 CMC, vol.3, no.3, pp.119-129, 2006

Nonlinear Dynamical Analysis in Incompressible Transversely Isotropic Nonlinearly Elastic Materials: Cavity Formation and Motion in Solid Spheres

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Abstract: In this paper, the problem of cavity formation and motion in an incompressible transversely isotropic nonlinearly elastic solid sphere, which is subjected to a uniform radial tensile dead load on its surface, is examined in the context of nonlinear elastodynamics. The strain energy density associated with the nonlinearly elastic material may be viewed as the generalized forms of some known material models. It is proved that some determinate conditions must be imposed on the form of the strain energy density such that the surface tensile dead load has a finite critical value. Correspondingly, as the surface tensile dead load exceeds the critical value, a cavity would form in the interior of the sphere and the motion of the formed cavity with time would present a class of singular nonlinear periodic oscillations. The effects of constitutive parameters on cavity formation and motion are discussed in detail, and the corresponding numerical examples are given simultaneously.

keyword: Nonlinear elastodynamics, Cavity formation and motion, Incompressible transversely isotropic nonlinearly elastic material, Classical (or generalized) periodic solution, Nonlinear periodic oscillation.

1 Introduction

In applications, many engineering materials form cavities (or voids) in various deformation processes as precursors to failure. Thus prediction of cavity formation and growth in materials has long been of concern for many investigators who engage in engineering and technology.

The phenomenon of sudden cavity formation (cavitation) has been observed experimentally in vulcanized rubber by Gent and Lindly (1958). Many similar phenomena have been observed since then. See Willams and Schapery (1965), Beahan et al. (1976), and so on. The impetus for nonlinear theories of solid mechanics framework on cavity formation and growth was supplied by the work of Ball (1982). Moreover, the notion bifurcation was emphasized by Ball (1982) who formulated the problem of modeling cavity formation in the context of nonlinear elastostatics. Thereafter, many significant works have been carried out. See the review articles, by Polignone and Horgan (1995) and by Yuan et al. (2005a), for comprehensive reviews for both incompressible and compressible materials. In particular, cavitation for a class of transversely isotropic nonlinearly elastic materials was studied by Polignone and Horgan (1993), and the effect of material anisotropy on cavity formation and growth in incompressible nonlinearly elastic solids was also examined in their work. Further representative references on this aspect are Murphy and Biwa (1997), Shang and Cheng (2001), Ren and Cheng (2002), Yuan and Zhang (2005). The above investigations show the advance of cavitation in the context of nonlinear elastostatics. However, the analogous dynamic problem is relatively unexplored. The problems of the radial oscillation was studied by Knowles (1960) for a cylindrical tube composed of an isotropic incompressible hyperelastic material, and an expression for the period of oscillation was given in terms of the strain energy function associated with the hyper-elastic materials in his work. The finite oscillation of nonlinear elastic spherical shells was examined by Calderer (1983), and the dynamical mechanisms of the motion of the shells were analyzed. Cavitation in nonlinear elastodynamics for isotropic neo-Hookean materials was investigated by Chou-Wang and Horgan (1989), and that the motion of the formed cavity is nonlinear oscillation was pointed out. Recently, the radial symmetric motion problem has been examined by Yuan et al. (2005b) for a spherical shell composed of a class of imperfect incompressible hyper-elastic materials, and the effects of the prescribed imperfection parameter of the material and the ratio of the inner and the outer radii of the undeformed shell on the motion of the inner surface of the shell have been discussed.

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The purpose of this paper is to study, in the context of nonlinear elastodynamics, the problem of cavity formation and motion in a solid sphere, composed of an incompressible transversely isotropic nonlinearly elastic material, which is subjected to a uniform radial tensile dead load on its surface. The strain energy density associated with the nonlinearly elastic material can be viewed as the generalized forms of some known material models, such as the neo-Hookean material model, the Mooney-Rivlin material model, the Rivlin-Saunders material model, the Gent-Thomas material model, and so on. In Section 2, the basic governing equations with the initial and the boundary conditions of the problem are proposed. A second-order nonlinear ordinary differential equation that describes cavity formation and motion with time is obtained. In Section 3, the existence conditions of the equilibrium points of the differential equation are first presented, correspondingly, some determinate conditions must be imposed on the form of the strain energy density. The classification of the equilibrium points is then carried out. To better understanding the conclusions obtained in this paper, we introduce an incompressible transversely isotropic Gent-Thomas material model. By using the phase diagrams of the differential equation, the classification of the periodic solutions are carried out for any initial conditions and for any given parameters. Meanwhile, the effects of constitutive parameters on periodic solutions are examined and the corresponding numerical examples are also carried out. In particular, for solutions satisfying the zero initial condition, the differential equation has the generalized periodic solution of the first kind only when the surface tensile dead load exceeds a certain critical value. In other words, as the surface tensile dead load exceeds the critical value, a cavity would form in the interior of the sphere and the motion of the formed cavity with time would present a class of singular nonlinear periodic oscillations.

2 Formulation

2.1 Basic governing equations

Assume that a solid sphere with radius b is composed of a homogeneous, incompressible, nonlinearly elastic material. Consider the radial symmetric motion of the sphere, which is subjected to a uniform radial tensile dead load on its surface. The resulting deformation takes the point with Cartesian coordinates $(R\sin\Theta\cos\Phi, R\sin\Theta\sin\Phi, R\cos\Theta)$ to the point $(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ at time t. Under the assumption of radial symmetric deformation, we have

$$r = r(R,t) > 0, 0 < R \le b; \quad \Theta = \theta, \Phi = \phi \tag{1}$$

where $r(R,t)$ is the radial deformation function to be determined. The principal stretches are given by

$$\lambda_r = \frac{\partial r(R,t)}{\partial R}, \lambda_\theta = \lambda_\phi = \frac{r(R,t)}{R} \tag{2}$$

For nonlinearly elastic materials, it is known that the response of the material can be described completely by the form of the strain energy density. Moreover, the strain energy density per unit undeformed volume for a nonlinearly elastic material which is transversely isotropic about the radial direction is given by (see e.g. Polignone and Horgan (1995))

$$W = W(I_1, I_2, I_3, I_5) \tag{3}$$

where

$$I_1 = tr\boldsymbol{C} = \lambda_r^2 + \lambda_\theta^2 + \lambda_\phi^2 \tag{4}$$

$$I_2 = \frac{1}{2}[(tr\boldsymbol{C})^2 tr\boldsymbol{C}^2] = \lambda_r^2\lambda_\theta^2 + \lambda_\theta^2\lambda_\phi^2 + \lambda_r^2\lambda_\phi^2 \tag{5}$$

$$I_3 = det\boldsymbol{C} = \lambda_r^2\lambda_\theta^2\lambda_\phi^2 \tag{6}$$

$$I_5 = C_{11} = \lambda_r^2 \tag{7}$$

are the four strain invariants of the right Cauchy-Green deformation tensor $\boldsymbol{C}$.

