 CMC, vol.5, no.1, pp.1-11, 2006

The Boundary Contour Method for Magneto-Electro-Elastic Media with Linear Boundary Elements

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Abstract: This paper presents a development of the boundary contour method (BCM) for magneto-electro-elastic media. Firstly, the divergence-free of the integrand of the magneto- electro-elastic boundary element is proved. Secondly, the boundary contour method formulations are obtained by introducing linear shape functions and Green's functions (*Computers & Structures*, **82**(2004):1599-1607) for magneto-electro-elastic media and using the rigid body motion solution to regularize the BCM and avoid computation of the corner tensor. The BCM is applied to the problem of magneto-electro-elastic media. Finally, numerical solutions for illustrative examples are compared with exact ones and those of the conventional boundary element method (BEM). The numerical results of the BCM coincide very well with the exact solution, and the feasibility and efficiency of the method are verified.

keyword: Boundary Element Method(BCM), Magneto-electro-elastic media, Shape Function, Divergence

1 Introduction

The conventional boundary element method (BEM) usually requires numerical evaluation of line integrals for two-dimensional problems and surface integrals for three-dimensional ones. So, more and more attention has been paid to those methods that do not require the use of internal cells. Atluri(2004) gave a detailed account of problems relating to application of the meshless method(MLPG) for domain & BIE dicretizations. Yoshihiro and Vladimir(2004) gave a method using arbitrary internal points instead of internal cells, based on a three-dimensional interpolation method by using a poly-harmonic function with volume distribution in a three-dimensional BIEM. Sladek et al. (2004) proposed meshless methods based on the local Petrov-Galerkin approach for solution of steady and transient heat conduction problem in a continuously non-homogeneous anisotropic medium. Han and Atluri(2004) developed three different truly Meshless Local Petrov- Galerkin (MLPG) methods for solving 3D elasto-static problems. Using the general MLPG concept, those methods were derived through the local weak forms of the equilibrium equations, by using different test functions, namely, the Heaviside function, the Dirac delta function, and the fundamental solutions. Reutskiy(2005) reduced the solution of an eigenvalue problem to a sequence of inhomogeneous problems with the differential operator studied using the method of fundamental solutions. Hokwon et al.(2004) presented a mesh-free approach to numerically solving a class of second order time dependent partial differential equations which include equations of parabolic, hyperbolic and parabolic-hyperbolic types. Two types of Trefftz bases were considered, F-Trefftz bases based on the fundamental solution of the modified Helmholtz equation, and T-Trefftz bases based on separation of variables solutions.

For magneto-electro-elastic media, the BEM have been devired [see, for example, Ding and Jiang(2004); Ding and Jiang(2003)]. But the boundary contour method (BCM) can achieve a further reduction in dimension by using the divergence-free property of the integrand of the conventional boundary element method. Using this method, three-dimensional problems can be reduced to numerical evaluation of line integrals over closed contours and two-dimensional problems to merely evaluation of functions at nodes on the boundary of the plane. This is true even for boundary elements of arbitrary shape with curved boundary lines (for two-dimensional problems) or curved surface (for three-dimensional problems).

Nagarajan et al.(1994) have proposed this novel approach, called the BCM for linear elasticity problems.

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Nagarajan et al.(1996) used the Stokes' theorem to transform surface integrals in the conventional boundary elements into line integrals in the bounding contours of these elements. Phan et al.(1997) derived a BCM formulation and implemented the method for two- dimensional problems of linear elasticity with quadratic boundary elements. Zhou et al.(2002) developed the BCM based on equivalent boundary integral equations and applied the traction BCM to crack problems and the bending problems of elastic thin plate. For piezoelectric materials, Wang et al.(2003) presented a development of the BCM by introducing linear shape functions and Green's functions in Ding et al.(1998) for piezoelectric media.

However, to the authors' knowledge, no attempts in the literature have been made to solve problems of magneto-electro-elastic media by the BCM. This paper presents a development of the BCM for magneto-electro-elastic problems. Firstly, the divergence-free of the integrand of the magneto-electro-elastic boundary element is proved, then, the BCM formulation is derived and potential functions are obtained by introducing linear shape functions and Green's functions[Ding and Jiang(2004)] for magneto-electro-elastic media and using the rigid body motion solution to regularize the BCM and avoid computation of the corner tensor. The BCM is applied to the problem of magneto-electro- elastic media. Finally, numerical solutions for illustrative examples are compared with exact ones and those of the conventional boundary element method (BEM). The numerical results of the BCM coincide very well with the exact solution, and the feasibility and efficiency of the method are verified.

2 General Integral Formulation for Magneto-Electro- Elastic Plane

For two-dimensional transversely isotropic magneto-electro-elastic media, we define the general displacement **u**, general surface traction **t**, general stress **T** and general strain **S** as follows[Ding and Jiang(2004), Pan(2001)]

$$\mathbf{u}=\begin{Bmatrix} u \\ w \\ \Phi \\ \Psi \end{Bmatrix}, \quad \mathbf{t}=\begin{Bmatrix} t_x \\ t_z \\ -\omega \\ -\eta \end{Bmatrix},$$

$$\mathbf{T}=\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \\ D_x \\ D_z \\ B_x \\ B_z \end{Bmatrix}, \quad \mathbf{S}=\begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \\ -E_x \\ -E_z \\ -H_x \\ -H_z \end{Bmatrix} \tag{1}$$

So the relation between general stress and general strain can be written as

$$\mathbf{T}=\mathbf{DS} \tag{2}$$

where

$$\mathbf{D}=\begin{bmatrix} c_{11} & c_{13} & 0 & 0 & e_{31} & 0 & d_{31} \\ c_{13} & c_{33} & 0 & 0 & e_{33} & 0 & d_{33} \\ 0 & 0 & c_{44} & e_{15} & 0 & d_{15} & 0 \\ 0 & 0 & e_{15} & -\varepsilon_{11} & 0 & -g_{11} & 0 \\ e_{31} & e_{33} & 0 & 0 & -\varepsilon_{33} & 0 & -g_{33} \\ 0 & 0 & d_{15} & -g_{11} & 0 & -\mu_{11} & 0 \\ d_{31} & d_{33} & 0 & 0 & -g_{33} & 0 & -\mu_{33} \end{bmatrix}$$

