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The Method of Fundamental Solutions for Eigenfrequencies of Plate Vibrations

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Abstract: This paper describes the method of fundamental solutions (MFS) to solve eigenfrequencies of plate vibrations by utilizing the direct determinant search method. The complex-valued kernels are used in the MFS in order to avoid the spurious eigenvalues. The benchmark problems of a circular plate with clamped, simply supported and free boundary conditions are studied analytically as well as numerically using the discrete and continuous versions of the MFS schemes to demonstrate the major results of the present paper. Namely only true eigenvalues are contained and no spurious eigenvalues are included in the range of direct determinant search method. Consequently analytical derivation is carried out by using the degenerate kernels and Fourier series to obtain the exact eigenvalues which are used to validate the numerical methods. The MFS is free from meshes, singularities, and numerical integrations. As a result, the proposed numerical method can be easily used to solve plate vibrations free from spurious eigenvalues in simply connected domains.

keyword: Method of fundamental solutions, Plate vibration analysis, Spurious eigenvalues, Circular plate, Clamped, Simply supported, Free boundary conditions

1 Introduction

Various analytic and numerical methods for the analysis of plate vibrations are available in the literature. A comprehensive survey for the plate vibration analysis has been given by Leissa (1969); Leissa, Laura, and Gutierrez (1979); Leissa and Narita (1980),but a general method for computing eigenvalues as well as eigenmodes, especially for a simple scheme free from spurious eigenvalues, is still lacking. Hutchinson (1988, 1991) conducted a series of studies for plate vibrations by utilizing the boundary integral equation (BIE) method with real-part kernel, but did not circumvent the problem of spurious eigenvalues. Chen, Lin, Chen, and Chen (2004) employed the same concept of BIE with real-part kernel but used the singular value decomposition (SVD) updating technique to suppress the occurrence of the spurious eigenvalues. Kang and Lee (2001); and Kang (2002) recently employed the non-dimensional dynamic influence function (NDIF) method to solve the plate vibrations with clamped and mixed boundary conditions. Chen, Kou, Chen, and Cheng (2003) commented that the NDIF method is a special case of the imaginary-part BIE after lumping the distribution of intensity function, therefore like the real–part BIE the spurious eigenvalues were observed. Chen, Chen, Chen, Lee, and Yeh (2004) further revisited this concept and used the imaginary-part fundamental solution as the radial basis function (RBF) to solve the plate vibration problems. Even though true instead of spurious eigensolutions were obtained by using SVD technique, more or same computational efforts (have to use two terms of imaginary-part fundamental solution) as the NDIF or complex-valued BIE were needed. Furthermore, only clamped boundary condition was considered in their work. Nevertheless, spurious eigensolutions are inherent in the NDIF, real-part BIE as well as imaginary-part BIE formulations as indicated by Chen, Kou, Chen, and Cheng (2003). Kitahara (1985) had already employed the complex-valued BIE as a general method to resolve the eigenvalues and eigenmodes for plate vibrations with various boundary conditions.

In the literature, various meshless numerical methods have been investigated to solve partial differential equations [Han, and Atluri (2004); Atluri, Han, and Rajendran (2004); Atluri, and Shen (2005)]. In particular, the works of Li, Soric, Jarak, and Atluri (2005), as well as Soric, Li, Jarak, and Atluri (2004) were devoted to the meshless local Petrov-Galerkin (MLPG) formulation for thick and thin plates. In this paper, we concentrated on the MFS, which was first developed by Kupradze

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and Aleksidze (1964) and has re-emerged as a promising meshless numerical scheme for solving various types of partial differential equations. The basic idea of the MFS is to decompose the solutions of the partial differential equations into a linear combination of the fundamental solutions, in which source points are located on a fictitious boundary outside the computational domain. Here, the intensities of the sources are the unknown parameters to be found. Excellent reviews of the MFS are available in the recent literature survey [Fairweather and Karageorghis (1998); Golberg and Chen (1998); Fairweather, Karageorghis, and Martin (2003); Cho, Golberg, Muleshkov, and Li (2004)].

The MFS has also been applied to solve the Helmholtz equation for scattering and radiating problems. Kondapalli, Shippy, and Fairweather (1992); Fairweather, Karageorghis, and Martin (2003) applied the MFS to solve acoustic scattering problems. Karageorghis (2001); Young, Hu, Chen, Fan, and Murugesan (2005); Young and Ruan (2005); Chen, Fan, Young, Murugesan, and Tsai (2005) all applied the complex-valued MFS for the 2D Helmholtz problems in simply connected domains. Young and Ruan (2005) further extended the MFS to the coupled 3D Helmholtz equations for electromagnetic wave propagation problems. In this paper, we will develop the complex-valued MFS for the bi-Helmholtz equation to obtain eigenfrequencies of free vibrations of thin plates with the clamped, simply supported and free boundary conditions for simply connected domains. The direct determinant search method is adopted to obtain eigenvalues of the resulted system matrix [Chen., Lin, Chen, and Chen (2004)]. The free vibration analysis of a circular plate is investigated both analytically and numerically using the discrete and continuous versions of the MFS schemes. It is found that only true eigenvalues are captured and no spurious eigenvalues are contaminated using the proposed complex-valued MFS. The MFS formulations, analytical derivations, numerical results, and conclusions are addressed in the following sections.