For incompressible materials, the incompressibility condition requires that $I_3 = J^2 = \lambda_r^2\lambda_\theta^2\lambda_\phi^2 = 1$, with (2), so we have

$$r = r(R,t) = \left[R^3 + c^3(t)\right]^{1/3} \tag{8}$$

where $c(t) \ge 0$ is to be determined, which denotes the value of cavity radius of the sphere at time t. $c(t) = 0$ implies that the sphere remains a solid sphere in the current configuration. If it is found that $c(t) > 0$, then it implies that there is a cavity with radius $r(0+,t) = c(t) > 0$ centered at the origin in the current configuration at time t.

For studying conveniently, (8) is rewritten as

$$R = \left[r^3 - c^3(t)\right]^{1/3} \tag{9}$$

and so (2) can be rewritten as

$$\lambda_r = (1-\frac{c(t)^3}{r^3})^{2/3}, \lambda_\theta = \lambda_\phi = (1-\frac{c(t)^3}{r^3})^{-1/3} \quad (10)$$

The nonzero components of the Cauchy stress tensor for transversely isotropic incompressible materials about radial direction are given by (Polignone and Horgan (1995))

$$\tau_{rr}(r,t) = -p(r,t) + 2(\lambda_r^2\frac{\partial W}{\partial I_1} - \lambda_r^{-2}\frac{\partial W}{\partial I_2} + \lambda_r^2\frac{\partial W}{\partial I_5}) \quad (11)$$

and

$$\tau_{\theta\theta}(r,t) = \tau_{\phi\phi}(r,t) = -p(r,t) + 2(\lambda_r^{-1}\frac{\partial W}{\partial I_1} - \lambda_r\frac{\partial W}{\partial I_2}) \quad (12)$$

where $p(r,t)$ is the hydrostatic pressure associated with the incompressible constraint $J = \lambda_r\lambda_\theta\lambda_\phi = 1$.

In the absence of body force, the equilibrium differential equation that describes the radial symmetric motion of the sphere is given by

$$\rho\ddot{r} = \frac{\partial\tau_{rr}(r,t)}{\partial r} + \frac{2}{r}\left[\tau_{rr}(r,t) - \tau_{\theta\theta}(r,t)\right] \quad (13)$$

where ρ is a constant mass density of the material.

In this work, we shall assume that the strain energy density function W has the form

$$W = f(I_1) + g(I_2) + h(I_5) \quad (14)$$

where the nonlinear functions f, g and h are assumed to be twice continuously differentiable.

It is worth noting here, as f, g and h take certain special functions, the strain energy density corresponds to some classical nonlinear elastic material models, such as the well-known neo-Hookean material model, the Mooney-Rivlin material model, the Rivlin-Saunders material model, the isotropic Gent-Thomas material model, and so on. (The detail discussions of the strain energy density function can be found in Yuan and Zhang (2005))

Next we carry out the initial and the boundary conditions of the problem.

At the center of the sphere, one of the boundary conditions is given by

$$r(0+,t)\tau_{rr}(r(0+,t),t) = 0, t \geq 0 \quad (15)$$

(15) here means that, if no cavity forms in the sphere, we have $r(0+,t) = 0$, if it is found that a cavity with radius $r(0+,t) = c(t) > 0$ forms, then the condition for the traction-free cavity surface, $\tau_{rr}(r(0+,t),t) = 0$ must hold.

The other boundary condition, for a prescribed load $p_0 > 0$ that is suddenly applied and maintained at the surface of the sphere, is given by

$$\tau_{rr}(r(b,t),t) = p_0\left[\frac{b}{r(b,t)}\right]^2, \quad t \geq 0 \quad (16)$$

Assume that the sphere is in an undeformed state and at rest at time $t \leq 0$, so we have the initial conditions

$$r(R,0) = R, \dot{r}(R,0) = 0 \quad (17)$$

Thus, the initial-boundary value problem, which describes the radial symmetric motion of a homogeneous, incompressible transversely isotropic nonlinearly elastic solid sphere under a uniform surface radial tensile dead load $p_0 > 0$, is composed of Eqs.(8), (11), (12), (13), (14), the initial-boundary conditions (15), (16) and (17).

2.2 *Solutions of the Problem*

By introducing the notation

$$\eta = \eta(r, c(t)) = (1-\frac{c(t)^3}{r^3})^{1/3} \quad (18)$$

we have $\lambda_r = \eta^2$, $\lambda_\theta = \lambda_\phi = \eta^{-1}$. Moreover,

$$I_1 = \eta^4 + 2\eta^{-2}, \quad I_2 = \eta^{-4} + 2\eta^2, \quad I_5 = \eta^4 \quad (19)$$

and thus the strain energy density (14) can be written as

$$W = \hat{W}(\eta) = f(\eta^4 + 2\eta^{-2}) + g(\eta^{-4} + 2\eta^2) + h(\eta^4) \quad (20)$$

Further, (11), (12) and (13) can be respectively written as

$$\tau_{rr}(r,t) = -p(r,t) + 2(\eta^4 f' - \eta^{-4}g' + \eta^4 h') \quad (21)$$

$$\tau_{\theta\theta}(r,t) = \tau_{\phi\phi}(r,t) = -p(r,t) + 2(\eta^{-2}f' - \eta^2 g') \quad (22)$$

and

$$\rho\ddot{r} = \frac{\partial\tau_{rr}(r,t)}{\partial r} + \frac{4}{r}\left[(\eta^4 - \eta^{-2})(f' + \eta^{-2}g') + \eta^4 h'\right] \quad (23)$$

From (8) and (17), the initial conditions become

$$c(0) = 0, \dot{c}(0) = 0 \quad (24)$$

Using (8), we also have

$$\ddot{r} = 2c(t)r^{-5}(r^3 - c(t)^3)\,(\dot{c}(t))^2 + c(t)^2 r^{-2}\ddot{c}(t) \tag{25}$$

on the other hand, it is not difficult to show that

$$\begin{aligned}\ddot{r} = \frac{\partial}{\partial r}\Bigg[&\left(\frac{c(t)^4}{2r^4} - \frac{2c(t)}{r} + \frac{3}{2}\right)(\dot{c}(t))^2 \\ &+ c(t)\left(1 - \frac{c(t)}{r}\right)\ddot{c}(t)\Bigg]\end{aligned} \tag{26}$$

