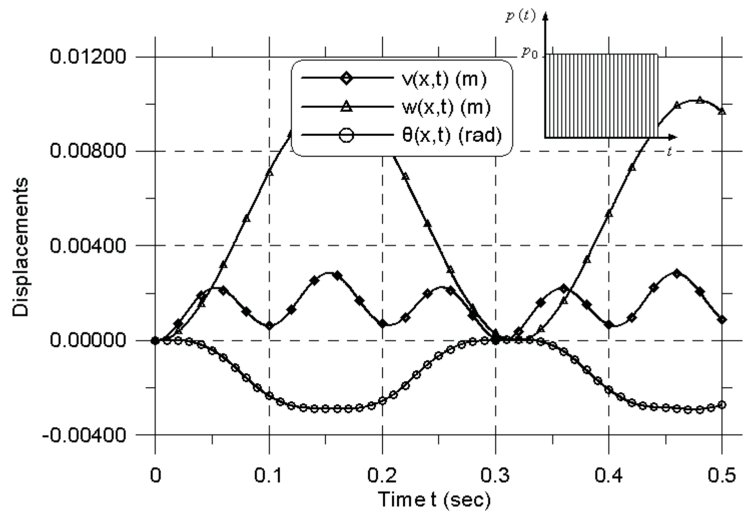

Figure 4: Time history of the displacements for the hinged-hinged beam of Example 2.

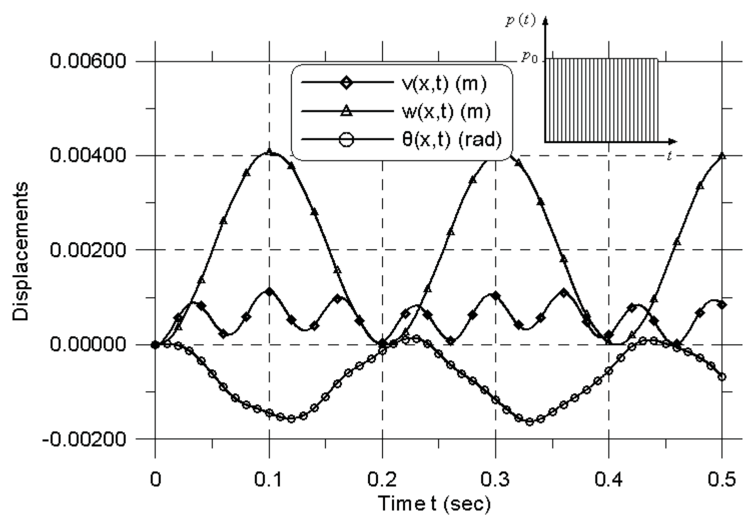

Figure 5: Time history of the displacements for the fixed-hinged beam of Example 2.

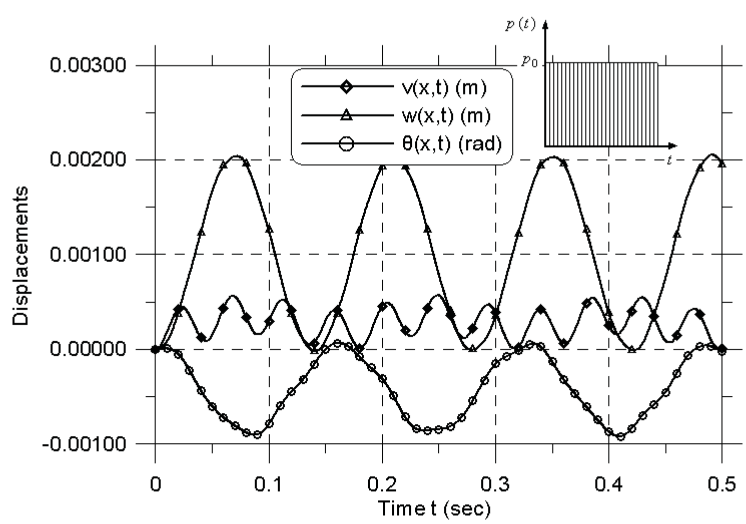

Figure 6: Time history of the displacements for the fixed-fixed beam of Example 2.

Table 5: Buckling load P of the composite beam of Example 2.

