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Effect of Constitutive Parameters on Cavity Formation and Growth in a Class of Incompressible Transversely Isotropic Nonlinearly Elastic Solid Spheres

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Abstract: Cavity formation and growth in a class of incompressible transversely isotropic nonlinearly elastic solid spheres are described as a bifurcation problem, for which the strain energy density is expressed as a nonlinear function of the invariants of the right Cauchy-Green deformation tensor. A bifurcation equation that describes cavity formation and growth is obtained. Some interesting qualitative properties of the bifurcation equation are presented. In particular, cavitated bifurcation is examined for a solid sphere composed of an incompressible anisotropic Gent-Thomas material model with a transversely isotropy about the radial direction. The effect of constitutive parameters on cavity formation and growth is then carried out. It is proved that a cavity forms in the interior of the sphere earlier or later than that in the isotropic Gent-Thomas sphere as the anisotropic parameter takes certain values. The condition for the bifurcation to the left or to the right of the cavity solution is proposed. The stability and the catastrophe of the equilibrium solutions are discussed by using the minimal potential energy principle. Whereas, in contrast to other isotropic nonlinear elastic spheres, cavitated bifurcation in the isotropic Gent-Thomas sphere is quite different, it is proved that the cavity solution can bifurcate locally to the left. The growth of a pre-existing micro-void in the sphere is examined, which interprets the physical implications of the preceding bifurcation problem.

keyword: Cavity formation and growth, bifurcation, incompressible transversely isotropic nonlinearly elastic material, stability and catastrophe, minimal potential energy principle.

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1 Introduction

Cavitation phenomenon, the sudden formation and growth of cavity in solid materials has given rise to many investigations in mechanics, applied mathematics, material science etc., due to its importance in understanding damage and failure mechanisms. The experimental observation of cavity formation in vulcanized rubber under tension loading was reported by Gent and Lindly (1958), moreover, a recent review on cavitation in rubber is that of Gent (1990). The nonlinear theoretical investigation in a solid mechanics framework for cavitation problems started with the work of Ball (1982), in which cavity formation and growth is modeled as a class of nonlinear bifurcation problems with discontinuous radial symmetric solutions in finite elasticity. On the other hand, an alternative interpretation for such problems in terms of the growth of a pre-existing micro-void has been given by Horgan and Abeyaratne (1986), see also Sivaloganathan (1986). As pointed out, for example, by Horgan and Abeyaratne (1986), cavitation is an inherently nonlinear phenomenon and cannot be modeled by using linearized solid mechanics theories. Many significant works have been carried out since then, see the review articles, by Polignone and Horgan (1995) for a comprehensive review of results up to 1995 for both incompressible and compressible materials. In particular, Chou-Wang and Horgan (1989) investigated the problem of void nucleation and growth for a class of incompressible isotropic nonlinearly elastic materials of power-law type. Polignone and Horgan (1993a, b) studied cavitation for (composite) anisotropic nonlinearly elastic materials, and they also examined the effect of material anisotropy (and inhomogeneity) on cavity formation and growth in incompressible nonlinearly elastic solids. Recently, Ren and Cheng (2002 a, b) studied the similar problem for the isotropic Valanis-Landel material and the transversely isotropic Ogden material. Yuan and Zhu (2004, 2005) carried out the qualitative analyses of cavitated bifurcation for the generalized Valanis-Landel material and the transversely isotropic modified Varga materials.

The purpose of this paper is to further study the bifurcation problem of cavity formation and growth in a class of incompressible transversely isotropic nonlinearly elastic solid spheres. The corresponding strain energy density is expressed as a nonlinear function of the invariants of the right Cauchy-Green deformation tensor, and it can be viewed as the generalized forms of some known material models, such as the neo-Hookean material model, the Mooney-Rivlin material model, the Gent-Thomas material model, etc. A bifurcation equation that describes cavity formation and growth is obtained. Some interesting qualitative properties of the bifurcation equation are carried out, such as the normal form, the condition of cavity formation, and the bifurcation direct of the cavity solution, and so on. In particular, cavitated bifurcation is considered for a sphere composed of an incompressible anisotropic Gent-Thomas material model with a transversely isotropy about the radial direction, and the effect of material anisotropy on cavity formation and growth is then examined. It is proved that a cavity occurs in the interior of the sphere earlier or later than that in the isotropic Gent-Thomas material as the anisotropic parameter takes different values. The distinguishing between the left-bifurcation and the right-bifurcation of the cavity solution at the critical point is made. The stability and the catastrophe of the equilibrium solutions are discussed by using the minimal potential energy principle. It is worth pointing out that cavitated bifurcation in the isotropic Gent-Thomas sphere is quite different from other isotropic nonlinear elastic spheres. It is proved that the cavity solution can bifurcate locally to the left and there exists a secondary turning point on the cavity solution that bifurcates locally to the left. Finally, to better understanding the physical implications of the preceding bifurcation problem, we examine the growth of a pre-existing micro-void in the sphere.

2 Formulation

2.1 Governing equation

Attention will be focused on the radial deformation of a solid sphere composed of a homogeneous, incompressible nonlinearly elastic material. The sphere is subjected to a uniform radial traction $p_0 > 0$ on its surface $R = b$. Assume that the resulting deformation is radially symmetric, then the coordinates of a material point in the undeformed state and a spatial point in the deformed state are respectively given by (the details of the formulation can be found in the text by Fu and Ogden (2001))

$$X_1 = R\sin\Theta\cos\Phi,\quad X_2 = R\sin\Theta\sin\Phi,\quad X_3 = R\cos\Theta \tag{1}$$

and

$$x_1 = r(R)\sin\theta\cos\phi,\quad x_2 = r(R)\sin\theta\sin\phi,\quad x_3 = r(R)\cos\theta \tag{2}$$

where $r = r(R)$ is the radial deformation function to be determined, $\Theta = \theta$ and $\Phi = \phi$. The principal stretch λ_i, $(i = r,\theta,\phi)$, the deformation gradient tensor $\mathbf{F}$ and the right Cauchy-Green deformation tensor $\mathbf{C}$, referred to spherical polar coordinates, are given by