Substituting (21), (22) into (23), and integrating it with respect to $r$from $c(t)$ to r, we then obtain

$$\begin{aligned}&\rho\left[\left(\frac{c(t)^4}{2r^4} - \frac{2c(t)}{r} + \frac{3}{2}\right)(\dot{c}(t))^2 + c(t)\left(1 - \frac{c(t)}{r}\right)\ddot{c}(t)\right] \\ &\quad = \tau_{rr}(r,t) - \tau_{rr}(c(t),t) \\ &\quad + 4\int_{c(t)}^{r} \frac{(\eta^4 - \eta^{-2})(f' + \eta^{-2}g') + \eta^4 h'}{\xi} d\xi\end{aligned} \tag{27}$$

where in the integral,

$$\eta = \eta(\xi, c(t)) = (1 - \frac{c(t)^3}{\xi^3})^{1/3} \tag{28}$$

Let $R = b$ in Eq.(27) and notice $r(0+,t) = c(t)$, from the boundary conditions (15) and (16), we have

$$\begin{aligned}\rho c(t)\Bigg[&\left(\frac{c(t)^4}{2S^4} - \frac{2c(t)}{S} + \frac{3}{2}\right)(\dot{c}(t))^2 \\ &+ c(t)\left(1 - \frac{c(t)}{S}\right)\ddot{c}(t)\Bigg] \\ &- 4c(t)\int_{c(t)}^{S} \frac{(\eta^4 - \eta^{-2})(f' + \eta^{-2}g') + \eta^4 h'}{\xi} d\xi \\ &- c(t)p_0\left(\frac{b}{S}\right)^2 = 0\end{aligned} \tag{29}$$

where $S = r(b,t) = (b^3 + c(t)^3)^{1/3}$.

In what follows, it is convenient to introduce the dimensionless quantities

$$x = x(t) = \frac{c(t)}{b}, \quad \dot{x} = \dot{x}(t) = \frac{\dot{c}(t)}{b} \tag{30}$$

so the initial conditions in (24) become

$$x(0) = 0, \quad \dot{x}(0) = 0 \tag{31}$$

Moreover, the dimensionless form of Eq.(29) is written as

$$F(x)\ddot{x} + G(x)\dot{x}^2 + H(x, p_0) = 0 \tag{32}$$

where

$$F(x) = \rho b^2 x^2\left(1 - \frac{x}{(1+x^3)^{1/3}}\right) = x\tilde{F}(x) \tag{33a}$$

$$\begin{aligned}G(x) &= \rho b^2 x\left(\frac{x^4}{2(1+x^3)^{4/3}} - \frac{2x}{(1+x^3)^{1/3}} + \frac{3}{2}\right) \\ &= x\tilde{G}(x)\end{aligned} \tag{33b}$$

and

$$\begin{aligned}H(x, p_0) &= -p_0 x(1+x^3)^{-2/3} + xq(x) \\ &= x\tilde{H}(x, p_0)\end{aligned} \tag{33c}$$

in (33c), $q(x)$ is given by

$$q(x) = -4\int_{bx}^{b(1+x^3)^{1/3}} \frac{(\eta^4 - \eta^{-2})(f' + \eta^{-2}g') + \eta^4 h'}{\xi} d\xi \tag{34}$$

By using the relation between $\eta(\xi, c(t))$ and ξ (see (28)), (34) can be rewritten as

$$q(x) = 4\int_{0}^{(1+x^3)^{-1/3}} \frac{(\eta^6 - 1)(f' + \eta^{-2}g') + \eta^6 h'}{\eta^3 - 1} d\eta \tag{35}$$

Obviously, for arbitrary prescribed $p_0 > 0$, $x \equiv 0$ is a solution of Eq.(32) and it corresponds to the homogeneous deformation solution $r(R,t) = R$ of the solid sphere. However, if there exists $x \geq 0$ satisfying Eq.(32) and the initial condition (31), then it implies that a cavity would form in the sphere, and then set into motion with time. Thus, it is necessary to investigate the motion rule of the formed cavity.

The second-order nonlinear ordinary differential equation (32) provides exactly a relationship between the tensile dead load p_0 and the cavity radius x**.** Thus we call Eq.(32) the **formation and motion equation of cavity.**

3 Nonlinear dynamical analyses of Eq.(32)

Let $y = \dot{x}$, then Eq.(32) is equivalent to the first order differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ L(x,y) \end{pmatrix} \tag{36}$$

where $L(x,y) = \frac{-\tilde{H}(x,p_0) - \tilde{G}(x)y^2}{\tilde{F}(x)}$.

3.1 Existence conditions of equilibrium point of Eq.(36)

Obviously, the equilibrium point of Eq.(36) is $(x,y) = (\bar{x},0)$, where $\bar{x}$ is a positive real solution of equation $\tilde{H}(x,p_0) = 0$. However, whether $\bar{x}$ exists or not depends exactly on the given value of $p_0 > 0$.

From (33c), we have

$$p_0 = (1+x^3)^{2/3} q(x) \tag{37}$$

Let $x \to 0+$, this leads to

$$p_{cr} = q(0) = 4\int_0^1 \frac{(\eta^6-1)(f'+\eta^{-2}g')+\eta^6 h'}{\eta^3-1} d\eta \tag{38}$$

Since the integral of Eq.(38) is improper, whether p_{cr} is finite or not depends strictly on the concrete form of the strain energy density.

By using the method proposed by Yuan and Zhang (2005), we obtain the following conditions that f,g and h must satisfy such that p_{cr} is finite.

From the strong convex condition of the strain energy density (see Ball (1982)), as $\eta \to 0$, we obtain

$$f(\eta^4+2\eta^{-2}) = O\left((\eta^4+2\eta^{-2})^{\alpha}\right) \tag{39}$$

$$g(\eta^{-4}+2\eta^2) = O\left((\eta^{-4}+2\eta^2)^{\beta}\right) \tag{40}$$

where $1/2 \le \alpha < 3/2, 0 < \beta < 3/4$.

