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A Three-Dimensional Asymptotic Theory of Laminated Piezoelectric Shells

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Abstract: An asymptotic theory of doubly curved laminated piezoelectric shells is developed on the basis of three-dimensional (3D) linear piezoelectricity. The twenty-two basic equations of 3D piezoelectricity are firstly reduced to eight differential equations in terms of eight primary variables of elastic and electric fields. By means of nondimensionalization, asymptotic expansion and successive integration, we can obtain recurrent sets of governing equations for various order problems. The two-dimensional equations in the classical laminated piezoelectric shell theory (CST) are derived as a first-order approximation to the 3D piezoelectricity. Higher-order corrections as well as the first-order solution can be determined by treating the CST equations at multiple levels in a systematic and consistent way. Several benchmark solutions for various piezoelectric laminates are given to demonstrate the performance of the theory.

keyword: Piezoelectric shells, 3D solutions; piezoelectricity, asymptotic expansion, perturbation, electroelastic analysis

1 Introduction

In recent years, the laminated plates and shells composed of piezoelectric materials were widely used in the engineering applications for sensing and actuation purposes. Exact solutions for the static and dynamic analyses of piezoelectric laminates are important for assessing a variety of the relevant approximate theories and numerical methodologies. Determination of those exact solutions therefore becomes an attractive research subject.

Exact solutions for the static analysis of laminated piezoelectric plates and shells with simple supports were presented by Heyliger (1994, 1997a and 1997b). In Heyliger's analysis, the primary field variables are expanded as Fourier series in the in-surface directions. By means of these doubly Fourier series functions, Heyliger reduced the three-dimensional (3D) basic equations as a set of ordinary differential equations. Exact solutions of the present problems were obtained using the Frobenious method. The coupling effect of electric and elastic fields on the static behavior of piezoelectric laminates was examined. Based on the linear theory of piezoelectricity, Lee and Jiang (1996) presented an exact analysis of coupled electroelastic behavior of piezoelectric plates using the state space approach. Within the framework of 3D piezoelectricity, Cheng et al. (2000) developed an asymptotic theory for anisotropic inhomogeneous and laminated piezoelectric plates. A benchmark problem for the cylindrical bending analysis of a hybrid laminate under specified thermal and electric-potential surface loading was proposed by Tauchert (1997). Comparison of the piezothermoelasticity results obtained from the higher-order shear deformation theory (HSDT) and classical plate theory (CPT) was made. The assessment of classical shell theory (CST) and first-order shear deformation theory (FSDT) for laminated piezoelectric cylindrical shells was also made in the literature [Kapuria et al. (1998)]. Comprehensive reviews of theoretical analysis and numerical modeling for piezoelectric laminates were presented [Gopinathan et al. (2000), Chee et al. (1998)].

Recently, several 3D solutions for the bending and stretching problems of laminated doubly curved shells were presented based on the 3D elasticity [Huang and Tauchert (1992), Fan and Zhang (1992), Bhimaraddi (1993)]. After a close literature survey, however, we found that 3D solutions for doubly curved laminated piezoelectric shells are scarce.

In several papers [Wu et al. (1996a, b), Wu and Chiu (2001, 2002), Wu and Liu (2001), Wu and Chi (2004)], a three-dimensional asymptotic theory was developed for the static, dynamic, buckling and nonlinear analyses of laminated composite shells by means of the method of perturbation. The purpose of this paper is to extend the asymptotic theory to doubly curved laminated piezoelectric shells. Due to the fact that the coupling effect of elas-

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tic and electric fields was involved in the basic equations of 3D piezoelectricity, the formulation is inherently more complicated than that in early papers. Nevertheless, we shall see that the derivation is consistent and the 3D solutions for the problem can be determined by treating the CST equations in a systematic and hierarchic way. The benchmark problems are demonstrated using the present asymptotic formulation.

2 Basic three-dimensional equations

Consider a doubly curved laminated piezoelectric shell as shown in Fig.1. The thickness of the shell is $2h$. A set of the orthogonal curvilinear coordinates $\alpha,\ \beta,\ \zeta$ is located on the middle surface. R_α and R_β denote the curvature radii to the middle surface, a_α *and* a_β are the curvilinear dimensions in α *and* β directions, respectively.

The linear constitutive equations of the piezoelectric material are given by

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\zeta \\ \tau_{\beta\zeta} \\ \tau_{\alpha\zeta} \\ \tau_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\zeta \\ \gamma_{\beta\zeta} \\ \gamma_{\alpha\zeta} \\ \gamma_{\alpha\beta} \end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{32} \\ 0 & 0 & e_{33} \\ e_{14} & e_{24} & 0 \\ e_{15} & e_{25} & 0 \\ 0 & 0 & e_{36} \end{bmatrix} \begin{Bmatrix} E_\alpha \\ E_\beta \\ E_\zeta \end{Bmatrix} \qquad (1)$$

$$\begin{Bmatrix} D_\alpha \\ D_\beta \\ D_\zeta \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & e_{14} & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & e_{25} & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & e_{36} \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\zeta \\ \gamma_{\beta\zeta} \\ \gamma_{\alpha\zeta} \\ \gamma_{\alpha\beta} \end{Bmatrix} + \begin{bmatrix} \eta_{11} & \eta_{12} & 0 \\ \eta_{12} & \eta_{22} & 0 \\ 0 & 0 & \eta_{33} \end{bmatrix} \begin{Bmatrix} E_\alpha \\ E_\beta \\ E_\zeta \end{Bmatrix}, \qquad (2)$$

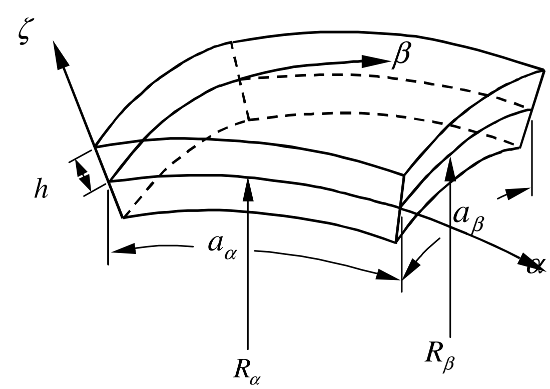

Figure 1 : Dimension and coordinate system for a doubly curved shell.

where $\sigma_\alpha,\ \sigma_\beta,\ \sigma_\zeta, \tau_{\alpha\zeta}, \tau_{\beta\zeta}, \tau_{\alpha\beta}$ *and* $\varepsilon_\alpha,\ \varepsilon_\beta,\ \varepsilon_\zeta, \gamma_{\alpha\zeta}, \gamma_{\beta\zeta}, \gamma_{\alpha\beta}$ denote the stress and strain components, respectively. D_α, D_β, D_ζ and E_α, E_β, $E_\zeta$denote the components of electric displacement and electric field, respectively. c_{ij}, e_{ij} and η_{ij} are the elastic coefficients, piezoelectric coefficients and dielectric coefficients, respectively, relative to the geometrical axes of the shell. The material is regarded to be heterogeneous through the thickness (i.e., $c_{ij}(\zeta)$, $e_{ij}(\zeta)$ and $\eta_{ij}(\zeta)$) and to be the layerwise step functions through the thickness direction.

The kinematic equations in terms of the curvilinear coordinates $\alpha,\ \beta$ *and* ζ are

$$\varepsilon_\alpha = \frac{1}{\gamma_\alpha}\left(\frac{\partial u_\alpha}{\partial \alpha} + \frac{u_\zeta}{R_\alpha}\right), \tag{3a}$$

$$\varepsilon_\beta = \frac{1}{\gamma_\beta}\left(\frac{\partial u_\beta}{\partial \beta} + \frac{u_\zeta}{R_\beta}\right), \tag{3b}$$

$$\varepsilon_\zeta = \frac{\partial u_\zeta}{\partial \zeta}, \tag{3c}$$

$$\gamma_{\beta\zeta} = \frac{1}{\gamma_\beta}\frac{\partial u_\zeta}{\partial \beta} + \frac{\partial u_\beta}{\partial \zeta} - \frac{u_\beta}{\gamma_\beta R_\beta}, \tag{3d}$$

$$\gamma_{\alpha\zeta} = \frac{1}{\gamma_\alpha}\frac{\partial u_\zeta}{\partial \alpha} + \frac{\partial u_\alpha}{\partial \zeta} - \frac{u_\alpha}{\gamma_\alpha R_\alpha}, \tag{3e}$$

$$\gamma_{\alpha\beta} = \frac{1}{\gamma_\alpha}\frac{\partial u_\beta}{\partial \alpha} + \frac{1}{\gamma_\beta}\frac{\partial u_\alpha}{\partial \beta} \tag{3f}$$

in which $\gamma_\alpha = 1+\frac{\zeta}{R_\alpha}$; $\gamma_\beta = 1+\frac{\zeta}{R_\beta}$; $u_\alpha,\ u_\beta$ *and* u_ζ are the displacement components.

The stress equilibrium equations without body forces are given by

$$\gamma_\beta\frac{\partial \sigma_\alpha}{\partial \alpha} + \gamma_\alpha\frac{\partial \tau_{\alpha\beta}}{\partial \beta} + \gamma_\alpha\gamma_\beta\frac{\partial \tau_{\alpha\zeta}}{\partial \zeta} + \left(\frac{2}{R_\alpha} + \frac{1}{R_\beta} + \frac{3\zeta}{R_\alpha R_\beta}\right)\tau_{\alpha\zeta} = 0, \tag{4}$$

$$\gamma_\alpha\frac{\partial \sigma_\beta}{\partial \beta} + \gamma_\beta\frac{\partial \tau_{\alpha\beta}}{\partial \alpha} + \gamma_\alpha\gamma_\beta\frac{\partial \tau_{\beta\zeta}}{\partial \zeta} + \left(\frac{1}{R_\alpha} + \frac{2}{R_\beta} + \frac{3\zeta}{R_\alpha R_\beta}\right)\tau_{\beta\zeta} = 0, \tag{5}$$

$$\gamma_\beta\frac{\partial \tau_{\alpha\zeta}}{\partial \alpha} + \gamma_\alpha\frac{\partial \tau_{\beta\zeta}}{\partial \beta} + \gamma_\alpha\gamma_\beta\frac{\partial \sigma_\zeta}{\partial \zeta} + \left(\frac{1}{R_\alpha} + \frac{1}{R_\beta} + \frac{2\zeta}{R_\alpha R_\beta}\right)\sigma_\zeta - \frac{\gamma_\beta}{R_\alpha}\sigma_\alpha - \frac{\gamma_\alpha}{R_\beta}\sigma_\beta = 0. \tag{6}$$

The charge equation of the piezoelectric material in curvilinear coordinates $\alpha,\ \beta$ *and* ζ is

$$\gamma_\beta\frac{\partial D_\alpha}{\partial \alpha} + \gamma_\alpha\frac{\partial D_\beta}{\partial \beta} + \gamma_\alpha\gamma_\beta\frac{\partial D_\zeta}{\partial \zeta} + \left(\frac{\gamma_\beta}{R_\alpha} + \frac{\gamma_\alpha}{R_\beta}\right)D_\zeta = 0. \tag{7}$$

The relations between the electric field and electric potential in curvilinear coordinates $\alpha,\ \beta$ *and* ζ are

$$E_\alpha = -\frac{1}{\gamma_\alpha}\frac{\partial \Phi}{\partial \alpha}, \quad E_\beta = -\frac{1}{\gamma_\beta}\frac{\partial \Phi}{\partial \beta}, \quad E_\zeta = -\frac{\partial \Phi}{\partial \zeta}, \tag{8}$$

where Φ denotes the electric potential.

The boundary conditions of the problem are specified as follows:

On the lateral surface the transverse load $\overline{q}_\zeta^{\pm}(\alpha,\ \beta)$ and electric potential $\overline{\Phi}_\zeta^{\pm}(\alpha,\ \beta)$ are prescribed,

$$\begin{bmatrix}\tau_{\alpha\zeta} & \tau_{\beta\zeta}\end{bmatrix} = \begin{bmatrix}0 & 0\end{bmatrix} \qquad on \quad \zeta = \pm h, \tag{9a}$$

$$\sigma_\zeta = \overline{q}_\zeta^{\pm}(\alpha,\ \beta) \qquad on \quad \zeta = h, \tag{9b}$$

$$\Phi = \overline{\Phi}_\zeta^{\pm}(\alpha,\ \beta) \qquad on \quad \zeta = -h. \tag{10}$$

The edge boundary conditions require one member of each pair of the following quantities be satisfied:

$$n_1\sigma_\alpha + n_2\tau_{\alpha\beta} = p_1, \quad \text{or} \quad u_\alpha = \overline{u}_\alpha; \tag{11a}$$

$$n_1\tau_{\alpha\beta} + n_2\sigma_\beta = p_2, \quad \text{or} \quad u_\beta = \overline{u}_\beta; \tag{11b}$$

$$n_1\tau_{\alpha\zeta} + n_2\tau_{\beta\zeta} = p_3, \quad \text{or} \quad u_\zeta = \overline{u}_\zeta; \tag{11c}$$

where $p_1,\ p_2,\ p_3$ are applied edge loads; $\overline{u}_\alpha$, $\overline{u}_\beta$ and $\overline{u}_\zeta$ are the prescribed edge displacements; n_1 and $n_2$denote the outward unit normal at a point along the edge.

In addition, the edges are suitably grounded so that the electric potential Φ at the edges are zero and given by

$$\Phi = 0. \tag{12}$$

According to Eqs.(1)-(8), it is listed that there are twenty-two basic equations for the present electroelastic analysis of doubly curved laminated piezoelectric shells. For a three-dimensional analysis, we have to determine the aforementioned set of twenty-two unknown variables that satisfies the basic equations (Eqs.(1)-(8)) in the shell domain, the boundary conditions at the top and bottom surfaces (Eqs. (9)-(10)) and the edge boundary conditions (Eqs. (11)-(12)). Based on the perturbation method, we present an asymptotic formulation for the 3D analysis of laminated piezoelectric shells. The detailed derivation is given as follows.