2 MFS Formulation

For free flexural vibration of a uniform thin plate, the governing equation is the bi-Helmholtz equation with boundary conditions [Kitahara (1985)]:

$$\begin{cases} \nabla^4 u(\mathbf{x}) - \lambda^4 u(\mathbf{x}) = 0 \ \mathbf{x} \in \Omega \\ B_1 u(\mathbf{x}) = B_2 u(\mathbf{x}) = 0 \ \mathbf{x} \in \Gamma \end{cases} \tag{1}$$

where $u(\mathbf{x})$is the function of lateral displacement, $\lambda^4 = \frac{\omega^2\rho_s}{D}$is the frequency parameter, ω is the circular frequency, ρ_s is the mass density per unit area, and h is the plate thickness. Also, $D = \frac{Eh}{12(1-\nu)}$ is the flexural rigidity of the thin plate, ν is the Poisson ratio,E is the Young's modulus and Ω is the domain of interest with boundary Γ. We assume homogeneous boundary conditions for simplicity. In equation (1), the boundary operators B_1 and B_2 are any two of the following operators:

$$\mathbf{K}_u(\bullet) = 1 \tag{2a}$$

$$\mathbf{K}_\theta(\bullet) = \frac{\partial(\bullet)}{\partial n_\mathbf{x}} \tag{2b}$$

$$\mathbf{K}_m(\bullet) = \nu\nabla^2_\mathbf{x}(\bullet) + (1-\nu)\frac{\partial^2(\bullet)}{\partial n^2_\mathbf{x}} \tag{2c}$$

$$\mathbf{K}_\nu(\bullet) = \frac{\partial\nabla^2_\mathbf{x}(\bullet)}{\partial n_\mathbf{x}} + (1-\nu)\frac{\partial}{\partial t_\mathbf{x}}\frac{\partial^2(\bullet)}{\partial n_\mathbf{x}\partial t_\mathbf{x}} \tag{2d}$$

where $\frac{\partial}{\partial n_\mathbf{x}}$ and $\frac{\partial}{\partial t_\mathbf{x}}$ are the normal and tangential derivatives, respectively, on the boundary point $\mathbf{x}$. In the above equations we denote $\mathbf{K}_u(u(\mathbf{x}))$, $\mathbf{K}_\theta(u(\mathbf{x}))$, $\mathbf{K}_m(u(\mathbf{x}))$, and $\mathbf{K}_\nu(u(\mathbf{x}))$ the lateral displacement, the slope, the normal moment, and the effective sheer force respectively. Therefore, $(2a) \sim (2b)$ are selected for plate vibrations with clamped boundary condition, $(2a)$and$(2c)$ for simply supported boundary condition, and $(2c) \sim (2d)$ for free boundary condition.

The fundamental solution of the equation (1) is defined by

$$\nabla^4 G_\lambda(\mathbf{x};\mathbf{s}) - \lambda^4 G_\lambda(\mathbf{x};\mathbf{s}) = -\delta(\mathbf{x}-\mathbf{s}) \tag{3}$$

where $\mathbf{x}$ are the coordinates of field points and $\mathbf{s}$ are the coordinates of source points. Then, the fundamental solution is given by [Kitahara (1985)]:

$$G_\lambda(\mathbf{x};\mathbf{s}) = \frac{1}{8\lambda^2}\left[-iH_0^{(1)}(\lambda r) + \frac{2}{\pi}K_0(\lambda r)\right] \tag{4}$$

where $H_0^{(1)}(\lambda r)$is the Hankel function of the first kind of order zero, and $K_0(\lambda r)$ is the modified Bessel function of the second kind of order zero, $r = |\mathbf{x}-\mathbf{s}|$ and $i^2 = -1$.

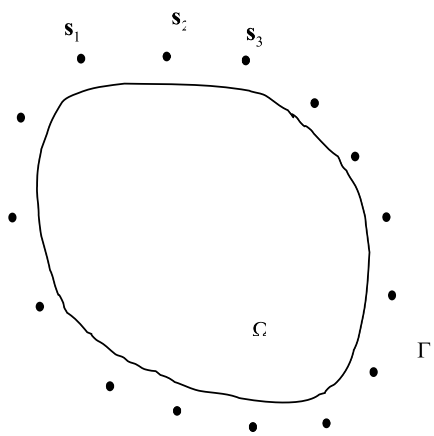

Figure 1 : Geometry configuration of the MFS for plate vibrations

In the MFS, the solution is assumed to be

$$u(\mathbf{x}) = \sum_{j=1}^{N} \{\alpha_j H_0^{(1)}(\lambda|\mathbf{x}-\mathbf{s}_j|) + \beta_j K_0(\lambda|\mathbf{x}-\mathbf{s}_j|)\} \qquad (5)$$

where α_j and β_j are the intensities of the source point at $\mathbf{s}_j$, and N is the number of source points as depicted in Fig. 1.

For simplicity, we define the following notations of kernels:

$$U_{1,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_u(H_0^{(1)}(\lambda|\mathbf{x}-\mathbf{s}|)) \qquad (6a)$$

$$U_{2,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_u(K_0(\lambda|\mathbf{x}-\mathbf{s}|)) \qquad (6b)$$

$$\Theta_{1,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_\theta(H_0^{(1)}(\lambda|\mathbf{x}-\mathbf{s}|)) \qquad (6c)$$

$$\Theta_{2,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_\theta(K_0(\lambda|\mathbf{x}-\mathbf{s}|)) \qquad (6d)$$

$$M_{1,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_m(H_0^{(1)}(\lambda|\mathbf{x}-\mathbf{s}|)) \qquad (6e)$$

$$M_{2,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_m(K_0(\lambda|\mathbf{x}-\mathbf{s}|)) \qquad (6f)$$

$$V_{1,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_v(H_0^{(1)}(\lambda|\mathbf{x}-\mathbf{s}|)) \qquad (6g)$$

$$V_{2,\lambda}(\mathbf{x},\mathbf{s}) = \mathbf{K}_v(K_0(\lambda|\mathbf{x}-\mathbf{s}|)) \qquad (6h)$$

