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The method of fundamental solutions for eigenproblems with Laplace and biharmonic operators

S.Yu. Reutskiy[1]

Abstract: In this paper a new meshless method for eigenproblems with Laplace and biharmonic operators in simply and multiply connected domains is presented. The solution of an eigenvalue problem is reduced to a sequence of inhomogeneous problems with the differential operator studied. These problems are solved using the method of fundamental solutions. The method presented shows a high precision in simply and multiply connected domains. The results of the numerical experiments justifying the method are presented.

keyword: Method of fundamental solutions, Membranes and Plates, Free vibration problem

1 Introduction

The goal of this paper is to present a new numerical technique for solution of the following eigenvalue problems:

$$\nabla^2 w + k^2 w = 0,\ \mathbf{x} \in \Omega \subset R^2, \qquad (1)$$
$$B_1[w] = 0,\ \mathbf{x} \in \partial\Omega$$

and

$$\nabla^4 w - k^4 w = 0,\ \mathbf{x} \in \Omega \subset R^2, \qquad (2)$$
$$w = 0,\ B_2[w] = 0,\ \mathbf{x} \in \partial\Omega.$$

Here Ω is a simply or multiply connected domain of interest with boundary $\partial\Omega$. The boundary operator in (1) $B_1[...]$ will be considered of the two types: the Dirichlet $B_1[w] = w$ and of the Neumann type $B_1[w] = \partial w/\partial n$; for biharmonic operator in (2), $B_2[w] = \partial w/\partial n$ or $B_2[w] = \partial^2 w/\partial n^2$. As a mechanical application, this corresponds to recovering the free vibration frequencies of membranes and plates. Such problems often arise in engineering applications.

The usual approach for eigenvalue problems with a self-adjoint operator is to use the Rayleigh minimal principle. In particular, the stationary points of the functional

$$R(w) = \int_\Omega \|\nabla w\|^2 d\Omega / \int w^2 d\Omega.$$

coincide with eigenfunctions of the Laplace operator. See [Courant (1943); Courant and Hilbert (1953); Morse and Feshbach (1953); Strang and Fix (1973)] for more details and references. Then, using an approximation for w with finite number of free parameters, one gets the same problem in a finite-dimensional subspace which can be solved by a standard procedure of linear algebra, e.g., see [Golub and Loan (1996); Strang (1976)]. However, a standard finite differences method can produce good results when dealing with a particular type of shapes defined on rectangular grids, while for other type of shapes the finite element method or the boundary element method are more appropriated. The method of fundamental solutions (MFS) [Fairweather and Karageorghis (1998); Golberg and Chen (1998, 1997)] is the convenient tool in this field. The similar technique is used in the boundary knot method (BKM) [Chen (2005); Chen and Tanaka (2002)]. Unlike the MFS, it employs nonsingular general solutions as the basis functions to avoid the fictitious boundary outside the physical domain.

In the framework of the boundary methods a general approach to solving these problems is as follows. First, using an integral representation of w in the BEM, or an approximation over fundamental solutions in MFS, one gets a homogeneous linear system $A(k)\,\mathbf{q} = \mathbf{0}$ with matrix elements depending on the wave number k. The determinant of this matrix must be zero to obtain the non-trivial solution:

$$\det[A(k)] = 0 \qquad (3)$$

This equation must be investigated analytically or numerically to get the eigenvalues. This technique is described in [Karageorghis (2001); Chen, Lin, Kuo, and Chyuan (2001); Chen, Liu, and Hong (2003); Chen, Chen, Chen,

[1] Laboratory of Magnetohydrodynamics, Timurovtzev, 29 D, ap. 51, 61142, Kharkov, Ukraine

Lee, and Yeh (2004); Chen, Chen, and Lee (2005)] with more details. In the two latest papers there is a complete bibliography on the subject considered. In [Alves and Antunes (2005)] the MFS based procedure is applied to eigenproblems in general simply connected shapes.

The method presented in this article uses the same MFS boundary technique. This is a mathematical model of physical measurements when the resonance frequencies of a system are determined by the amplitude of response to some external excitation. As a result, e.g., instead of (1) we solve a sequence of *inhomogeneous* boundary value problems (BVP):

$$\nabla^2 w + k^2 w = f(\mathbf{x}),\ \mathbf{x} \in \Omega \subset R^2, \qquad (4)$$
$$B_1[w] = 0,\ \mathbf{x} \in \partial\Omega,$$

where f describes some source placed outside the solution domain. Let $F(k)$ be some norm of the solution w. As it will be shown below, this function of k has sharp maximums at the eigenvalues and, under some conditions can be used for their determining. Certainly such behaviour of $F(k)$ near the eigenvalues is a consequence of (3). Techniques of numerical solution of linear BVPs like (4) are well developed. It should be emphasized that any Helmholtz (or biharmonic) equation solver can be used in the framework of the method presented. However, the MFS technique seems to be a more suitable one for this goal in the case of an arbitrary domain.

The outline of this paper is as follows: for the sake of simplicity we begin by describing the 1D case in Section 2. In Section 3, we present the algorithm of MFS in application to problem (1). Here we present numerical examples to illustrate the method presented for simply and multiple connected domains. In Section 4, the same technique is described in application to problem (2). Some generalization of the technique and the fields of its development are discussed in Section 5.

2 One-dimensional eigenproblem

To illustrate the method presented let us consider the wave equation [Morse and Feshbach (1953)]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (5)$$

with the Dirichlet conditions at the endpoints of the interval $[0,1]$, i.e., $u(0,t) = u(1,t) = 0$. Considering the free harmonic vibrations $u(x,t) = e^{-ikt} w(x)$, we get the following 1D $Sturm - Liouville$ problem on the interval $[0,1]$:

$$\frac{d^2 w}{dx^2} + k^2 w = 0, \quad w(0) = w(1) = 0. \qquad (6)$$

The well known solution is: $k_n = n\pi$, $w_n = \sin(n\pi x)$, $n = 1, 2, ..., \infty$.

Following the boundary approach, let us consider the fundamental solution

$$\Psi(x, \xi, k) = \frac{1}{2k} \exp(ik|x - \xi|), \qquad (7)$$

which satisfies the homogeneous equation everywhere except the singular point $x = \xi$. A general solution of the homogeneous equation in the interval $[0,1]$ can be written in the form:

$$w = q_1 \Psi(x, \xi_1, k) + q_2 \Psi(x, \xi_2, k).$$

Here ξ_1, ξ_2 are two source points placed outside the solution domain $[0,1]$, e.g., $\xi_1 < 0$, $\xi_2 > 1$; q_1, q_2 are free parameters. Using the boundary conditions $w(0) = w(1) = 0$, one gets the linear system:

$$A(k)\,\mathbf{q} = \begin{cases} q_1 e^{(-ik\xi_1)} + q_2 e^{(ik\xi_2)} = 0 \\ q_1 e^{(-ik(1-\xi_1))} + q_2 e^{(ik(\xi_2 - 1))} = 0 \end{cases}$$

The wave numbers k_n can be determined from the condition: $\det[A(k)] = 0$. After simple transforms we get: $\exp(2ik) = 1$, or $k = n\pi$. Thus, MFS gets the exact solution. Note that in multidimensional cases such computations are time consuming and not so simple.