$$\lambda_r = \dot{r}(R), \quad \lambda_\theta = \lambda_\phi = r(R)/R \tag{3}$$

$$\mathbf{F} = \mathrm{diag}(\lambda_r,\lambda_\theta,\lambda_\phi) = \mathrm{diag}\left(\dot{r}(R), r(R)/R, r(R)/R\right) \tag{4}$$

and

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathrm{diag}(\dot{r}^2, r^2/R^2, r^2/R^2) \tag{5}$$

where $\dot{r}(R)$ denotes the derivative with respect to R. Under the assumption of the radial deformation, the four strain invariants are given by

$$I_1 = tr\mathbf{C} = \lambda_r^2+\lambda_\theta^2+\lambda_\phi^2 \tag{6}$$

$$I_2 = \frac{1}{2}[(tr\mathbf{C})^2 tr\mathbf{C}^2] = \lambda_r^2\lambda_\theta^2+\lambda_\theta^2\lambda_\phi^2+\lambda_r^2\lambda_\phi^2 \tag{7}$$

$$I_3 = det\mathbf{C} = \lambda_r^2\lambda_\theta^2\lambda_\phi^2 \tag{8}$$

$$I_5 = C_{11} = \lambda_r^2 \tag{9}$$

As is well known, the response of an elastic material is described completely by the form of its strain energy density. Moreover, the strain energy density per unit undeformed volume for an elastic material which is transversely isotropic about the radial direction is given by (see e.g. Polignone and Horgan (1995))

$$W = W(I_1,I_2,I_3,I_5) \tag{10}$$

For incompressible materials, $I_3 = J^2 = \lambda_r^2\lambda_\theta^2\lambda_\phi^2 = 1$. The corresponding nonzero components of the Cauchy stress tensor for transversely isotropic incompressible materials about radial direction are given by

$$\tau_{rr}(R) = -p + 2(\dot{r}^2 W_1 - \dot{r}^{-2} W_2 + \dot{r}^2 W_5) \tag{11}$$

and

$$\tau_{\theta\theta}(R) = \tau_{\phi\phi}(R) = -p + 2(\dot{r}^{-1} W_1 - \dot{r} W_2) \tag{12}$$

where p is the unknown hydrostatic pressure associated with the incompressible constraint $J = \lambda_r\lambda_\theta\lambda_\phi = 1$, and $W_j = \partial W/\partial I_j$, $(j = 1,2,5)$. It is assumed that, in the undeformed state where $I_1 = 3, I_2 = 3, I_3 = 1$ and $I_5 = 1$, W varnishes and the initial stress is a hydrostatic pressure so that one has the following normalization conditions for the strain energy density

$$W(3,3,1,1) = 0, \quad W_5(3,3,1,1) = 0 \tag{13}$$

In this work, we shall assume that the strain energy density function W has the form

$$W = f(I_1) + g(I_2) + h(I_5) \tag{14}$$

where the nonlinear functions f, g and h are assumed to be twice continuously differentiable. Obviously, if $h(I_5) \equiv 0$, the expression (14) then reduces to the isotropic material model.

By using the normalization conditions in (13), we obtain certain conditions that f, g and h must hold, i.e.,

$$f(3) = g(3) = h(1) = 0, \quad h'(1) = 0 \tag{15}$$

In fact, as f, g and h take some special functions, the strain energy density corresponds to the following classical nonlinear elastic material models:

1. If $f(I_1) = \frac{\mu}{2}(I_1 - 3)$ and $g(I_2) = h(I_5) \equiv 0$, (14) then reduces to the well-known isotropic neo-Hookean material model (see Chow-Wang and Horgan (1989)). Further, If $g(I_2) \equiv 0$ and $h(I_5) = a(I_5 - 1)^2$, (14) is the neo-Hookean material model which is transversely isotropic about radial direction given by Polignone and Horgan (1993).

2. If f and g are linear functions, and $h(I_5) \equiv 0$, (14) is the isotropic Mooney-Rivlin material model (see Chow-Wang and Horgan (1989)).

3. If f is a linear function, $g(I_2)$ is a nonlinear function, and $h(I_5) \equiv 0$, (14) then reduces to the isotropic Rivlin-Saunders material model (see Chow-Wang and Horgan (1989)).

4. If f is a linear function, $g(I_2) = \mu_2(I_2 - 2)/2$, and $h(I_5) \equiv 0$, (14) is the isotropic Gent-Thomas material model (see Gent and Thomas (1958)).

Due to the radial deformation and the absence of body force, the equilibrium equation reduces to

$$\dot{\tau}_{rr}(R) + 2\frac{\dot{r}(R)}{r(R)}[\tau_{rr}(R) - \tau_{\theta\theta}(R)] = 0 \tag{16}$$

On the other hand, from the incompressibility constraint $\lambda_r\lambda_\theta\lambda_\phi = 1$ and (3), we have

$$\dot{r}(R) = \frac{R^2}{r^2(R)} \tag{17}$$

At the center of the sphere, we have the condition

$$r(0+)\tau_{rr}(0+) = 0 \tag{18}$$

(18) here means that if no cavity occurs, we have $r(0+) = 0$, if a cavity forms at the center of the sphere, for the traction-free cavity, $\tau_{rr}(0+) = 0$.

Since the surface of the sphere is subjected to a prescribed uniform radial tensile dead load $p_0 > 0$, the surface condition requires that

$$\tau_{rr}(b) = p_0 \left[\frac{b}{r(b)}\right]^2 \tag{19}$$

Thus, under a prescribed uniform surface radial tensile dead load $p_0 > 0$, the governing equations, which describes the radial deformation of a homogeneous, incompressible transversely isotropic nonlinearly elastic solid sphere, with respect to $\tau_{rr}(R)$, $\tau_{\theta\theta}(R)$, the radial deformation function $r(R)$, and the hydrostatic pressure $p(R)$, is composed of Eqs.(3), (11), (12), (14), (16), (17), and the boundary conditions (18) and (19).