3 Nondimensionalization

A set of dimensionless coordinates and variables are defined as

$$x=\frac{\alpha}{\sqrt{Rh}},\quad y=\frac{\beta}{\sqrt{Rh}},\quad z=\frac{\zeta}{h}; \tag{13a}$$

$$u=\frac{u_\alpha}{\sqrt{Rh}},\quad v=\frac{u_\beta}{\sqrt{Rh}},\quad w=\frac{u_\zeta}{R}; \tag{13b}$$

$$R_x=\frac{R_\alpha}{R},\quad R_y=\frac{R_\beta}{R}; \tag{13c}$$

$$\sigma_x=\frac{\sigma_\alpha}{Q},\quad \sigma_y=\frac{\sigma_\beta}{Q},\quad \tau_{xy}=\frac{\tau_{\alpha\beta}}{Q}; \tag{13d}$$

$$\tau_{xz}=\frac{\tau_{\alpha\zeta}}{Q\varepsilon},\quad \tau_{yz}=\frac{\tau_{\beta\zeta}}{Q\varepsilon},\quad \sigma_z=\frac{\sigma_\zeta}{Q\varepsilon^2}; \tag{13e}$$

$$D_x=D_\alpha/e\varepsilon,\quad D_y=D_\beta/e\varepsilon,\quad D_z=D_\zeta/e; \tag{13f}$$

$$\phi=\Phi e/RQ\varepsilon^2; \tag{13g}$$

where $\varepsilon^2=h/R$; R, Q and e denote a characteristic length of the shell, a reference elastic moduli and a reference piezoelectric moduli, respectively.

In order to make the previous complicated formulation (Eqs. (1)-(12)) suitable for mathematical treatment, we eliminate the in-surface stresses (σ_α, $\sigma_\beta, \tau_{\alpha\beta}$) and electric displacements (D_α and D_β), the components of strain (ε_α, ε_β, ε_ζ, $\gamma_{\alpha\zeta}$, $\gamma_{\beta\zeta}$, $\gamma_{\alpha\beta}$) and electric field (E_α, E_β, E_ζ) from Eqs.(1)-(8), introduce the set of dimensionless coordinates and variables (Eq.(13)) in the resulting equations, and then express the basic equations as follows:

$$w_{,z}=-\varepsilon^2\mathbf{L}_1\mathbf{u}-\varepsilon^2\tilde{l}_{33}w+\varepsilon^4\tilde{l}_{34}\sigma_z+\varepsilon^2\tilde{l}_{35}D_z \tag{14}$$

$$\mathbf{u}_{,z}=-\mathbf{D}w+\varepsilon^2\mathbf{L}_2\mathbf{u}+\varepsilon^2\mathbf{S}\sigma_s+\varepsilon^4\mathbf{L}_3\sigma_s+\varepsilon^2\mathbf{L}_4\phi, \tag{15}$$

$$D_{z,z}=-\varepsilon^2\mathbf{L}_{11}\mathbf{d}-\varepsilon^2\tilde{l}_{71}D_z, \tag{16}$$

$$\sigma_{s,z}=-\mathbf{L}_5\mathbf{u}-\mathbf{L}_6 w-\varepsilon^2\mathbf{L}_7\sigma_s-\varepsilon^2\left(\gamma_\alpha\gamma_\beta\right)\mathbf{L}_1^T\sigma_z-\mathbf{L}_8 D_z \tag{17}$$

$$\sigma_{z,z}=\mathbf{L}_9\mathbf{u}+\tilde{l}_{63}w-\mathbf{D}^T\sigma_s-\varepsilon^2\mathbf{L}_{10}\sigma_s-\varepsilon^2\tilde{l}_{64}\sigma_z+\tilde{l}_{65}D_z \tag{18}$$

$$\phi_{,z}=-\left(\gamma_\alpha\gamma_\beta\right)\mathbf{L}_8^T\mathbf{u}-\left(1/\gamma_\alpha\gamma_\beta\right)\tilde{l}_{65}w+\varepsilon^2\tilde{l}_{35}\sigma_z+\tilde{l}_{81}D_z \tag{19}$$

where

$$\mathbf{u}=\begin{Bmatrix} u \\ v \end{Bmatrix},\ \mathbf{D}=\begin{Bmatrix} \partial_x \\ \partial_y \end{Bmatrix},$$

$$\mathbf{S}=Q\begin{bmatrix} l_{14} & l_{15} \\ l_{15} & l_{25} \end{bmatrix},\ \sigma_s=\begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix},$$

$$\mathbf{d}=\begin{Bmatrix} D_x \\ D_y \end{Bmatrix},\ \mathbf{L}_1=\begin{bmatrix} \tilde{l}_{31} & \tilde{l}_{32} \end{bmatrix},$$

$$\mathbf{L}_2=\begin{bmatrix} \tilde{l}_{11} & 0 \\ 0 & \tilde{l}_{22} \end{bmatrix},\ \mathbf{L}_3=\begin{bmatrix} \tilde{l}_{16} & \tilde{l}_{17} \\ \tilde{l}_{26} & \tilde{l}_{27} \end{bmatrix},$$

$$\mathbf{L}_4=\begin{bmatrix} \tilde{l}_{18} \\ \tilde{l}_{28} \end{bmatrix},\ \mathbf{L}_5=\begin{bmatrix} \tilde{l}_{41} & \tilde{l}_{42} \\ \tilde{l}_{51} & \tilde{l}_{52} \end{bmatrix},$$

$$\mathbf{L}_6=\begin{bmatrix} \tilde{l}_{43} \\ \tilde{l}_{53} \end{bmatrix},\ \mathbf{L}_7=\begin{bmatrix} \tilde{l}_{44} & 0 \\ 0 & \tilde{l}_{55} \end{bmatrix},$$

$$\mathbf{L}_8=\begin{bmatrix} \tilde{l}_{46} \\ \tilde{l}_{56} \end{bmatrix},\ \mathbf{L}_9=\begin{bmatrix} \tilde{l}_{61} & \tilde{l}_{62} \end{bmatrix},$$

$$\mathbf{L}_{10}=\begin{bmatrix} \frac{z}{R_y}\partial_x & \frac{z}{R_x}\partial_y \end{bmatrix},\ \mathbf{L}_{11}=\begin{bmatrix} \frac{1}{\gamma_\alpha}\partial_x & \frac{1}{\gamma_\beta}\partial_y \end{bmatrix},$$

and $\tilde{l}_{ij}$ are given in Appendix A.

After the elimination process, we obtain the resulting equations (Eqs.(14)-(19)) where all the differential operators on the left-hand sides are with respect to z only, whereas on the right-hand sides are with respect to x and y.

The in-surface stresses and electric displacements are dependent field variables that can be expressed in terms of the primary variables in the following form

$$\sigma_p=\mathbf{B}_1\mathbf{u}+\mathbf{B}_2 w+\varepsilon^2\mathbf{B}_3\sigma_z+\mathbf{B}_4 D_z \tag{20}$$

$$\mathbf{d}=\mathbf{B}_5\sigma_s+\mathbf{B}_6\phi, \tag{21}$$

where

$$\sigma_p=\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix},\ \mathbf{B}_1=\begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} \\ \tilde{b}_{21} & \tilde{b}_{22} \\ \tilde{b}_{31} & \tilde{b}_{32} \end{bmatrix},$$

$$\mathbf{B}_2=\begin{bmatrix} \tilde{b}_{13} \\ \tilde{b}_{23} \\ \tilde{b}_{33} \end{bmatrix},\ \mathbf{B}_3=\begin{bmatrix} \tilde{b}_{14} \\ \tilde{b}_{24} \\ \tilde{b}_{34} \end{bmatrix},$$

$$\mathbf{B}_4=\begin{bmatrix} \tilde{b}_{15} \\ \tilde{b}_{25} \\ \tilde{b}_{35} \end{bmatrix},\ \mathbf{B}_5=\begin{bmatrix} \tilde{b}_{41} & \tilde{b}_{42} \\ \tilde{b}_{51} & \tilde{b}_{52} \end{bmatrix},$$

$$\mathbf{B}_6=\begin{bmatrix} \tilde{b}_{43} \\ \tilde{b}_{53} \end{bmatrix},$$

and $\tilde{b}_{ij}$ are given in Appendix A.

The dimensionless form of boundary conditions of the problem are specified as follows:

On the lateral surface the transverse load and electric potential are prescribed,

$$[\tau_{xz} \quad \tau_{yz}] = [0 \quad 0] \qquad \text{on} \quad \zeta = \pm 1, \tag{22a}$$

$$\sigma_z = \overline{q}_z^{\pm}(\alpha, \beta) \qquad \text{on} \quad \zeta = \pm 1, \tag{22b}$$

$$\phi = \overline{\phi}_z^{\pm}(x, y) \qquad \text{on} \quad z = \pm 1. \tag{23}$$

At the edges one member of each pair of the following quantities is satisfied:

$$n_1\sigma_x + n_2\tau_{xy} = p_{nx}, \quad \text{or} \quad u = \overline{u}; \tag{24a}$$

$$n_1\tau_{xy} + n_2\sigma_y = p_{ny}, \quad \text{or} \quad v = \overline{v}; \tag{24b}$$

$$n_1\tau_{nz} + n_2\tau_{yz} = p_{nz}, \quad \text{or} \quad w = \overline{w}; \tag{24c}$$

In addition,

$$\phi = 0; \tag{25}$$

where
$\overline{q}_z^{\pm} = \overline{q}_\zeta^{\pm}/Q\varepsilon^2$;
$\overline{\phi}_z^{\pm} = \overline{\Phi}_\zeta^{\pm} e/RQ\varepsilon^2$;
$(p_{nx},\ p_{ny},\ p_{nz}) = (p_1/Q, \quad p_2/Q, \quad p_3/Q\varepsilon)$;
$(\overline{u},\ \overline{v}, \quad \overline{w}) = \left(\overline{u}_\alpha/\sqrt{Rh},\ \overline{u}_\beta/\sqrt{Rh}, \quad \overline{u}_\zeta/R\right)$.

4 Asymptotic expansion

Since Eqs.(14)-(19) contain terms involving only even powers of ε, we therefore asymptotically expand the primary variables in the powers ε^2 as given by

$$f(x,\ y,\ z,\ \varepsilon) = f^{(0)}(x,\ y,\ z) + \varepsilon^2 f^{(1)}(x,\ y,\ z) + \varepsilon^4 f^{(2)}(x,\ y,\ z) + \ldots \tag{26}$$

Substituting Eq.(26) into Eqs.(14)-(19) and collecting coefficients of equal powers of ε, we obtain the following sets of recurrence equations.
Order ε^0 :

$$w_{,z}^{(0)} = 0, \tag{27}$$

$$u_{,z}^{(0)} = -L_2 w^{(0)}, \tag{28}$$

$$D_{z,z}^{(0)} = 0, \tag{29}$$

$$\sigma_{s,z}^{(0)} = -\mathbf{L}_5\mathbf{u}^{(0)} - \mathbf{L}_6 w^{(0)} - \mathbf{L}_8 D_z^{(0)}, \tag{30}$$

$$\sigma_{z,z}^{(0)} = L_9 u^{(0)} + \tilde{l}_{63} w^{(0)} - D^T\sigma_s^{(0)} + \tilde{l}_{65} D_z^{(0)}, \tag{31}$$

$$\phi^{(0)}{}_{,z} = -\left(\gamma_\alpha\gamma_\beta\right)\mathbf{L}_8^T\mathbf{u}^{(0)} - \left(1/\gamma_\alpha\gamma_\beta\right)\tilde{l}_{65} w^{(0)} + \tilde{l}_{81} D_z^{(0)}, \tag{32}$$

$$\sigma_p^{(0)} = \mathbf{B}_1\mathbf{u}^{(0)} + \mathbf{B}_2 w^{(0)} + \mathbf{B}_4 D_z^{(0)}, \tag{33}$$

$$\mathbf{d}^{(0)} = \mathbf{B}_5\sigma_s^{(0)} + \mathbf{B}_6\phi^{(0)}, \tag{34}$$

Order ε^{2k} (k=1, 2, 3,...):

$$w_z^{(k)} = -\mathbf{L}_1\mathbf{u}^{(k-1)} - \tilde{l}_{33} w^{(k-1)} + \tilde{l}_{34}\sigma_z^{(k-2)} + \tilde{l}_{35} D_z^{(k-1)}, \tag{35}$$

$$\mathbf{u}^{(k)}{}_{,z} = -\mathbf{D}w^{(k)} + \mathbf{L}_2\mathbf{u}^{(k-1)} + \mathbf{S}\sigma_s^{(k-1)} + \mathbf{L}_3\sigma_s^{(k-2)} + \mathbf{L}_4\phi^{(k-1)}, \tag{36}$$

$$\sigma_s^{(k)}{}_{,z} = -\mathbf{L}_5\mathbf{u}^{(k)} - \mathbf{L}_6 w^{(k)} - \mathbf{L}_7\sigma_s^{(k-1)} - \left(\gamma_\alpha\gamma_\beta\right)\mathbf{L}_1^T\sigma_z^{(k-1)} - \mathbf{L}_8 D_z^{(k)}, \tag{37}$$

$$\sigma_z^{(k)}{}_{,z} = \mathbf{L}_9\,\mathbf{u}^{(k)} + \tilde{l}_{63} w^{(k)} - \mathbf{D}^T\sigma_s^{(k)} - \mathbf{L}_{10}\sigma_s^{(k-1)} - \tilde{l}_{64}\sigma_z^{(k-1)} + \tilde{l}_{65} D_z^{(k)}, \tag{38}$$

$$D_z^{(k)}{}_{,z} = -\mathbf{L}_{11}\,\mathbf{d}^{(k-1)} - \tilde{l}_{71} D_z^{(k-1)}, \tag{39}$$

$$\phi^{(k)}{}_{,z} = -\left(\gamma_\alpha\gamma_\beta\right)\mathbf{L}_8^T\mathbf{u}^{(k)} - \left(1/\gamma_\alpha\gamma_\beta\right)\tilde{l}_{65} w^{(k)} + \tilde{l}_{35}\sigma_z^{(k-1)} + \tilde{l}_{81} D_z^{(k)}, \tag{40}$$

$$\sigma_p^{(k)} = \mathbf{B}_1\mathbf{u}^{(k)} + \mathbf{B}_2 w^{(k)} + \mathbf{B}_3\sigma_z^{(k-1)} + \mathbf{B}_4 D_z^{(k)}, \tag{41}$$

$$\mathbf{d}^{(k)} = \mathbf{B}_5\sigma_s^{(k)} + \mathbf{B}_6\phi^{(k)}, \tag{42}$$

The transverse loads and electric potential at the lateral surfaces are given as
Order ε^0 :

$$\left[\tau^{(0)}xz \quad \tau^{(0)}yz\right] = [0 \quad 0] \qquad \text{on} \quad z = \pm 1, \tag{43a}$$

$$\sigma_z^{(0)} = \overline{q}_z^{\pm}(x,\ y) \qquad \text{on} \quad z = 1, \tag{43b}$$

$$\phi^{(0)} = \overline{\phi}_z^{\pm}(x,\ y) \qquad \text{on} \quad z = \pm 1. \tag{44}$$