Moreover, we define

$$\mathbf{U}^*=\begin{bmatrix} u_{11}^* & u_{12}^* & \Phi_1^* & \Psi_1^* \\ u_{21}^* & u_{22}^* & \Phi_2^* & \Psi_2^* \\ u_{31}^* & u_{32}^* & \Phi_3^* & \Psi_3^* \\ u_{41}^* & u_{42}^* & \Phi_4^* & \Psi_4^* \end{bmatrix},$$

$$\mathbf{T}^*=\begin{bmatrix} t_{11}^* & t_{12}^* & \omega_1^* & \eta_1^* \\ t_{21}^* & t_{22}^* & \omega_2^* & \eta_2^* \\ t_{31}^* & t_{32}^* & \omega_3^* & \eta_3^* \\ t_{41}^* & t_{42}^* & \omega_4^* & \eta_4^* \end{bmatrix} \tag{3}$$

where u_{ij}^* and t_{ij}^* $(i,j=1,2)$ are, respectively, displacements and surface tractions at a field point Q in the X_j $(X_1=x,\ X_2=z)$ coordinate directions due to a unit load acting in one of the X_i directions at a source point P on the boundary, $u_{3j}^*$and t_{3j}^* $(j=1,2)$ are, respectively, displacement components and surface tractions in the X_j coordinate directions at Q due to a unit electric charge at P, u_{4j}^* and t_{4j}^* $(j=1,2)$ are, respectively, displacement components and surface tractions in the X_j coordinate directions at Q due to a unit current at P, Φ_i^*, Ψ_i^*, ω_i^* and η_i^* $(i=1,2)$ are, respectively, electric potential, magnetic potential, surface charge and surface magnetic induction at Q due to a unit load acting in one of the X_i directions at P, Φ_3^*, Ψ_3^*, ω_3^* and η_3^* are, respectively, electric potential, magnetic potential, surface charge and surface magnetic induction at Q due to a unit electric charge at P, Φ_4^*, Ψ_4^*, ω_4^* and η_4^* are, respectively, electric potential, magnetic potential, surface charge and surface magnetic induction at Q due to a unit current at P. The full statement of $\mathbf{U}^*$ and $\mathbf{T}^*$ can be seen in Appendix A. It is assumed that there is neither body force nor electric charge. Based on the extended Somigliana equation, the boundary integral formulation is obtained

$$\mathbf{C}(P)\mathbf{u}(P) = \int_S \mathbf{U}^*(P,Q)\mathbf{t}(Q)d\mathbf{s} - \int_S \mathbf{T}^*(P,Q)\mathbf{u}(Q)d\mathbf{s} \quad (4)$$

The general surface $\mathbf{t}$ and the matrix $\mathbf{T}^*$ can be written as

$$\left\{ \begin{array}{c} t_x \\ t_z \\ -\omega \\ -\eta \end{array} \right\} = \left[\begin{array}{cc} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \\ D_x & D_z \\ B_x & B_z \end{array} \right] \left\{ \begin{array}{c} n_x \\ n_z \end{array} \right\} \quad (5)$$

$$\begin{bmatrix} t_{11}^* & t_{21}^* & t_{31}^* & t_{41}^* \\ t_{12}^* & t_{22}^* & t_{32}^* & t_{42}^* \\ \omega_1^* & \omega_2^* & \omega_3^* & \omega_4^* \\ \eta_1^* & \eta_2^* & \eta_3^* & \eta_4^* \end{bmatrix} = \begin{bmatrix} \sigma_{1x} & \tau_{1xz} & \sigma_{2x} & \tau_{2xz} & \sigma_{3x} & \tau_{3xz} & \sigma_{4x} & \tau_{4xz} \\ \tau_{1xz} & \sigma_{1z} & \tau_{2xz} & \sigma_{2z} & \tau_{3xz} & \sigma_{3z} & \tau_{4xz} & \sigma_{4z} \\ -D_{1x} & -D_{1z} & -D_{2x} & -D_{2z} & -D_{3x} & -D_{3z} & -D_{4x} & -D_{4z} \\ -B_{1x} & -B_{1z} & -B_{2x} & -B_{2z} & -B_{3x} & -B_{3z} & -B_{4x} & -B_{4z} \end{bmatrix} \begin{bmatrix} n_x & 0 & 0 & 0 \\ n_z & 0 & 0 & 0 \\ 0 & n_x & 0 & 0 \\ 0 & n_z & 0 & 0 \\ 0 & 0 & n_x & 0 \\ 0 & 0 & n_z & 0 \\ 0 & 0 & 0 & n_x \\ 0 & 0 & 0 & n_z \end{bmatrix} \quad (6)$$

It is more convenient to use the index notation rather than the matrix representation

$$t_i = T_{ij}n_j \quad , \quad T_{ki}^* = \Sigma_{kij}n_j \quad (7)$$

where $\mathbf{T}$ is the general stress tensor and Σ is the Green's function stress tensor. Then, Eq.(4) can be rewritten as

$$c_{ki}(P)u_i(P) = \int_S \{u_{ki}^*(P,Q)T_{ij}(Q) - \Sigma_{kij}(P,Q)u_i(Q)\}\mathbf{e}_j \cdot d\mathbf{s} \quad (8)$$

where $\mathbf{e}_j$ are global Cartesian unit vectors.