2.2 Solutions of the Problem

On integration of Eq.(17) with respect to R, this leads to

$$r(R) = (R^3 + k^3)^{1/3} \tag{20}$$

where $k \geq 0$ is a constant to be determined, it denotes the cavity radius. If $k > 0$, then Eq.(20) means that a cavity of radius k forms at the center of the sphere, while if $k = 0$, the body remains a solid sphere.

For convenience, we introduce the notation

$$\omega = \omega(R,k) = (1 + \frac{k^3}{R^3})^{1/3} \tag{21}$$

From Eqs.(3) and (21), we have

$$\lambda_r = \omega^{-2}, \quad \lambda_\theta = \lambda_\phi = \omega \tag{22}$$

and

$$I_1 = \omega^{-4} + 2\omega^2, \quad I_2 = \omega^4 + 2\omega^{-2}, \quad I_5 = \omega^{-4} \tag{23}$$

and thus the strain energy density (14) can be written as

$$W = f(\omega^{-4} + 2\omega^2) + g(\omega^4 + 2\omega^{-2}) + h(\omega^{-4}) \tag{24}$$

Substituting Eqs.(11), (12) and (14) into Eq.(16), and then integrating the obtained equation from 0 to R, one obtains

$$\tau_{rr}(R) - \tau_{rr}(0+) = -4M(R,k) \tag{25}$$

where

$$M(R,k) = \int_0^R \frac{(\omega^{-7} - \omega^{-1})f' + (\omega^{-5} - \omega)g' + \omega^{-7}h'}{\xi} d\xi \tag{26}$$

in Eq.(26), ω denotes $\omega = \omega(\xi,k) = (1 + k^3/\xi^3)^{1/3}$.

From Eqs.(11) and (14), we obtain

$$\begin{aligned} p(R,k) = {} & 2\left(\omega^{-4}f' - \omega^4 g' + \omega^{-4}h'\right)\big|_{(R,k)} \\ & + 4M(R,k) - \tau_{rr}(0+) \end{aligned} \tag{27}$$

where $r(0+) = k$ and $\tau_{rr}(0+)$ can be determined by the boundary conditions (18) and (19).

Let $R = b$ in Eq.(25), we have

$$\tau_{rr}(0+) = 4M(b,k) + p_0 \left[\frac{b^3}{b^3 + k^3}\right]^{2/3} \tag{28}$$

On multiplication both sides of Eq.(28) by k, it leads to

$$kp_0 = -4k(1 + \frac{k^3}{b^3})^{2/3} M(b,k) \tag{29}$$

In sum, for the prescribed dead load $p_0 > 0$, if there exists values of k satisfying Eq.(29), then $\tau_{rr}(0+)$ can be obtained by Eq.(28). Obviously, for $p_0 > 0$, $k \equiv 0$ is a solution of Eq.(29) and $\tau_{rr}(0+)$ is given by (28), so

$$p(R) = 2f'(3) - 2g'(3) - p_0, \quad r(R) = R \tag{30}$$

are the homogeneous solutions of the radial symmetric deformation of the sphere, while if $k > 0$ is a solution of Eq.(29), we get $\tau_{rr}(0+) = 0$ from Eq.(28), then Eqs.(20) and (27) are the nontrivial solutions of the problem.

It is not difficult to see that Eq.(29) describes exactly the relationship between k andp_0, so we call it **the bifurcation equation of cavity**.

3 Existent conditions and qualitative properties of cavity solutions of Eq.(29)

3.1 Existent conditions of cavity solutions

In what follows, it is convenient to introduce the dimensionless quantity

$$x = k/b \tag{31}$$

Using the relation between $\omega(\xi,k)$ and ξ, we have

$$M(x) = \int_{(1+x^3)^{\frac{1}{3}}}^{\infty} \frac{(\omega - \omega^{-5})f' + (\omega^3 - \omega^{-3})g' - \omega^{-5}h'}{\omega^3 - 1} d\omega \tag{32}$$

By introducing the function

$$\Psi(\omega) = f(\omega^{-4} + 2\omega^2) + g(\omega^4 + 2\omega^{-2}) + h(\omega^{-4}) \tag{33}$$

it is easy to show that

$$\Psi_1 = \frac{d\Psi}{d\omega} = 4\left[(-\omega^{-5} + \omega)f' + (\omega^3 - \omega^{-3})g' - \omega^{-5}h'\right] \tag{34}$$

Thus, the dimensionless form of Eq.(29) can be rewritten as

$$F(x,p_0) = x(1+x^3)^{2/3} \int_{(1+x^3)^{\frac{1}{3}}}^{\infty} \frac{\Psi_1(\omega)}{\omega^3 - 1} d\omega - xp_0 = 0 \tag{35}$$

Obviously, for any prescribed $p_0 > 0$, $x \equiv 0$ is a solution of Eq.(29). Corresponding to it, the homogeneous deformation solutions of the problem are given by Eq.(30), and thus we call $x \equiv 0$ the trivial solution of Eq.(35). Moreover, if there exists $x > 0$ satisfying Eq.(35), i.e.,

$$p_0 = (1+x^3)^{2/3} \int_{(1+x^3)^{\frac{1}{3}}}^{\infty} \frac{\Psi_1(\omega)}{\omega^3 - 1} d\omega \tag{36}$$

this implies a cavity forms in the interior of the sphere, so we call Eq.(36) **the cavity solution**.

Let $x \to 0+$, we then obtain

$$p_{cr} = \int_1^{\infty} \frac{\Psi_1(\omega)}{\omega^3 - 1} d\omega \tag{37}$$

it is called the critical dead-load at which a internal cavity may be initiated.