As $\eta \to 1$, from the normalization conditions (Polignone and Horgan (1995)), the following expressions must be valid

$$h(\eta^4) = O\left((\eta^4-1)^{\gamma}\right) \tag{41}$$

$$\frac{d\hat{W}(1)}{d\eta} = 0, \frac{d^2\hat{W}(1)}{d\eta^2} = 24f'(3)+24g'(3)+16h''(1) \tag{42}$$

where $\gamma \ge 2$. Moreover, it is required that $f'(3), g'(3)$ and $h''(1)$ must be positive finite values as $\eta \to 1$.

In summary, for the strain energy density (20), if f,g and h respectively satisfy the conditions (39)∼(42), one can see that p_{cr} is finite.

We now consider the local character of Eq.(37) at $x = 0$ by analyzing the curve of $p_0 = p_0(x)$. A Taylor expansion of the right hand of Eq.(37) shows that

$$p_0 = q(0) + \frac{2}{3}Mx^3 + o\left(x^3\right) \quad \text{as } x \to 0 \tag{43}$$

where

$$M = q(0) - \frac{1}{6}\frac{d^2\hat{W}(1)}{d\eta^2} \tag{44}$$

From the above analyses, we have

Conclusion 1 (i) As $M > 0$, the curve of $p_0 = p_0(x)$ (Eq.(37)) increases monotonously with respect to x, this means that Eq.(36) has a unique equilibrium point $(x_1,0)$ only when $p_0 > p_{cr}$;

(ii) As $M < 0$, the curve of $p_0 = p_0(x)$ decreases monotonously with respect to x in the sufficient small neighborhood of $x = 0$. However, it can be shown that $(1+x^3)^{2/3}q(x) \to \infty$ as $x \to \infty$, that is to say, there exists a minimal point, written as (x_m, p_m), on the curve, and thus the curve decreases monotonously with respect to x as $0 < x < x_m$ and increases monotonously as $x > x_m$. This implies that Eq.(36) has two equilibrium points $(x_2,0)$ and $(x_3,0)$ $(x_2 < x_m < x_3)$ as $p_m < p_0 < p_{cr}$ and has a unique equilibrium point $(x_4,0)$ as $p_0 > p_{cr}$.

To better understanding the conclusions obtained in this paper, assume that the sphere is composed of a transversely isotropic Gent-Thomas material model, in which the corresponding strain energy density function is given by

$$W(I_1,I_2,I_5) = \frac{\mu_1}{2}[(I_1-3)+\delta\ln(I_2-2) \\ +\varepsilon(I_5^3-3I_5+2)] \tag{45}$$

where$\delta = \mu_2/\mu_1$, and $\mu_1,\mu_2 > 0$ are material constants in the state of infinitesimal deformations. $\varepsilon \ge 0$ is a dimensionless parameter which serves as a measure of the anisotropic degree about radial direction of the material. If $\varepsilon = 0$, the corresponding nonlinearly elastic material is isotropic (Gent and Thomas (1958)). While if $\varepsilon \ne 0$, the corresponding nonlinearly elastic material is called the transversely isotropic Gent-Thomas material.

Obviously, the strain energy density (45) satisfies the conditions (39)∼(42).

It is worth pointing out here, for other transversely isotropic hyper-elastic material models satisfying the conditions (39)∼(42), the conclusions are similar to those obtained in this paper.

Using the notation (18), we have

$$\hat{W}(\eta) = \frac{\mu_1}{2}[(\eta^4+2\eta^{-2}-3)+\delta\ln(\eta^{-4}+2\eta^2-2) \\ +\varepsilon(\eta^{12}-3\eta^4+2)] \tag{46}$$

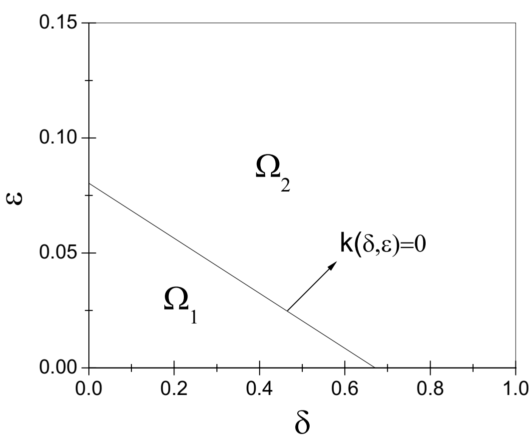

Figure 1 : Regions partitioned by $k(\delta,\varepsilon) = 0$.

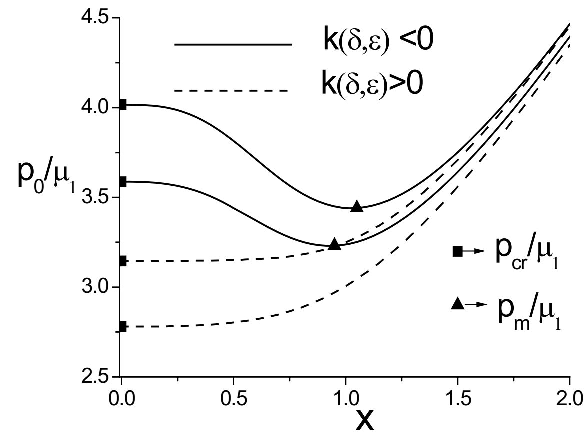

Figure 2 : Example curves for $p_0(x) \sim x$.

Substitution from (46) into (38) yields

$$p_{cr}/\mu_1 \doteq 2.5 + 1.2551\delta + 1.7763\varepsilon \tag{47}$$

Moreover, (43) becomes

$$p_0/\mu_1 = (2.5 + 1.2551\delta + 1.7763\varepsilon + \frac{2}{3}k(\delta,\varepsilon)x^3 + o(x^4)) \quad \text{as } x \to 0+ \tag{48}$$

where

$$k(\delta,\varepsilon) = 0.5 - 0.7449\delta - 6.2237\varepsilon \tag{49}$$

In this case, $M >$(or < 0) in (43) is equivalent to $k(\delta,\varepsilon) >$(or < 0) in (49) for the strain energy density (45).

We divide the parameter plane (δ,ε) into two regions by the line $k(\delta,\varepsilon) = 0$, as shown in Fig.1. The regions are denoted by

$$\Omega_1 = \{(\delta,\varepsilon)\,|\,0 \le \delta \le 0.6712, 0 \le \varepsilon \le 0.0803, k(\delta,\varepsilon) > 0\} \tag{50}$$

$$\Omega_2 = \{(\delta,\varepsilon)\,|\,\delta \ge 0, \varepsilon \ge 0, k(\delta,\varepsilon) < 0\} \tag{51}$$

Corresponding to Conclusion 1, the curves of $p_0 = p_0(x)$ are shown in Fig.2 for the strain energy density (45) as the parameters, δ,ε, satisfy $k(\delta,\varepsilon) >$(or < 0).