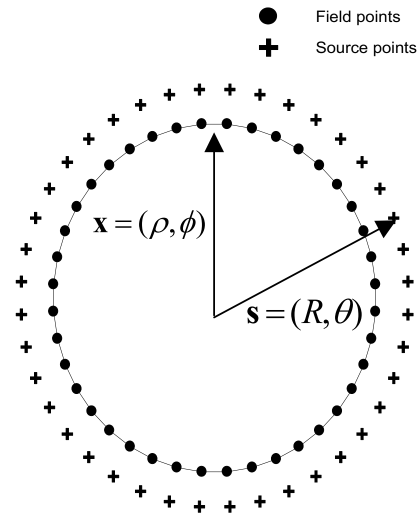

Figure 2 : Geometry configuration of the MFS for circular plate vibrations

Equation (5) then becomes:

$$u(\mathbf{x}) = \sum_{j=1}^{N} \{\alpha_j U_{1,\lambda}(\mathbf{x},\mathbf{s}_j) + \beta_j U_{2,\lambda}(\mathbf{x},\mathbf{s}_j)\} \qquad (7)$$

In order to obtain α_j and β_j, $1 \le j \le N$, N boundary field points are chosen to satisfy the specified boundary conditions in equation(1). That is,

$$\mathbf{K}_u(u(\mathbf{x}_i)) = \sum_{j=1}^{N} \{\alpha_j U_{1,\lambda}(\mathbf{x}_i,\mathbf{s}_j) + \beta_j U_{2,\lambda}(\mathbf{x}_i,\mathbf{s}_j)\} = 0 \quad \text{for } \{\mathbf{x}_i\}_1^N \in \Gamma \qquad (8a)$$

$$\mathbf{K}_\theta(u(\mathbf{x}_i)) = \sum_{j=1}^{N} \{\alpha_j \Theta_{1,\lambda}(\mathbf{x}_i,\mathbf{s}_j) + \beta_j \Theta_{2,\lambda}(\mathbf{x}_i,\mathbf{s}_j)\} = 0 \quad \text{for } \{\mathbf{x}_i\}_1^N \in \Gamma \qquad (8b)$$

$$\mathbf{K}_m(u(\mathbf{x}_i)) = \sum_{j=1}^{N} \{\alpha_j M_{1,\lambda}(\mathbf{x}_i,\mathbf{s}_j) + \beta_j M_{2,\lambda}(\mathbf{x}_i,\mathbf{s}_j)\} = 0 \quad \text{for } \{\mathbf{x}_i\}_1^N \in \Gamma \qquad (8c)$$

$$\mathbf{K}_v(u(\mathbf{x}_i)) = \sum_{j=1}^{N} \{\alpha_j V_{1,\lambda}(\mathbf{x}_i,\mathbf{s}_j) + \beta_j V_{2,\lambda}(\mathbf{x}_i,\mathbf{s}_j)\} = 0 \quad \text{for } \{\mathbf{x}_i\}_1^N \in \Gamma \qquad (8d)$$

where $\{\mathbf{x}_i\}_1^N$ are the collocated boundary field points. For simplicity, we assume $\mathbf{s}_j$ to be *a priori* distributed source points and the number of collocated boundary field point is equal to the number of source points. This results in a 2Nx2N linear system;

$$\begin{bmatrix} A(\lambda,\mathbf{x}_1,\mathbf{s}_1) & \cdots & A(\lambda,\mathbf{x}_1,\mathbf{s}_N) & B(\lambda,\mathbf{x}_1,\mathbf{s}_{21}) & \cdots & B(\lambda,\mathbf{x}_1,\mathbf{s}_N) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A(\lambda,\mathbf{x}_N,\mathbf{s}_1) & \cdots & A(\lambda,\mathbf{x}_N,\mathbf{s}_N) & B(\lambda,\mathbf{x}_N,\mathbf{s}_1) & \cdots & B(\lambda,\mathbf{x}_N,\mathbf{s}_N) \\ C(\lambda,\mathbf{x}_1,\mathbf{s}_1) & \cdots & C(\lambda,\mathbf{x}_1,\mathbf{s}_N) & D(\lambda,\mathbf{x}_1,\mathbf{s}_{21}) & \cdots & D(\lambda,\mathbf{x}_1,\mathbf{s}_N) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ C(\lambda,\mathbf{x}_N,\mathbf{s}_1) & \cdots & C(\lambda,\mathbf{x}_N,\mathbf{s}_N) & D(\lambda,\mathbf{x}_N,\mathbf{s}_1) & \cdots & D(\lambda,\mathbf{x}_N,\mathbf{s}_N) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \\ \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{9}$$

where $A(\lambda,\mathbf{x}_i,\mathbf{s}_j) = U_{1,\lambda}(\mathbf{x}_i,\mathbf{s}_j)$, $B(\lambda,\mathbf{x}_i,\mathbf{s}_j) = U_{2,\lambda}(\mathbf{x}_i,\mathbf{s}_j)$, $C(\lambda,\mathbf{x}_i,\mathbf{s}_j) = \Theta_{1,\lambda}(\mathbf{x}_i,\mathbf{s}_j)$, and $D(\lambda,\mathbf{x}_i,\mathbf{s}_j) = \Theta_{2,\lambda}(\mathbf{x}_i,\mathbf{s}_j)$ for clamped boundary condition, and similarly for other types of boundary conditions. Equation (9) is an eigenproblem for λ and we are searching for eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ and their corresponding eigenvectors $[\alpha_1,...,\alpha_N,\beta_1,...,\beta_N]^T$ such that equation (9) has nontrivial solutions. In this paper, we use the direct determinant search method to obtain eigenvalues [Chen, Lin, Chen, and Chen (2004)]. Details will be addressed in the Section of Numerical Results.