As it is mentioned above, the method suggested is a mathematical model of physical measurements, when a mechanical or acoustic system is excited by an external source and resonance frequencies can be determined using an increase of amplitude of oscillations near these frequencies. So, instead of (6) we solve the inhomogeneous problem:

$$\frac{d^2 w}{dx^2} + k^2 w = f(x), \quad w(0) = w(1) = 0. \qquad (8)$$

The general solution can be written in the form:

$$w = q_1 \Psi(x, \xi_1, k) + q_2 \Psi(x, \xi_2, k) + w_p. \qquad (9)$$

When the excitation is performed by the point source with the same wave number k which is placed at the point

ξ_0 outside the solution domain, then $f(x) = i\delta(x-\xi_0)$ and the particular solution is:

$$w_p = \Psi(x,\xi_0,k) = \frac{1}{2k}\exp(ik|x-\xi_0|). \tag{10}$$

Using again the same homogeneous boundary conditions $w(0) = w(1) = 0$, now we get an *inhomogeneous* linear system for each k. Let us introduce the norm of the solution as

$$F(k) = \sqrt{\frac{1}{N}\sum_{n=1}^{N}|w(x_n)|^2},\ F_d(k) = F(k)/F(k_0), \tag{11}$$

where $F_d(k)$ is the dimensionless value, k_0 is a reference wave number and the points x_n are randomly distributed in $[0,1]$. In all the calculations presented in this section we use $N = 5$. This function characterizes the value of the response of the system to the outer excitation. Note that the right hand side f corresponding to (10) equals to zero identically inside $[0,1]$ and BVP (8) has a unique solution $w = 0$ for all k except $k = k_n$ - eigenvalues when the solution is not unique.

In Fig. 1 the value F_d as a function of the wave number k is shown.

The graph contains large sharp peaks at the positions of eigenvalues. Generally speaking, this resonance curve can be used to determine the eigenvalues in the same way as $\det[A(k)]$ in the technique described above. However, the graph $F_d(k)$ is a non smooth one, as it is shown in the lower part of Fig. 1 with more details. This can be explained by the following reasons. Problem (8), (9) with w_p given in (10) has the *exact solution* $q_1 = 0$, $q_2 = -e^{ik(\xi_0-\xi_2)}$ and so the total solution $w(x) = 0$, for $x \in [0,1]$. So, here we have $F(k)$ which is equal to zero with machine precision accuracy when k is far from eigenvalues; $F(k)$ grows considerably in a neighbourhood of the eigenvalues when the linear system becomes almost degenerated. And a smoothing procedure is needed to get an appropriate curve which is convenient for applying an optimization procedure. The following two smoothing procedures are used in the paper.

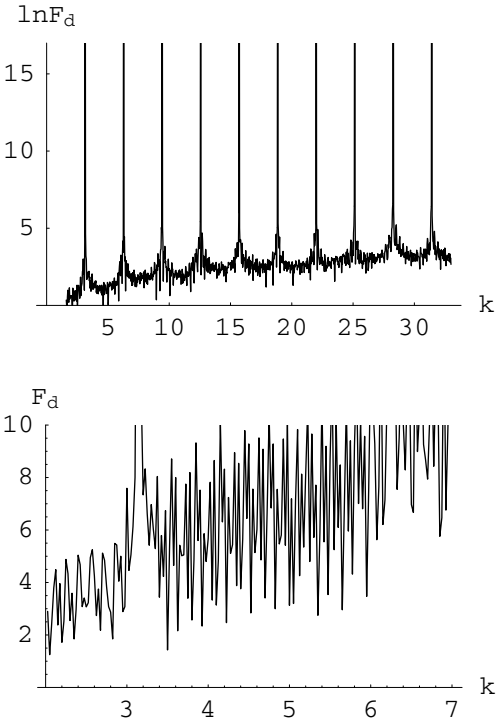

Figure 1 : Resonance curve in 1D eigenproblem.

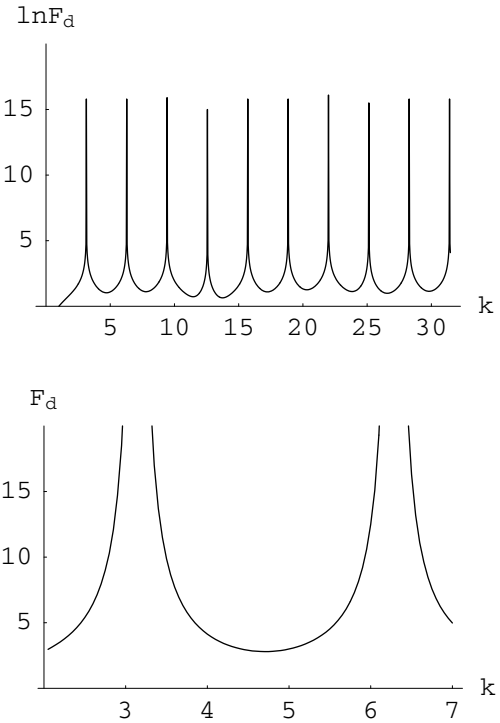

Figure 2 : Resonance curve in 1D eigenproblem. Smoothing by a friction term.

2.1 smoothing by a dissipative term

The first procedure consists of introducing an additional dissipative term in the governing equation. And instead of (8) we consider the problem:

$$\frac{d^2w}{dx^2} + \left(k^2 + i\varepsilon k\right)w = f, \quad w(0) = w(1) = 0. \tag{12}$$

Here ε is a small parameter. This means that the initial wave equation (5) is changed by the equation $\partial^2_{tt}u = \partial^2_{xx}u - \varepsilon\partial_t u$ which describes vibration of a homogeneous string with a friction [Morse and Feshbach (1953)]. The fundamental solution is:

$$\Psi(x,\xi,k,\varepsilon) = \frac{1}{2\chi}\exp(i\chi|x-\xi|),$$
$$\chi = \sqrt{k^2+i\varepsilon k}. \tag{13}$$

Now the system $w(0)=0,\ w(1)=0$ with w_p given in (10) has a unique non zero solution for all real k. The resonance curve corresponding to $\varepsilon = 10^{-6}$ is shown in Fig. 2.