Since the integral of Eq.(37) is improper, p_{cr} may or may not be finite, and thus cavitation may or may not take place, in other words, p_{cr} depends strictly on the concrete form of the strain energy density. Further, from Eq.(34), it is easy to show that (the details of the formulation can be found in Polignone and Horgan (1993))

$$\left.\frac{d\Psi(\omega)}{d\omega}\right|_{\omega=1} = 0, \left.\frac{d^2\Psi(\omega)}{d\omega^2}\right|_{\omega=1} = 4(a_1 - a_2 + 4a_3) \tag{38}$$

where a_1, a_2, a_3 are elastic constants associated with infinitesimal deformations of the transversely isotropic material. Thus, we know that the right hand of Eq.(37) is finite as $\omega \to 1$ by using the l'Hôpital's rule. On the other hand, from the properties of the integrand, if it is required that p_{cr} is finite, the following expression must be valid as $\omega \to \infty$, i.e.,

$$\Psi(\omega) = O(\omega^{1+\kappa}), (0 < \kappa < 1) \tag{39}$$

From the above analyses, we now present the conditions that f, g and h must satisfy such that p_{cr} is finite.

From the strong convex condition (see Ball(1982)), and the normalization conditions (15), as $\omega \to \infty$, we obtain

$$f(\omega^{-4} + 2\omega^2) = O\left((\omega^{-4} + 2\omega^2)^{\alpha}\right) \tag{40}$$

$$g(\omega^4 + 2\omega^{-2}) = O\left((\omega^4 + 2\omega^{-2})^{\beta}\right) \tag{41}$$

$$h(\omega^{-4}) = O\left((\omega^{-4} - 1)^{\eta}\right) \tag{42}$$

where $1/2 \le \alpha < 3/2, 0 < \beta < 3/4, \eta \ge 2$.

Further, from Eq.(34), we have

$$\frac{d\Psi(1)}{d\omega} = 0, \frac{d^2\Psi(1)}{d\omega^2} = 24f'(3) + 24g'(3) + 16h''(1) \tag{43}$$

So it is required that $f'(3)$,$g'(3)$ and $h''(1)$ must be positive finite values as $\omega \to 1$.

All in all, for the strain energy density (14), if f, g and h respectively satisfy the conditions (40)~(43), one can see that p_{cr} is finite, that is to say, a cavity would form in the sphere associated with the material model (14) as the surface tensile dead load p_0 exceeds the critical value p_{cr}.

3.2 *qualitative properties of cavity solutions*

In this subsection, assume that f, g and h respectively satisfy the conditions (40) $\sim$ (43). From

$$F_{xp_0}(0, p_{cr}) = -1 \tag{44}$$

we obtain a unique bifurcation point $(x, p_0) = (0, p_{cr})$, namely, the cavity solution bifurcates from the trivial solution $x \equiv 0$. Further, at the bifurcation point $(0, p_{cr})$, it is easy to show that

$$F_x(0, p_{cr}) = F_{xx}(0, p_{cr}) = F_{xxx}(0, p_{cr}) = F_{p_0}(0, p_{cr}) = 0 \tag{45}$$

and

$$F_{xxxx}(0, p_{cr}) = 16 \left[p_{cr} - \frac{1}{6} \frac{d^2\Psi(1)}{d\omega^2} \right] \tag{46}$$

Conclusion 1 From the distinguished conditions of the bifurcation equation in singularity theory, one can know that $F(x, p_0)$ is equivalent to the normal form $\pm x^4 + \delta x$ with single-sided constraint conditions in certain neighborhood of the critical point $(0, p_{cr})$. Further, From Eq.(44), we know that there is a unique bifurcation point on the trivial solution of the equation $F(x, p_0) = 0$. As $p_{cr} - \frac{1}{6}\frac{d^2\Psi(1)}{d\omega^2} > 0$ (or < 0), the nontrivial solution of Eq.(35) bifurcates locally to the right (or to the left) from the trivial solution at $(0, p_{cr})$.

4 An example

In this section, assume that the sphere is composed of a class of transversely isotropic Gent-Thomas material models (Gent and Thomas (1958)), in which the corresponding strain energy density function is given by

$$W(I_1, I_2, I_5) = \frac{\mu_1}{2} \left[(I_1 - 3) + \gamma \ln(I_2 - 2) + h(I_5) \right] \tag{47}$$

where$\gamma = \mu_2/\mu_1$, and $\mu_1, \mu_2 > 0$ are material constants in the state of infinitesimal deformations. From the normalization conditions (15) (or (42)), take

$$h(I_5) = h(\omega^{-4}) = a(I_5 - 1)^2 + b(I_5 - 1)^3 \tag{48}$$

where $a, b \geq 0$ are dimensionless parameters which serve as measures of the degree of anisotropic about radial direction of the material about the radial direction. If $a = b = 0$, the corresponding hyper-elastic material is isotropic. If at least value of a and b is nonzero, the corresponding hyper-elastic material is called the transversely isotropic Gent-Thomas material. Maybe, Polignone and Horgan (1993) presented the first paper of the problem of cavity formation and growth for this kind of materials**.** In their work, the strain energy functions of anisotropic materials were discussed in detail**.** Using the notation (21), we have

$$\hat{W}(\omega) = \frac{\mu_1}{2}[(\omega^{-4} + 2\omega^2 - 3) + \gamma \ln(\omega^4 + 2\omega^{-2} - 2) \\ + a(\omega^{-4} - 1)^2 + b(\omega^{-4} - 1)^3] \tag{49}$$

On substitution of Eq.(49) into Eq.(35), **the bifurcation equation of cavity** associated with the transversely isotropic Gent-Thomas material model is given by

$$F(x, p_0) = x(G(x, \gamma, a, b) - p_0) = 0 \tag{50}$$

where

$$G(x, \gamma, a, b) = (1 + x^3)^{2/3} \int_{(1+x^3)^{\frac{1}{3}}}^{\infty} \frac{\hat{W}_1(\omega)}{\omega^3 - 1} d\omega \tag{51}$$