(HH)	(FH)	(F-FTS) $k_R = 10^5$	(F-FTS) $k_R = 3\times10^5$	(F-FTS) $k_R = 5\times10^5$	(FF)
1013656	1750987	1464110	2121313	2738484	2998312

Table 6: Eigenfrequencies of the composite beam of Example 2.

(HH)	(FH)	(F-FTS) $k_R = 10^7$	(F-FTS) $k_R = 10^8$	(F-FTS) $k_R = 10^9$	(FF)
3.146	4.500	5.640	6.070	6.112	6.117
3.248	4.915	7.132	7.132	7.132	7.132
9.816	13.364	12.869	15.900	16.202	16.235
10.824	15.263	19.666	19.666	19.666	19.666
12.588	15.932	21.787	22.094	22.099	22.100
23.328	27.147	22.602	29.956	31.170	31.290
28.334	33.254	37.241	38.569	38.569	38.569

From the aforementioned tables it can be pointed out that all eigenfrequencies are increasing also monotonically with the ratio E_2/E_1.

Example 2

The composite beam of length $l = 40.0$ m having a cross section consisting of a concrete C20/25 rectangular part stiffened by three concrete C35/45 beams (Fig. 3) has been studied. The data are $E_1 I_Y = 1.64321\times10^8$ kNm^2, $E_1 I_Z = 1.17524\times10^9$ kNm^2, $E_1 C_s = 9.96332\times10^8$ kNm^4, $G_1 I_t = 4.46380\times10^6$ kNm^2, $A = 6.40092$ m^2, $I_S = 60.38196$ m^4, $\rho_1 = \rho_2 = 2.5$ ton/m^3, $z_C = 1.48902$ m. Three types of boundary conditions, namely hinged-hinged (HH), fixed-hinged (FH) and fixed-fixed (FF) are considered, as well as a fixed-fixed with a torsional elastic support (F-FTS) at $x = l$ (all coefficients in boundary conditions are set to zero except from $\alpha_1 = \beta_1 = \overline{\alpha}_1 = \overline{\beta}_1 = 1$ at $x = 0, l$, $\gamma_1 = \overline{\gamma}_1 = 1$ at $x = 0$ and $\gamma_1 = k_R$, $\gamma_2 = \overline{\gamma}_1 = 1$ at $x = l$). In Tables 5 and 6 the buckling load P and the first seven eigenfrequencies of the aforementioned beam are presented. Moreover, the forced vibrations arising from the application of the dynamic loading $p_Z(t) = p_{Z0}$ and $p_Y(t) = p_{Y0}$ with zero initial conditions are examined. In Figs. 4 through 6 the time histories of the displacements at the cross-section $x = 20.0$ m, for the three aforementioned types of boundary conditions and $p_{Y0} = 50$ kN/m, $p_{Z0} = 25$ kN/m are presented. The influence of the boundary conditions on the buckling load, the eigenfrequencies and the response of the beam is pronounced.

5 Conclusions

In this paper a boundary element method is developed for the general flexural-torsional buckling and vibration analysis of composite Euler-Bernoulli beams of arbitrarily shaped cross section. The composite beam consists of materials in contact each of which can surround a finite number of inclusions. It is subjected to a compressive centrally applied load together with arbitrarily transverse and/or torsional distributed or concentrated loading, while its edges are restrained by the most general linear boundary conditions. The main conclusions that can be drawn from this investigation are:

a. The general character of the proposed method is verified from the fact that all basic equations are formulated with respect to an arbitrary coordinate system, which is not restricted to the principal one.

b. The proposed method can treat composite beams of both thin and thick walled cross sections taking into account the warping along the thickness of the walls.

c. The developed method overcomes the shortcoming of a possible thin tube theory (TTT) solution, which its utilization has been proven to be prohibitive even in thin walled homogeneous sections.

d. The method permits the inclusion of the torsion-bending and flexural coupling stiffnesses which play an important role in the response of the beam and have to be taken always into account.

e. The influence of the boundary conditions on the buckling load is significant, while the buckling load is increasing monotonically with the ratio E_2/E_1.

f. All eigenfrequencies are increasing also monotonically with the ratio E_2/E_1.

g. The computer program, based on the numerical implementation described above, can be used with no modifications for both homogeneous and composite beams of arbitrary cross section, subjected to any linear boundary conditions and to an arbitrarily dynamic loading.

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