Now this is a smooth curve with separated maximums at the positions of eigenvalues. To find the eigenvalues we use the following algorithm through the paper. Let us look for the eigenvalues on the interval $[a,b]$ Then:

(A)

step 0: *Choose* $h>0$;
if $F(a) > F(a+h)$ *goto step 5;*
step 1: $x_1 = a$; $F1 = F(x_1)$;
step 2: $x_2 = x_1 + h$; $F2 = F(x_2)$;
if $x_2 > b$ *stop;*
step 3: *if* $F2 > F1$ *then* $[F1 = F2; x_1 = x_2]$;
goto step 2;
step 4: *find the maximum point* x_m *of* $F(x)$
on $[x_2 - 2h, x_2]$;
step 5: $x_1 = a$; $F1 = F(x_1)$;
step 6: $x_2 = x_1 + h$; $F2 = F(x_2)$;
if $x_2 > b$ *stop;*
step 7: *if* $F2 < F1$ *then* $[F1 = F2; x_1 = x_2;$
goto step 6$]$;
else goto step 2.

Note that any univariate optimization procedure can be used at *step 4*. In particular, we applied Brent's method based on a combination of parabolic interpolation and bisection of the function near to the extremum(see [Press, Teukolsky, Vetterling, and Flannery (2002)], Ch. 10 and [Brent (1973)], Ch. 5). The step is taken $h = 0.01$ through the paper. The data placed in Tab. 1 are obtained by applying this technique with $\varepsilon = 0.1,\ 10^{-3},\ 10^{-6}$. Other parameters are: $\xi_1 = -0.5, \xi_2 = 1.5, \xi_0 = 5$. Here we place the relative errors

$$e_r = |k_i - k_i^{(ex)}|/k_i^{(ex)} \tag{14}$$

in the calculation of the first five eigenvalues.

Table 1 : One dimensional eigenproblem. The relative errors in calculations of the eigenvalues. Smoothing by the friction term.

$k_i^{(ex)}$	$\varepsilon = 0.1$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-6}$
π	$1.3\cdot10^{-4}$	$1.3\cdot10^{-8}$	$1.7\cdot10^{-12}$
2π	$3.2\cdot10^{-5}$	$3.1\cdot10^{-9}$	$1.6\cdot10^{-12}$
3π	$1.4\cdot10^{-5}$	$1.4\cdot10^{-9}$	$1.5\cdot10^{-12}$
4π	$7.9\cdot10^{-6}$	$7.9\cdot10^{-10}$	$9.7\cdot10^{-13}$
5π	$5.1\cdot10^{-6}$	$5.0\cdot10^{-10}$	$9.0\cdot10^{-13}$

2.2 smoothing by a shift of wave numbers

The second smoothing technique is as following. Let us introduce the constant shift Δk between the exciting source and the studied mode, i.e., instead of (10), we take the particular solution in the form:

$$w_p = \Psi(x,\xi_0,k+\Delta k)$$
$$= \frac{1}{2(k+\Delta k)}\exp(i(k+\Delta k)|x-\xi_0|). \tag{15}$$

Now the linear system $w(0) = w(1) = 0$ has non zero solutions for all k except the eigenvalues k_n when the system becomes degenerate. However, due to iterative procedure of solution and rounding errors we never solve the system with the *exact* k_n. And we observe degeneration of the system as a considerable growth of the solution in a neighbourhood of the eigenvalues. The resonance curve corresponding to $\Delta k = 1$ is shown in Fig. 3.

Some results of the calculations we got using the second smoothing technique are presented in Tab. 2. The values ξ_1, ξ_2, ξ_0 are the same as above.

Below we will name these procedures as $\varepsilon-$procedure and $k-$procedure.

Table 2 : One dimensional eigenproblem. The relative errors in calculation of the eigenvalues. Smoothing by shift of the wave numbers.

$k_i^{(ex)}$	$\triangle k = 0.1$	$\triangle k = 1$	$\triangle k = 10$
π	$1.4\cdot10^{-11}$	$9.1\cdot10^{-12}$	$7.8\cdot10^{-12}$
2π	$5.8\cdot10^{-13}$	$3.5\cdot10^{-12}$	$5.5\cdot10^{-12}$
3π	$6.4\cdot10^{-12}$	$1.3\cdot10^{-12}$	$3.5\cdot10^{-12}$
4π	$3.3\cdot10^{-13}$	$2.8\cdot10^{-12}$	$2.3\cdot10^{-12}$
5π	$5.3\cdot10^{-12}$	$3.5\cdot10^{-12}$	$5.9\cdot10^{-13}$

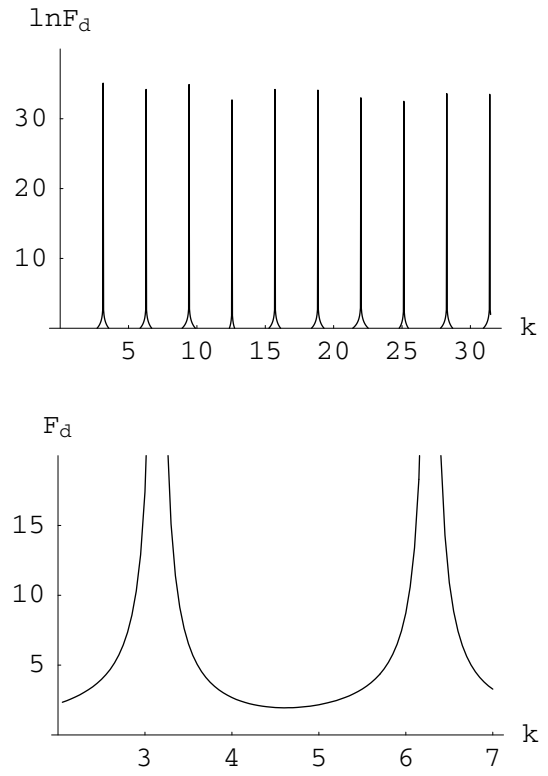

Figure 3 : Resonance curve in 1D eigenproblem. Smoothing by a shift between the wave numbers.

3 Helmholtz eigenproblem

Applying the MFS to problem (4) we look for an approximation solution in the form of a linear combination:

$$w(\mathbf{x}|\mathbf{q}) = w_p(\mathbf{x}) + \sum_{n=1}^{N} q_n \Phi_n(\mathbf{x}), \tag{16}$$

where w_p is the particular solution corresponding to f, and the trial functions

$$\Phi_n(\mathbf{x}) = H_0^{(1)}(k|\mathbf{x}-\zeta_n|) \tag{17}$$

satisfy the homogeneous PDE. This is the so-called Kupradze basis [Kupradze (1967)]. The singular points ζ_n are located outside the solution domain. The free parameters q_n should be chosen to satisfy the boundary condition $B_1[w(\mathbf{x}|\mathbf{q})] = 0,\ \mathbf{x} \in \partial\Omega$. In particular the unknowns q_n are taken as a solution of the minimization problem:

$$\min_{\mathbf{q}} \sum_{i=1}^{N_c} \left\{ B_1[w_p(\mathbf{x}_i)] + \sum_{n=1}^{N} q_n B_1[\Phi_n(\mathbf{x}_i)] \right\}^2 \tag{18}$$

Here the points $\mathbf{x}_i$, $i = 1,...,N_c$ are distributed uniformly on the boundary. We take N_c approximately twice as large as the number of free parameters N. The problem is solved by the standard least squares procedure. Note that we get (18) as a result of discretization of the integral condition:

$$\min_{w} \int_{\partial\Omega} \{B_1[w(\mathbf{x}|\mathbf{q})]\}^2 ds$$

More details of this technique can be found, e.g., in [Fairweather and Karageorghis (1998); Golberg and Chen (1998)].