Consider an arbitrary rigid body translation where $u_i(Q) = u_i(P)$=constant. Thus, $T_{ij}(Q) = 0$. Use of this rigid body motion solution in Eq.(8) gives

$$c_{ki}(P)u_i(P) = -\int_S \Sigma_{kij}(P,Q)u_i(P)\mathbf{e}_j \cdot d\mathbf{s} \quad (9)$$

Substituting Eq.(9) into Eq.(8) yields a new BEM equation

$$\int_S \{u_{ki}^*(P,Q)T_{ij}(Q) - \Sigma_{kij}(P,Q)[u_i(Q) - u_i(P)]\}\mathbf{e}_j \cdot d\mathbf{s} \quad (10)$$

Thus, the corner tensor c_{ki} is now eliminated from the BEM equation. Its evaluation is avoided and this is first advantage of using the rigid body motion technique.

Now let

$$\mathbf{F}_k = \{u_{ki}^*(P,Q)T_{ij}(Q) - \Sigma_{kij}(P,Q)[u_i(Q) - u_i(P)]\}\mathbf{e}_j \quad (11)$$

It is easy to show that when we take the divergence of $\mathbf{F}_k$ at a field point Q, this vector is divergence free everywhere except at the source point P, i.e.

$$\begin{aligned} \nabla_Q \cdot \mathbf{F}_k &= \{u_{ki}^*(P,Q)T_{ij}(Q) - \Sigma_{kij}(P,Q)[u_i(Q) - u_i(P)]\}_{,j} \\ &= [S_{kij}^*(P,Q)T_{ij}(Q) - \Sigma_{kij}(P,Q)S_{ij}(Q)] \\ &\quad + u_{ki}^*(P,Q)T_{ij,j}(Q) - \Sigma_{kij,j}(P,Q)[u_i(Q) - u_i(P)] \\ &= 0 \end{aligned} \quad (12)$$

Where $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, for $i = 1,2$; $S_{ij} = u_{i,j}$, for $i = 3,4$, and similarly for S_{kij}^*.

Eq.(12) shows the existence of a function Φ_k such that

$$\mathbf{F}_k = \frac{\partial \Phi_k}{\partial z}\mathbf{e}_1 - \frac{\partial \Phi_k}{\partial x}\mathbf{e}_2 \quad (13)$$

The boundary is now discretized into n elements, and each general curved boundary element with e_1 and e_2 as the end nodes and m the middle node of the element (Figure 1).

Then, Eq.(10) becomes

$$\begin{aligned} \int_S \mathbf{F}_k \cdot d\mathbf{S} &= \sum_{e=1}^{n} \int_{e_1}^{e_2} \mathbf{F}_k \cdot d\mathbf{S} \\ &= \sum_{e=1}^{n} \int_{e_1}^{e_2} \mathbf{F}_k \cdot \mathbf{n} dS = \sum_{e=1}^{n} \int_{e_1}^{e_2} d\Phi_k \\ &= \sum_{e=1}^{n} [\Phi_k^e(e_2) - \Phi_k^e(e_1)] \end{aligned} \quad (14)$$

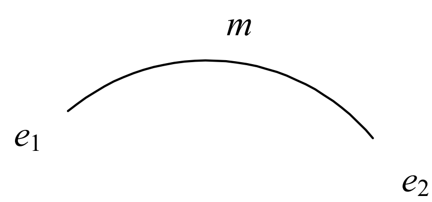

Figure 1 :

which means that there is no need for any numerical integration for two dimensional magneto- electro-elastic problems.

It is important to observe that the above integrand contains unknown functions $\mathbf{u}$ and $\mathbf{t}$ on $d\mathbf{S}$ which must satisfy the basic equations of magneto-electro-elastic media. Thus, local shape functions for $\mathbf{u}$ must be chosen such that they satisfy the general Navier-Cauchy equations and the shape functions for $\mathbf{t}$ must be derived from those of $\mathbf{u}$.

3 Two-Dimensional Magneto-Electro-Elastic Plane Strain with Linear Shape Functions

Linear shape functions for $\mathbf{u}$ are chosen for each element as in Figure 1 such that

$$
\begin{aligned}
u &= a_1 + b_1 x + c_1 z && \text{for the displacement in } x \text{ direction}\\
w &= a_2 + b_2 x + c_2 z && \text{for the displacement in } z \text{ direction}\\
\Phi &= a_3 + b_3 x + c_3 z && \text{for electric potential}\\
\Psi &= a_4 + b_4 x + c_4 z && \text{for magnetic potential}
\end{aligned}
\tag{15}
$$

where x and z are co-ordinate with respect to a global co-ordinate system.

From the equations of magneto-electro-elastic media, we have

$$
\left\{\begin{array}{c} t_x \\ t_z \\ -\omega \\ -\eta \end{array}\right\} = \left[\begin{array}{cc}
c_{11}b_1 + c_{13}c_2 + e_{31}c_3 + d_{31}c_4 & c_{44}c_1 + c_{44}b_2 + e_{15}b_3 + d_{15}b_4 \\
c_{44}c_1 + c_{44}b_2 + e_{15}b_3 + d_{15}b_4 & c_{13}b_1 + c_{33}c_2 + e_{33}c_3 + d_{33}c_4 \\
e_{15}c_1 + e_{15}b_2 - \varepsilon_{11}b_3 - g_{11}b_4 & e_{31}b_1 + e_{33}c_2 - \varepsilon_{33}c_3 - g_{33}c_4 \\
d_{15}c_1 + d_{15}b_2 - g_{11}b_3 - \mu_{11}b_4 & d_{31}b_1 + d_{33}c_2 - g_{33}c_3 - \mu_{33}c_4
\end{array}\right] \left\{\begin{array}{c} n_x \\ n_z \end{array}\right\}
\tag{16}
$$

The twelve constants in Eq.(15) are related to twelve physical quantities on a boundary element (e) by a transformation matrix $\mathbf{T}^{(e)}$

$$
\left\{\mathbf{P}^{(e)}\right\} = [\mathbf{T}^{(e)}(x,z)] \left\{\mathbf{C}^{(e)}\right\}
\tag{17}
$$

where $\mathbf{P}^{(e)} = [u^{(e_1)}\, w^{(e_1)} \; -\Phi^{(e_1)} \; -\Psi^{(e_1)} \; t_x^{(m)}\, t_z^{(m)} \; -\omega^{(m)} \; -\eta^{(m)}\, u^{(e_2)}\, w^{(e_2)} \; -\Phi^{(e_2)} \; -\Psi^{(e_2)}]^{\mathbf{T}}$, and $\mathbf{C}^{(e)} = [a_1\; b_1\; c_1\; a_2\; b_2\; c_2\; a_3\; b_3\; c_3\; a_4\; b_4\; c_4]^{\mathbf{T}}$. The element of the matrix $\mathbf{T}^{(e)}$ depend on the nodal co-ordinates and the normal at m. The variables in $\mathbf{P}^{(e)}$are chosen such that the matrix $\mathbf{T}^{(e)}$ is invertible.