3.2 *Classification of equilibrium points*

Corresponding to the equilibrium point $(x_i, 0)$, we consider the eigenvalues of the linearization equation of Eq.(36), i.e.,

$$\lambda_{1,2} = \pm\left[\frac{-\tilde{H}_x(x_i,p_0)}{\tilde{F}(x_i)}\right]^{\frac{1}{2}} \tag{52}$$

where x_i is a positive real solution of $\tilde{H}(x,p_0) = 0$.

The following conclusions are valid:

Conclusion 2 For $M > 0$ and for the prescribed $p_0 > p_{cr}$, the unique equilibrium point $(x_1,0)$ of Eq.(36) is a center.

Proof. For $p_0 > p_{cr}$, Eq.(36) has a unique equilibrium point $(x_1,0)$. From Conclusion 1, we know that $\tilde{H}_x(x_1,p_0) > 0$, that is to say, λ_1 and λ_2 are two pure imaginary eigenvalues with opposite sign, and thus the equilibrium point $(x_1,0)$ is a center of the linearized equation. Since $L(x,-y) = L(x,y)$ and $(x_1,0)$ is the unique equilibrium point in its sufficient small neighborhood, from the Symmetry Principle, $(x_1,0)$ is also a center of Eq.(36).

Conclusion 3 For $M < 0$ and for $p_m < p_0 < p_{cr}$, the equilibrium point $(x_2,0)$ is a saddle point and the equilibrium point $(x_3,0)$ is a center of Eq.(36); For $p_0 > p_{cr}$, the unique equilibrium point $(x_4,0)$ is a center.

Proof. For $p_m < p_0 < p_{cr}$, Eq.(36) has two equilibrium points, $(x_2,0)$ and $(x_3,0)$ $(x_2 < x_m < x_3)$. From Conclusion 1, we know that $\tilde{H}_x(x_2,p_0) < 0$ and $\tilde{H}_x(x_3,p_0) > 0$. For $(x_2,0)$, λ_1 and λ_2 are two real eigenvalues with opposite sign, and thus $(x_2,0)$ is a saddle point of the linearized equation and is also a saddle point of Eq.(36). For $(x_3,0)$, similar to the proof of Conclusion 2, λ_1 and λ_2 are pure imaginary eigenvalues with opposite sign, in this case, $(x_3,0)$ is a center of Eq.(36).

Similarly, for $p_0 > p_{cr}$, $(x_4, 0)$ is a center of Eq.(36). (57)

3.3 Qualitative properties of solutions

In this subsection, we first define three classes of periodic solutions, as follows,

Definition 1 If $x = x(t)$ is a periodic solution of period T, and is smoothing enough at any time t, we then call it **the classical periodic solution**.

Definition 2 If $x = x(t)$ is a periodic solution of period T, and if the left- and right-limit of $\dot{x} = dx/dt$ exist but do not equal each other at certain times, we then call it **the generalized periodic solution of the first kind**.

Definition 3 If $x = x(t)$ is a periodic solution of period T, and if at least a value of $\dot{x} = dx/dt$ does not exist at certain times, we then call it **the generalized periodic solution of the second kind**.

To further study the qualitative properties of the solutions of Eq.(32), we now present some necessary information.

It is not difficult to shown that $\frac{d(x^2\tilde{F}(x))}{dx} = 2x^2\tilde{G}(x)$. Multiplying both sides of (32) by $x\dot{x}$, and then integrating it with respect to t, we obtain the first integral

$$E = \frac{1}{2}x^2\tilde{F}(x)\dot{x}^2 + V(x, p_0) \tag{53}$$

where E is a energy constant which is related to the initial conditions, and

$$V(x, p_0) = \int_0^x \xi^2\tilde{H}(\xi, p_0)d\xi \tag{54}$$

From (53), we know that, if there exists $x > 0$ such that $E - V(x, p_0) > 0$, the implicit solution of Eq.(32) is then given by

$$\pm\int_{x_0}^{x}\left(\frac{z^2\tilde{F}(z)}{2(E - V(z, p_0))}\right)^{1/2} dz = t - t_0 \tag{55}$$

where $x(t_0) = x_0$ is the initial condition.

If the solutions of Eq.(32) satisfy the initial condition $x(0) = 0$, we get $E = 0$ form (53), i.e.,

$$\frac{1}{2}x^2\tilde{F}(x)\dot{x}^2 + V(x, p_0) = 0 \tag{56}$$

It is worth noting here, from (32) and (56), the following expressions are valid as $t \to 0+$

$$\dot{x}(0+) = \pm\left(\frac{2(p_0 - q(0))}{3\rho b^2}\right)^{1/2}, \ddot{x}(0+) = \frac{2(p_0 - q(0))}{6\rho b^2}$$

namely, $\dot{x}(0+)$ and $\ddot{x}(0+)$ are determined uniformly. That is to say, the first derivative of $x(t)$ is discontinuous at the initial moment $t = 0$ (see (31)). In this case, we call Eq.(32) the singular second-order nonlinear ordinary differential equation with the initial condition $x(0) = 0$. On the other hand, if the solutions satisfy the initial condition $x(0) = x_0 \neq 0$, to determine the solutions of Eq.(32) completely, another initial condition $\dot{x}(0) = \dot{x}_0$ must also be given. In this case, for the prescribed p_0 and for the initial conditions x_0 and $\dot{x}_0$, we have

$$E_0 = \frac{1}{2}x_0^2\tilde{F}(x_0)\dot{x}_0^2 + V(x_0, p_0) \tag{58}$$

and x is then determined implicitly by Eq.(55).

It is necessary to study the relationship between $V(x, p_0)$ and p_0. For this, we examine the equation $V_x(x, p_0) = x^2\tilde{H}(x, p_0) = 0$ for $p_0 > 0$. On the other hand, from (54), we know that $V_{xx}(x, p_0)$ and $\tilde{H}_x(x, p_0)$ have the same sign for $p_0 > 0$and x satisfying $\tilde{H}(x, p_0) = 0$.

Obviously, $V(0, p_0) = 0$ and $\lim_{x\to+\infty} V(x, p_0) = +\infty$ are valid for any prescribed $p_0 > 0$.