As a particular solution corresponding to the exciting source we take the same fundamental solution

$$w_p(\mathbf{x}) = \Phi_{ex}(\mathbf{x}, \zeta_{ex}, k) \equiv H_0^{(1)}(k|\mathbf{x}-\zeta_{ex}|) \tag{19}$$

with ζ_{ex} placed outside the solution domain.

When dealing with problems in multiply connected domains, the same trial functions can be used. And the source points should be placed also inside each hole. As an alternative approach one can use the special trial functions associated with each hole:

$$\begin{aligned} &\Phi_{s,1}(\mathbf{x}) = H_0^{(1)}(kr_s), \\ &\Phi_{s,2n+1}(\mathbf{x}) = H_n^{(1)}(kr_s)\cos n\theta_s, \\ &\Phi_{s,2n}(\mathbf{x}) = H_n^{(1)}(kr_s)\sin n\theta_s. \end{aligned} \tag{20}$$

Here $r_s = |\mathbf{x}-\mathbf{x}_s|, \theta_s$ is the local polar coordinate system with the origin at $\mathbf{x}_s$. This is so-called Vekua basis [Vekua (1957); Hafner (1990)], or multipole expansion. It is proven that every regular solution of the 2D Helmholtz equation in a domain with holes can be approximated with any desired accuracy by linear combinations of such functions if the origin $\mathbf{x}_s$ of a multipole is inside every hole. In this case instead of (16) we use:

$$\begin{aligned} w(\mathbf{x}|\mathbf{q},\mathbf{p}_s) = w_p(\mathbf{x}) &+ \sum_{n=1}^{N} q_n \Phi_n(\mathbf{x}) \\ &+ \sum_{s=1}^{S}\sum_{m=1}^{M} p_{s,m}\Psi_{s,m}(\mathbf{x}), \end{aligned} \tag{21}$$

where S is the number of holes and M is the number of terms in each multipole expansion.

When the $\varepsilon-$smoothing procedure is applied, then instead of (4) we consider the problem:

$$\begin{aligned} &\nabla^2 w_h + \left(k^2 + i\varepsilon k\right) w_h = 0,\ \mathbf{x} \in \Omega, \\ &B_1[w_h(\mathbf{x})] = -B_1[w_p(\mathbf{x})],\ \mathbf{x} \in \partial\Omega. \end{aligned} \tag{22}$$

with some small $\varepsilon > 0$. Note that this problem has a unique nonzero solution for all real k. Then the trial functions (17) should be also modified:

$$\Phi_n(\mathbf{x}) = H_0^{(1)}(\chi\,|\mathbf{x}-\zeta_n|),$$
$$\chi(k,\varepsilon) = \sqrt{k^2 + i\varepsilon k}. \qquad (23)$$

Applying the $k-$procedure we modify the particular solution which should be taken in the form:

$$w_p(\mathbf{x}) = \Phi_{ex}(\mathbf{x},\widetilde{k}) \equiv H_0^{(1)}(\widetilde{k}\,|\mathbf{x}-\zeta_{ex}|),\ \widetilde{k} = k + \Delta k. \qquad (24)$$

3.1 numerical examples

Here the results of the numerical experiments are given to illustrate the method presented. In all the cases considered below the resonance curve $F(k)$ is computed using N_t testing points $\mathbf{x}_{t,l} \in \Omega$: $F(k) = \sqrt{1/N_t \sum_{l=1}^{N_t} |w(\mathbf{x}_{t,l})|^2}$. In all the calculations we use 15 testing points distributed inside Ω with the help of RNUF generator of pseudorandom numbers from the Microsoft IMSL Library. To get the eigenvalues we look for the maxima of $F(k)$ using the Brent's procedure mentioned.

Example 1) A circular domain with the radius $r = 1$ subjected to Dirichlet or Neumann boundary condition is considered. The exciting source is placed at the position $\zeta_{ex} = (5,5)$; the singular points ζ_n of the fundamental solutions (17) are located on the circle with the radius $R = 2$. The results shown in Tab. 3 correspond to $\varepsilon = 10^{-6}$. Here we place the relative errors (14) in the calculation of the first 5 eigenvalues. The line $-$ in a cell indicates that the solution process failed with these parameters. The exact eigenvalues $k_i^{(ex)}$ are the roots of the equation $J_n(k) = 0$ (Dirichlet) or $J_n'(k) = 0$ (Neumann).

Example 2) The role of the parameter ε is shown in Tab. 4. We solve the same problem as above with Dirichlet condition. Here we fix the number of free parameters $N = 25$ and vary the parameter ε. The parameter ε coarsens the system. For a large ε we can calculate all the eigenvalues $k_i,\ i = 1,...,10$ but the precision is not very high. When ε decreases, the precision in determining of k_i increases but it fails for large i.

The figures Fig. 4, Fig. 5, Fig. 6 correspond to the data placed in Tab. 4. For $\varepsilon = 10^{-2}$ the resonance peaks are spread because the friction. When ε decreases the peaks become more sharp and narrow. Besides for $\varepsilon = 10^{-8}$ the peaks corresponding to $k_i, i > 2$ are placed on the rising sharply part of the resonance curve. As a result the algorithm (A) 'jumps over' the eigenvalues and one should decrease the step parameter h to capture the maxima. As it is shown in Tab. 4, for $\varepsilon = 10^{-8}$ the algorithm finds k_1 and k_2 with $h = 0.01$. When h is reduced to 0.001 then the algorithm also gives the eigenvalues k_3 and k_4. To get $k_i, i = 1,...,10$ one should take $h = 0.0001$. However, the algorithm becomes highly expansive in the CPU time.

Example 3) Next, we consider the case when Ω is the unit square with the same Dirichlet or Neumann boundary condition.This problem has an analytical solution:

Table 3 : Circular domain with the radius $r = 1$. The relative errors in calculations of the eigenvalues. $\varepsilon-$procedure; $\varepsilon = 10^{-6}$.