The critical dead load is given by

$$p_{cr}/\mu_1 \doteq 2.5 + 1.2551\gamma + 0.7184a - 0.3789b \tag{52}$$

Moreover, we have

$$F_{xxxx}(0, p_{cr}) \doteq 16\mu_1(0.5 - 0.7449\gamma \\ - 1.9482a - 0.3789b) \tag{53}$$

To further study the qualitative properties of the cavity solution of Eq.(50), by expanding Eq. (51) at $x = 0$, we get the Taylor expansion as follows:

$$G(x, \gamma, a, b) = \mu_1 \left(2.5 + q_1(\gamma, a, b) + q_2(\gamma, a, b)x^3 \\ + o\left(x^4\right)\right) \text{ as } x \to 0+ \tag{54}$$

where

$$q_1(\gamma, a, b) = 1.2551\gamma + 0.7184a - 0.3789b \tag{55}$$

and

$$q_2(\gamma, a, b) = 0.5 - 0.7449\gamma - 1.9482a - 0.3789b \tag{56}$$

4.1 Isotropic case

Let $a = b = 0$ in (49), the material model then reduces to an isotropic one. From Eq.(52), we know that a cavity would form at the center of the sphere composed of the isotropic Gent-Thomas (1958) material model as the surface tensile dead load exceeds the finite critical value. However, another interesting result, which is different from some classical material models, such as the isotropic neo-Hookean material, Ogden material, valanis-Landel material, etc., is obtained in this paper.

Conclusion 2 There exists a value of $\gamma = \mu_2/\mu_1$, written as $\gamma_0 = 0.6712$, as $0 \leq \gamma \leq \gamma_0$, the cavity solution of Eq.(50), i.e., $p_0 = G(x, \gamma, 0, 0)$, bifurcates locally to the right from the trivial solution$x = 0$, however, as $\gamma > \gamma_0$, the cavity solution bifurcates locally to the left from the trivial solution, and there also exists a secondary turning bifurcation point on the cavity solution, which bifurcates locally to the left.

Proof. From Eq.(56), we know that $q_2(\gamma, 0, 0) > 0$ as $0 \leq \gamma \leq \gamma_0$, this means that the cavity solution bifurcates locally to the right near the critical point $(0, p_{cr})$. It can be shown that $G_x(x, \gamma, 0, 0) > 0$ for all $x > 0$, so the cavity solution increases monotonously as $p_0 > p_{cr}$. Whereas, from Eq.(54), we have $G_x(0, \gamma, 0, 0) = G_{xx}(0, \gamma, 0, 0) = 0$ and $G_{xxx}(0, \gamma, 0, 0) = 6q_2(\gamma, 0, 0) < 0$ as $\gamma > \gamma_0$, so we have $G_x(x, \gamma, 0, 0) < 0$ for sufficient small values of $x > 0$, this implies that the cavity solution bifurcates locally to the left near the critical point. On the other hand, it is easy to show that $G_x(x, \gamma, 0, 0) > 0$ for sufficient large values of $x > 0$. Thus we can conclude that there must exist a value x_n, such that $G_x(x_n, \gamma, 0, 0) = 0$. Let $p_n = G(x_n, \gamma, 0, 0)$, we then obtain a secondary turning bifurcation point (x_n, p_n). For different values of γ, curves for $p_0/\mu_1 \sim x$ are shown in Fig.1.

4.2 Anisotropic case

For the solid sphere composed of the transversely isotropic Gent-Thomas model (49), namely, at least value of a and b is nonzero, cavitation will be very interesting.

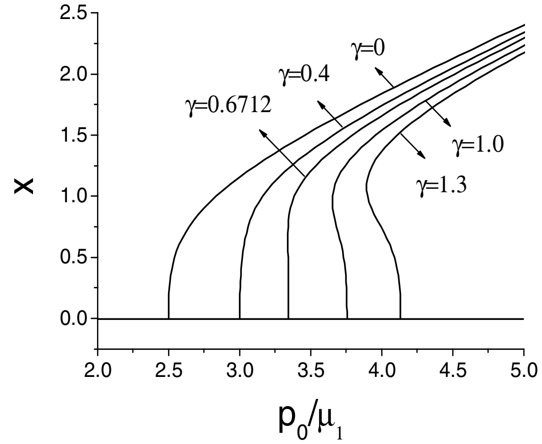

Figure 1 : Solution curves of Eq.(50) in isotropic case for different values of γ.

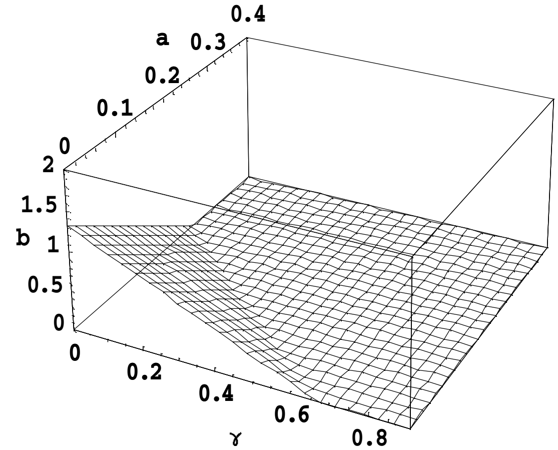

Figure 2 : Regions partitioned by $q_2(\gamma, a, b) = 0$.