3 Analytic Solutions

In general, the eigenfrequencies of plate vibrations obtained by the MFS cannot be solved analytically except for some special geometric shapes. We therefore derive analytically the MFS solutions for the following problems: (I) circular plate with clamped boundary condition, (II) circular plate with simply supported boundary condition and (III) circular plate with free boundary condition. In the following derivations, we assume the source points $\mathbf{s}$ are dense enough so that the equations of the MFS equation (8) can be transformed into the integral forms as follows:

$$\mathbf{K}_u(u(\mathbf{x})) = \int \{\alpha(\mathbf{s})U_{1,\lambda}(\mathbf{x},\mathbf{s}) + \beta(\mathbf{s})U_{2,\lambda}(\mathbf{x},\mathbf{s})\}d\mathbf{s} \tag{10a}$$

$$\mathbf{K}_\theta(u(\mathbf{x})) = \int \{\alpha(\mathbf{s})\Theta_{1,\lambda}(\mathbf{x},\mathbf{s}) + \beta(\mathbf{s})\Theta_{2,\lambda}(\mathbf{x},\mathbf{s})\}d\mathbf{s} \tag{10b}$$

$$\mathbf{K}_m(u(\mathbf{x})) = \int \{\alpha(\mathbf{s})M_{1,\lambda}(\mathbf{x},\mathbf{s}) + \beta(\mathbf{s})M_{2,\lambda}(\mathbf{x},\mathbf{s})\}d\mathbf{s} \tag{10c}$$

$$\mathbf{K}_v(u(\mathbf{x})) = \int \{\alpha(\mathbf{s})V_{1,\lambda}(\mathbf{x},\mathbf{s}) + \beta(\mathbf{s})V_{2,\lambda}(\mathbf{x},\mathbf{s})\}d\mathbf{s} \tag{10d}$$

We consider the circular plate, as described in Fig. 2, and let $\mathbf{s} = (R,\theta)$ and $\mathbf{x} = (\rho,\phi)$ denote the polar coordinates of source and boundary field points respectively. To analytically derive the MFS solutions in a circular domain, it is necessary to decompose the kernels $(6a) \sim (6b)$ into circular harmonics, which will result in the following degenerate kernels [Abramowitz and Stegun (1972); Chen, Lin, Chen, and Chen (2004); Chen, Chen, Chen, Lee, and Yeh (2004)]:

$$U_{1,\lambda}(\mathbf{x},\mathbf{s}) = \sum_{m=-\infty}^{m=\infty} J_m(\lambda\rho)H_m^{(1)}(\lambda R)\cos(m(\theta-\phi)) \tag{11a}$$

$$U_{2,\lambda}(\mathbf{x},\mathbf{s}) = \sum_{m=-\infty}^{m=\infty} I_m(\lambda\rho)K_m(\lambda R)\cos(m(\theta-\phi)) \tag{11b}$$

where $J_m(\lambda\rho)$ is the Bessel function of the first kind of order m, $H_m^{(1)}(\lambda R)$ is the Hankel function of the first kind of order m, $I_m(\lambda\rho)$ is the modified Bessel function of the first kind of order m, and $K_m(\lambda R)$ is the modified Bessel function of the second kind of order m. Moreover, $R > \rho$ is assumed in the above equations for the interior acoustics problems. Similar equations can be obtained for equations $(6c) \sim (6h)$:

$$\Theta_{1,\lambda}(\mathbf{x},\mathbf{s}) = \sum_{m=-\infty}^{m=\infty} \lambda J'_m(\lambda\rho)H_m^{(1)}(\lambda R)\cos(m(\theta-\phi)) \tag{11c}$$

$$\Theta_{2,\lambda}(\mathbf{x},\mathbf{s}) = \sum_{m=-\infty}^{m=\infty} \lambda I'_m(\lambda\rho)K_m^{(}\lambda R)\cos(m(\theta-\phi)) \tag{11d}$$

$$M_{1,\lambda}(\mathbf{x},\mathbf{s}) = \sum_{m=-\infty}^{m=\infty} H_m^{(1)}(\lambda R)\cos(m(\theta-\phi)) \times (\lambda^2 J_m''(\lambda\rho) + \nu\frac{\lambda}{\rho}J_m'(\lambda\rho) - \nu(\frac{m}{\rho})^2 J_m(\lambda\rho)) \quad (11e)$$

$$M_{2,\lambda}(\mathbf{x},\mathbf{s}) = \sum_{m=-\infty}^{m=\infty} K_m(\lambda R)\cos(m(\theta-\phi)) \times (\lambda^2 I_m''(\lambda\rho) + \nu\frac{\lambda}{\rho}I_m'(\lambda\rho) - \nu(\frac{m}{\rho})^2 I_m(\lambda\rho)) \quad (11f)$$