Order ε^{2k} (k = 1, 2, 3,...):

$$\left[\tau_{xz}^{(k)} \quad \tau_{yz}^{(k)} \quad \sigma_z^{(k)}\right] = [0 \quad 0 \quad 0] \quad \text{on} \quad z = \pm 1, \tag{45}$$

$$\phi^{(k)} = 0 \qquad \text{on} \quad z = \pm 1. \tag{46}$$

Along the edges one member of each pair of the following quantities must be satisfied:

Order ε^0 :

$$n_1\sigma_x^{(0)} + n_2\tau_{xy}^{(0)} = p_{nx}, \quad \text{or} \quad u_{(0)} = \overline{u}, \tag{47a}$$

$$n_1\tau_{xy}^{(0)} + n_1\sigma_y^{(0)} = p_{ny}, \quad \text{or} \quad v_{(0)} = \overline{v}, \tag{47b}$$

$$n_1\sigma_{xz}^{(0)} + n_2\tau_{yz}^{(0)} = p_{nz}, \quad \text{or} \quad w_{(0)} = \overline{w}, \tag{47c}$$

$$\phi^{(0)} = 0. \tag{48}$$

Order ε^{2k} (k = 1, 2, 3,...):

$$n_1\sigma_x^{(k)} + n_2\tau_{xy}^{(k)} = 0, \quad \text{or} \quad u^{(k)} = 0, \tag{49a}$$

$$n_1\tau_{xy}^{(k)} + n_1\sigma_y^{(k)} = 0, \quad \text{or} \quad v^{(k)} = 0, \tag{49b}$$

$$n_1\tau_{xz}^{(k)} + n_2\tau_{yz}^{(k)} = 0, \quad \text{or} \quad w^{(k)} = 0, \tag{49c}$$

$$\phi^{(k)} = 0 \quad . \tag{50}$$

5 Asymptotic integration

Examination of the sets of asymptotic equations, it is found that the analysis can be carried on by integrating those equations through the thickness direction. We therefore integrate Eqs.(27)-(29) to obtain

$$w^{(0)} = w^0(x,y) \tag{51}$$

$$\mathbf{u}^{(0)} = \mathbf{u}^0 - z\mathbf{D}w^0 \tag{52}$$

$$D_z^{(0)} = D_z^0(x,\ y), \tag{53}$$

where $w^0(x,y)$, $\mathbf{u}^0 = [u_0(x,y)\ v_0(x,\ y)]^T$ and $D_z^0(x,y)$ represent the displacements and piezoelectric displacement on the middle surface and those are also of the Kirchhoff-Love type in CST.

With the lateral boundary conditions on z=-1 (i.e., Eqs.(43)-(44)), we then proceed to successively integrate Eqs.(30)-(32) and it yields

$$\sigma_s^{(0)} = -\int_{-1}^{z} \left[\mathbf{L}_5\left(\mathbf{u}^0 - \eta\mathbf{D}w^0\right) + \mathbf{L}_6 w^0 + \mathbf{L}_8 D_z^0\right] d\eta, \tag{54}$$

$$\begin{aligned}\sigma_z^{(0)} = {} & \overline{q}_z^- + \int_{-1}^{z} \left[\mathbf{L}_9\left(\mathbf{u}^0 - \eta\mathbf{D}w^0\right) + \tilde{l}_{63} w^0\right] d\eta \\ & + \int_{-1}^{z} (z-\eta)\mathbf{D}^T \left[\mathbf{L}_5\left(\mathbf{u}^0 - \eta\mathbf{D}w^0\right) + \mathbf{L}_6 w^0 + \mathbf{L}_8 D_z^0\right] d\eta\end{aligned} \tag{55}$$

$$\phi^{(0)} = \overline{\phi}_z^- - \int_{-1}^{z} \left[\begin{array}{l}\gamma_\alpha\gamma_\beta \mathbf{L}_8^T\left(\mathbf{u}^0 - \eta\,\mathbf{D}\,w^0\right) \\ + \left(1/\gamma_\alpha\gamma_\beta\right)\tilde{l}_{65}\,w^0 + \tilde{l}_{81} D_z^0\end{array}\right] d\eta. \tag{56}$$

Imposing the remaining lateral boundary conditions on z=1 (i.e., Eqs.(43)-(44)) in Eqs.(54)-(56), we obtain

$$K_{11}\,u^0 + K_{12}\,v^0 + K_{13}\,w^0 + K_{14} D_z^0 = 0, \tag{57}$$

$$K_{21}\,u^0 + K_{22}\,v^0 + K_{23}\,w^0 + K_{24} D_z^0 = 0, \tag{58}$$

$$K_{31}\,u^0 + K_{32}\,v^0 + K_{33}\,w^0 + K_{34} D_z^0 = \overline{q}_z^+ - \overline{q}_z^-, \tag{59}$$

$$K_{41}\,u^0 + K_{42}\,v^0 + K_{43}\,w^0 + K_{44} D_z^0 = \overline{\phi}_z^+ - \overline{\phi}_z^-, \tag{60}$$

in which

$$K_{11} = \hat{A}_{11}\partial_{xx} + \left(\tilde{A}_{16} + \tilde{A}_{61}\right)\partial_{xy} + \overline{A}_{66}\partial_{yy},$$

$$K_{12} = \hat{A}_{16}\partial_{xx} + \left(\tilde{A}_{12} + \tilde{A}_{66}\right)\partial_{xy} + \overline{A}_{62}\partial_{yy},$$

$$\begin{aligned}K_{13} = {} & -\hat{B}_{11}\partial_{xxx} - \left(\tilde{B}_{16} + \hat{B}_{16} + \tilde{B}_{61}\right)\partial_{xxy} \\ & - \left(\tilde{B}_{12} + \tilde{B}_{66} + \overline{B}_{66}\right)\partial_{xyy} - \overline{B}_{62}\partial_{yyy} \\ & + \left(\hat{A}_{11}/R_x + \tilde{A}_{12}/R_y\right)\partial_x + \left(\tilde{A}_{61}/R_x + \overline{A}_{62}/R_y\right)\partial_y,\end{aligned}$$

$$K_{14} = \hat{E}_{31}\partial_x + \overline{E}_{36}\partial_y,$$

$$K_{21} = \hat{A}_{61}\partial_{xx} + \left(\tilde{A}_{21} + \tilde{A}_{66}\right)\partial_{xy} + \overline{A}_{26}\partial_{yy},$$

$$K_{22} = \hat{A}_{66}\partial_{xx} + \left(\tilde{A}_{26} + \tilde{A}_{62}\right)\partial_{xy} + \overline{A}_{22}\partial_{yy},$$

$$\begin{aligned}K_{23} = {} & -\hat{B}_{16}\partial_{xxx} - \left(\tilde{B}_{21} + \tilde{B}_{66} + \hat{B}_{66}\right)\partial_{xxy} \\ & - \left(\overline{B}_{26} + \tilde{B}_{26} + \tilde{B}_{62}\right)\partial_{xyy} - \overline{B}_{22}\partial_{yyy} \\ & + \left(\hat{A}_{61}/R_x + \tilde{A}_{62}/R_y\right)\partial_x + \left(\tilde{A}_{21}/R_x + \overline{A}_{22}/R_y\right)\partial_y,\end{aligned}$$

$$K_{24} = \hat{E}_{36}\partial_x + \overline{E}_{32}\partial_y,$$

$$\begin{aligned}K_{31} = {} & -\hat{B}_{11}\partial_{xxx} - \left(\tilde{B}_{16} + \tilde{B}_{61} + \hat{B}_{61}\right)\partial_{xxy} \\ & - \left(\tilde{B}_{21} + \tilde{B}_{66} + \overline{B}_{66}\right)\partial_{xyy} - \overline{B}_{26}\partial_{yyy} \\ & + \left(\hat{A}_{11}/R_x + \tilde{A}_{21}/R_y\right)\partial_x + \left(\tilde{A}_{16}/R_x + \overline{A}_{26}/R_y\right)\partial_y,\end{aligned}$$

$$\begin{aligned}K_{32} = {} & -\hat{B}_{16}\partial_{xxx} - \left(\tilde{B}_{12} + \tilde{B}_{66} + \hat{B}_{66}\right)\partial_{xxy} \\ & - \left(\tilde{B}_{26} + \tilde{B}_{62} + \overline{B}_{62}\right)\partial_{xyy} - \overline{B}_{22}\partial_{yyy} \\ & + \left(\hat{A}_{16}/R_x + \tilde{A}_{26}/R_y\right)\partial_x + \left(\tilde{A}_{12}/R_x + \overline{A}_{22}/R_y\right)\partial_y,\end{aligned}$$

$$\begin{aligned}K_{33} = {} & \hat{D}_{11}\partial_{xxxx} + (\tilde{D}_{16} + \hat{D}_{16} + \tilde{D}_{61} + \hat{D}_{61})\partial_{xxxy} \\ & + \left(\tilde{D}_{12} + \tilde{D}_{21} + \overline{D}_{66} + 2\tilde{D}_{66} + \hat{D}_{66}\right)\partial_{xxyy} \\ & + \left(\overline{D}_{26} + \tilde{D}_{26} + \overline{D}_{62} + \tilde{D}_{62}\right)\partial_{xyyy} + \overline{D}_{22}\partial_{yyyy} \\ & - \left[2\hat{B}_{11}/R_x + \left(\tilde{B}_{12} + \tilde{B}_{21}\right)/R_y\right]\partial_{xx} \\ & - \left[\begin{array}{l}\left(\tilde{B}_{16} + \tilde{B}_{61} + \hat{B}_{16} + \hat{B}_{61}\right)/R_x \\ + \left(\overline{B}_{26} + \overline{B}_{62} + \tilde{B}_{26} + \tilde{B}_{62}\right)/R_y\end{array}\right]\partial_{xy} \\ & - \left[\left(\tilde{B}_{12} + \tilde{B}_{21}\right)/R_x + 2\overline{B}_{22}/R_y\right]\partial_{yy} \\ & + \left[\hat{A}_{11}/R_x^2 + \left(\tilde{A}_{12} + \tilde{A}_{21}\right)/R_xR_y + \overline{A}_{22}/R_y^2\right]\end{aligned}$$

$$K_{34} = -\hat{F}_{31}\partial_{xx} - \left(\hat{F}_{36} + \overline{F}_{36}\right)\partial_{xy} - \overline{F}_{32}\partial_{yy} + \left(\hat{E}_{31}/R_x + \overline{E}_{32}/R_y\right),$$

$$K_{41} = -\tilde{E}_{31}\partial_x - \breve{E}_{36}\partial_y, \quad K_{42} = -\tilde{E}_{36}\partial_x - \breve{E}_{32}\partial_y,$$

$$K_{43} = \tilde{F}_{31}\partial_{xx} + \left(\breve{F}_{36} + \tilde{F}_{36}\right)\partial_{xy} + \breve{F}_{32}\partial_{yy} - \left(\tilde{E}_{31}/R_x + \breve{E}_{32}/R_y\right),$$

$$K_{44} = -E_{30},$$

$$\hat{A}_{ij} = \int_{-1}^{1} \left(\tilde{Q}_{ij}\gamma_\beta/\gamma_\alpha\right) dz$$

$$\tilde{A}_{ij} = \int_{-1}^{1} \tilde{Q}_{ij}\, dz,$$

$$\overline{A}_{ij} = \int_{-1}^{1} \left(\tilde{Q}_{ij}\gamma_\alpha/\gamma_\beta\right) dz,$$

$$\hat{B}_{ij} = \int_{-1}^{1} z\left(\tilde{Q}_{ij}\gamma_\beta/\gamma_\alpha\right) dz$$

$$\tilde{B}_{ij} = \int_{-1}^{1} z\, \tilde{Q}_{ij}\, dz,$$

$$\overline{B}_{ij} = \int_{-1}^{1} z\left(\tilde{Q}_{ij}\gamma_\alpha/\gamma_\beta\right) dz,$$

$$\hat{D}_{ij} = \int_{-1}^{1} z^2\left(\tilde{Q}_{ij}\gamma_\beta/\gamma_\alpha\right) dz$$

$$\tilde{D}_{ij} = \int_{-1}^{1} z^2\tilde{Q}_{ij}\, dz,$$

$$\overline{D}_{ij} = \int_{-1}^{1} z^2\left(\tilde{Q}_{ij}\gamma_\alpha/\gamma_\beta\right) dz,$$

$$\begin{pmatrix} \hat{E}_{3i} & \hat{F}_{3i} \end{pmatrix} = \int_{-1}^{1} \begin{pmatrix} 1 & z \end{pmatrix} \frac{\gamma_\beta e}{Q}\left(\frac{e_{33}c_{3i} - e_{3i}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right) dz,$$

$$\begin{pmatrix} \overline{E}_{3i} & \overline{F}_{3i} \end{pmatrix} = \int_{-1}^{1} \begin{pmatrix} 1 & z \end{pmatrix} \frac{\gamma_\alpha e}{Q}\left(\frac{e_{33}c_{3i} - e_{3i}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right) dz,$$

$$\begin{pmatrix} \breve{E}_{3i} & \breve{F}_{3i} \end{pmatrix} = \int_{-1}^{1} \begin{pmatrix} 1 & z \end{pmatrix} \frac{e}{\gamma_\beta Q}\left(\frac{e_{33}c_{3i} - e_{3i}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right) dz,$$

$$\begin{pmatrix} \tilde{E}_{3i} & \tilde{F}_{3i} \end{pmatrix} = \int_{-1}^{1} \begin{pmatrix} 1 & z \end{pmatrix} \frac{e}{\gamma_\alpha Q}\left(\frac{e_{33}c_{3i} - e_{3i}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right) dz,$$

$$E_{30} = \int_{-1}^{1} \frac{e^2}{Q}\left(\frac{c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right) dz.$$

Following a similar process in the early paper [Wu et al. (1996a)], it can be shown that the governing equations for the displacements in CST are recovered from Eqs.(57)-(60) by introducing a geometry assumption of the thin shell: $z/R_\alpha << 1$ and$z/R_\beta << 1$. Thus, the CST equations have been derived as the first-order approximation to the three-dimensional theory. Solution of Eqs.(57)-(60) must be supplemented with the edge boundary conditions Eqs.(47)-(48) to constitute a well-posed boundary value problem. Once u^0, v^0, w^0 and D_z^0 are determined, the leading-order displacements are given by Eqs.(51)-(53), the transverse shear and normal stresses by Eqs.(54)-(55), the in-surface stresses by Eq.(33), the in-surface electric displacements by Eq.(34) and the electric potential by Eq.(56).