Dirichlet condition			
i	$N = 15$	$N = 20$	$N = 25$
1	$8\cdot10^{-11}$	$8\cdot10^{-12}$	$7\cdot10^{-12}$
2	$2\cdot10^{-3}$	$5\cdot10^{-11}$	$2\cdot10^{-11}$
3	$3\cdot10^{-9}$	$1\cdot10^{-9}$	$1\cdot10^{-9}$
4	$2\cdot10^{-3}$	$4\cdot10^{-11}$	$1\cdot10^{-11}$
5	$6\cdot10^{-7}$	$2\cdot10^{-3}$	$1\cdot10^{-9}$
Neumann condition			
1	$2\cdot10^{-9}$	$2\cdot10^{-9}$	$2\cdot10^{-9}$
2	$4\cdot10^{-9}$	$2\cdot10^{-9}$	$2\cdot10^{-9}$
3	$9\cdot10^{-12}$	$1\cdot10^{-11}$	$6\cdot10^{-12}$
4	$7\cdot10^{-8}$	$9\cdot10^{-10}$	$8\cdot10^{-10}$
5	$2\cdot10^{-6}$	$6\cdot10^{-10}$	$3\cdot10^{-10}$

Table 4 : Circular domain with the radius $r = 1$. Dirichlet condition. The relative errors in calculations of the eigenvalues. $\varepsilon-$procedure with varying ε, $N = 25$.

i	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$
1	$6.4\cdot10^{-6}$	$6.0\cdot10^{-10}$	$7.3\cdot10^{-12}$	$4.9\cdot10^{-11}$
2	$2.4\cdot10^{-6}$	$1.9\cdot10^{-10}$	$2.0\cdot10^{-11}$	$4.3\cdot10^{-11}$
3	$3.2\cdot10^{-6}$	$1.4\cdot10^{-9}$	$1.0\cdot10^{-9}$	–
4	$9.0\cdot10^{-7}$	$1.6\cdot10^{-10}$	$1.3\cdot10^{-11}$	–
5	$1.1\cdot10^{-6}$	$1.6\cdot10^{-9}$	$1.4\cdot10^{-9}$	–
6	$6.5\cdot10^{-7}$	$1.5\cdot10^{-10}$	–	–
7	$4.9\cdot10^{-7}$	$4.8\cdot10^{-10}$	–	–
8	$2.7\cdot10^{-6}$	$1.1\cdot10^{-9}$	–	–
9	$4.9\cdot10^{-7}$	$5.9\cdot10^{-9}$	–	–
10	$5.2\cdot10^{-6}$	–	–	–

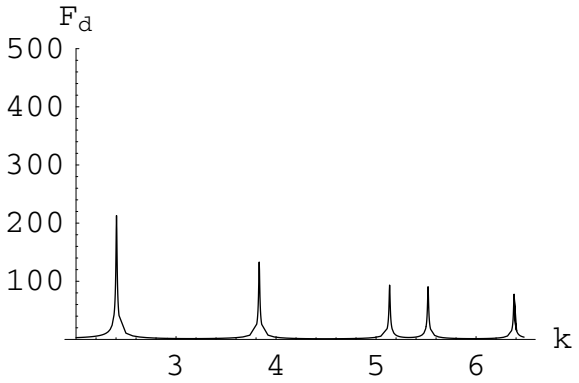

Figure 4 : Circular membrane with the radius 1. Dirichlet conditions. ε - procedure with $\varepsilon = 10^{-2}$.

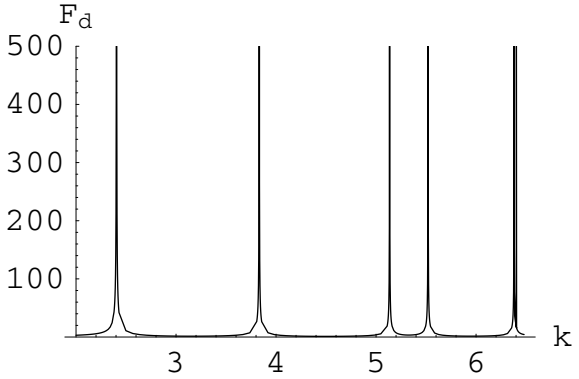

Figure 5 : Circular membrane with the radius 1. Dirichlet conditions. ε - procedure with $\varepsilon = 10^{-4}$.

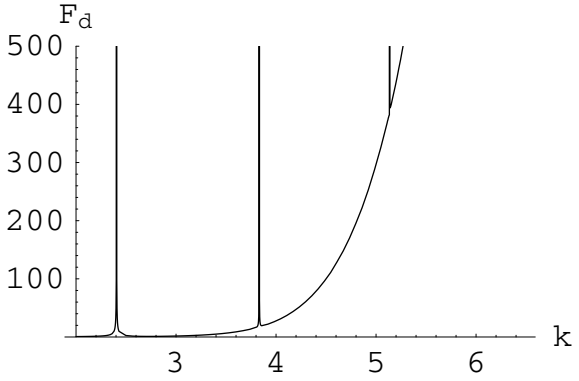

Figure 6 : Circular membrane with the radius 1. Dirichlet conditions. ε - procedure with $\varepsilon = 10^{-8}$.

$k^{(ex)} = \pi\sqrt{i^2+j^2}$, $i,j = 1,2,..$(Dirichlet condition) and $i,j = 0,1,2,..$(Neumann condition). In Tab. 5, we show the results of calculation of the first 5 eigenvalues with $\varepsilon = 10^{-6}$. The placement of the singular points ζ_n and the exciting source are the same as above.

Example 4) For the next example, we consider an annular case of the doubly connected domain between the two circles: $\Omega = \left\{(x_1,x_2) \mid r_1^2 \le x_1^2+x_2^2 \le r_2^2\right\}$ The inner and outer radii of an annular domain are $r_1 = 0.5$ and $r_2 = 2$, respectively. We take Dirichlet condition on the outer boundary and Neumann on the inner one. The singular points are distributed at the circles with the radii $a = 5$(outside the domain) and $b = 0.3$ (inside the hole). The number of the singular points on each auxiliary contour is equal to N. The exciting source is placed at $\zeta_{ex} = (10,10)$. In Tab. 6 we present the relative errors (14) in calculation of the first 5 eigenvalues of the problem described with $\varepsilon = 10^{-5}$. The values $k_i^{(ex)}$ are obtained numerically as the roots of the equation: $J_n'(r_1k)Y_n(r_2k) - J_n(r_2k)Y_n'(r_1k) = 0$.

Table 5 : Square with the side $a = 1$. The relative errors in calculations of the eigenvalues. ε−procedure; $\varepsilon = 10^{-6}$.