$$V_{1,\lambda}(\mathbf{x},\mathbf{s}) = \sum_{m=-\infty}^{m=\infty} H_m^{(1)}(\lambda R)\cos(m(\theta-\phi)) \times (\lambda^3 J_m'''(\lambda\rho) + \frac{\lambda^2}{\rho}J_m''(\lambda\rho) + \frac{(\nu-2)m^2-1}{\rho^2}\lambda J_m'(\lambda\rho) + \frac{(3-\nu)m^2}{\rho^3}J_m(\lambda\rho)) \quad (11g)$$

$$V_{2,\lambda}(\mathbf{x},\mathbf{s}) = \sum_{m=-\infty}^{m=\infty} K_m(\lambda R)\cos(m(\theta-\phi)) \times (\lambda^3 I_m'''(\lambda\rho) + \frac{\lambda^2}{\rho}I_m''(\lambda\rho) + \frac{(\nu-2)m^2-1}{\rho^2}\lambda I_m'(\lambda\rho) + \frac{(3-\nu)m^2}{\rho^3}I_m(\lambda\rho)) \quad (11h)$$

We likewise decompose the source intensities (in equation(10)) into the Fourier series

$$\alpha(\mathbf{s}) = \sum_{n=-\infty}^{\infty} A_n\cos n\theta + B_n\sin n\theta \quad (12a)$$

$$\beta(\mathbf{s}) = \sum_{n=-\infty}^{\infty} C_n\cos n\theta + D_n\sin n\theta \quad (12b)$$

Substituting equations $(11) \sim (12)$ into the integral forms of equation(10), we then obtain

$$\mathbf{K}_u(u(\mathbf{x})) = \sum_{m=-\infty}^{\infty} \left\{\pi A_m H_m^{(1)}(\lambda R) J_m(\lambda\rho) + \pi C_m K_m(\lambda R) I_m(\lambda\rho)\right\} \quad (13a)$$

$$\mathbf{K}_\theta(u(\mathbf{x})) = \sum_{m=-\infty}^{\infty} \left\{\pi A_m H_m^{(1)}(\lambda R)\lambda J_m'(\lambda\rho) + \pi C_m K_m(\lambda R)\lambda I_m'(\lambda\rho)\right\} \quad (13b)$$

$$\mathbf{K}_m(u(\mathbf{x})) = \sum_{m=-\infty}^{\infty} \left\{\pi A_m H_m^{(1)}(\lambda R)(\lambda^2 J_m''(\lambda\rho) + \nu\frac{\lambda}{\rho}J_m'(\lambda\rho) - \nu(\frac{m}{\rho})^2 J_m(\lambda\rho)) + \pi C_m K_m(\lambda R)(\lambda^2 I_m''(\lambda\rho) + \nu\frac{\lambda}{\rho}I_m'(\lambda\rho) - \nu(\frac{m}{\rho})^2 I_m(\lambda\rho))\right\} \quad (13c)$$

$$\mathbf{K}_v(u(\mathbf{x})) = \sum_{m=-\infty}^{\infty} \left\{\pi A_m H_m^{(1)}(\lambda R)(\lambda^3 J_m'''(\lambda\rho) + \frac{\lambda^2}{\rho}J_m''(\lambda\rho) + \frac{(\nu-2)m^2-1}{\rho^2}\lambda J_m'(\lambda\rho) + \frac{(3-\nu)m^2}{\rho^3}J_m(\lambda\rho)) + \pi C_m K_m(\lambda R)(\lambda^3 I_m'''(\lambda\rho) + \frac{\lambda^2}{\rho}I_m''(\lambda\rho) + \frac{(\nu-2)m^2-1}{\rho^2}\lambda I_m'(\lambda\rho) + \frac{(3-\nu)m^2}{\rho^3}I_m(\lambda\rho))\right\} \quad (13d)$$

in which the following orthogonal relations are applied:

$$\begin{cases} \int_0^{2\pi} \sin[n\theta]\cos[m\theta]\, d\theta = 0 \\ \int_0^{2\pi} \cos[n\theta]\cos[m\theta]\, d\theta = \pi\delta_{mn} \\ \int_0^{2\pi} \sin[n\theta]\sin[m\theta]\, d\theta = \pi\delta_{mn} \end{cases} \quad (14)$$

where δ_{mn} is the Kronecker delta symbol.

3.1 Case I: circular plate with clamped boundary condition

We are now ready to derive the analytic solution of plate vibrations with clamped boundary condition for the continuous version of the MFS. The clamped boundary condition is defined by

$$\begin{cases} \mathbf{K}_u(u(\mathbf{x})) = 0 \\ \mathbf{K}_\theta(u(\mathbf{x})) = 0 \end{cases} \quad (15)$$

Applying equations $(13a)$ and $(13b)$ and the fact that A_n, B_n, C_n, and D_n are arbitrary constants, we obtain

$$H_n^{(1)}(\lambda R)J_n(\lambda\rho)K_n(\lambda R)I_n'(\lambda\rho) = H_n^{(1)}(\lambda R)J_n'(\lambda\rho)K_n(\lambda R)I_n(\lambda\rho) \quad (16)$$

Since $H_n^{(1)}(\lambda R) \neq 0$ and $Kn_n(\lambda R) \neq 0$ for any real numberλ, equation (16) can be further simplified to

$$J_n(\lambda\rho)I_{n+1}(\lambda\rho) + J_{n+1}(\lambda\rho)I_n(\lambda\rho) = 0 \tag{17}$$

Equation (17) is just the eigenequation using the continuous version of the MFS scheme for clamped circular plate.