A new coordinate system (ξ,η) centered at the source point j is introduced. This is done in order to make the shape function variables conform to those of u^*_{ki} and Σ_{kij}(which are functions of ξ and η only). The ξ and η axes are parallel to the global x and z axes, thus

$$
\xi = x(Q) - x(P) \quad , \quad \eta = z(Q) - z(P)
\tag{18}
$$

By instituting Eq.(18) into Eq.(15) the displacements shape functions can be written as

$$
\{\mathbf{u}\} = [\mathbf{T}_u(\xi,\eta)][\mathbf{B}_j]\{\mathbf{C}\} = [\mathbf{T}_u(\xi,\eta)]\left\{\hat{\mathbf{C}}\right\}
\tag{19}
$$

in which $[\mathbf{B}_j]$ is a transformation matrix that depends only on the coordinates of the source point j. The traction remains the same form as in Eq.(16).

If (h) is the element containing the source point at its first node, with this new coordinate system $u_1(P) = \hat{C}_1^{(h)}$, $u_2(P) = \hat{C}_4^{(h)}$, $u_3(P) = \hat{C}_7^{(h)}$ and $u_4(P) = \hat{C}_{10}^{(h)}$. So, for the element (e), we have

$$
\{\mathbf{u}(Q) - \mathbf{u}(P)\} = [\mathbf{T}_u(\xi,\eta)]\left\{\tilde{\mathbf{C}}^{(e)}\right\}
\tag{20}
$$

where the columns of $[\mathbf{T}_u(\xi,\eta)]$ are the twelve shape functions

$$
\left\{\begin{array}{c}1\\0\\0\\0\end{array}\right\},
\left\{\begin{array}{c}\xi\\0\\0\\0\end{array}\right\},
\left\{\begin{array}{c}\eta\\0\\0\\0\end{array}\right\},
\left\{\begin{array}{c}0\\1\\0\\0\end{array}\right\},
\left\{\begin{array}{c}0\\\xi\\0\\0\end{array}\right\},
\left\{\begin{array}{c}0\\\eta\\0\\0\end{array}\right\},
$$

$$
\left\{\begin{array}{c}0\\0\\1\\0\end{array}\right\},
\left\{\begin{array}{c}0\\0\\\xi\\0\end{array}\right\},
\left\{\begin{array}{c}0\\0\\\eta\\0\end{array}\right\},
\left\{\begin{array}{c}0\\0\\0\\1\end{array}\right\},
\left\{\begin{array}{c}0\\0\\0\\\xi\end{array}\right\},
\left\{\begin{array}{c}0\\0\\0\\\eta\end{array}\right\}
\tag{21}
$$

and

$$\{\tilde{\mathbf{C}}^{(e)}\} = \left\langle [\hat{C}_1^{(e)} - \hat{C}_1^{(h)}]\, \hat{C}_2^{(e)}\, \hat{C}_3^{(e)}\, [\hat{C}_4^{(e)} - \hat{C}_4^{(h)}]\, \hat{C}_5^{(e)} \times \hat{C}_6^{(e)}\, [\hat{C}_7^{(e)} - \hat{C}_7^{(h)}]\, \hat{C}_8^{(e)}\, \hat{C}_9^{(e)}\, [\hat{C}_{10}^{(e)} - \hat{C}_{10}^{(h)}]\, \hat{C}_{11}^{(e)}\, \hat{C}_{12}^{(e)} \right\rangle^T \quad (22)$$

Potential functions must be obtained for each of twelve states, for each of the direction, electric potentials and magnetic potentials corresponding to $k = 1 \sim 4$ in Eq.(14). Let $\varphi_{1,l}(l = 1 \sim 12)$ denote potentials for the above twelve states, for $k = 1$. These corresponding to the unit force in the x direction. The function in Eq.(14) is a linear combination of $\varphi_{1,l}(l = 1 \sim 12)$. $\varphi_{1,l}$ can be calculated individually by using Eq.(13), with $\varphi_{1,l}$ replacing Φ_1, ξ and η replacing x and z, respectively.

Similarly, let $\varphi_{2,l}$, $\varphi_{3,l}$ and $\varphi_{4,l}(l = 1 \sim 12)$ denote potentials for the twelve states, for $k = 2,3,4$ corresponding to the unit force in the x direction, the unit electric charge and the unit current, respectively. The potentials $\varphi_{k,l}(k = 1 \sim 4 \quad, l = 1 \sim 12)$ are given in Appendix B. It should be noted that Φ_k in Eq.(14) is composed of $\varphi_{k,l}(l = 1 \sim 12)$, respectively.

Now, with the potential functions already derived, the BCM discretized equations are developed as follows.