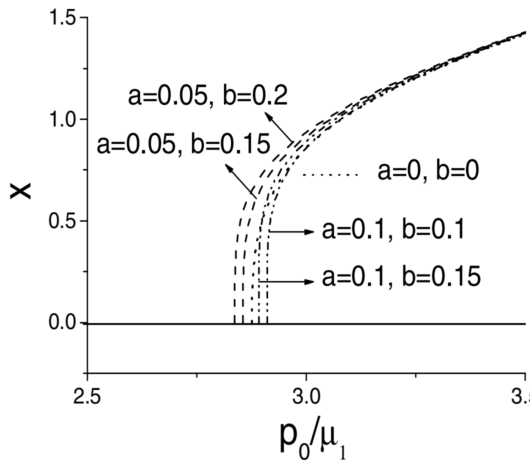

Figure 3 : Solution curves of Eq.(50) in Ω_1 as $\gamma = 0.3$

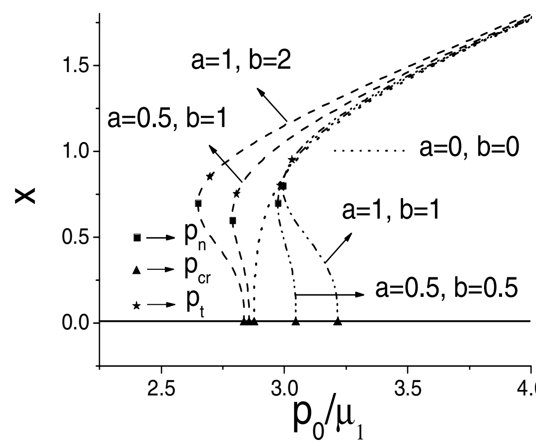

Figure 4 : Solution curves of Eq.(50) in Ω_2 as $\gamma = 0.3$

First of all, let $a = b = 0$ in (52), $\hat{p}_{cr}/\mu_1 = 2.5 + 1.2551\gamma$ then denotes the critical dead-load associated with an initial cavity centered at the origin of the sphere, in which the sphere is composed of isotropic Gent-Thomas model (1958). So we have

Conclusion 3 For the given γ, as $0.7184a - 0.3789b < 0$ (or > 0), from Eq.(52) we know that $p_{cr}/\mu_1 <$(or $>$)$\hat{p}_{cr}/\mu_1$, that is to say, a cavity forms in the interior of the sphere composed of the transversely isotropic Gent-Thomas model is earlier (or later) than that for the isotropic material model.

Next we divide the first quadrant of the parameter space (γ, a, b) into two regions by the plane $q_2(\gamma, a, b) = 0$, as shown in Fig.2. The regions are denoted by

$$\Omega_1 = \{(\gamma, a, b) \,|\, 0 \leq \gamma \leq 0.6712, 0 \leq a \leq 0.2566, \\ 0 \leq b \leq 1.3196, q_2(\gamma, a, b) < 0\} \tag{57}$$

$$\Omega_2 = \{(\gamma, a, b) \,|\, \gamma \geq 0, a \geq 0, b \geq 0, q_2(\gamma, a, b) > 0\} \tag{58}$$

Conclusion 4 As the parameter (γ, a, b) belongs to the region Ω_1, the cavity solution of Eq.(50), i.e., $p_0 = G(x, \gamma, a, b)$, bifurcates locally to the right from the trivial solution $x = 0$. However, as the parameter (γ, a, b) belongs to the region Ω_2, the cavity solution bifurcates locally to the left from the trivial solution, and there also exists a secondary turning bifurcation point on the cavity solution, which bifurcates locally to the left.

The proof is similar to that of the Conclusion 2.

As the parameter (γ, a, b) belongs to the region Ω_1 and Ω_2, curves for $p_0/\mu_1 \sim x$ are respectively shown in Fig.3 and Fig.4.

Moreover, in Fig.3 and Fig.4, curves denoted by dashed (or dash dot dot line) also shows that a cavity forms in the interior of the sphere composed of transversely isotropic Gent-Thomas material earlier (or later) than that of the isotropic material.

4.3 Stability of solutions

We now carry out an energy analysis to examine the stability of the solutions of Eq.(50).

For the transversely isotropic Gent-Thomas material model (49), the total energy corresponding to any equilibrium configuration of the body is given by

$$\begin{aligned} E(c) &= 4\pi\int_0^b WR^2 dR - 4\pi b^2 p_0[r(b)-b] \\ &= 4\pi k^3 \int_{(1+k^3/b^3)^{1/3}}^{\infty} \frac{\omega^2 \hat{W}(\omega)}{(\omega^3-1)^2} d\omega \\ &\quad - 4\pi b^2 p_0[(b^3+k^3)^{1/3} - b] \end{aligned} \tag{59}$$

where $\hat{W}(\omega)$ is given by Eq.(49). The first term in Eq.(59) is the total strain energy while the second term is the work done by the tensile dead load. The dimensionless form of Eq.(59) is denoted by

$$\begin{aligned} \Lambda(x) &= \frac{E(x)}{(4/3)\pi A^3 \mu} = \frac{3x^3}{\mu}\int_{(1+x^3)^{1/3}}^{\infty} \frac{\omega^2 \Sigma(\omega)}{(\omega^3-1)^2} d\omega \\ &\quad - 3[(1+x^3)^{1/3} - 1] \end{aligned} \tag{60}$$

For the trivial solution $x = 0$, we have

$$\Lambda(0) = 0 \tag{61}$$

For the cavity solution, i.e., $x > 0$, $\Lambda(x)$ is given by Eq.(60). Further, it is not difficult to obtain

$$\begin{aligned} &\Lambda(0+) = \Lambda'(0+) = \Lambda''(0+) = 0, \\ &\Lambda'''(0+) = 6(p_{cr} - p)/\mu_1 \end{aligned} \tag{62}$$

where Λ' denotes derivative with respect to x.

Obviously, $\Lambda(0)$is a local minimum as $p_{cr} < p_0$, thus the trivial solution $x = 0$ here is stable. Moreover, $\Lambda(0)$ is a local maximum as $p_{cr} > p_0$, the trivial solution $x = 0$ here is unstable.

As the parameter (γ, a, b) belongs to the region Ω_1, from the Conclusion 3, we know that the cavity solution solution (also called the nonzero equilibrium solution) to the right from the trivial solution at the critical point $(0, p_{cr})$, and $G_x(x, \gamma, a, b) > 0$ for any $x > 0$. It can be shown that $\Lambda_x(x) = 0$ and $\Lambda_{xx}(x) > 0$, namely, $\Lambda(x)$ takes the minimum as $p_0 > p_{cr}$. Thus we can say that the cavity solution, which increases monotonically, is stable.