3.2 Case II: circular plate with simply supported boundary condition

The derivation of analytic solution for simply supported circular plate can be similarly performed in a similar fashion. The simply supported boundary condition is defined by

$$\begin{cases} \mathbf{K}_u(u(\mathbf{x})) = 0 \\ \mathbf{K}_m(u(\mathbf{x})) = 0 \end{cases} \tag{18}$$

Applying equations $(13a)$ and $(13c)$ and the fact that A_n, B_n, C_n, and D_n are arbitrary and $H_n^{(1)}(\lambda R) \neq 0$ and $K_n(\lambda R) \neq 0$ for any real number λ, we obtain

$$(1-\nu)\left[J_{n+1}(\lambda\rho)I_n(\lambda\rho) + J_n(\lambda\rho)I_{n+1}(\lambda\rho)\right] - 2\lambda\rho J_n(\lambda\rho)I_n(\lambda\rho) = 0. \tag{19}$$

Equation (19) is the eigenequation using the continuous version of the MFS for simply supported circular plate.

3.3 Case III: circular plate with free boundary condition

The above procedure for the derivation of analytic solution can also be extended to free boundary condition, which is defined by

$$\begin{cases} \mathbf{K}_m(u(\mathbf{x})) = 0 \\ \mathbf{K}_v(u(\mathbf{x})) = 0 \end{cases} \tag{20}$$

Applying equations $(13c)$ and $(13d)$, we similarly have

$$\lambda\rho(1-\nu)[-4n^2(n-1)I_n(\lambda\rho)J_n(\lambda\rho) - 2\lambda^2\rho^2 I_{n+1}(\lambda\rho)J_{n+1}(\lambda\rho)] + 2n\lambda^2\rho^2(1-\nu)(1-n)[I_{n+1}(\lambda\rho)J_n(\lambda\rho) - I_n(\lambda\rho)J_{n+1}(\lambda\rho)] + [n^2(1-\nu)^2(n^2-1) + \lambda^4\rho^4] \times [I_{n+1}(\lambda\rho)J_n(\lambda\rho) + I_n(\lambda\rho)J_{n+1}(\lambda\rho)] = 0 \tag{21}$$

Table 1 : The eigenvalues (λ) for the clamped circular plate

	eigenvalues for n of					
m	0	1	2	3	4	5
1	3.19	4.61	5.90	7.14	8.34	9.52
2	6.30	7.80	9.19	10.53	11.83	13.10
3	9.44	10.95	12.40	13.79	15.15	16.47
4	12.57	14.10	15.58	17.00	18.39	19.75

Table 2 : The eigenvalues (λ) for the simply supported circular plate

	eigenvalues for n of					
m	0	1	2	3	4	5
1	2.23	3.73	5.06	6.32	7.54	8.73
2	5.45	6.96	8.37	9.72	11.03	12.31
3	8.61	10.14	11.59	12.98	14.34	15.67
4	11.76	13.29	14.77	16.20	17.59	18.96

Table 3 : The eigenvalues (λ) for the free circular plate

	eigenvalues for n of					
m	0	1	2	3	4	5
1			2.29	3.50	4.64	5.75
2	3.01	4.53	5.93	7.27	8.56	9.82
3	6.20	7.73	9.18	10.57	11.93	13.25
4	9.37	10.91	12.38	13.80	15.19	16.55

Equation (21) is the eigenequation using the continuous formulation of the MFS for free circular plate.

It is noted that the resulting eigenequations of equations(17), (19) and (21) for the three different boundary conditions of the circular plate are all exactly agreement with the results in the literature [Leissa (1969); Ito and Crandall (1979); Leissa and Narita (1980); Chen, Lin, Chen, and Chen. (2004)]. The first few eigenvalues for the three cases ($\rho = 1$ and $\nu = 0.33$) are shown in Tab. 1 $\sim$ Tab. 3, in which m refers to the number of nodal circles and n is the number of nodal diameters, and will be used to validate the numerical results obtained in the following section.

4 Numerical Results

In this section, numerical experiments are carried out for the discrete version of the MFS and results are discussed and compared with the analytical solutions obtained from using the continuous version of the MFS in the last section. The following numerical experiments are carried out: (I) circular plate with clamped boundary condition, (II) circular plate with simply supported boundary condi-

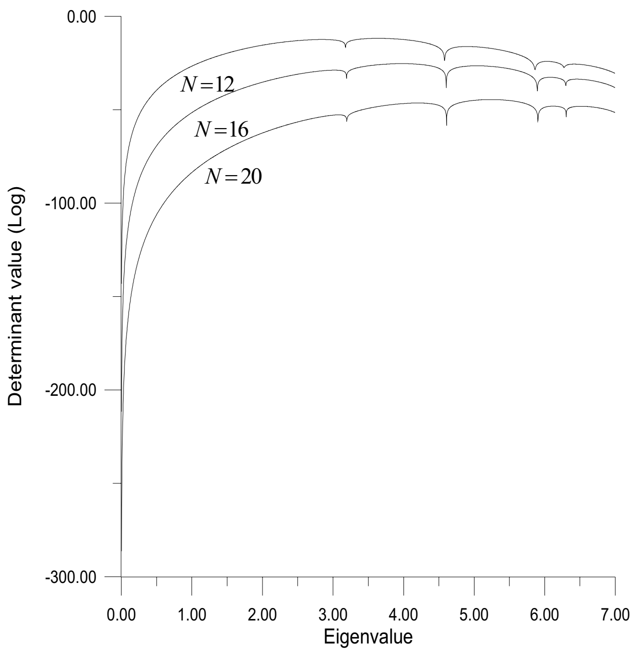

Figure 3 : Direct determinant search method for different node numbers of clamped circular plate ($R = 1.6$)