For the source point j(source points are only placed at the ends of each boundary element, see Figure 1.)

$$\sum_{e=1}^{n} [\Phi_k^e(e_2) - \Phi_k^e(e_1)] = \sum_{e=1}^{n} \sum_{l=1}^{12} [\varphi_{k,l}^j(e_2) - \varphi_{k,l}^j(e_1)] \tilde{C}_l^{(e)} = 0 \quad (23)$$

It should be noted that the potential functions $\varphi_{k,1}(\xi,\eta)$,$\varphi_{k,4}(\xi,\eta)$,$\varphi_{k,7}(\xi,\eta)$ and $\varphi_{k,10}(\xi,\eta)(k = 1 \sim 4)$corresponding to constant shape functions, are singular when a field point $Q \rightarrow$the source point P, i.e. when $(\xi,\eta) \rightarrow (0,0)$. But in this case $u_k(Q) - u_k(P) = 0(r)$, and Eq.(20) lead to

$$\begin{cases} \tilde{C}_1^{(e)} = [\hat{C}_1^{(e)} - \hat{C}_1^{(h)}] = 0 \\ \tilde{C}_4^{(e)} = [\hat{C}_4^{(e)} - \hat{C}_4^{(h)}] = 0 \\ \tilde{C}_7^{(e)} = [\hat{C}_7^{(e)} - \hat{C}_7^{(h)}] = 0 \\ \tilde{C}_{10}^{(e)} = [\hat{C}_{10}^{(e)} - \hat{C}_{10}^{(h)}] = 0 \end{cases} \quad (24)$$

so the evaluation of these potential functions can be avoided, i.e. expression(23) is now completely regular. This is the second advantage of the approach using the rigid body motion technique.

With n source pints corresponding to n displacement nodes on the boundary ds, one can get the final BCM linear system of equations

$$[\mathbf{K}]\{\mathbf{P}\} = \{\mathbf{0}\} \quad (25)$$

where $\{\mathbf{P}\}$ are degrees of freedom on the whole boundary ds. The global system of equations is condensed in accordance with continuity of displacements across element in the usual way.

Finally, the system of equation (25) needs to be reordered in accordance with the boundary conditions to form

$$[\mathbf{A}]\{\mathbf{X}\} = \{\mathbf{Z}\} \quad (26)$$

where $\{\mathbf{X}\}$ contains the unknown boundary quantities and $\{\mathbf{Z}\}$ is known in terms of prescribed boundary quantities and geometrical and material data of the problem.

After the solution of the global equation system (26) is obtained, one can easily derive the article variables $\{\mathbf{C}^{(e)}\}$ from Eq.(17). At this stage, the remaining physical variables at any node on the boundary can be easily calculated from Eq.(15) and the corresponding relations for stresses and tractions in terms of their shape functions.

Evaluation of strain components at points inside a body requires transformation of equations of the strain BEM at an internal point to an integrated form analogous to Eq.(14). This can be done since the integrand is again divergence free. Stress calculations would then follow from magneto-electro-elastic fundamental equations.

4 Numerical Example

Consider a magneto-electro-elastic column of size $a \times b$ under three load cases, i.e. uniform axial tension, electric displacement or magnetic induction as in Ding and Jiang(2004). The problem is treated as a plane-strain one.

For numerical calculation, we consider the column with the same geometrical and material constants as in Ding and Jiang(2004) for which totally twenty linear elements are used.

Table 1 : Comparison of BCM results with the exact (*) and BEM results (**)

Load	u	w	Φ	Ψ
1	-0.9500E-11 -0.9501E-11(*) -0.9500E-11(**)	+0.5681E-12 +0.5680E-12(*) +0.5683E-12(**)	+0.9495E-3 +0.9495E-3(*) +0.9495E-3(**)	+0.2138-4E-4 +0.2138E-4(*) +0.2139E-4(**)
2	-0.2108E-12 -0.2107E-12(*) -0.2108E-12(**)	+0.9493E-14 +0.9492E-14(*) +0.9495E-14(**)	-0.6289E-4 -0.6289E-4(*) -0.6289E-4(**)	+0.2566E-6 +0.2566E-6(*) +0.2567E-6(**)
3	+0.5077E-12 +0.5077E-12(*) +0.5077E-12(**)	+0.2140E-13 +0.2140E-13(*) +0.2139E-13(**)	+0.2565E-4 +0.2564E-4(*) +0.2567E-4(**)	-0.7521E-5 -0.7520E-5(*) -0.7521E-5(**)

Because of the linearity property, the corresponding results are compared with the exact ones(*) and BEM results(**) only at the corner point ($a/2$, $b/2$) for example, which is shown in Table 1.

5 Conclusions

In this paper, the BCM is presented for 2D magneto-electro-elasticity based on the fundamental solution of an infinite magneto-electro-elastic plane. This approach does not require any numerical integration at all for 2D problem, even with curved boundary elements, and it requires only numerical evaluation of contour integrals for 3D problems. Numerical results for 2D problem show that the BCM performs very well.

Acknowledgement: The work was supported by the National Natural Science Foundation of China (No. 10472102, 10432030) and Zhejiang Natural Science Foundation (No.Y605040).

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Appendix A: Fundamental solutions

$$u_{11}^* = \sum_{j=1}^{4} \lambda_j \ln R_j,$$

$$u_{12}^* = \sum_{j=1}^{4} s_j k_{1j} \lambda_j \arctan \frac{x}{z_j},$$

$$\Phi_1^* = \sum_{j=1}^{4} s_j k_{2j} \lambda_j \arctan \frac{x}{z_j}$$

$$\Psi_1^* = \sum_{j=1}^{4} s_j k_{3j} \lambda_j \arctan \frac{x}{z_j},$$

$$u_{21}^* = -\sum_{j=1}^{4} \alpha_j \arctan \frac{x}{z_j},$$

$$u_{22}^* = \sum_{j=1}^{4} s_j k_{1j} \alpha_j \ln R_j$$

$$\Phi_2^* = \sum_{j=1}^{4} s_j k_{2j} \alpha_j \ln R_j,$$

$$\Psi_2^* = \sum_{j=1}^{4} s_j k_{3j} \alpha_j \ln R_j,$$

$$u_{31}^* = -\sum_{j=1}^{4} \beta_j \arctan \frac{x}{z_j}$$

$$u_{32}^* = \sum_{j=1}^{4} s_j k_{1j} \beta_j \ln R_j, \Phi_3^* = \sum_{j=1}^{4} s_j k_{2j} \beta_j \ln R_j,$$