Proceed to order ε^2 following the same line as was done before, we readily obtain

$$w^{(1)} = w^1\ (x,\ y) + \varphi_{31}(x,\ y,\ z), \tag{61}$$

$$\mathbf{u}^{(1)} = \mathbf{u}^1 - z\,\mathbf{D}\, w^1 + \varphi^1, \tag{62}$$

$$D_z^{(1)} = D_z^1(x,\ y) + \varphi_{41}(x,\ y,\ z), \tag{63}$$

$$\sigma_s^{(1)} = -\int_{-1}^{z}\left[\mathbf{L}_5\left(\mathbf{u}^1 - \eta\mathbf{D}w^1\right) + \mathbf{L}_6 w^1 + \mathbf{L}_8 D_z^1\right] d\eta + \mathbf{f}^1(x,y,z), \tag{64}$$

$$\sigma_z^{(1)} = \int_{-1}^{z}\left[\mathbf{L}_9\left(\mathbf{u}^1 - \eta\mathbf{D}w^1\right) + \tilde{l}_{63}w^1\right] d\eta + \int_{-1}^{z}(z-\eta)\mathbf{D}^T\begin{bmatrix} \mathbf{L}_5\left(\mathbf{u}^1 - \eta\mathbf{D}w^1\right) \\ +\mathbf{L}_6 w^1 + \mathbf{L}_8 D_z^1 \end{bmatrix} d\eta - f_{31}(x,y,z), \tag{65}$$

$$\phi^{(1)} = -\int_{-1}^{z}\begin{bmatrix} \left(\gamma_\alpha\gamma_\beta\right)\mathbf{L}_8^T\left(\mathbf{u}^1 - \eta\mathbf{D}w^1\right) \\ +\left(1/\gamma_\alpha\gamma_\beta\right)\tilde{l}_{65}w^1 + \tilde{l}_{81}D_z^1 \end{bmatrix} d\eta - f_{41}(x,y,z), \tag{66}$$

where $\varphi_{31}(x,y,z) = -\int_0^z \left[\mathbf{L}_1\mathbf{u}^{(0)} + \tilde{l}_{33}w^{(0)} - \tilde{l}_{35}D_z^{(0)}\right] d\eta$,

$$\mathbf{u}^1 = \left[u^1(x,y) \quad v^1(x,y)\right]^T,$$

$$\varphi^1 = \begin{Bmatrix} \varphi_{11}(x,y,z) \\ \varphi_{21}(x,y,z) \end{Bmatrix} = \int_0^z \left(\mathbf{L}_2\mathbf{u}^{(0)} + \mathbf{S}\sigma_s^{(0)} + \mathbf{L}_4\phi^{(0)} - \mathbf{D}\varphi_{31}\right) d\eta,$$

$$\varphi_{41} = -\int_0^z \left(\mathbf{L}_{11}\mathbf{d}^{(0)} + \tilde{l}_{71}D_z^{(0)}\right) d\eta,$$

$$\mathbf{f}^1 = -\int_{-1}^{z} \begin{bmatrix} \mathbf{L}_5\,\varphi^1 + \mathbf{L}_6\,\varphi_{31} + \mathbf{L}_8\varphi_{41} \\ +\mathbf{L}_7\sigma_s^{(0)} + \gamma_\alpha\gamma_\beta\,\mathbf{L}_1^T\sigma_z^{(0)} \end{bmatrix} d\eta,$$

$$f_{31} = -\int_{-1}^{z} \begin{bmatrix} \mathbf{L}_9\varphi^1 + \tilde{l}_{63}\,\varphi_{31} - \mathbf{L}_{10}\sigma_s^{(0)} \\ -\tilde{l}_{64}\sigma_z^{(0)} + \tilde{l}_{65}\varphi_{41} - \mathbf{D}^T\mathbf{f}^1 \end{bmatrix} d\eta,$$

$$f_{41} = \int_{-1}^{z} \begin{bmatrix} \gamma_\alpha\gamma_\beta\,\mathbf{L}_8^T\,\varphi^1 + \left(1/\gamma_\alpha\gamma_\beta\right)\tilde{l}_{65}\,\varphi_{31} \\ -\tilde{l}_{35}\,\sigma_z^{(0)} - \tilde{l}_{81}\varphi_{41} \end{bmatrix} d\eta.$$

$w^1, \mathbf{u}^1$ and D_z^1 represent the modifications to the elastic and electric displacements on the middle surface. Upon imposing the associated lateral boundary conditions Eqs.(45)-(46) on Eqs.(64) and (66), we arrive again at the CST-type equations, only with nonhomogeneous terms that are known from the leading-order solution. The resulting equations are

$$K_{11}\,u^1 + K_{12}\,v^1 + K_{13}\,w^1 + K_{14}D_z^1 = f_{11}(x,\,y,\,1), \quad (67)$$

$$K_{21}\,u^1 + K_{22}\,v^1 + K_{23}\,w^1 + K_{24}D_z^1 = f_{21}(x,\,y,\,1), \quad (68)$$

$$K_{31}\,u^1 + K_{32}\,v^1 + K_{33}\,w^1 + K_{34}D_z^1 = f_{31}(x,\,y,\,1) - \frac{\partial f_{11}(x,\,y,\,1)}{\partial x} - \frac{\partial f_{21}(x,\,y,\,1)}{\partial y}, \quad (69)$$

$$K_{41}\,u^1 + K_{42}\,v^1 + K_{43}\,w^1 + K_{44}D_z^1 = f_{41}(x,\,y,\,1). \quad (70)$$

The governing equations for the ε^2-order modifications to $u^k(x,y)$, $v^k(x,y)$ and $w^k(x,y)$ of order ε^{2k} (k = 2,3,...) are obtained in a similar way by integrating the higher-order equations Eqs.(25)-(28) in succession. The equations are given by

$$K_{11}\,u^k + K_{12}\,v^k + K_{13}\,w^k + K_{14}D_z^k = f_{1k}(x,\,y,\,1), \quad (71)$$

$$K_{21}\,u^k + K_{22}\,v^k + K_{23}\,w^k + K_{24}D_z^k = f_{2k}(x,\,y,\,1), \quad (72)$$

$$K_{31}\,u^k + K_{32}\,v^k + K_{33}\,w^k + K_{34}D_z^k = f_{3k}(x,\,y,\,1) - \frac{\partial f_{1k}(x,\,y,\,1)}{\partial x} - \frac{\partial f_{2k}(x,\,y,\,1)}{\partial y}, \quad (73)$$

$$K_{41}\,u^k + K_{42}\,v^k + K_{43}\,w^k + K_{44}D_z^k = f_{4k}(x,\,y,\,1), \quad (74)$$

in which

$$\varphi_{3k}(x,y,z) = -\int_0^z \begin{bmatrix} \mathbf{L}_1\,\mathbf{u}^{(k-1)} + \tilde{l}_{33}\,w^{(k-1)} \\ -\tilde{l}_{35}D_z^{(k-1)} - \tilde{l}_{34}\sigma_z^{(k-2)} \end{bmatrix} d\eta,$$

$$\mathbf{u}^k = \left[u^k(x,y) \quad v^k(x,y)\right]^T,$$

$$\varphi^k = \begin{Bmatrix} \varphi_{1k}(x,y,z) \\ \varphi_{2k}(x,y,z) \end{Bmatrix} = -\int_0^z \begin{pmatrix} \mathbf{D}\varphi_{3k} - \mathbf{L}_2\,\mathbf{u}^{(k)} - \mathbf{S}\sigma_s^{(k-1)} \\ -\mathbf{L}_3\sigma_s^{(k-2)} - \mathbf{L}_4\,\phi^{(k-1)} \end{pmatrix} d\eta,$$

$$\varphi_{4k} = -\int_0^z \left(\mathbf{L}_{11}\mathbf{d}^{(k-1)} + \tilde{l}_{71}D_z^{(k-1)}\right) d\eta,$$

$$\mathbf{f}^k = -\int_{-1}^{z} \begin{bmatrix} \mathbf{L}_5\,\varphi^k + \mathbf{L}_6\,\varphi_{3k} + \mathbf{L}_8\varphi_{4k} \\ +\mathbf{L}_7\sigma_s^{(k-1)} + \gamma_\alpha\gamma_\beta\,\mathbf{L}_1^T\sigma_z^{(k-1)} \end{bmatrix} d\eta,$$

$$f_{3k} = -\int_{-1}^{z} \begin{bmatrix} \mathbf{L}_9\varphi^k + \tilde{l}_{63}\,\varphi_{3k} - \mathbf{L}_{10}\sigma_s^{(k-1)} \\ -\tilde{l}_{64}\sigma_z^{(k-1)} + \tilde{l}_{65}\varphi_{4k} - \mathbf{D}^T\mathbf{f}^k \end{bmatrix} d\eta,$$

$$f_{4k} = \int_{-1}^{z} \begin{bmatrix} \gamma_\alpha\gamma_\beta\mathbf{L}_8^T\,\phi^k + \left(1/\gamma_\alpha\gamma_\beta\right)\tilde{l}_{65}\,\varphi_{3k} \\ -\tilde{l}_{35}\,\sigma_z^{(k-1)} - \tilde{l}_{81}\varphi_{4k} \end{bmatrix} d\eta.$$

6 Applications to benchmark problems

The benchmark problems of simply supported, doubly curved laminated piezoelectric shells under mechanical or electric loads on lateral surfaces are studied using the present asymptotic theory. The material of cross-ply laminated piezoelectric shells is considered so that the elastic moduli for orthotropic materials are

$$(Q_{16})_i = (Q_{26})_i = (Q_{36})_i = (Q_{45})_i = (e_{36})_i = 0 \quad (75)$$

The boundary conditions on the four edges are of a shear diaphragm type specified by

$$\sigma_\alpha = u_\beta = u_\zeta = \Phi = 0 \quad \text{on } \alpha = 0 \text{ and } \alpha = a_\alpha, \quad (76)$$

$$\sigma_\beta = u_\alpha = u_\zeta = \Phi = 0 \quad \text{on } \beta = 0 \text{ and } \beta = a_\beta. \quad (77)$$

The mechanical or electric loads acting on lateral surface of the shell ($\zeta = h$) is considered. The mechanical and electric loads are expressed by the double Fourier series in the dimensionless form

$$\tilde{q}(x,\,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{q}_{mn}\,\sin\tilde{m}x\,\sin\tilde{n}y, \quad (78)$$

$$\tilde{\phi}(x,\,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{\phi}_{mn}\,\sin\tilde{m}x\,\sin\tilde{n}y, \quad (79)$$

where $\tilde{m} = m\pi\sqrt{Rh}/a_\alpha$ *and* $\tilde{n} = n\pi\sqrt{Rh}/a_\beta$.

For this problem the governing equations of the leading-order problem (i.e., Eqs.(57)-(60)) can be easily solved by letting

$$u^0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u^0_{mn}\cos\tilde{m}x\ \sin\tilde{n}y, \tag{80}$$

$$v^0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v^0_{mn}\sin\tilde{m}x\ \cos\tilde{n}y, \tag{81}$$

$$w^0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w^0_{mn}\sin\tilde{m}x\ \sin\tilde{n}y, \tag{82}$$

$$D^0_z = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D^0_{zmn}\sin\tilde{m}x\ \sin\tilde{n}y. \tag{83}$$

Substituting Eqs.(80)-(83) into Eqs.(57)-(60) gives

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u^0_{mn} \\ v^0_{mn} \\ w^0_{mn} \\ D^0_{zmn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\tilde{q}_{mn} \\ \tilde{\phi}_{mn} \end{Bmatrix}, \tag{84}$$

where

$$k_{11} = -\tilde{m}^2\hat{A}_{11} - \tilde{n}^2\overline{A}_{66}$$

$$k_{12} = -\tilde{m}\tilde{n}\left(\tilde{A}_{12}+\tilde{A}_{66}\right)$$

$$k_{13} = \tilde{m}^3\hat{B}_{11} + \tilde{m}\tilde{n}^2\left(\tilde{B}_{12}+\overline{B}_{66}+\tilde{B}_{66}\right) + \tilde{m}\left(\hat{A}_{11}/R_x + \tilde{A}_{12}/R_y\right)$$

$$k_{14} = \tilde{m}\hat{E}_{31}$$

$$k_{21} = -\tilde{m}\tilde{n}\left(\tilde{A}_{21}+\tilde{A}_{66}\right)$$

$$k_{22} = -\tilde{m}^2\hat{A}_{66} - \tilde{n}^2\overline{A}_{22}$$

$$k_{23} = \tilde{m}^2\tilde{n}\left(\tilde{B}_{21}+\tilde{B}_{66}+\hat{B}_{66}\right) + \tilde{n}^3\overline{B}_{22} + \tilde{n}\left(\tilde{A}_{21}/R_x + \overline{A}_{22}/R_y\right)$$

$$k_{24} = \tilde{n}\overline{E}_{32}$$

$$k_{31} = \tilde{m}^3\hat{B}_{11} + \tilde{m}\tilde{n}^2\left(\tilde{B}_{21}+\overline{B}_{66}+\tilde{B}_{66}\right) + \tilde{m}\left(\hat{A}_{11}/R_x + \tilde{A}_{21}/R_y\right)$$

$$k_{32} = \tilde{m}^2\tilde{n}\left(\tilde{B}_{12}+\tilde{B}_{66}+\hat{B}_{66}\right) + \tilde{n}^3\overline{B}_{22} + \tilde{n}\left(\tilde{A}_{12}/R_x + \overline{A}_{22}/R_y\right)$$

$$\begin{aligned} k_{33} = &-\tilde{m}^4\hat{D}_{11} - \tilde{m}^2\tilde{n}^2\left(\tilde{D}_{12}+\tilde{D}_{21}+\overline{D}_{66}+2\tilde{D}_{66}+\hat{D}_{66}\right) \\ &-\tilde{n}^4\overline{D}_{22} - \tilde{m}^2\left[2\hat{B}_{11}/R_x + \left(\tilde{B}_{12}+\tilde{B}_{21}\right)/R_y\right] \\ &-\tilde{n}^2\left[\left(\tilde{B}_{12}+\tilde{B}_{21}\right)/R_x + 2\overline{B}_{22}/R_y\right] \\ &-\left[\hat{A}_{11}/R_x^2 + \left(\tilde{A}_{12}+\tilde{A}_{21}\right)/R_xR_y + \overline{A}_{22}/R_y^2\right] \end{aligned}$$

$$k_{34} = -\tilde{m}^2\hat{F}_{31} - \tilde{n}^2\overline{F}_{32} - \left(\hat{E}_{31}/R_x + \overline{E}_{32}/R_y\right)$$

$$k_{41} = \tilde{m}\tilde{E}_{31}$$

$$k_{42} = \tilde{n}\breve{E}_{32}$$

$$k_{43} = -\tilde{m}^2\tilde{F}_{31} - \tilde{n}^2\breve{F}_{32} - \left(\tilde{E}_{31}/R_x + \breve{E}_{32}/R_y\right)$$

$$k_{44} = -E_{30}$$

$u^0_{(mn)}$, $v^0(mn)$ and $w^0(mn)$ can be obtained by solving the simultaneously algebraic equations (84). Once u^0_{mn}, v^0_{mn} and w^0_{mn} have been determined, the solution of ε^0 order is obtained by introducing Eqs.(51)-(53) into Eqs.(54)-(56) and Eqs.(20)-(21). The explicit expressions are given in Appendix B. The summation signs have been dropped for brevity.