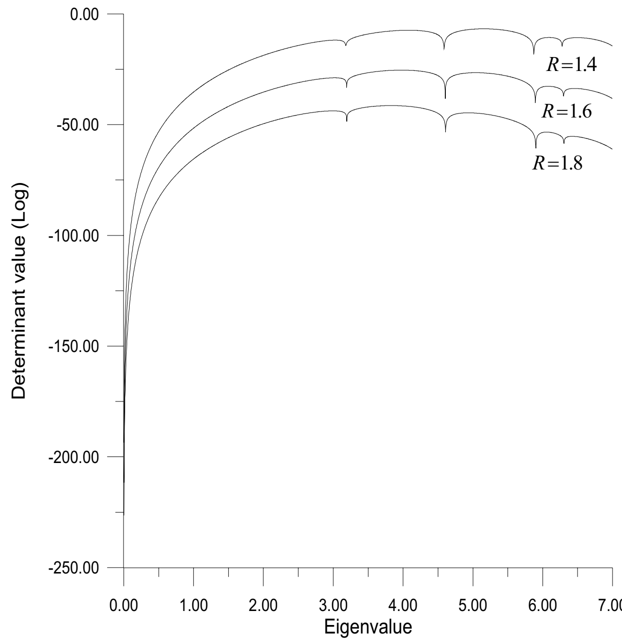

Figure 4 : Direct determinant search method for different source distances of clamped circular plate ($Nodes = 16$)

tion and (III) circular plate with free boundary condition. Since the three cases all have exact solutions, we can validate our numerical calculations by the discrete version of the MFS.

In this section, we choose all the fictitious boundaries as a circle with radius *R*. All the source points are evenly distributed around the fictitious boundary. We also denote ρ as the radius of the circular plate.

4.1 Case I: circular plate with clamped boundary condition

A circular plate with a unit radius, $\rho = 1$, and $\nu = 0.33$ subjected to clamped boundary condition is considered. Fig. 3 shows the results of the direct determinant search method for different numbers of nodes (*Nodes* =12, 16, and 20). In Fig. 3, the value λ, in which there is a cusp, is an eigenvalue since the corresponding determinant value is suddenly minified [Chen, Lin, Chen, and Chen (2004)]. The numerical eigenvalues are in excellent $N = 16N = 20N = 24R = 2.0R = 1.8R = 2.0R = 2.2Nodes = 20$ are in excellent agreement (up to 0.01) with the exact solutions as shown in Tab. 1. Note that there is no spurious eigenvalues in the range of direct determinant search method as we have studied analytically in the last section. The determinant for various source locations (*R*=1.4, 1.6, and 1.8) is shown in Fig. 4. The results demonstrate that the MFS is insensitive with respect to the locations of source points.

4.2 Case II: circular plate with simply supported boundary condition

We next consider the case of simply supported boundary condition, for $\rho = 1$ and $\nu = 0.33$. Fig. 5 depicts the results of the determinant for *Nodes* = 16, 20, and 24. The numerical eigenvalues are accurate up to 0.01 with the exact solutions as shown in Table 2 and no spurious eigenvalues are observed in the range of direct determinant search method. Furthermore, the direct determinant search method for different source locations with *R*=1.8, 2.0, and 2.2 is described in Fig. 6. The MFS is also found to be insensitive with respect to the locations of source points.

4.3 Case III: circular plate with free boundary condition

The application of MFS scheme for the circular plate with free boundary condition is straightforward. Here we choose $\rho = 1$ and $\nu = 0.33$. In Fig. 7, we show the direct

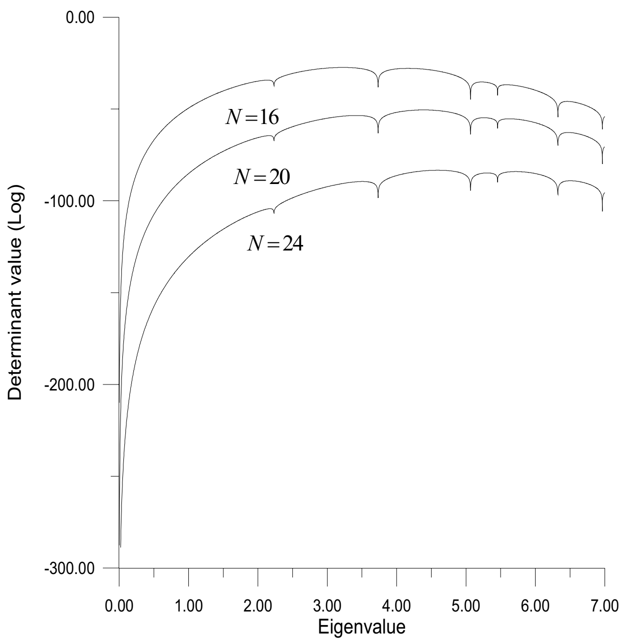

Figure 5 : Direct determinant search method for different node numbers of simply supported circular plate ($R = 2.0$)