$$\Psi_3^* = \sum_{j=1}^{4} s_j k_{3j} \beta_j \ln R_j$$

$$u_{41}^* = -\sum_{j=1}^{4} \gamma_j \arctan \frac{x}{z_j},$$

$$u_{42}^* = \sum_{j=1}^{4} s_j k_{1j} \gamma_j \ln R_j,$$

$$\Phi_4^* = \sum_{j=1}^{4} s_j k_{2j} \gamma_j \ln R_j$$

$$\Psi_4^* = \sum_{j=1}^{4} s_j k_{3j} \gamma_j \ln R_j,$$

$$t_{11}^* = \sum_{j=1}^{4} \omega_{1j} s_j^2 \frac{\lambda_j}{R_j^2} (x n_x + z n_z),$$

$$t_{12}^* = \sum_{j=1}^{4} \omega_{1j} \frac{\lambda_j}{R_j^2} (s_j^2 z n_x - x n_z)$$

$$\omega_1^* = -\sum_{j=1}^{4} \omega_{2j} \frac{\lambda_j}{R_j^2} (s_j^2 z n_x - x n_z) \quad ,$$

$$\eta_1^* = -\sum_{j=1}^{4} \omega_{3j} \frac{\lambda_j}{R_j^2}(s_j^2 z n_x - x n_z)$$

$$t_{21}^* = \sum_{j=1}^{4} \omega_{1j} s_j \frac{\alpha_j}{R_j^2}(-s_j^2 z n_x + x n_z),$$

$$t_{22}^* = \sum_{j=1}^{4} \omega_{1j} s_j \frac{\alpha_j}{R_j^2}(x n_x + z n_z)$$

$$\omega_2^* = -\sum_{j=1}^{4} \omega_{2j} s_j \frac{\alpha_j}{R_j^2}(x n_x + z n_z) \quad ,$$

$$\eta_2^* = -\sum_{j=1}^{4} \omega_{3j} s_j \frac{\alpha_j}{R_j^2}(x n_x + z n_z)$$

$$t_{31}^* = \sum_{j=1}^{4} \omega_{1j} s_j \frac{\beta_j}{R_j^2}(-s_j^2 z n_x + x n_z),$$

$$t_{32}^* = \sum_{j=1}^{4} \omega_{1j} s_j \frac{\beta_j}{R_j^2}(x n_x + z n_z)$$

$$\omega_3^* = -\sum_{j=1}^{4} \omega_{2j} s_j \frac{\beta_j}{R_j^2}(x n_x + z n_z),$$

$$\eta_3^* = -\sum_{j=1}^{4} \omega_{3j} s_j \frac{\beta_j}{R_j^2}(x n_x + z n_z)$$

$$t_{41}^* = \sum_{j=1}^{4} \omega_{1j} s_j \frac{\gamma_j}{R_j^2}(-s_j^2 z n_x + x n_z),$$

$$t_{42}^* = \sum_{j=1}^{4} \omega_{1j} s_j \frac{\gamma_j}{R_j^2}(x n_x + z n_z)$$

$$\omega_4^* = -\sum_{j=1}^{4} \omega_{2j} s_j \frac{\gamma_j}{R_j^2}(x n_x + z n_z) \quad ,$$

$$\eta_4^* = -\sum_{j=1}^{4} \omega_{3j} s_j \frac{\gamma_j}{R_j^2}(x n_x + z n_z)$$

where $z_j = s_j z$, $R_j = \sqrt{x^2 + z_j^2}$ and $s_j, k_{ij}, \omega_{ij}, \lambda_j, \alpha_j, \beta_j, \gamma_j$ are as in Ding and Jiang(2004).

Appendix B: Potential functions

$$\varphi_{1,1} = \sum_{j=1}^{4} \omega_{1j} s_j \lambda_j \arctan \frac{\xi}{\eta_j}$$

$$\varphi_{1,2} = \sum_{j=1}^{4} [c_{11}\lambda_j(\eta \ln R_j - \eta) - \frac{c_{11} - \omega_{1j} s_j^2}{s_j}\lambda_j \xi \arctan \frac{\xi}{\eta_j}]$$

$$\varphi_{1,3} = \sum_{j=1}^{4} [s_j(\omega_{1j} - c_{44})\lambda_j(\eta \arctan \frac{\xi}{\eta_j} + \frac{\xi}{s_j}\ln R_j) - \omega_{1j}\lambda_j \xi \ln R_j + c_{44}\lambda_j \xi]$$

$$\varphi_{1,4} = -\sum_{j=1}^{4} \omega_{1j}\lambda_j \ln R_j$$

$$\varphi_{1,5} = \sum_{j=1}^{4} [s_j(\omega_{1j} - c_{44})\lambda_j(\eta \arctan \frac{\xi}{\eta_j} + \frac{\xi}{s_j}\ln R_j) - \omega_{1j}\lambda_j \xi \ln R_j - (\omega_{1j} - c_{44})\lambda_j \xi]$$

$$\varphi_{1,6} = \sum_{j=1}^{4} [c_{13}\lambda_j(\eta \ln R_j - \eta - \frac{\xi}{s_j}\arctan \frac{\xi}{\eta_j}) - \frac{\omega_{1j}\lambda_j}{s_j}\xi \arctan \frac{\xi}{\eta_j} - \omega_{1j}\lambda_j \eta]$$

$$\varphi_{1,7} = \sum_{j=1}^{4} \omega_{2j}\lambda_j \ln R_j$$

$$\varphi_{1,8} = -\sum_{j=1}^{4} [s_j(\omega_{2j} - e_{15})\lambda_j(\eta \arctan \frac{\xi}{\eta_j} + \frac{\xi}{s_j}\ln R_j) - \omega_{2j}\lambda_j \xi \ln R_j - (\omega_{2j} - e_{15})\lambda_j \xi]$$