From Conclusion 1, we know that:

(i) As $M > 0$, from the dashed shown in Fig. 2, we have **(a)** For the prescribed $p_0 < p_{cr}$, $\tilde{H}(x, p_0) = 0$ has no positive nonzero real solution, and thus $V(x, p_0)$ has no critical point and increases monotonously relating to x, moreover, $V(x, p_0) > 0$ for any $x > 0$; **(b)** For $p_0 > p_{cr}$, since $\tilde{H}(x, p_0) = 0$ has a unique positive real solution x_1, and $\tilde{H}_x(x_1, p_0) > 0$, $V(x, p_0)$takes the minimum at x_1, moreover, from $V(0, p_0) = 0$, we have $V(x_1, p_0) < 0$.

Further, from $\lim_{x\to+\infty} V(x, p_0) = +\infty$, we can conclude that there must exist a nonzero value of $x \in (x_1, +\infty)$, written as x_u, such that $V(x_u, p_0) = 0$. That is to say, for $p_0 > p_{cr}$, we have $V(x, p_0) < 0$ as $0 < x < x_u$.

Fig.3 shows the curves of $V(x, p_0) \sim x$ associated with the strain energy density (45) for different values of p_0/μ_1 as $(\delta, \varepsilon) = (0.5, 0.01) \in \Omega_1$.

For the prescribed value of $p_0 < p_{cr}$, the phase diagrams of Eq.(32) are shown in Fig.4. For any prescribed initial conditions $x_0 > 0$ and $\dot{x}_0$, it is easy to show that $E_0 > 0$ and $\lim_{x\to 0+} \dot{x} = +\infty$ from (53) and (58). In this case, the phase diagrams of Eq.(32) are not closed.

For the prescribed $p_0 > p_{cr}$, the phase diagrams of Eq.(32) are shown in Fig.5. The trajectories correspond-

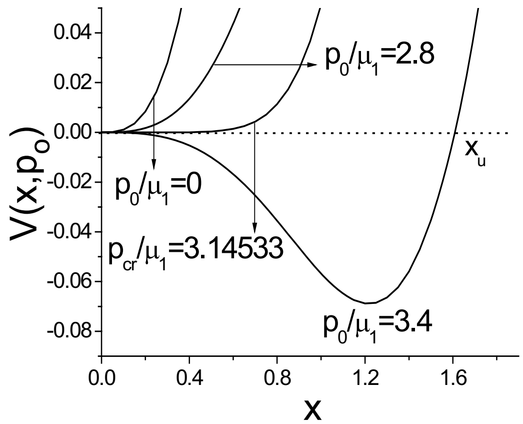

Figure 3 : Example curves of $V(x,p_0) \sim x$ in Ω_1 as $(\delta,\varepsilon) = (0.5, 0.01)$.

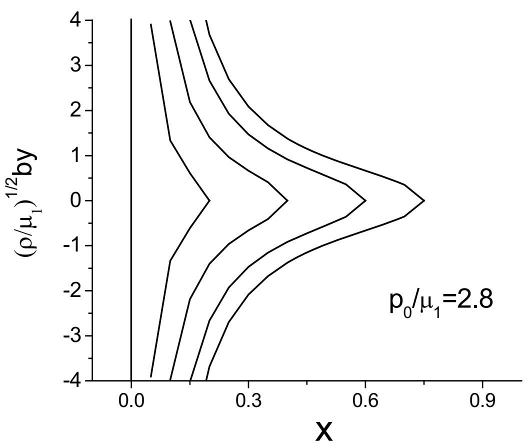

Figure 4 : Example phase diagrams of Eq.(36) in Ω_1 as $p_0 < p_{cr}$.

ing to $E = 0$ are composed of $\Gamma_1 : x \equiv 0$ and the nonzero phase trajectory Γ_2 of Eq.(32). The interior region enclosed by Γ_1 and Γ_2 is denoted by Π_1, and the outer region ($x > 0$) is denoted by Π_2. If the phase diagrams given by Eq.(32) are all belong to the region Π_1, we have $E < 0$, and the phase diagrams are all simple, smooth, closed and convex curves; while if the phase diagrams are all belong to the region Π_2, we have $E > 0$ and $\lim\limits_{x\to 0+} \dot{x} = +\infty$, and thus the phase diagrams are not closed.

Thus, as the parameter $(\delta,\varepsilon) \in \Omega_1$, we have

Conclusion 4 (a) For the initial condition $x(0) = 0$, Eq.(32) has only zero solution as $p_0 < p_{cr}$. **(b)** For the initial conditions $x(0) = x_0 > 0$, $\dot{x}(0) = \dot{x}_0$, it can be shown that the integral of (55) is finite, and thus the solutions of Eq.(32) are the generalized periodic solutions of the second kind.

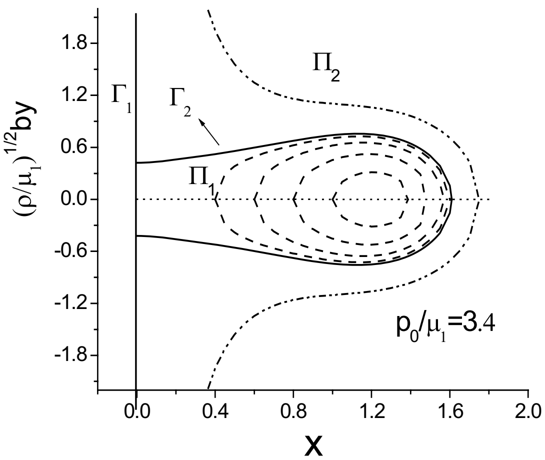

Figure 5 : Example phase diagrams of Eq.(32) in Ω_1 as $p_0 > p_{cr}$.

Conclusion 5 For $p_0 > p_{cr}$, **(a)** if $(x_0,\dot{x}_0) \in \Pi_1$, the solutions of Eq.(32) are the classical periodic solutions; **(b)** If the initial condition is taken as $x(0) = 0$, the solutions are the generalized periodic solutions of the first kind; **(c)** If $(x_0,\dot{x}_0) \in \Pi_2$, the solutions are the generalized periodic solutions of the second kind.

Proof. (a) Since the phase diagrams of Eq.(32) are simple, smooth, closed and convex curves, thus we know that the solutions of Eq.(32) are classical periodic solutions, moreover, the period of the solutions can be obtained by the implicit solution (55).