Carrying on the solution to order ε^2, we find that the non-homogeneous terms for fixed values of m and n in the ε^2-order equations are

$$f_{11}(x,\ y,\ 1) = \tilde{f}_{11}(1)\ \cos\tilde{m}x\ \sin\tilde{n}y \tag{85}$$

$$f_{21}(x,\ y,\ 1) = \tilde{f}_{21}(1)\ \sin\tilde{m}x\ \cos\tilde{n}y \tag{86}$$

$$f_{31}(x,\ y,\ 1) = \tilde{f}_{31}(1)\ \sin\tilde{m}x\ \sin\tilde{n}y \tag{87}$$

$$f_{41}(x,\ y,\ 1) = \tilde{f}_{41}(1)\ \sin\tilde{m}x\ \sin\tilde{n}y \tag{88}$$

where $\tilde{f}_{11}$, $\tilde{f}_{21}$, $\tilde{f}_{31}$, $\tilde{f}_{41}$ are given in Appendix B.

In view of the recurrence of the equations, the ε^2-order solution can be obtained by letting

$$u^1 = u^1_{mn}\cos\tilde{m}x\ \sin\tilde{n}y \tag{89}$$

$$v^1 = v^1_{mn}\sin\tilde{m}x\ \cos\tilde{n}y \tag{90}$$

$$w^1 = w^1_{mn}\sin\tilde{m}x\ \sin\tilde{n}y \tag{91}$$

$$D^1_z = D^1_{zmn}\sin\tilde{m}x\ \sin\tilde{n}y \tag{92}$$

Substituting Eqs.(89)-(92) into Eqs.(67)-(70) gives

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} u^1_{mn} \\ v^1_{mn} \\ w^1_{mn} \\ D^1_{zmn} \end{Bmatrix} = \begin{Bmatrix} \tilde{f}_{11}(1) \\ \tilde{f}_{21}(1) \\ -\left(\tilde{f}_{31}(1) + \tilde{m}\tilde{f}_{11}(1) + \tilde{n}\tilde{f}_{21}(1)\right) \\ \tilde{f}_{41}(1) \end{Bmatrix}, \tag{93}$$

$u^1(mn)$, $v^1(mn)$, $w^1(mn)$ and $D^1_{zmn}$are easily obtained by solving the system of algebraic equations (83). Afterwards, the ε^2-order corrections are determined from Eqs.(61)-(66) and Eqs.(41)-(42). The explicit expressions are given in Appendix B.

Examining Eqs.(57)-(60) and Eqs.(67)-(70), we find that the solutions are in recurrent forms. The higher-order corrections can be determined using the lower-order solutions in a hierarchic manner.

The performance of the asymptotic solution will be illustrated in the following examples.

6.1 Laminated piezoelectric plates and strips

Several benchmark solutions of laminated piezoelectric plates and strips were given in the literature [Heyliger (1994, 1997a), Cheng et al. (2000)]. The laminated piezoelectric plates and strips are regarded as a special case of doubly curved shells in which R_α and R_β approach ∞ and $1/R_\alpha = 1/R_\beta = 0$. To facilitate numerical comparisons, we first compute the results for a simply supported, two-layer plate of dissimilar piezoelectric materials where PZT-4 is on the top and PVDF on the bottom [Heyliger (1994, 1997a)], and then compute those of a three-layer cross-ply strip of PVDF [Cheng et al. (2000)].

The laminated piezoelectric plates considered are subjected to either mechanical or electrical loads (i.e., $\overline{q}^+_\zeta(\alpha,\ \beta) = q_0\ \sin\pi\alpha/a_\alpha\ \sin\pi\beta/a_\beta$, $q_0 = 1$ N / m^2 or $\overline{\Phi}^+_\zeta(\alpha,\ \beta) = \Phi_0\ \sin\pi\alpha/a_\alpha\ \sin\pi\beta/a_\beta$, $\phi_0 = 1$ V) on the lateral surfaces. The elastic, piezoelectric and dielectric properties of piezoelectric materials used in Tables 2-3 are given in Table 1. For comparison purpose, the geometry parameters are taken as $a_\alpha/a_\beta = 2$, $a_\alpha/2h$=10 and $2h$=0.005 m. Tables 2-3 show the asymptotic solutions of various orders for mechanical and electrical field variables of laminated piezoelectric plates under lateral loads and lateral potentials, respectively. The asymptotic solution is computed up to the ε^1-order level in order to closely examine the convergence of the present asymptotic theory. It is shown that the convergent solution is obtained at the ε^4-order level where $a_\alpha/2h$=10. The convergent solution is also compared with the 3D piezoelectricity solution [Heyliger (1997a)] and the solution of discrete-layer theory [Heyliger et al. (1994)] available in the literature. It is shown that the convergent solution of the present asymptotic theory is in excellent agreement with the 3D piezoelectricity solution.

Table 1 : Elastic, piezoelectric and dielectric properties of piezoelectric Materials

Moduli	PVDF	PZT-4
c_{11}(GPa)	238.00	139.00
c_{22}(GPa)	23.60	139.00
c_{33}(GPa)	10.60	115.00
c_{12}(GPa)	3.98	77.80
c_{13}(GPa)	2.19	74.30
c_{23}(GPa)	1.92	74.30
c_{44}(GPa)	2.15	25.60
c_{55}(GPa)	4.40	25.60
c_{66}(GPa)	6.43	30.60
e_{24}(C / m^2)	-0.01	12.72
e_{15}(C / m^2)	-0.01	12.72
e_{31}(C / m^2)	-0.13	-5.20
e_{32}(C / m^2)	-0.14	-5.20
e_{33}(C / m^2)	-0.28	15.08
η_{11}/ε_0	12.50	1475.00
η_{22}/ε_0	11.98	1475.00
η_{33}/ε_0	11.98	1300.00

$\varepsilon_0 = 8.854\mathrm{X}10^{-12}$ (F / m)

A [0/90/0] laminated PVDF piezoelectric strip (i.e., $a_\beta \to \infty$ and $1/a_\beta = 0$) is considered in Tables 4-5. The mechanical and electrical loads applied on the lateral surfaces are specified as $\overline{q}^+_\zeta(\alpha,\ \beta) = q_0\ \sin\pi\alpha/a_\alpha$ and $\overline{\Phi}^+_\zeta(\alpha) = \phi_0\ \sin\pi\alpha/a_\alpha$. The elastic, piezoelectric and dielectric properties of piezoelectric materials used in Tables 4-5 [Cheng et al. (2000)] are slightly different from those properties in Table 1 where the values of c_{11}, c_{33} are replaced by 238.24 GPa, 10.64 GPa, and those of e_{24}, e_{15}, e_{32}, e_{33} by -0.009 C/m^2, -0.135 C/m^2, -0.145 C/m^2, -0.276 C/m^2. The geometry parameters are taken as $a_\alpha/2h$=4, 10, 100. The mechanical and electrical field variables are nondimensionalized in the same form as used in Cheng et al. (2000) where $\overline{u}_i = u_i c^*/q_0 a_\alpha$, $\overline{\tau}_{ij} = \tau_{ij}/q_0$, $\overline{\Phi} = \Phi/q_0 a_\alpha$, $\overline{D}_i = D_i c^*/q_0 e^*$ are used for the applied load cases; $\overline{u}_i = u_i c^*/\phi_0 e^*$, $\overline{\tau}_{ij} = \tau_{ij} a_\alpha/\phi_0 e^*$, $\overline{\Phi} = \Phi/\phi_0$, $\overline{D}_i = D_i c^* a_\alpha/\phi_0\ (e^*)^2$ for the applied potential cases and $c^* = 1$ N / m^2, $e^* = 1$ C / m^2. The present asymptotic solution of mechanical and electric field variables for the applied load and potential cases is presented in Tables 4-5, respectively. By comparing the present asymptotic solution with the 3D piezoelectricity solution,

Table 2 : Mechanical and electric field variables of laminated piezoelectric plates under lateral loads $(a_\alpha/2h = 10)$

ε	$u_\zeta(\frac{a_\alpha}{2}, \frac{a_\beta}{2}, -h)$	$D_\zeta(\frac{a_\alpha}{2}, \frac{a_\beta}{2}, -h)$	$\tau_{\alpha\zeta}(0, \frac{a_\beta}{2}, 0)$	$\tau_{\beta\zeta}(\frac{a_\alpha}{2}, 0, 0)$	$\sigma_\zeta(\frac{a_\alpha}{2}, \frac{a_\beta}{2}, 0)$	$\Phi(\frac{a_\alpha}{2}, \frac{a_\beta}{2}, 0)$
ε^0	5.0319e-12	1.11805e-11	1.2317	1.4519	0.40063	1.3735e-4
ε^2	6.6920e-12	0.77623e-11	1.1402	1.3223	0.35130	1.1332e-4
ε^4	6.4964e-12	0.81738e-11	1.1506	1.3390	0.35797	1.1747e-4
ε^6	6.5219e-12	0.81205e-11	1.1493	1.3367	0.35706	1.1685e-4
ε^{14}	6.5188e-12	0.81269e-11	1.1494	1.3370	0.35716	1.1693e-4
3D solutions [Heyliger (1997a)]	6.5167e-12	0.82096e-11	1.1491	1.3367	0.35705	1.1675e-4
Discrete-layer theory [Heyliger et al. (1994)]	6.5158e-12	0.82092e-11	1.1501	1.3385	0.35681	1.1675e-4

we conclude that the convergent solution yields at the ε^2-order level in the cases of thin laminates $(a_\alpha\ 2h = 100)$, at the ε^4-order level in the cases of moderately thick laminates $(a_\alpha/2h = 10)$ and at the ε^6-order level in the cases of thick laminates $(a_\alpha/2h = 4)$.

6.2 Doubly curved laminated piezoelectric shells

Tables 6-7 consider the doubly curved [0/90/0] laminated PVDF piezoelectric shells under either mechanical or electrical loads (i.e., $\overline{q}_\zeta^+(\alpha, \beta) = q_0\ \sin\pi\alpha/a_\alpha\ \sin\pi\beta/a_\beta$, $q_0 = 1\ \mathrm{N/m^2}$ or $\overline{\Phi}_\zeta^+(\alpha, \beta) = \Phi_0\ \sin\pi\alpha/a_\alpha\ \sin\pi\beta/a_\beta$, $\phi_0 = 1\ \mathrm{V}$) on the lateral surfaces. The elastic, piezoelectric and dielectric properties of piezoelectric materials used in Cheng et al. (2000) are adopted in the numerical applications. The geometry parameters are taken as $a_\alpha/a_\beta = 1$; $a_\alpha/2h$=10, 20, 100; R_α/a_α=1; R_β/a_β=1, 5, 10 and a_α=0.2 m. Based on the previous results, we realize that the ε^{14}-order solution is merely the 3D piezoelectricity solution. The present ε^{14}-order solution for the mechanical and electric field variables are given in Tables 6-7. For a fixed value of $a_\alpha/2h$, it is shown that the central deflection u_ζ increases as the shell is getting flat (i.e., the curvature radius increases) in the applied load cases, and variation of the central electric displacement with the curvature radius in the applied load cases is much more sensitive than that in the applied potential cases. To the best of the authors' knowledge, the 3D piezoelectricity solution of doubly curved laminated piezoelectric shells is lacking in the literature. The present solution in Tables 6-7 can be provided as the benchmark solution for assessing various approximate 2D shell theories. To have a more clear picture in the laminates, we present the distributions of the mechanical and electric field variables through the thickness of the shell in Figs 2-5 for the applied load cases and in Figs 6-9 for the applied potential cases. The geometry parameters are taken as $a_\alpha/a_\beta = 1$, $a_\alpha/2h$=10, R_α/a_α=1, R_β/a_β=1 and $2h$=0.0075 m. It is shown that the asymptotic solution yields continuous interlaminar mechanical and electric field variables and the lateral boundary conditions are satisfied exactly. It is observed from Figs. 5 and 9 that the electric potential is a higher-degree polynomial function through the thickness direction in the applied load cases and an almost linear function in the applied potential cases. The distributions of the transverse stresses through the thickness direction in the applied load cases are different from those in the applied potential cases.