$$\varphi_{1,9} = -\sum_{j=1}^{4} [e_{31}\lambda_j(\eta \ln R_j - \eta - \frac{\xi}{s_j}\arctan \frac{\xi}{\eta_j}) - \frac{\omega_{2j}\lambda_j}{s_j}\xi \arctan \frac{\xi}{\eta_j} - \omega_{2j}\lambda_j \eta]$$

$$\varphi_{1,10}=\sum_{j=1}^{4}\omega_{3j}\lambda_j\ln R_j$$

$$\varphi_{1,11}=-\sum_{j=1}^{4}[s_j(\omega_{3j}-d_{15})\lambda_j(\eta\arctan\frac{\xi}{\eta_j}+\frac{\xi}{s_j}\ln R_j) -\omega_{3j}\lambda_j\xi\ln R_j-(\omega_{3j}-d_{15})\lambda_j\xi]$$

$$\varphi_{1,12}=-\sum_{j=1}^{4}[d_{31}\lambda_j(\eta\ln R_j-\eta-\frac{\xi}{s_j}\arctan\frac{\xi}{\eta_j}) -\frac{\omega_{3j}\lambda_j}{s_j}\xi\arctan\frac{\xi}{\eta_j}-\omega_{3j}\lambda_j\eta]$$

$$\varphi_{2,1}=\sum_{j=1}^{4}\omega_{1j}s_j\alpha_j\ln R_j$$

$$\varphi_{2,2}=\sum_{j=1}^{4}[-c_{11}\alpha_j(\eta\arctan\frac{\xi}{\eta_j}+\frac{\xi}{s_j}\ln R_j) +\omega_{1j}s_j\alpha_j\xi\ln R_j+\frac{c_{11}\alpha_j}{s_j}\xi]$$

$$\varphi_{2,3}=\sum_{j=1}^{4}[s_j(\omega_{1j}-c_{44})\alpha_j(\eta\ln R_j-\eta) +c_{44}\alpha_j\xi\arctan\frac{\xi}{\eta_j}+\omega_{1j}s_j\alpha_j\eta]$$

$$\varphi_{2,4}=\sum_{j=1}^{4}\omega_{1j}\alpha_j\arctan\frac{\xi}{\eta_j}$$

$$\varphi_{2,5}=\sum_{j=1}^{4}[s_j(\omega_{1j}-c_{44})\alpha_j(\eta\ln R_j-\eta-\frac{\xi}{s_j}\arctan\frac{\xi}{\eta_j}) +\omega_{1j}\alpha_j\xi\arctan\frac{\xi}{\eta_j}]$$

$$\varphi_{2,6}=\sum_{j=1}^{4}[-c_{13}\alpha_j(\eta\arctan\frac{\xi}{\eta_j}+\frac{\xi}{s_j}\ln R_j) -\frac{\omega_{1j}\alpha_j}{s_j}\xi\ln R_j+\frac{(c_{13}+\omega_{1j})\alpha_j}{s_j}\xi]$$

$$\varphi_{2,7}=-\sum_{j=1}^{4}\omega_{2j}\alpha_j\arctan\frac{\xi}{\eta_j}$$

$$\varphi_{2,8}=-\sum_{j=1}^{4}[s_j(\omega_{2j}-e_{15})\alpha_j(\eta\ln R_j-\eta-\frac{\xi}{s_j}\arctan\frac{\xi}{\eta_j}) +\omega_{2j}\alpha_j\xi\arctan\frac{\xi}{\eta_j}]$$

$$\varphi_{2,9}=-\sum_{j=1}^{4}[-e_{31}\alpha_j(\eta\arctan\frac{\xi}{\eta_j}+\frac{\xi}{s_j}\ln R_j) -\frac{\omega_{2j}\alpha_j}{s_j}\xi\ln R_j+\frac{(e_{31}+\omega_{2j})\alpha_j}{s_j}\xi]$$

$$\varphi_{2,10}=-\sum_{j=1}^{4}\omega_{3j}\alpha_j\arctan\frac{\xi}{\eta_j}$$

$$\varphi_{2,11}=-\sum_{j=1}^{4}[s_j(\omega_{3j}-d_{15})\alpha_j(\eta\ln R_j-\eta-\frac{\xi}{s_j}\arctan\frac{\xi}{\eta_j}) +\omega_{3j}\alpha_j\xi\arctan\frac{\xi}{\eta_j}]$$

$$\varphi_{2,12}=-\sum_{j=1}^{4}[-d_{31}\alpha_j(\eta\arctan\frac{\xi}{\eta_j}+\frac{\xi}{s_j}\ln R_j) -\frac{\omega_{3j}\alpha_j}{s_j}\xi\ln R_j+\frac{(d_{31}+\omega_{3j})\alpha_j}{s_j}\xi]$$

$$\varphi_{3,1}=\sum_{j=1}^{4}\omega_{1j}s_j\beta_j\ln R_j$$

$$\varphi_{3,2}=\sum_{j=1}^{4}[-c_{11}\beta_j(\eta\arctan\frac{\xi}{\eta_j}+\frac{\xi}{s_j}\ln R_j) +\omega_{1j}s_j\beta_j\xi\ln R_j+\frac{c_{11}\beta_j}{s_j}\xi]$$

$$\varphi_{3,3}=\sum_{j=1}^{4}[s_j(\omega_{1j}-c_{44})\beta_j(\eta\ln R_j-\eta) +c_{44}\beta_j\xi\arctan\frac{\xi}{\eta_j}+\omega_{1j}s_j\beta_j\eta]$$

$$\varphi_{3,4} = \sum_{j=1}^{4} \omega_{1j}\beta_j \arctan\frac{\xi}{\eta_j}$$

$$\varphi_{3,5} = \sum_{j=1}^{4} [s_j(\omega_{1j} - c_{44})\beta_j(\eta \ln R_j - \eta - \frac{\xi}{s_j}\arctan\frac{\xi}{\eta_j}) + \omega_{1j}\beta_j\xi\arctan\frac{\xi}{\eta_j}]$$