(b) If the solution of Eq.(32) satisfies the initial condition $x(0) = 0$, in this case, the corresponding energy constant $E = 0$, and the phase diagram of Eq.(32) is composed of Γ_1 and Γ_2. Since the first-derivative of $x(t)$ is discontinuous at $t = 0$, i.e., $\dot{x}(0+) = \left(\frac{2(p_0-q(0))}{3\rho b^2}\right)^{1/2}$ and $\dot{x}(0) = 0$, we can conclude that the solution of Eq.(32) bifurcates from the trivial solution $x(t) \equiv 0$ for $p_0 > p_{cr}$. Let $T_0 = \int_0^{x_u} \left(\frac{z^2\tilde{F}(z)}{2(-V(z,p_0))}\right)^{1/2} dz$, then $x(T_0) = x_u$ and $\dot{x}(T_0) = 0$, namely, $x(t)$ increases monotonously on $[0,T_0]$. From the symmetry of T_0, we have $x(2T_0^-) = 0$ and $\dot{x}(2T_0^-) = -\left(\frac{2(p_0-q(0))}{3\rho b^2}\right)^{1/2}$. Since $x(t) \geq 0$, we get $x(2T_0^+) = 0$, and $\dot{x}(2T_0^+) = \left(\frac{2(p_0-q(0))}{3\rho b^2}\right)^{1/2}$. Although we can conclude that the solution $x(t)$ is still a periodic solution with period $2T_0$, the left- and right-limit of $\dot{x} = dx/dt$

exist but do not equal each other at $2kT_0$. In this case, the solution is called the generalized periodic solution of first kind.

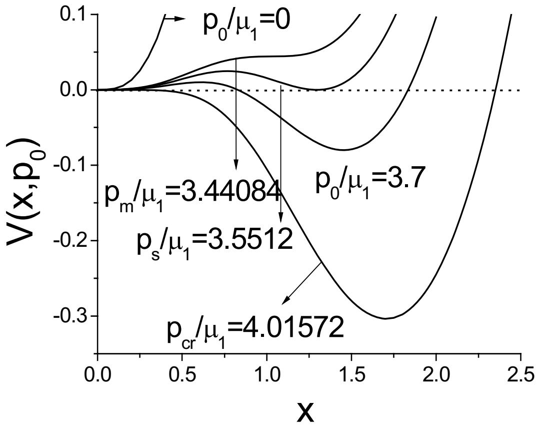

Figure 6 : Example curves of $V(x,p_0) \sim x$ in Ω_2 as $(\delta,\varepsilon) = (0.5,0.5)$

(ii) As $M < 0$, from the solid line shown in Fig.1, we have **(a)** For the prescribed $p_0 < p_m$, $V(x,p_0)$ has no critical point and increases monotonously relating to x, moreover, $V(x,p_0) > 0$ for any $x > 0$; **(b)** For $p_0 = p_m$, x_m is an inflexion point of $V(x,p_0)$ and $V(x_m,p_0) > 0$, because $\tilde{H}_x(x,p_0) < 0$ as $x < x_m$ and $\tilde{H}_x(x,p_0) > 0$ as $x > x_m$; **(c)** However, for $p_m < p_0 < p_{cr}$, $V(x,p_0)$ has two local positive critical points, written as x_2 and x_3 $(0 < x_2 < x_3)$, that is to say, as p_0 increases from p_m, the inflexion point of $V(x,p_0)$ splits into a local maximum x_2 and a local minimum x_3, moreover, x_2 and x_3 also vary with the increasing p_0. In this case, we can conclude that $V(x_2,p_0) > 0$ and $V(x_3,p_0) > 0$. The local maximum and the local minimum both decrease gradually as p_0 increases. However, as p_0 attains a certain value, written as p_s $(< p_{cr})$, we have $V(x_3,p_s) = 0$. As p_0 increases more, the local minimum is negative and turns into the global minimum, moreover, the local maximum is still positive, and x_2 closes to the origin gradually; **(d)** As $p_0 = p_{cr}$, x_2 degenerates into the origin, but $V(x,p_0)$ has two local critical points, one is $x_2 = 0$, the other is $x_3 = x_c$; **(e)** As $p_0 > p_{cr}$, $V(x,p_0)$ has a unique critical point, written as x_4, i.e., $V(x,p_0)$ takes the minimum at x_4 and $V(x_4,p_0) < 0$.

Fig.6 shows the curves of $V(x,p_0) \sim x$ associated with the strain energy density (45) for different values of p_0/μ_1 as $(\delta,\varepsilon) = (0.5,0.5) \in \Omega_2$.

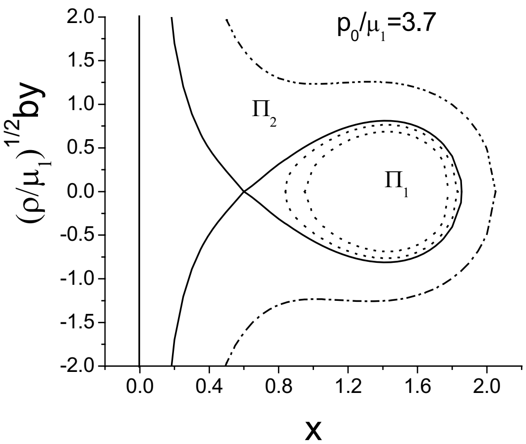

Figure 7 : Example phase diagrams of Eq.(32) in Ω_2 as $p_m < p_0 < p_{cr}$

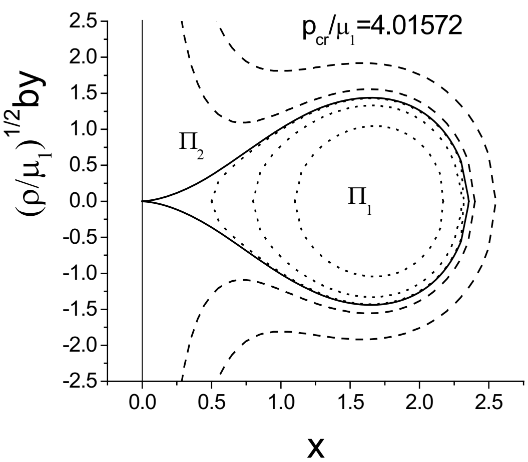

Figure 8 : Example phase diagrams of Eq.(32) in Ω_2 as $p_0 = p_{cr}$

For $p_0 < p_m$, the phase diagrams of Eq.(32) are similar to those in Fig.4. For $p_m < p_0 < p_{cr}$, $p_0 = p_{cr}$ and $p_0 > p_{cr}$, the phase diagrams of Eq.(32) are respectively shown in Fig.7, Fig.8, and Fig.9.