7 Conclusions

The 3D asymptotic solutions for the mechanical and electric field variables of doubly curved laminated piezoelectric shells under either the lateral loads or the lateral potentials have been presented. The derivation is based on 3D linear piezoelectricity and requires neither kinematic nor static assumptions. The present asymptotic formulation is reduced to that of laminated piezoelectric plates by letting $1/R_\alpha = 1/R_\beta = 0$, and further to that of laminated piezoelectric strips by letting $1/a_\beta$=0. Applications to the benchmark problems show that accurate results are obtained by carrying out only two steps of the solution in the case of thin laminates $(a_\alpha/2h = 100)$, three steps in the case of moderately thick laminats$(a_\alpha/2h = 10)$ and

Table 3 : Mechanical and electric field variables of laminated piezoelectric plates under lateral potentials ($a_\alpha/2h = 10$)

ε	$u_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},-h)$	$D_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},-h)$	$\tau_{\alpha\zeta}(0,\frac{a_\beta}{2},0)$	$\tau_{\beta\zeta}(\frac{a_\alpha}{2},0,0)$	$\sigma_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},0)$	$\Phi(\frac{a_\alpha}{2},\frac{a_\beta}{2},0)$
ε^0	-0.1118e-10	-0.45167e-7	-1.5469	3.2518	0.2207	0.99308
ε^2	-2.0693e-10	-0.40982e-7	-8.1938	-6.2405	-4.6090	0.91784
ε^4	-2.0797e-10	-0.41223e-7	-6.6418	-4.2433	-3.5897	0.92288
ε^6	-2.0522e-10	-0.41214e-7	-6.9396	-4.5508	-3.7514	0.92251
ε^{14}	-2.0570e-10	-0.41214e-7	-6.8920	-4.5154	-3.7307	0.92253
3D solutions [Heyliger (1997a)]	-2.0599e-10	-0.41175e-7	-6.9023	-4.5247	-3.7377	0.92243
Discrete-layer theory [Heyliger et al. (1994)]	-2.0607e-10	-0.41174e-7	-6.9095	-4.5102	-3.7486	0.92243

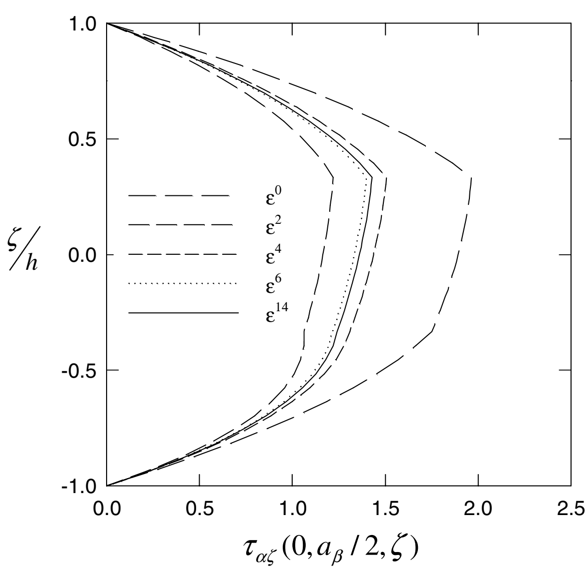

Figure 2 : Distribution of transverse shear stress through the thickness of [0/90/0] laminated PVDF piezoelectric shells under applied load.

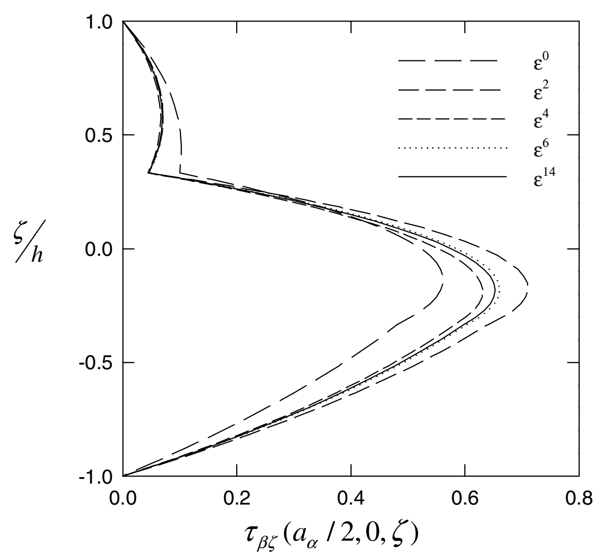

Figure 3 : Distribution of transverse shear stress through the thickness of [0/90/0] laminated PVDF piezoelectric shells under applied load.

Table 4 : Mechanical and electric field variables of [0/90/0] laminated PVDF piezoelectric strips under lateral loads

$a_\alpha/2h$	ε	$\bar{u}_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},0)$	$\bar{D}_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},0)$	$\bar{\tau}_{\alpha\zeta}(0,\frac{a_\beta}{2},\frac{h}{3})$	$\bar{\sigma}_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},\frac{h}{3})$	$\bar{\Phi}(\frac{a_\alpha}{2},\frac{a_\beta}{2},\frac{h}{3})$
4	ε^0	0.3429e-10	-0.13551e-23	1.7563	0.73178	0.0015194
	ε^2	2.0517e-10	-0.12922e-10	1.3607	0.67784	0.0041849
	ε^4	1.8120e-10	-0.12908e-10	1.5414	0.69730	0.0032791
	ε^6	1.9174e-10	-0.12771e-10	1.4587	0.68802	0.0036750
	ε^{14}	1.8857e-10	-0.12810e-10	1.4835	0.69088	0.0035560
	3D solutions [Cheng et al. (2000)]	1.8841e-10	-0.12811e-10	1.4847	0.69102	0.0035504
10	ε^0	5.3587e-10	-0.54991e-21	4.3908	0.73178	0.0037985
	ε^2	9.6306e-10	-0.12922e-10	4.2325	0.72315	0.0048647
	ε^4	9.5348e-10	-0.12920e-10	4.2441	0.72365	0.0048067
	ε^6	9.5415e-10	-0.12916e-10	4.2432	0.72361	0.0048108
	ε^{14}	9.5410e-10	-0.12916e-10	4.2433	0.72361	0.0048105
	3D solutions [Cheng et al. (2000)]	9.5409e-10	-0.12916e-10	4.2433	0.72360	0.0048103
100	ε^0	5358.7e-10	-0.55378e-17	43.908	0.73178	0.037985
	ε^2	5401.4e-10	-0.12922e-10	43.892	0.73169	0.038091
	ε^4	5401.4e-10	-0.12922e-10	43.892	0.73169	0.038091
	ε^6	5401.4e-10	-0.12922e-10	43.892	0.73169	0.038091
	ε^{14}	5401.4e-10	-0.12922e-10	43.892	0.73169	0.038091
	3D solutions [Cheng et al. (2000)]	5401.4e-10	-0.12922e-10	43.892	0.73168	0.038091

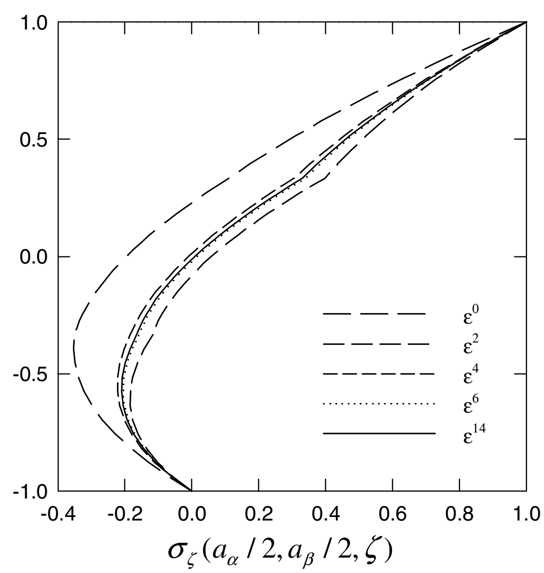

Figure 4 : Distribution of transverse normal stress through the thickness of [0/90/0] laminated PVDF piezoelectric shells under applied load.

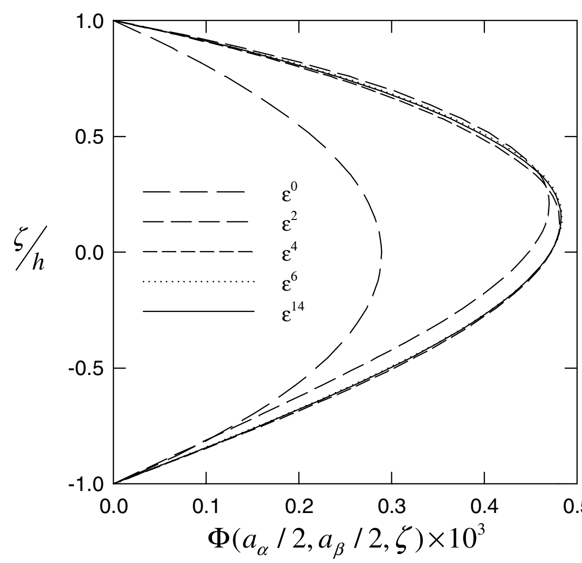

Figure 5 : Distribution of electric potential through the thickness of [0/90/0] laminated PVDF piezoelectric shells under applied load.

Table 5 : Mechanical and electric field variables of [0/90/0] laminated PVDF piezoelectric strips under lateral potentials

$^{a_\alpha}/_{2h}$	ε	$\overline{u}_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},0)$	$\overline{D}_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},0)$	$\overline{\tau}_{\alpha\zeta}(0,\frac{a_\beta}{2},\frac{h}{3})$	$\overline{\sigma}_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},\frac{h}{3})$	$\overline{\Phi}(\frac{a_\alpha}{2},\frac{a_\beta}{2},\frac{h}{3})$
4	ε^0	0.01480e-21	-4.5308e-10	0.043945	-0.0057523	0.66666
	ε^2	0.09357e-10	-4.4137e-10	0.062773	0.0023293	0.62940
	ε^4	0.10146e-10	-4.4169e-10	0.051545	-0.0013450	0.63157
	ε^6	0.09407e-10	-4.4164e-10	0.056991	0.0002296	0.63140
	ε^{14}	0.09644e-10	-4.4165e-10	0.055326	-0.0002255	0.63143
	3D solutions [Cheng et al. (2000)]	0.09654e-10	-4.4166e-10	0.055241	-0.0002430	0.63130
10	ε^0	0.05577e-20	-11.327e-10	0.043945	-0.0023009	0.66666
	ε^2	0.09357e-10	-11.280e-10	0.046957	-0.0017837	0.66070
	ε^4	0.09483e-10	-11.280e-10	0.046670	-0.0018213	0.66075
	ε^6	0.09464e-10	-11.280e-10	0.046692	-0.0018188	0.66075
	ε^{14}	0.09464e-10	-11.280e-10	0.046692	-0.0018188	0.66075
	3D solutions [Cheng et al. (2000)]	0.09466e-10	-11.281e-10	0.046689	-0.0018187	0.66083
100	ε^0	0.05539e-16	-113.27e-10	0.043944	-0.00023009	0.66666
	ε^2	0.09358e-10	-113.26e-10	0.043975	-0.00022958	0.66660
	ε^4	0.09358e-10	-113.26e-10	0.043975	-0.00022958	0.66660
	ε^6	0.09358e-10	-113.26e-10	0.043975	-0.00022958	0.66660
	ε^{14}	0.09358e-10	-113.26e-10	0.043975	-0.00022958	0.66660
	3D solutions [Cheng et al. (2000)]	0.09361e-10	-113.27e-10	0.043975	-0.00022958	0.66675

Table 6 : Mechanical and electric field variables of [0/90/0] laminated PVDF piezoelectric shells under lateral loads. ($a_\alpha = 0.2m$)

$\frac{a_\alpha}{2h}$	$\frac{R_\alpha}{a_\alpha}$	$\frac{R_\beta}{a_\beta}$	$u_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},0)$	$D_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},0)$	$\tau_{\alpha\zeta}(0,\frac{a_\beta}{2},\frac{h}{3})$	$\tau_{\beta\zeta}(\frac{a_\alpha}{2},0,\frac{h}{3})$	$\sigma_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},\frac{h}{3})$	$\Phi(\frac{a_\alpha}{2},\frac{a_\beta}{2},\frac{h}{3})$
100	1	1	8.5956e-10	-0.7952e-10	0.5529	-0.33215	0.61317	0.1954e-3
		5	23.428e-10	-1.1988e-10	1.2677	-0.47467	0.31751	0.4320e-3
		10	27.747e-10	-1.2887e-10	1.4659	-0.49984	0.24177	0.4996e-3
20	1	1	1.5313e-10	-1.7161e-11	1.4100	-0.11444	0.33514	0.8271e-3
		5	3.2646e-10	-1.3352e-11	2.8411	0.07839	-0.14013	1.4331e-3
		10	3.6440e-10	-1.2262e-11	3.1484	0.12810	-0.23384	1.5627e-3
10	1	1	0.6315e-10	-9.4087e-12	1.4294	0.04605	0.33011	1.2302e-3
		5	1.0403e-10	-5.5618e-12	2.2602	0.29502	0.06833	1.6126e-3
		10	1.1081e-10	-4.9377e-12	2.3953	0.33872	0.03270	1.6730e-3

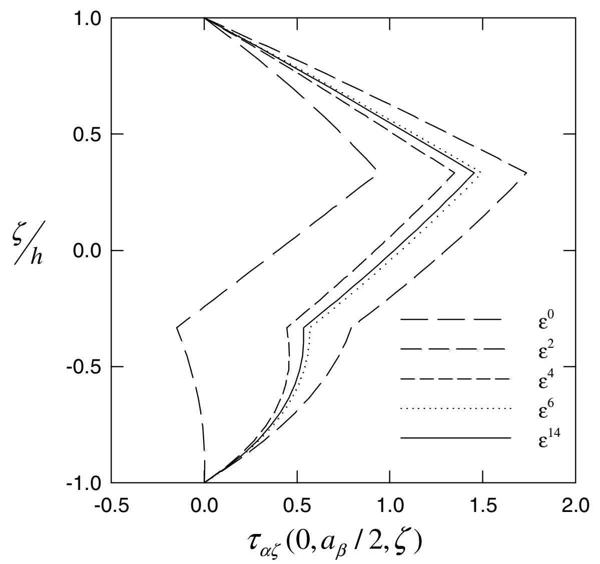

Figure 6 : Distribution of transverse shear stress through the thickness of [0/90/0] laminated PVDF piezoelectric shells under applied potential.

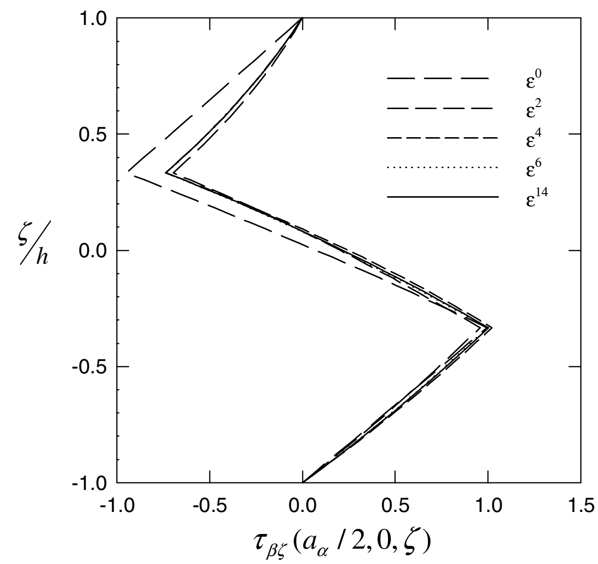

Figure 7 : Distribution of transverse shear stress through the thickness of [0/90/0] laminated PVDF piezoelectric shells under applied potential.