$$\varphi_{3,6} = \sum_{j=1}^{4} [-c_{13}\beta_j(\eta\arctan\frac{\xi}{\eta_j} + \frac{\xi}{s_j}\ln R_j) - \frac{\omega_{1j}\beta_j}{s_j}\xi\ln R_j + \frac{(c_{13}+\omega_{1j})\beta_j}{s_j}\xi]$$

$$\varphi_{3,7} = -\sum_{j=1}^{4} \omega_{2j}\beta_j \arctan\frac{\xi}{\eta_j}$$

$$\varphi_{3,8} = -\sum_{j=1}^{4} [s_j(\omega_{2j} - e_{15})\beta_j(\eta \ln R_j - \eta - \frac{\xi}{s_j}\arctan\frac{\xi}{\eta_j}) + \omega_{2j}\beta_j\xi\arctan\frac{\xi}{\eta_j}]$$

$$\varphi_{3,9} = -\sum_{j=1}^{4} [-e_{31}\beta_j(\eta\arctan\frac{\xi}{\eta_j} + \frac{\xi}{s_j}\ln R_j) - \frac{\omega_{2j}\beta_j}{s_j}\xi\ln R_j + \frac{(e_{31}+\omega_{2j})\beta_j}{s_j}\xi]$$

$$\varphi_{3,10} = -\sum_{j=1}^{4} \omega_{3j}\beta_j \arctan\frac{\xi}{\eta_j}$$

$$\varphi_{3,11} = -\sum_{j=1}^{4} [s_j(\omega_{3j} - d_{15})\beta_j(\eta \ln R_j - \eta - \frac{\xi}{s_j}\arctan\frac{\xi}{\eta_j}) + \omega_{3j}\beta_j\xi\arctan\frac{\xi}{\eta_j}]$$

$$\varphi_{3,12} = -\sum_{j=1}^{4} [-d_{31}\beta_j(\eta\arctan\frac{\xi}{\eta_j} + \frac{\xi}{s_j}\ln R_j) - \frac{\omega_{3j}\beta_j}{s_j}\xi\ln R_j + \frac{(d_{31}+\omega_{3j})\beta_j}{s_j}\xi]$$

$$\varphi_{4,1} = \sum_{j=1}^{4} \omega_{1j}s_j\gamma_j \ln R_j \tag{27}$$

$$\varphi_{4,2} = \sum_{j=1}^{4} [-c_{11}\gamma_j(\eta\arctan\frac{\xi}{\eta_j} + \frac{\xi}{s_j}\ln R_j) + \omega_{1j}s_j\gamma_j\xi\ln R_j + \frac{c_{11}\gamma_j}{s_j}\xi]$$

$$\varphi_{4,3} = \sum_{j=1}^{4} [s_j(\omega_{1j} - c_{44})\gamma_j(\eta\ln R_j - \eta) + c_{44}\gamma_j\xi\arctan\frac{\xi}{\eta_j} + \omega_{1j}s_j\gamma_j\eta]$$

$$\varphi_{4,4} = \sum_{j=1}^{4} \omega_{1j}\gamma_j \arctan\frac{\xi}{\eta_j}$$

$$\varphi_{4,5} = \sum_{j=1}^{4} [s_j(\omega_{1j} - c_{44})\gamma_j(\eta \ln R_j - \eta - \frac{\xi}{s_j}\arctan\frac{\xi}{\eta_j}) + \omega_{1j}\gamma_j\xi\arctan\frac{\xi}{\eta_j}]$$

$$\varphi_{4,6} = \sum_{j=1}^{4} [-c_{13}\gamma_j(\eta\arctan\frac{\xi}{\eta_j} + \frac{\xi}{s_j}\ln R_j) - \frac{\omega_{1j}\gamma_j}{s_j}\xi\ln R_j + \frac{(c_{13}+\omega_{1j})\gamma_j}{s_j}\xi]$$

$$\varphi_{4,7} = -\sum_{j=1}^{4} \omega_{2j}\gamma_j \arctan\frac{\xi}{\eta_j}$$

$$\varphi_{4,8} = -\sum_{j=1}^{4} [s_j(\omega_{2j} - e_{15})\gamma_j(\eta \ln R_j - \eta - \frac{\xi}{s_j}\arctan\frac{\xi}{\eta_j}) + \omega_{2j}\gamma_j\xi\arctan\frac{\xi}{\eta_j}]$$

$$\varphi_{4,9} = -\sum_{j=1}^{4} [-e_{31}\gamma_j(\eta\arctan\frac{\xi}{\eta_j} + \frac{\xi}{s_j}\ln R_j) - \frac{\omega_{2j}\gamma_j}{s_j}\xi\ln R_j + \frac{(e_{31}+\omega_{2j})\gamma_j}{s_j}\xi]$$

$$\varphi_{4,10} = -\sum_{j=1}^{4} \omega_{3j}\gamma_j \arctan\frac{\xi}{\eta_j}$$

$$\varphi_{4,11} = -\sum_{j=1}^{4}[s_j(\omega_{3j} - d_{15})\gamma_j(\eta \ln R_j - \eta - \frac{\xi}{s_j}\arctan\frac{\xi}{\eta_j})$$
$$+\omega_{3j}\gamma_j\xi \arctan\frac{\xi}{\eta_j}]$$

$$\varphi_{4,12} = -\sum_{j=1}^{4}[-d_{31}\gamma_j(\eta \arctan\frac{\xi}{\eta_j} + \frac{\xi}{s_j}\ln R_j)$$
$$-\frac{\omega_{3j}\gamma_j}{s_j}\xi \ln R_j + \frac{(d_{31} + \omega_{3j})\gamma_j}{s_j}\xi]$$

where $\eta_j = s_j\eta$, $R_j = \sqrt{\xi^2 + \eta_j^2}$.