Dirichlet condition			
i	$N = 15$	$N = 20$	$N = 25$
1	$1\cdot10^{-6}$	$3\cdot10^{-8}$	$1\cdot10^{-9}$
2	$1\cdot10^{-5}$	$9\cdot10^{-8}$	$1\cdot10^{-8}$
3	$8\cdot10^{-5}$	$3\cdot10^{-8}$	$8\cdot10^{-9}$
4	$3\cdot10^{-4}$	$1\cdot10^{-6}$	$3\cdot10^{-9}$
5	$3\cdot10^{-3}$	$4\cdot10^{-5}$	$6\cdot10^{-7}$
Neumann condition			
1	$4\cdot10^{-7}$	$5\cdot10^{-8}$	$8\cdot10^{-12}$
2	$1\cdot10^{-6}$	$3\cdot10^{-8}$	$3\cdot10^{-9}$
3	$4\cdot10^{-5}$	$1\cdot10^{-7}$	$3\cdot10^{-10}$
4	$1\cdot10^{-4}$	$6\cdot10^{-6}$	$5\cdot10^{-9}$
5	$5\cdot10^{-4}$	$2\cdot10^{-5}$	$6\cdot10^{-7}$

Table 6 : Annular domain. The relative errors in calculations of the eigenvalues. ε−procedure; $\varepsilon = 10^{-5}$.

i	$k_i^{(ex)}$	$N = 15$	$N = 20$	$N = 25$
1	1.3339427880	$5\cdot10^{-11}$	$2\cdot10^{-11}$	$2\cdot10^{-11}$
2	1.7388632616	$6\cdot10^{-8}$	$7\cdot10^{-12}$	$5\cdot10^{-12}$
3	2.4753931967	–	$7\cdot10^{-11}$	$8\cdot10^{-12}$
4	3.1645013237	–	$7\cdot10^{-8}$	$5\cdot10^{-11}$
5	3.2899912986	–	–	$7\cdot10^{-11}$

Example 5) In this example, doubly connected region with the inner region of vanishing maximal dimension is considered. The geometry of the problem is the same as in *Example 3*. However, here we consider the case of very small inner holes. In particular, we take $r_1 = 10^{-1}, 10^{-2}, 10^{-3}$ with the same fixed $r_2 = 2$. Now, the Kupradze type basis functions (17) are unfit to approximate the solution in a neighbourhood of the hole. Here we use a combined basis which includes the trial functions (17) with the singular points placed on an auxiliary circular contour outside the solution domain and a multipole expansion with the origin at the center of the hole. Thus, we look for an approximate solution in the form:

$$w(\mathbf{x}|\mathbf{q},\mathbf{p}) = w_p(\mathbf{x}) + \sum_{n=1}^{N} q_n \Phi_n(\mathbf{x}) + \sum_{m=1}^{M} p_m \Psi_m(\mathbf{x}).$$

The data presented in Tab. 7, Tab. 8, Tab. 9 correspond to the number of sources on the outer auxiliary circular contour $N = 50$. The number of terms in multipole expansion M varies from $M = 11 (r_1 = 10^{-1})$ to $M = 5 (r_1 = 10^{-3})$. The exciting source is placed at the position $\zeta_{ex} = (10, 10)$. We use the $k-$procedure with the shift $\Delta k = 1$. We would like to draw the readers' attention to the fact that the method presented can separate very close eigenvalues: $k_4^{(ex)} = 3.1900833197$ and $k_5^{(ex)} = 3.2126996563$(see data corresponding to $r_1 = 10^{-1}$). Here the step in the algorithm (A) is taken $h = 0.001$. The detailed discussion of Vecua basis for Helmholtz equation can be found in [Hafner (1990)].

Table 7 : Circle with a small hole. Dirichlet boundary condition. The outer radius: $r_2 = 2$. The relative errors in calculation of the first ten eigenvalues. $k-$procedure with $\Delta k = 1$.

	$r_1 = 0.1, N = 50, M = 11$	
i	$k_i^{(ex)}$	e_r
1	1.5322036536	$1.9 \cdot 10^{-8}$
2	1.9301625755	$5.8 \cdot 10^{-9}$
3	2.5680354360	$1.6 \cdot 10^{-9}$
4	3.1900833197	$1.3 \cdot 10^{-11}$
5	3.2126996563	$7.4 \cdot 10^{-9}$
6	3.5522743165	$3.7 \cdot 10^{-10}$
7	3.7941712382	$1.2 \cdot 10^{-11}$
8	4.2101115868	$9.0 \cdot 10^{-12}$
9	4.3857419081	$4.4 \cdot 10^{-12}$
10	4.8805392651	$1.0 \cdot 10^{-11}$

Table 8 : Circle with a small hole. Dirichlet boundary condition. The outer radius: $r_2 = 2$. The relative errors in calculation of the first ten eigenvalues. $k-$procedure with $\Delta k = 1$.

	$r_1 = 0.01, N = 50, M = 7$	
i	$k_i^{(ex)}$	e_r
1	1.3709447159	$2.5 \cdot 10^{-8}$
2	1.9160005377	$5.4 \cdot 10^{-9}$
3	2.5678112121	$1.6 \cdot 10^{-9}$
4	2.9632630840	$5.3 \cdot 10^{-9}$
5	3.1900809955	$2.9 \cdot 10^{-12}$
6	3.5082790790	$2.3 \cdot 10^{-12}$
7	3.7941712738	$1.0 \cdot 10^{-9}$
8	4.2086222910	$7.6 \cdot 10^{-12}$
9	4.3857419733	$1.1 \cdot 10^{-11}$
10	4.5543927267	$1.3 \cdot 10^{-9}$

4 Eigenproblems with biharmonic operator

According to the technique proposed, instead of (2) let's consider BVP

$$\nabla^4 w - k^4 w = f,\ \mathbf{x} \in \Omega \subset R^2, \qquad (25)$$
$$w = 0,\ B_2[w] = 0,\ \mathbf{x} \in \partial\Omega.$$

In application to this problem, the MFS technique is similar to the one considered in the previous section. The trial functions now are of the two types: the fundamental solutions of the Helmholtz operator $\nabla^2 + k^2$:

$$\Phi_n^{(1)}(\mathbf{x}) = H_0^{(1)}(k|\mathbf{x} - \zeta_n|) \qquad (26)$$

considered above and the fundamental solutions of the modified Helmholtz operator $\nabla^2 - k^2$:

$$\Phi_n^{(2)}(\mathbf{x}) = H_0^{(1)}(ik|\mathbf{x} - \zeta_n|) = -i\frac{2}{\pi} K_0(k|\mathbf{x} - \zeta_n|), \qquad (27)$$

where $H_0^{(1)}$ is the Hankel function and K_0 is the modified Bessel function of the second kind and of order zero. So, an approximate solution is sought in the form of the linear combination:

$$w(\mathbf{x}|\mathbf{q}_1,\mathbf{q}_2) = w_p(\mathbf{x}) + \sum_{n=1}^{N} q_{1,n} \Phi_n^{(1)}(\mathbf{x}) + \sum_{n=1}^{N} q_{2,n} \Phi_n^{(2)}(\mathbf{x}). \qquad (28)$$

where $w_p(\mathbf{x})$ is a particular solution corresponding to the external source. We take it in the same form as in the

Table 9 : Circle with a small ole. Dirichlet boundary condition. The outer radius: $r_2 = 2$. The relative errors in calculation of the first ten eigenvalues. $k-$procedure with $\Delta k = 1$.