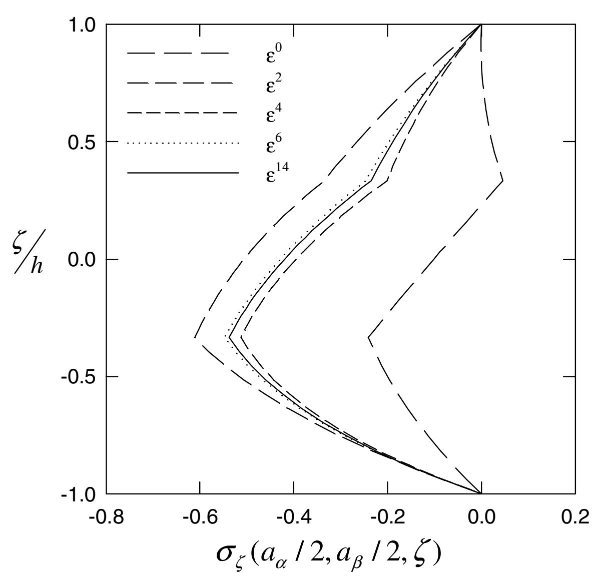

Figure 8 : Distribution of transverse normal stress through the thickness of [0/90/0] laminated PVDF piezoelectric shells under applied potential.

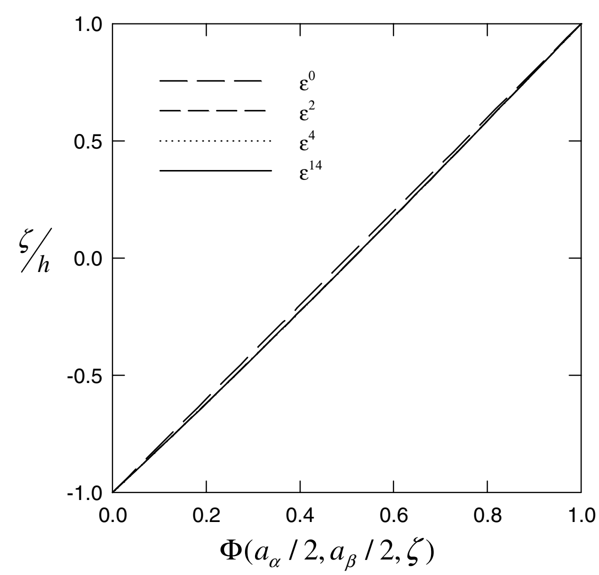

Figure 9 : Distribution of electric potential through the thickness of [0/90/0] laminated PVDF piezoelectric shells under applied potential.

Table 7 : Mechanical and electric field variables of [0/90/0] laminated PVDF piezoelectric shells under lateral potentials. ($a_\alpha = 0.2m$)

$\frac{a_\alpha}{2h}$	$\frac{R_\alpha}{a_\alpha}$	$\frac{R_\beta}{a_\beta}$	$u_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},0)$	$D_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},0)$	$\tau_{\alpha\zeta}(0,\frac{a_\beta}{2},\frac{h}{3})$	$\tau_{\beta\zeta}(\frac{a_\alpha}{2},0,\frac{h}{3})$	$\sigma_\zeta(\frac{a_\alpha}{2},\frac{a_\beta}{2},\frac{h}{3})$	$\Phi(\frac{a_\alpha}{2},\frac{a_\beta}{2},\frac{h}{3})$
100	1	1	66.064e-12	-5.66669e-8	0.34327	-0.35936	0.00694	0.66883
		5	106.84e-12	-5.66676e-8	0.35618	-0.36530	-0.08813	0.66793
		10	115.67e-12	-5.66676e-8	0.35917	-0.36572	-0.10070	0.66781
20	1	1	5.5091e-12	-1.13040e-8	0.60787	-0.29727	-0.09360	0.67494
		5	3.9319e-12	-1.13076e-8	0.58524	-0.33845	-0.15893	0.67042
		10	3.3073e-12	-1.13078e-8	0.57878	-0.34821	-0.16757	0.66985
10	1	1	-0.0219e-12	-0.56144e-8	0.54564	-0.27606	-0.08794	0.67706
		5	-0.9711e-12	-0.56199e-8	0.52481	-0.36087	-0.14170	0.66818
		10	-1.1367e-12	-0.56200e-8	0.52113	-0.37229	-0.15125	0.66706

four steps in the case of thick laminats($a_\alpha/2h = 4$). In general, performing two steps of the asymptotic solution is sufficient to yield acceptable results and the present ε^{14}-order solution is merely the 3D piezoelectricity solution for a wide range of various geometry parameters.

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Appendix A:

The relevant functions $\tilde{l}_{ij}$ in Eqs.(14)-(19) are given as follows:

$$\tilde{l}_{31} = \left(\frac{e_{31}e_{33}+c_{13}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{1}{\gamma_\alpha}\frac{\partial}{\partial x} + \left(\frac{e_{33}e_{36}+c_{36}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{1}{\gamma_\beta}\frac{\partial}{\partial y}, \tag{94}$$

$$\tilde{l}_{32} = \left(\frac{e_{36}e_{33}+c_{36}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{1}{\gamma_\alpha}\frac{\partial}{\partial x} + \left(\frac{e_{33}e_{32}+c_{23}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{1}{\gamma_\beta}\frac{\partial}{\partial y}, \tag{95}$$

$$\tilde{l}_{33} = \left(\frac{e_{31}e_{33}+c_{13}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{1}{\gamma_\alpha R_x} + \left(\frac{e_{33}e_{32}+c_{23}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{1}{\gamma_\beta R_y}, \tag{96}$$

$$\tilde{l}_{34} = l_{34}Q, \quad \tilde{l}_{35} = l_{35}e, \quad \tilde{l}_{11} = (1-z\partial_z)/R_x, \quad \tilde{l}_{22} = (1-z\partial_z)/R_y\,, \tag{97}$$

$$\tilde{l}_{16} = l_{14}Qz/R_x \quad \tilde{l}_{17} = l_{15}Qz/R_x, \quad \tilde{l}_{26} = l_{15}Qz/R_y \quad \tilde{l}_{27} = l_{25}Qz/R_y, \tag{98}$$

$$\tilde{l}_{18} = \left(\frac{c_{45}e_{14}-c_{44}e_{15}}{c_{44}c_{55}-c_{45}^2}\right)\frac{Q}{e}\frac{\partial}{\partial x} + \left(\frac{c_{45}e_{24}-c_{44}e_{25}}{c_{44}c_{55}-c_{45}^2}\right)\frac{Q\gamma_\alpha}{e\gamma_\beta}\frac{\partial}{\partial y}, \tag{99}$$

$$\tilde{l}_{28} = \left(\frac{c_{45}e_{15}-c_{55}e_{14}}{c_{44}c_{55}-c_{45}^2}\right)\frac{Q\gamma_\beta}{e\gamma_\alpha}\frac{\partial}{\partial x} + \left(\frac{c_{45}e_{25}-c_{55}e_{24}}{c_{44}c_{55}-c_{45}^2}\right)\frac{Q}{e}\frac{\partial}{\partial y}, \tag{100}$$

$$\tilde{l}_{41} = \frac{\tilde{Q}_{11}\gamma_\beta}{\gamma_\alpha}\frac{\partial^2}{\partial x^2} + \left(\tilde{Q}_{16}+\tilde{Q}_{61}\right)\frac{\partial^2}{\partial x\partial y} + \frac{\tilde{Q}_{66}\gamma_\alpha}{\gamma_\beta}\frac{\partial^2}{\partial y^2}, \tag{101}$$

$$\tilde{l}_{42} = \frac{\tilde{Q}_{16}\gamma_\beta}{\gamma_\alpha}\frac{\partial^2}{\partial x^2} + (\tilde{Q}_{12}+\tilde{Q}_{66})\frac{\partial^2}{\partial x\partial y} + \frac{\tilde{Q}_{62}\gamma_\alpha}{\gamma_\beta}\frac{\partial^2}{\partial y^2}, \tag{102}$$

$$\tilde{l}_{51} = \frac{\tilde{Q}_{61}\gamma_\beta}{\gamma_\alpha}\frac{\partial^2}{\partial x^2} + (\tilde{Q}_{21}+\tilde{Q}_{66})\frac{\partial^2}{\partial x\partial y} + \frac{\tilde{Q}_{26}\gamma_\alpha}{\gamma_\beta}\frac{\partial^2}{\partial y^2}, \tag{103}$$

$$\tilde{l}_{52} = \frac{\tilde{Q}_{66}\gamma_\beta}{\gamma_\alpha}\frac{\partial^2}{\partial x^2} + \left(\tilde{Q}_{26}+\tilde{Q}_{62}\right)\frac{\partial^2}{\partial x\partial y} + \frac{\tilde{Q}_{22}\gamma_\alpha}{\gamma_\beta}\frac{\partial^2}{\partial y^2}, \tag{104}$$

$$\tilde{l}_{43} = \left(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{12}}{R_y}\right)\frac{\partial}{\partial x} + \left(\frac{\tilde{Q}_{61}}{R_x} + \frac{\tilde{Q}_{62}\gamma_\alpha}{R_y\gamma_\beta}\right)\frac{\partial}{\partial y}, \tag{105}$$

$$\tilde{l}_{53} = \left(\frac{\tilde{Q}_{61}\gamma_\beta}{R_x\gamma_\alpha} + \frac{\tilde{Q}_{62}}{R_y}\right)\frac{\partial}{\partial x} + \left(\frac{\tilde{Q}_{21}}{R_x} + \frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta}\right)\frac{\partial}{\partial y}, \tag{106}$$

$$\tilde{l}_{44} = \frac{2}{R_x} + \frac{1}{R_y} + \frac{3hz}{R_xR_yR} + \left(\frac{z}{R_x}+\frac{z}{R_y}\right)\frac{\partial}{\partial z} + \left(\frac{hz^2}{R_xR_yR}\right)\frac{\partial}{\partial z}, \tag{107}$$

$$\tilde{l}_{55} = \frac{1}{R_x} + \frac{2}{R_y} + \frac{3hz}{R_xR_yR} + \left(\frac{z}{R_x}+\frac{z}{R_y}\right)\frac{\partial}{\partial z} + \left(\frac{hz^2}{R_xR_yR}\right)\frac{\partial}{\partial z}, \tag{108}$$

$$\tilde{l}_{46} = \left(\frac{c_{13}e_{33}-e_{31}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{\gamma_\beta e}{Q}\frac{\partial}{\partial x} + \left(\frac{c_{36}e_{33}-e_{36}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{\gamma_\alpha e}{Q}\frac{\partial}{\partial y}, \tag{109}$$

$$\tilde{l}_{56} = \left(\frac{c_{36}e_{33}-e_{36}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{\gamma_\beta e}{Q}\frac{\partial}{\partial x} + \left(\frac{c_{23}e_{33}-e_{32}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{\gamma_\alpha e}{Q}\frac{\partial}{\partial y}, \tag{110}$$

$$\tilde{l}_{61}=(\frac{\tilde{Q}_{11}\gamma_\beta}{R_x\gamma_\alpha}+\frac{\tilde{Q}_{21}}{R_y})\frac{\partial}{\partial x}+(\frac{\tilde{Q}_{16}}{R_x}+\frac{\tilde{Q}_{26}\gamma_\alpha}{R_y\gamma_\beta})\frac{\partial}{\partial y}, \quad (111)$$

$$\tilde{l}_{62}=(\frac{\tilde{Q}_{16}\gamma_\beta}{R_x\gamma_\alpha}+\frac{\tilde{Q}_{26}}{R_y})\frac{\partial}{\partial x}+(\frac{\tilde{Q}_{12}}{R_x}+\frac{\tilde{Q}_{22}\gamma_\alpha}{R_y\gamma_\beta})\frac{\partial}{\partial y}, \quad (112)$$

$$\tilde{l}_{63}=\frac{\tilde{Q}_{11}\gamma_\beta}{R_x^2\gamma_\alpha}+\frac{(\tilde{Q}_{12}+\tilde{Q}_{21})}{R_xR_y}+\frac{\tilde{Q}_{22}\gamma_\alpha}{R_y^2\gamma_\beta}\quad , \quad (113)$$

$$\begin{aligned}\tilde{l}_{64}=&\left(\frac{1}{R_x}+\frac{1}{R_y}+\frac{2\,hz}{R_xR_yR}\right)-\frac{\gamma_\beta}{R_x}\left(\frac{e_{31}e_{33}+c_{13}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\\&-\frac{\gamma_\alpha}{R_y}\left(\frac{e_{32}e_{33}+c_{23}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\\&+\left(\frac{z}{R_x}+\frac{z}{R_y}\right)\frac{\partial}{\partial z}+\frac{h}{R}\left(\frac{z^2}{R_xR_y}\right)\frac{\partial}{\partial z}, \quad (114)\end{aligned}$$

$$\begin{aligned}\tilde{l}_{65}=&\frac{\gamma_\beta e}{R_xQ}\left(\frac{c_{13}e_{33}-e_{31}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\\&+\frac{\gamma_\alpha e}{R_yQ}\left(\frac{c_{23}e_{33}-e_{32}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right), \quad (115)\end{aligned}$$

$$\tilde{l}_{71}=\frac{1}{\gamma_\alpha R_x}+\frac{1}{\gamma_\beta R_y},\quad \tilde{l}_{81}=-\left(\frac{c_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)\frac{e^2}{Q}, \quad (116)$$

$$\begin{aligned}&\tilde{Q}_{ij}=\frac{Q_{ij}}{Q},\\&Q_{ij}=c_{ij}-\left(\frac{e_{33}e_{3j}+c_{j3}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}\right)c_{i3}\\&-\left(\frac{e_{33}c_{j3}-c_{33}e_{3j}}{e_{33}^2+c_{33}\eta_{33}}\right)e_{3i}\quad (i,j=1,2,6). \quad (117)\end{aligned}$$