As the parameter (γ, a, b) belongs to the region Ω_2, however, there exists a secondary turning bifurcation point on the cavity solution (see Fig.4), this implies that the number of the equilibrium solutions depends on the values of p_0 in the following way: Eq.(50) has only a stable trivial solution $x = 0$ as $p_0 < p_n$; there are exactly three solutions as $p_0 \in (p_n, p_{cr})$, that is, a stable trivial solution, and two additional nonzero equilibrium solutions, written as x_1 and x_2, respectively, in which $x_1 < x_n < x_2$. It is not difficult to show that $\Lambda(x)$ takes the local maximum and minimum respectively at x_1 and x_2, and thus x_1 and x_2 are respectively unstable and stable; as $p_0 > p_{cr}$, there are two solutions, namely, an unstable trivial solution and a stable cavity solution. In particular, there two stable equilibrium states as $p_0 \in (p_n, p_{cr})$. It is easy to show that there exists a value of p_0, written as p_t, the trivial solution makes the total energy minimized locally as $p_0 \in (p_n, p_t)$ and thus it is actual stable; the cavity solution makes the total energy maximized locally as $p_0 \in (p_t, p_{cr})$, and thus it is actual stable. Example energy curves are show in Fig.5 and Fig.6.

On the other hand, in region Ω_2, from the above analyses, we know that the solutions of Eq.(50) can also occur catastrophic phenomena. As shown in Fig.4, if the surface tensile dead load p_0 changes quasi-statically from small to large, then for $p_0 < p_t$, no cavity forms in the interior of the sphere; but when $p_0 > p_t$, the state of the sphere breaks suddenly, namely, a relatively larger cavity occurs. Whereas, if the dead load p_0 changes quasi-statically from large to small, only when reduces lower thanp_t, the cavity radius reduces suddenly to zero.

4.4 Growth of micro-void

To better understand the physical implications of the preceding bifurcation problem, here we consider the growth

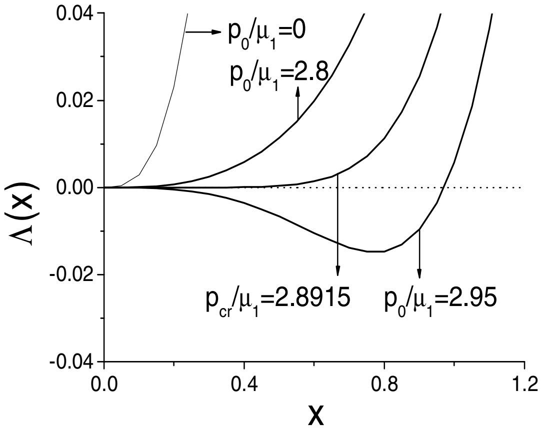

Figure 5 : Energy curves of the sphere in Ω_1 as (γ ,a,b) = (0.3,0.1,0.15)

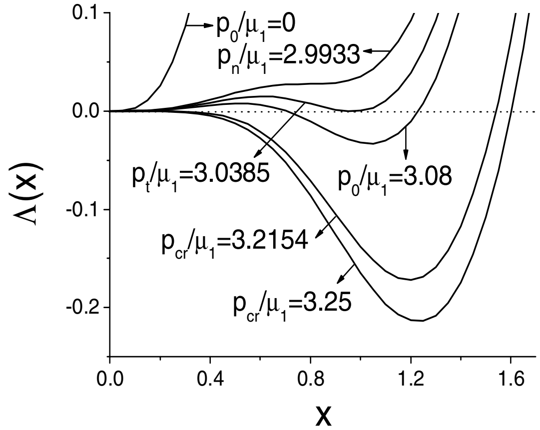

Figure 6 : Energy curves of the sphere in Ω_2 as (γ ,a,b) = (0.3,1,1)

of a small pre-existing void under the surface tensile dead-load.

Assume that there exists a micro-void with radius ε at the center of the sphere in its undeformed configuration. The sphere is subjected to a prescribed uniform radial tensile dead-load at its outer surface.

The governing equations and the outer boundary condition of the finite deformation problem are the same as those in ***Subsection 2.1***, only the inner boundary condition becomes

$$\tau_{rr}(\varepsilon)=0 \tag{63}$$

Using the similar method in ***Subsection 2.2***, we then obtain the expression between the tensile dead-load and the growth of the pre-existing micro-void

$$p_0=(1+\frac{k^3}{b^3})^{2/3}\int_{(1+\frac{k^3}{b^3})^{\frac{1}{3}}}^{(1+\frac{k^3}{\varepsilon^3})^{\frac{1}{3}}}\frac{\hat{W}_1(\omega)}{\omega^3-1}d\omega \tag{64}$$

in this case, $k\geq 0$ denotes the increasing value of the initial void with radius ε along withp_0.

Let

$$k/b=\rho,\varepsilon/b=\delta \tag{65}$$

the dimensionless expression of Eq.(64) is then given by

$$p_0=H(\rho,\delta,\gamma,a,b) \tag{66}$$

where

$$H(\rho,\delta,\gamma,a,b)=(1+\rho^3)^{2/3}\int_{(1+\rho^3)^{\frac{1}{3}}}^{(1+\frac{\rho^3}{\delta^3})^{\frac{1}{3}}}\frac{\hat{W}_1(\omega)}{\omega^3-1}d\omega \tag{67}$$

The qualitative analyses of Eq.(66) are similar to those of Eq.(51). In fact, as the parameter (γ,a,b) belongs to the region Ω_1 (given by Eq.(57)), from the Conclusion 4, it is easy to show that $H_\rho(\rho,\delta,\gamma,a,b)>0$ for any $\rho>0$ and $\delta>0$. However, as the parameter (γ,a,b) belongs to the region Ω_2 (given by Eq.(58)), the number of the real root with respect to ρ of the equation $H_\rho(\rho,\delta,\gamma,a,b)=0$ depends on the values of δ. It is not difficult to show that there exists a value of δ, written as δ_0, such that the equation $H_\rho(\rho,\delta,\gamma,a,b)=0$ has only one real root as $\delta=\delta_0$, has no real root as $\delta>\delta_0$, and has two real roots as $\delta<\delta_0$.