	$r_1 = 0.001, N = 50, M = 5$	
i	$k_i^{(ex)}$	e_r
1	1.3148533741	$2.0 \cdot 10^{-8}$
2	1.9158544900	$5.4 \cdot 10^{-9}$
3	2.5678111892	$1.5 \cdot 10^{-9}$
4	2.8883437835	$2.8 \cdot 10^{-9}$
5	3.1900809955	$1.1 \cdot 10^{-10}$
6	3.5077982552	$3.0 \cdot 10^{-11}$
7	3.7941712738	$1.2 \cdot 10^{-11}$
8	4.2086221329	$5.9 \cdot 10^{-12}$
9	4.3857419733	$1.2 \cdot 10^{-12}$
10	4.4650868082	$3.6 \cdot 10^{-10}$

Table 10 : A circular plate with the radius: $r = 1$;the relative errors in calculation of the first eight eigenvalues. $k-$procedure, $\Delta k = 0.1$.

	$w = \partial w/\partial n = 0$		
i	$N = 20$	$N = 25$	$N = 30$
1	$3 \cdot 10^{-9}$	$3 \cdot 10^{-9}$	$3 \cdot 10^{-9}$
2	$7 \cdot 10^{-7}$	$3 \cdot 10^{-9}$	$2 \cdot 10^{-9}$
3	$1 \cdot 10^{-5}$	$8 \cdot 10^{-8}$	$6 \cdot 10^{-10}$
4	$6 \cdot 10^{-10}$	$6 \cdot 10^{-10}$	$6 \cdot 10^{-10}$
5	$7 \cdot 10^{-5}$	$9 \cdot 10^{-7}$	$1 \cdot 10^{-8}$
6	$1.2 \cdot 10^{-5}$	$1 \cdot 10^{-7}$	$2 \cdot 10^{-9}$
7	$9.4 \cdot 10^{-4}$	$8 \cdot 10^{-6}$	$8 \cdot 10^{-8}$
8	$1.3 \cdot 10^{-4}$	$2 \cdot 10^{-6}$	$2 \cdot 10^{-8}$
	$w = \partial^2 w/\partial n^2 = 0$		
1	$2 \cdot 10^{-9}$	$3 \cdot 10^{-9}$	$3 \cdot 10^{-9}$
2	$5 \cdot 10^{-5}$	$3 \cdot 10^{-6}$	$1 \cdot 10^{-7}$
3	$3 \cdot 10^{-4}$	$2 \cdot 10^{-5}$	$8 \cdot 10^{-7}$
4	$4 \cdot 10^{-9}$	$5 \cdot 10^{-9}$	$4 \cdot 10^{-9}$
5	$2 \cdot 10^{-3}$	$9 \cdot 10^{-5}$	$4 \cdot 10^{-6}$
6	$3 \cdot 10^{-4}$	$1 \cdot 10^{-5}$	$6 \cdot 10^{-7}$
7	$9 \cdot 10^{-3}$	$3 \cdot 10^{-4}$	$2 \cdot 10^{-5}$
8	$9 \cdot 10^{-2}$	$6 \cdot 10^{-5}$	$3 \cdot 10^{-6}$

previous section, i.e. (19). The free parameters are determined from the boundary conditions.

We apply the same ε and k smoothing procedure. When the $\varepsilon-$procedure is applied the governing equation should be replaced by the following one:

$$\nabla^4 w(\mathbf{x}) - \left(k^4 + i\varepsilon k^2\right) w(\mathbf{x}) = f \tag{29}$$

and so the arguments of the trial functions $\Phi_n^{(1)}(\mathbf{x})$, $\Phi_n^{(2)}(\mathbf{x})$ should be modified. Applying the $k-$procedure we modify the external source and take it in the form (24).

4.1 numerical examples

Example 6) A circular plate with the radius $r = 1$ subjected to the boundary conditions: a) $w = \partial w/\partial n = 0$ (clamped boundary) and b) $w = \partial^2 w/\partial n^2 = 0$ is considered. The exciting source is placed at the position $\zeta_{ex} = (5,5)$; the singular points ζ_n of the fundamental solutions (26), (27) are located on the circle with the radius $R = 2$. Remark that now the number of free parameters is $2N$. The data presented in Tab. 10 are obtained using $k-$procedure with $\Delta k = 0.1$. Here we place the relative errors (14). The exact eigenvalues $k_i^{(ex)}$ are the roots of the equation $J_n'(k) I_n(k) - J_n(k) I_n'(k) = 0$ (conditions a)) or $J_n''(k) I_n(k) - J_n(k) I_n''(k) = 0$ (conditions b)).

Example 7) Next, we consider a square plate with the side $a = 1$ subjected to the boundary conditions $w = \partial^2 w/\partial n^2 = 0$. This problem has an analytical solution: $k^{(ex)} = \pi\sqrt{i^2 + j^2}$, $i, j = 1, 2, ..$ The results placed in Tab. 11 are obtained using $k-$procedure with $\Delta k = 0.1$.

Example 8) A rectangular 1.2×0.9 plate subjected to the boundary conditions $w = \partial w/\partial n = 0$ (clamped boundary) is considered. The results placed in Tab. 12 are obtained using $k-$procedure with $\Delta k = 0.1$. In this case, the analytic solution is not available. The results obtained in [Chen, Chen, Chen, Lee, and Yeh (2004)] and [Kang and Lee (2001)] are used for comparison. These data are placed in the last two columns of the table. Note that using $\varepsilon-$procedure with $\varepsilon = 0.01$ and $N = 56$, we get the following eigenvalues: $k_1 = 5.95263, k_2 = 7.70983, k_3 = 9.12854, k_4 = 10.27133, k_5 = 11.96763, k_6 = 12.49617$

5 Concluding remarks

In this paper, a new meshfree method for eigenproblems with Laplace and biharmonic operators is proposed. This is a mathematical model of physical measurements, when a mechanical or acoustic system is excited by an external source and resonance frequencies can be determined using the growth of amplitude of oscillations near these

Table 11 : A square plate. The relative errors in calculation of the first six eigenvalues. $k-$procedure, $\Delta k = 0.1$.