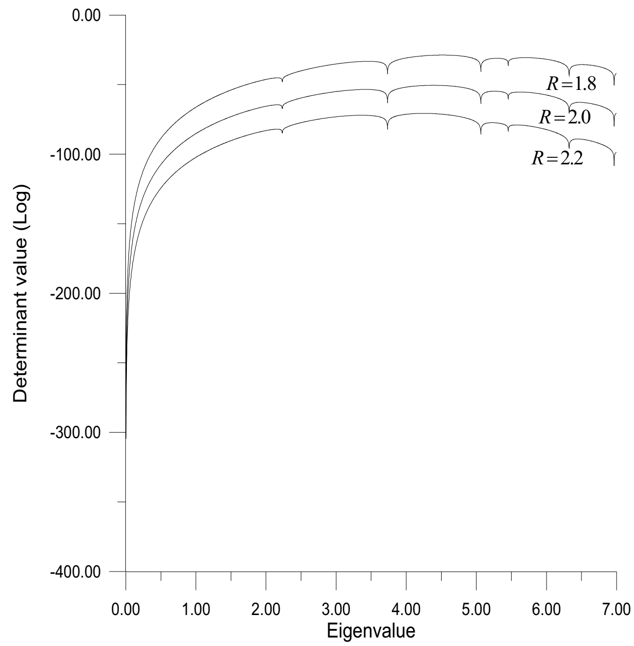

Figure 6 : Direct determinant search method for different source distances of simply supported circular plate ($Nodes = 20$)

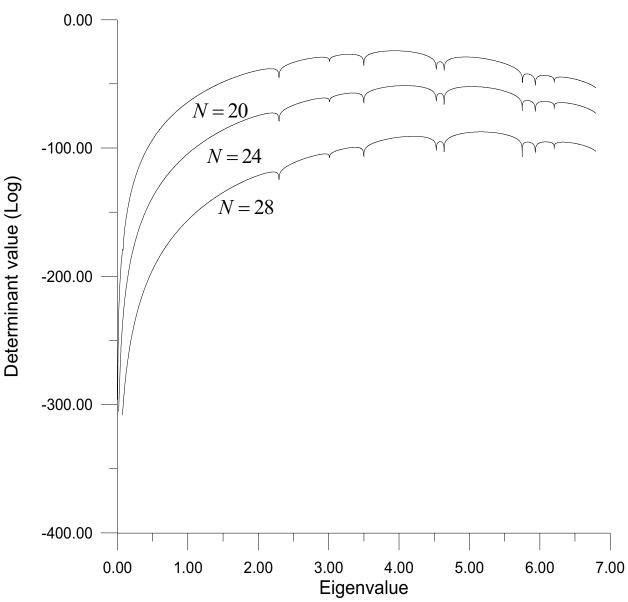

Figure 7 : Direct determinant search method for different node numbers of clamped circular plate ($R = 2.6$)

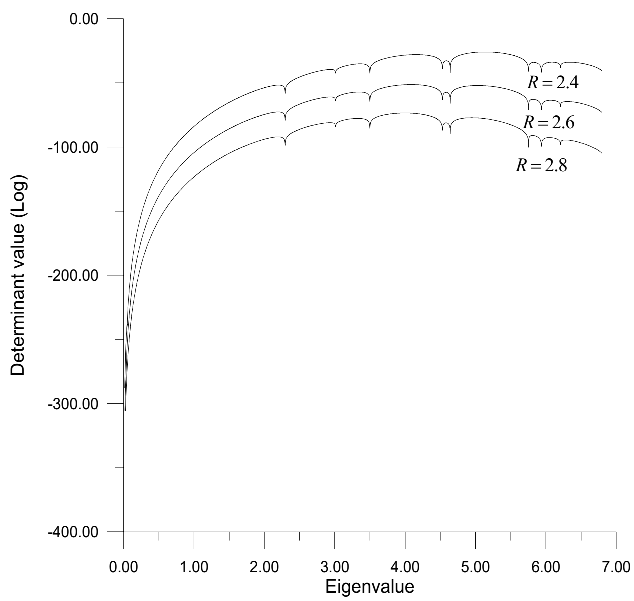

Figure 8 : Direct determinant search method for different source distances of free circular plate ($Nodes = 24$)

determinant search method for *Nodes* = 20, 24, and 26. The numerical eigenvalues are in good agreement (up to 0.01) with the exact solutions as shown in Table 3. In Fig. 7, no spurious eigenvalues is observed in the range of direct determinant search method. The determinant for various source locations is shown in Fig. 8. The results reveal that the MFS performs well for *R*= 2.4, 2.6 and 2.8.

From the above numerical results, it is found that more nodes are required in order to capture all true eigenvalues for the case of free boundary condition than the simply supported and clamped boundary conditions (free>simply supported>clamped). Similarly, it is desirable to have farther source point locations for the free boundary condition than the other two boundary conditions (free>simply supported>clamped). This is due to the reason that for the eigenvalue problem higher accuracy is required for plate vibrations with free boundary condition than clamped and simply supported boundary conditions according to the physics of free vibration of plates.

5 Conclusions

An analytical study is carried out to demonstrate that if the complex-value kernel is adopted for the MFS scheme in a simply connected domain, then there is no spurious eigenvalue. Numerical experiments are also consistent with the above statement. The direct determinant search method is utilized to obtain the eigenvalues, with the results showing good agreement with the exact solutions for very few nodes. This implies that the MFS is a very efficient numerical method. Moreover, numerical experiments also indicate that the MFS is insensitive to the locations of source points. According to our numerical studies, it is also found that more and farther source points are necessary for the free boundary condition than the simply supported and clamped boundary conditions. The MFS is a meshless numerical method that is free from meshes, singularities, numerical integrations, and the treatment of spurious eigenvalues. Thus, through this work we have demonstrated the simplicity and efficiency of the MFS to solve free vibrations of circular plate with clamped, simply supported and free boundary conditions.

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