We now carry out two numerical examples to show the effect of material parameters on the growth of pre-existing void in the sphere.

Take$(\gamma,a,b)=(0.3,0.1,0.15)$ and $(\gamma,a,b)=(0.3,2,3)$, curves of $p_0/\mu_1\sim\delta+\rho$ are respectively shown in Fig.7 and Fig.8 for different values of δ. Example energy curves of the sphere with micro-void are show in Fig.9 as $(\gamma,a,b)=(0.3,2,3)$.

As seen from Fig.7 and Fig.8, as the parameter belongs to the region Ω_1 for any $\delta>0$ and the region Ω_2 for $\delta>\delta_0$, the micro-void grows continuously with respect to the surface tensile dead-loadp_0. However, as the parameter belongs to the region Ω_2 for $\delta<\delta_0$, it can be

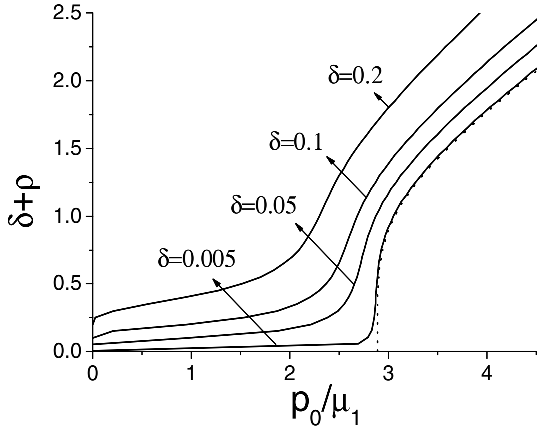

Figure 7 : Growth curves of micro-void in the sphere as (γ ,a,b) = (0.3,0.1,0.15)

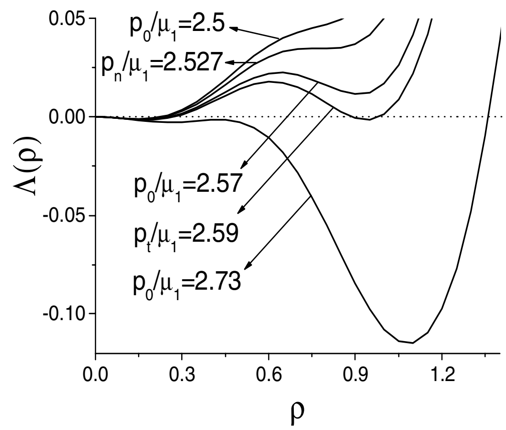

Figure 9 : Energy curves of the sphere with micro-void as (γ ,a,b) = (0.3,2,3)

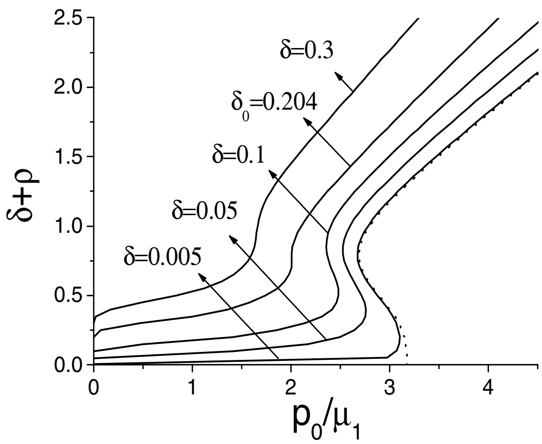

Figure 8 : Growth curves of micro-void in the sphere as (γ ,a,b) = (0.3,2,3)

shown that the growth of the micro-void is discontinuous and a jump may occur by using the minimal potential energy principle (as shown in Fig.8). Thus, the bifurcation model can be interpreted as describing sudden rapid growth of a pre-existing micro-void as was first shown in Horgan and Abeyaratne (1986).

5 Conclusions

In this work, we first give out the condition of cavitation in the interior of the solid sphere composed of a class of incompressible transversely isotropic nonlinearly elastic materials, in which the corresponding strain energy density may be viewed as the generalized forms of some known material models, such as the neo-Hookean material model, the Mooney-Rivlin material model, the Gent-Thomas material model, etc. In Conclusion 1, we present some interesting qualitative properties of the bifurcation equation, such as the normal form, the condition of cavity formation, and the bifurcation direct of the cavity solution, and so on. As an example, we examine the effect of material anisotropy on cavity formation and growth in the incompressible transversely isotropic Gent-Thomas solid sphere, see Conclusion 3 and 4. By using the minimal potential energy principle, we examine the actual stable state and catastrophe of the equilibrium solutions. In particular, the cavity solution associated with the isotropic Gent-Thomas sphere can bifurcate locally to the left and there exists a secondary turning point on the cavity solution that bifurcates locally to the left, which is quite different from other isotropic nonlinear elastic spheres, see Conclusion 2. To further understanding the physical implications of the preceding bifurcation problem, we finally examine the growth of a pre-existing micro-void in the sphere. We also carry out the corresponding numerical figures simultaneously.

It is worth pointing out that, via introducing the quadratic and cubic term (serve as measures of the degree of anisotropic about radial direction of the material) into the strain energy function, the qualitative properties of cavitation can be described completely, such as, a cavity forms in the transversely isotropic Gent- Thomas sphere earlier or later than that in the isotropic Gent- Thomas sphere; the cavity solution bifurcates locally to the right of to the left, and so on. On the other hand, according to singularity theory, if some higher-order terms about radial direction of the material are introduced into the strain-energy function (48), the qualitative properties of the solutions of the bifurcation equation of cavity are also similar.

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