i	$N=20$	$N=25$	$N=30$	$N=40$
1	$3.3\cdot10^{-6}$	$2.0\cdot10^{-7}$	$1.8\cdot10^{-8}$	$1.7\cdot10^{-8}$
2	$6.9\cdot10^{-4}$	$1.5\cdot10^{-5}$	$3.2\cdot10^{-8}$	$1.7\cdot10^{-8}$
3	–	$7.9\cdot10^{-5}$	$3.7\cdot10^{-6}$	$1.7\cdot10^{-8}$
4	–	$9.2\cdot10^{-5}$	$5.5\cdot10^{-7}$	$1.7\cdot10^{-8}$
5	–	$4.5\cdot10^{-2}$	$1.5\cdot10^{-5}$	$2.2\cdot10^{-8}$
6	–	–	$1.4\cdot10^{-3}$	$1.2\cdot10^{-7}$

Table 12 : A rectangular plate 1.2×0.9 with clamped boundary.

i	$N=35$	$N=42$	$N=49$	I	II
1	5.9515	5.9529	5.9527	5.952	5.952
2	7.7125	7.7116	7.7104	7.703	7.703
3	9.1333	9.1319	9.1316	9.129	9.131
4	9.9466	9.9510	9.9493	9.947	9.955
5	10.2692	10.2717	10.2742	10.266	10.27
6	11.9501	11.9552	11.9565	11.95	11.95
7	12.3849	12.3719	12.3710	–	–

Table 13 : The BKM solution. Circular domain with Dirichlet conditions. The relative errors in calculations of the eigenvalues. $k-$procedure; $\Delta k = 0.1$.

i	$N=10$	$N=14$	$N=20$	$N=30$
1	$2\cdot10^{-4}$	$2\cdot10^{-6}$	$4\cdot10^{-9}$	$7\cdot10^{-9}$
2	$3\cdot10^{-4}$	$4\cdot10^{-7}$	$1\cdot10^{-10}$	$1\cdot10^{-8}$
3	–	$9\cdot10^{-5}$	$2\cdot10^{-8}$	$1\cdot10^{-8}$
4	–	–	$4\cdot10^{-7}$	$4\cdot10^{-9}$
5	–	–	$1\cdot10^{-6}$	$8\cdot10^{-9}$

frequencies. The method shows a high precision in simply and multiply connected domains. The idea can be extended quite simply to the 3D case.

The method presented is based on the MFS solution of the problem. However, it can be combined with other boundary techniques. The BKM mentioned in Section 1 seems to be perspective in this connection. For example, if the BKM is applied to Helmholtz equation, the approximation solution is looked for in the form:

$$w(\mathbf{x}|\mathbf{q}) = w_p(\mathbf{x}) + \sum_{n=1}^{N} q_n J_0(k|\mathbf{x}-\zeta_n|)$$

cf. (16). Here the source points ζ_n can be placed inside the solution domain.

To test BKM in the framework of the method presented we solve the same problem as the one described in *Example 1* with Dirichlet condition. The half of the source points $\zeta_n,\ n=1,...,1/2N$ are placed uniformly on the boundary $\partial\Omega$. The rest source points $\zeta_n,\ n=1/2N+1,...,N$ are distributed inside Ω with the help of the generator of pseudorandom numbers. The data presented in Tab. 13 are obtained using $k-$procedure with $\Delta k = 0.1$. The parameters of the exciting source are the same as above in *Example 1*.

It should be noted that the BKM and the MFS, as well as the all methods of the Trefftz type in general, have a narrow field of application. It is restricted by the cases when there exists a representative set of known exact solutions of PDEs under consideration, i.e. by the problems posed by linear PDEs with constant coefficients. See, however, [Reutskiy (2002)], where a Trefftz type technique is developed for PDEs with varying coefficients.

Besides the Trefftz type techniques produce the systems of equations with unsymmetric fully populated matrices. As a result, the MFS is highly ill conditioned. In some cases one can overcomes this drawback by the use of matrices of the special block circulant structure and an efficient matrix decomposition technique [Tsangaris, Smyrlis, and Karageorghis (2004)].

However, taking in mind further applications of the method presented in the paper to eigenproblems with PDEs of general type in irregular domains, one should combine it with meshless methods based on the local approximation of the solution like the Meshless Local Petrov-Galerkin Method [Atluri (2004), Han and Atluri (2003), Han and Atluri (2004)]. The comparison between global and local approximation, e.g. BEM and FEM, and they combination see in [Grannell and Atluri (1978)].

Comparing the method with the technique based on computations of the determinant of the system, the following circumstances should be taken into account. Since the MFS is highly ill conditioned, the determinant is very small. Indeed, let us consider again the same eigenvalue problem which is described in *Example 1* , i.e. Helmholtz equation in the circle with the radius 1 and Dirichlet boundary condition. We take the number of the sources N equal to the number of the collocation points on the boundary. Thus, we get a square matrix of the problem

Table 14 : Circular domain with Dirichlet conditions. The number of the source points $N = 30$; $\varepsilon-$procedure.

	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-6}$	
i	e_r	$F(k_i)$	e_r	$F(k_i)$	e_r	$F(k_i)$
1	4×10^{-4}	0.701	4×10^{-10}	0.701	5×10^{-12}	0.701
2	2×10^{-4}	0.652	1×10^{-10}	0.654	6×10^{-11}	0.654
3	9×10^{-5}	0.509	9×10^{-10}	0.516	1×10^{-9}	0.516

$A(k,N)$ and can calculate the determinant $|\det A(k,N)|$. Placing the sources on the circle with radius 2 and taking $k = 1$ we get: $|\det A(1,20)| = 3\times10^{-47}$, $|\det A(1,30)| = 4\times10^{-117}$, $|\det A(1,40)| = 3\times10^{-217}$. The wave number $k = 1$ is not the eigenvalue of the problem. This is the "background" value between extremums and one looks for the minima of $|\det A(k,N)|$ on such background. So, using this technique one operates with values of the order $\sim 10^{-50} - 10^{-500}$, see [Alves and Antunes (2005); Chen, Chen, and Lee (2005)] for more detailed information.

At the same time let us calculate the norm function $F(k,N)$ which is used to obtain the eigenvalues in the method presented. We get for $\varepsilon = 0.0001$: $F(1,20) = 2.13\times10^{-5}$, $F(1,30) = 2.13\times10^{-5}$, $F(1,40) = 2.13\times10^{-5}$. We present the values of the norm function $F(k)$ when k is close to eigenvalue in Tab. 14.

Here the number of the sources is fixed $N = 30$ and the smoothing parameter ε is varied. e_r is the relative error in determining of the approximated eigenvalue k_i and $F(k_i)$ denotes the value of the norm function at this approximated eigenvalue. So, in the framework of the method presented we always deal with the values which can be handled on PC with a single precision.

The method is easy to program and not expensive in the CPU time. The all calculations presented in the paper were performed using 366 MHz PC.

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