Similar to Conclusion 4 and Conclusion 5, as the parameter $(\delta,\varepsilon) \in \Omega_2$, we have

Conclusion 6 For $p_m < p_0 < p_{cr}$, **(a)** if $(x_0,\dot{x}_0) \in \Pi_1$, the solutions of Eq.(32) are the classical periodic solutions; **(b)** If the initial condition is taken as $x(0) = 0$, Eq.(32) has only zero solution; **(c)** If $(x_0,\dot{x}_0) \in \Pi_2$, the solutions are the generalized periodic solutions of the second kind.

Conclusion 7 For $p_0 = p_{cr}$, **(a)** if $(x_0,\dot{x}_0) \in \Pi_1$, the solutions of Eq.(32) are the classical periodic solutions;

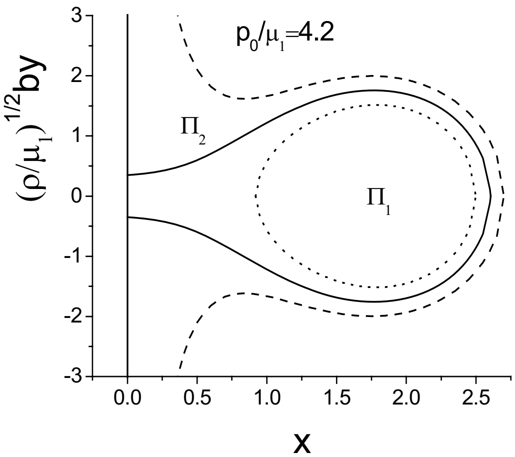

Figure 9 : Example phase diagrams of Eq.(32) in Ω_2 as $p_0 > p_{cr}$

(b) If the initial condition is taken as $x(0) = 0$, we have $\dot{x}(0) = 0$ from (57), and thus the solutions are the classical periodic solutions; **(c)** If $(x_0, \dot{x}_0) \in \Pi_2$, the solutions are the generalized periodic solutions of the second kind.

Conclusion 8 For $p_0 > p_{cr}$, **(a)** if $(x_0, \dot{x}_0) \in \Pi_1$, the solutions of Eq.(32) are the classical periodic solutions; **(b)** If the initial condition is taken as $x(0) = 0$, the solutions are the generalized periodic solutions of the first kind; **(c)** If $(x_0, \dot{x}_0) \in \Pi_2$, the solutions are the generalized periodic solutions of the second kind.

In summary, as the initial condition is taken as $x(0) = 0$, the solution of Eq.(32) then describes the cavity formation and motion with time in the solid sphere, composed of the incompressible transversely isotropic nonlinearly elastic material, which is subjected to a prescribed surface tensile load. From Conclusion 4, Conclusion 6, and the corresponding phase diagrams shown in Fig.4 and Fig.7, we know that, as the prescribed tensile load $p_0 < p_{cr}$, Eq.(32) has only zero solution, that is to say, the sphere remains solid. However, as the prescribed tensile load $p_0 > p_{cr}$, from Conclusion 5, Conclusion 8, and the corresponding phase diagrams shown in Fig.5 and Fig.9, Eq.(32) has the generalized periodic solutions of the first kind, in other words, a cavity forms in the sphere and will expand until its radius reaches the maximum value x_u at time T_0. However, the expanding velocity $\dot{x}(t)$ of the cavity radius reaches directly to $\left(\frac{2(p_0-q(0))}{3\rho b^2}\right)^{1/2}$ from 0 as a cavity forms suddenly, and will reduce to zero as the cavity radius reaches the maximum value. Thereafter, the cavity will contract and the contracting velocity will reach $-\left(\frac{2(p_0-q(0))}{3\rho b^2}\right)^{1/2}$ as the cavity reduces to zero at time $t = 2T_0^-$. Along with the time increases, the expanding velocity will leap directly from $-\left(\frac{2(p_0-q(0))}{3\rho b^2}\right)^{1/2}$ to $\left(\frac{2(p_0-q(0))}{3\rho b^2}\right)^{1/2}$, and then the cyclic will repeat. Thus we can say that a cavity forms at the center of the sphere as the surface tensile dead load $p_0 > p_{cr}$, and the motion rule of the formed cavity with respect to time presents a class of singular periodic oscillations.

It is worth pointing out that, as the parameter $(\delta, \varepsilon) \in \Omega_1$, for the prescribed $p_0 = p_{cr}$, the value of x, corresponding to p_{cr} given by (47), is zero, in other words, no cavity forms in the interior of the sphere, and the sphere is in the critical state of cavity formation. However, as the parameter $(\delta, \varepsilon) \in \Omega_2$, there are two values of x, i.e., $0+$and x_c, corresponding to p_{cr}, since p_0 increases continuously, we can conclude that a cavity has formed in the sphere and then presented a classical nonlinear periodic oscillation, x_c is the oscillation center, see the phase diagrams shown in Fig.8.

4 Conclusions

In this work, we first present a second-order nonlinear ordinary differential equation that describes cavity formation and motion with time. Via anaylzing the dynamical properties of the differential equation, we then carry out the existence conditions of the equilibrium points of the differential equation, see Conclusion 1, correspondingly, some determinate conditions must be imposed on the form of the strain energy density. To better understanding the conclusions obtained in this paper, we introduce an incompressible transversely isotropic Gent-Thomas material model. We divide the constitutive parameters into two regions, and discuss the effects of constitutive parameters on the qualitative properties of the solutions in detail. Further, we classify the equilibrium points, see Conclusion 2, 3. By using the phase diagrams of the differential equation, we also classify the periodic solutions for any initial conditions in different regions partitioned by the constitutive parameters, see Conclusion 4∼9. In particular, as the parameter $(\delta, \varepsilon) \in \Omega_1$, it is proved that a cavity would form in the interior of the sphere and the motion of the formed cavity with time would present a class of singular nonlinear periodic oscillations only when the surface tensile dead load exceeds the critical value; However, as the parameter $(\delta, \varepsilon) \in \Omega_2$, a cavity has formed in the sphere, and then presented a classical nonlinear periodic oscillation as the surface tensile dead load attains the critical value i.e., $p_0 = p_{cr}$.

Acknowledgement: This work was supported by the National Natural Science Foundation of China (10272069) and (10272084).

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