The relevant functions $\tilde{b}_{ij}$ in Eqs.(20)-(21) are given by

$$\tilde{b}_{11}=\frac{\tilde{Q}_{11}}{\gamma_\alpha}\frac{\partial}{\partial x}+\frac{\tilde{Q}_{16}}{\gamma_\beta}\frac{\partial}{\partial y},\quad \tilde{b}_{12}=\frac{\tilde{Q}_{16}}{\gamma_\alpha}\frac{\partial}{\partial x}+\frac{\tilde{Q}_{12}}{\gamma_\beta}\frac{\partial}{\partial y}, \quad (118)$$

$$\tilde{b}_{21}=\frac{\tilde{Q}_{21}}{\gamma_\alpha}\frac{\partial}{\partial x}+\frac{\tilde{Q}_{26}}{\gamma_\beta}\frac{\partial}{\partial y},\quad \tilde{b}_{22}=\frac{\tilde{Q}_{26}}{\gamma_\alpha}\frac{\partial}{\partial x}+\frac{\tilde{Q}_{22}}{\gamma_\beta}\frac{\partial}{\partial y}, \quad (119)$$

$$\tilde{b}_{31}=\frac{\tilde{Q}_{61}}{\gamma_\alpha}\frac{\partial}{\partial x}+\frac{\tilde{Q}_{66}}{\gamma_\beta}\frac{\partial}{\partial y},\quad \tilde{b}_{32}=\frac{\tilde{Q}_{66}}{\gamma_\alpha}\frac{\partial}{\partial x}+\frac{\tilde{Q}_{62}}{\gamma_\beta}\frac{\partial}{\partial y}, \quad (120)$$

$$\begin{aligned}&\tilde{b}_{13}=\frac{\tilde{Q}_{11}}{\gamma_\alpha R_x}+\frac{\tilde{Q}_{12}}{\gamma_\beta R_y},\quad \tilde{b}_{23}=\frac{\tilde{Q}_{21}}{\gamma_\alpha R_x}+\frac{\tilde{Q}_{22}}{\gamma_\beta R_y},\\&\tilde{b}_{33}=\frac{\tilde{Q}_{61}}{\gamma_\alpha R_x}+\frac{\tilde{Q}_{62}}{\gamma_\beta R_y}, \quad (121)\end{aligned}$$

$$\begin{aligned}&\tilde{b}_{14}=\frac{e_{31}e_{33}+c_{13}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}},\quad \tilde{b}_{24}=\frac{e_{32}e_{33}+c_{23}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}},\\&\tilde{b}_{34}=\frac{e_{36}e_{33}+c_{36}\eta_{33}}{e_{33}^2+c_{33}\eta_{33}}, \quad (122)\end{aligned}$$

$$\begin{aligned}&\tilde{b}_{15}=\frac{e}{Q}\frac{c_{13}e_{33}-e_{31}c_{33}}{e_{33}^2+c_{33}\eta_{33}},\quad \tilde{b}_{25}=\frac{e}{Q}\frac{c_{23}e_{33}-e_{32}c_{33}}{e_{33}^2+c_{33}\eta_{33}},\\&\tilde{b}_{35}=\frac{e}{Q}\frac{c_{36}e_{33}-e_{36}c_{33}}{e_{33}^2+c_{33}\eta_{33}}, \quad (123)\end{aligned}$$

$$\tilde{b}_{41}=\frac{e}{Q}\frac{c_{44}e_{15}-c_{45}e_{14}}{c_{44}c_{55}-c_{45}^2},\quad \tilde{b}_{42}=\frac{e}{Q}\frac{c_{55}e_{15}-c_{45}e_{15}}{c_{44}c_{55}-c_{45}^2}, \quad (124)$$

$$\tilde{b}_{51}=\frac{e}{Q}\frac{c_{44}e_{25}-c_{45}e_{24}}{c_{44}c_{55}-c_{45}^2},\quad \tilde{b}_{52}=\frac{e}{Q}\frac{c_{55}e_{24}-c_{45}e_{25}}{c_{44}c_{55}-c_{45}^2}, \quad (125)$$

$$\begin{aligned}\tilde{b}_{43}=&\left(\begin{array}{l}\frac{c_{45}e_{15}-c_{55}e_{14}}{c_{44}c_{55}-c_{45}^2}e_{14}\\+\frac{c_{45}e_{14}-c_{44}e_{15}}{c_{44}c_{55}-c_{45}^2}e_{15}-\eta_{11}\end{array}\right)\frac{Q}{e^2\gamma_\alpha}\frac{\partial}{\partial x}\\&+\left(\begin{array}{l}\frac{c_{45}e_{25}-c_{55}e_{24}}{c_{44}c_{55}-c_{45}^2}e_{14}\\+\frac{c_{45}e_{24}-c_{44}e_{25}}{c_{44}c_{55}-c_{45}^2}e_{15}-\eta_{12}\end{array}\right)\frac{Q}{e^2\gamma_\beta}\frac{\partial}{\partial y} \quad (126)\end{aligned}$$

$$\begin{aligned}\tilde{b}_{53}=&\left(\begin{array}{l}\frac{c_{45}e_{15}-c_{55}e_{14}}{c_{44}c_{55}-c_{45}^2}e_{24}\\+\frac{c_{45}e_{14}-c_{44}e_{15}}{c_{44}c_{55}-c_{45}^2}e_{25}-\eta_{12}\end{array}\right)\frac{Q}{e^2\gamma_\alpha}\frac{\partial}{\partial x}\\&+\left(\begin{array}{l}\frac{c_{45}e_{25}-c_{55}e_{24}}{c_{44}c_{55}-c_{45}^2}e_{24}\\+\frac{c_{45}e_{24}-c_{44}e_{25}}{c_{44}c_{55}-c_{45}^2}e_{25}-\eta_{22}\end{array}\right)\frac{Q}{e^2\gamma_\beta}\frac{\partial}{\partial y}. \quad (127)\end{aligned}$$

Appendix B:

The mechanical and electric field variables of the ε^{2k}-order corrections (k=0,1,2,...) are given by

$$u^{(k)}=\tilde{u}_{mn}^{(k)}\cos\tilde{m}x\ \sin\tilde{n}y, \quad (128)$$

$$v^{(k)}=\tilde{v}_{mn}^{(k)}\ \sin\tilde{m}x\ \cos\tilde{n}y, \quad (129)$$

$$w^{(k)}=\tilde{w}_{mn}^{(k)}\ \sin\tilde{m}x\ \sin\tilde{n}y, \quad (130)$$

$$D_x^{(k)}=\tilde{D}_{xmn}^{(k)}\ \cos\tilde{m}x\ \sin\tilde{n}y, \quad (131)$$

$$D_y^{(k)}=\tilde{D}_{ymn}^{(k)}\ \sin\tilde{m}x\ \cos\tilde{n}y, \quad (132)$$

$$D_z^{(k)}=\tilde{D}_{zmn}^{(k)}\ \sin\tilde{m}x\ \sin\tilde{n}y, \quad (133)$$

$$\sigma_x^{(k)} = \tilde{\sigma}_{xmn}^{(k)} \; \sin\tilde{m}x \; \sin\tilde{n}y, \tag{134}$$

$$\sigma_y^{(k)} = \tilde{\sigma}_{ymn}^{(k)} \sin\tilde{m}x \; \sin\tilde{n}y, \tag{135}$$

$$\sigma_z^{(k)} = \tilde{\sigma}_{zmn}^{(k)} \; \sin\tilde{m}x \; \sin\tilde{n}y, \tag{136}$$

$$\tau_{xz}^{(k)} = \tilde{\tau}_{xzmn}^{(k)} \; \cos\tilde{m}x \; \sin\tilde{n}y, \tag{137}$$

$$\tau_{yz}^{(k)} = \tilde{\tau}_{yzmn}^{(k)} \; \sin\tilde{m}x \; \cos\tilde{n}y, \tag{138}$$

$$\tau_{xy}^{(k)} = \tilde{\tau}_{xymn}^{(k)} \; \cos\tilde{m}x \cos\tilde{n}y, \tag{139}$$

$$\phi^{(k)} = \tilde{\phi}_{mn}^{(k)} \sin\tilde{m}x \; \sin\tilde{n}y, \tag{140}$$

and the relevant functions are

$$\varphi_{1k} = \tilde{\varphi}_{1k} \cos\tilde{m}x \; \sin\tilde{n}y, \tag{141}$$

$$\varphi_{2k} = \tilde{\varphi}_{2k} \sin\tilde{m}x \; \cos\tilde{n}y, \tag{142}$$

$$\varphi_{3k} = \tilde{\varphi}_{3k} \sin\tilde{m}x \; \sin\tilde{n}y, \tag{143}$$

$$\varphi_{4k} = \tilde{\varphi}_{4k} \sin\tilde{m}x \; \sin\tilde{n}y, \tag{144}$$

$$f_{1k} = \tilde{f}_{1k} \cos\tilde{m}x \; \sin\tilde{n}y, \tag{145}$$

$$f_{2k} = \tilde{f}_{2k} \sin\tilde{m}x \; \cos\tilde{n}y, \tag{146}$$

$$f_{3k} = \tilde{f}_{3k} \sin\tilde{m}x \; \sin\tilde{n}y, \tag{147}$$

$$f_{4k} = \tilde{f}_{4k} \sin\tilde{m}x \; \sin\tilde{n}y, \tag{148}$$

where

$$\tilde{u}^{(k)} = \tilde{u}^k - z\tilde{m}\tilde{w}^k + \tilde{\varphi}_{1k},$$

$$\tilde{v}^{(k)} = \tilde{v}^k - z\tilde{n}\tilde{w}^k + \tilde{\varphi}_{2k},$$

$$\tilde{w}^{(k)} = \tilde{w}^k + \tilde{\varphi}_{3k},$$

$$\tilde{D}_x^{(k)} = \frac{e_{15}Q}{c_{55}e}\tilde{\tau}_{xz}^{(k)} - \tilde{m}\left(\frac{e_{15}^2}{c_{55}} + \eta_{11}\right)\frac{Q}{\gamma_\alpha e^2}\tilde{\phi}^{(k)},$$

$$\tilde{D}_y^{(k)} = \frac{e_{24}Q}{c_{44}e}\tilde{\tau}_{yz}^{(k)} - \tilde{n}\left(\frac{e_{24}^2}{c_{44}} + \eta_{22}\right)\frac{Q}{\gamma_\beta e^2}\tilde{\phi}^{(k)},$$

$$\tilde{D}_z^{(k)} = \tilde{D}_z^k + \tilde{\varphi}_{4k},$$

$$\begin{aligned}\tilde{\sigma}_x^{(k)} = &-\frac{\tilde{Q}_{11}\tilde{m}}{\gamma_\alpha}\tilde{u}^{(k)} - \frac{\tilde{Q}_{12}\tilde{n}}{\gamma_\beta}\tilde{v}^{(k)} \\ &+\left(\frac{\tilde{Q}_{11}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{12}}{\gamma_\beta R_y}\right)\tilde{w}^{(k)} \\ &+\frac{e}{Q}\left(\frac{e_{33}c_{31} - e_{31}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)\tilde{D}_z^{(k)} \\ &+\left(\frac{e_{33}e_{31} + \eta_{33}c_{13}}{e_{33}^2 + \eta_{33}c_{33}}\right)\tilde{\sigma}_z^{(k-1)},\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}_y^{(k)} = &-\frac{\tilde{Q}_{21}\tilde{m}}{\gamma_\alpha}\tilde{u}^{(k)} - \frac{\tilde{Q}_{22}\tilde{n}}{\gamma_\beta}\tilde{v}^{(k)} \\ &+\left(\frac{\tilde{Q}_{21}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{22}}{\gamma_\beta R_y}\right)\tilde{w}^{(k)} \\ &+\frac{e}{Q}\left(\frac{e_{33}c_{32} - e_{32}c_{33}}{e_{33}^2 + \eta_{33}c_{33}}\right)\tilde{D}_z^{(k)} \\ &+\left(\frac{e_{33}e_{32} + \eta_{33}c_{23}}{e_{33}^2 + \eta_{33}c_{33}}\right)\tilde{\sigma}_z^{(k-1)},\end{aligned}$$

$$\tilde{\tau}_{xy}^{(k)} = \frac{\tilde{Q}_{66}\tilde{n}}{\gamma_\beta}\tilde{u}^{(k)} + \frac{\tilde{Q}_{66}\tilde{m}}{\gamma_\alpha}\tilde{v}^{(k)},$$

$$\tilde{\tau}_{xz}^{(k)} = \tilde{f}_{1k} + \int_{-1}^{z}\left\{\begin{array}{l}\left(\gamma_\beta\tilde{m}^2\tilde{Q}_{11}/\gamma_\alpha + \gamma_\alpha\tilde{n}^2\tilde{Q}_{66}/\gamma_\beta\right)\tilde{u}^{(k)} \\ +\tilde{m}\tilde{n}\left(\tilde{Q}_{12} + \tilde{Q}_{66}\right)\tilde{v}^{(k)} \\ -\tilde{m}\left(\gamma_\beta\tilde{Q}_{11}/\gamma_\alpha R_x + \tilde{Q}_{12}/R_y\right)\tilde{w}^{(k)} \\ -\tilde{m}\frac{e\gamma_\beta}{Q}\left(\frac{e_{33}c_{13} - e_{31}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\tilde{D}_z^{(k)}\end{array}\right\}dz,$$

$$\tilde{\tau}_{yz}^{(k)} = \tilde{f}_{2k} + \int_{-1}^{z}\left\{\begin{array}{l}\tilde{m}\tilde{n}\left(\tilde{Q}_{21} + \tilde{Q}_{66}\right)\tilde{u}^{(k)} \\ +\left(\gamma_\beta\tilde{m}^2\tilde{Q}_{66}/\gamma_\alpha + \gamma_\alpha\tilde{n}^2\tilde{Q}_{22}/\gamma_\beta\right)\tilde{v}^{(k)} \\ -\tilde{n}\left(\tilde{Q}_{21}/R_x + \gamma_\alpha\tilde{Q}_{22}/\gamma_\beta R_y\right)\tilde{w}^{(k)} \\ -\tilde{n}\frac{e\gamma_\alpha}{Q}\left(\frac{e_{33}c_{23} - e_{32}c_{33}}{e_{33}^2 + c_{33}\eta_{33}}\right)\tilde{D}_z^{(k)}\end{array}\right\}dz,$$

$$\begin{aligned}\tilde{\sigma}_z^{(k)} = &-\left(\frac{z}{R_x} + \frac{z}{R_y} + \frac{hz^2}{RR_xR_y}\right)\tilde{\sigma}_z^{(k-1)} \\ &+\int_{-1}^{z}\left\{\begin{array}{l}-\tilde{m}\left(\frac{\gamma_\beta\tilde{Q}_{11}}{\gamma_\alpha R_x} + \frac{\tilde{Q}_{21}}{R_y}\right)\tilde{u}^{(k)} \\ -\tilde{n}\left(\frac{\tilde{Q}_{12}}{R_x} + \frac{\gamma_\alpha\tilde{Q}_{22}}{\gamma_\beta R_y}\right)\tilde{v}^{(k)} \\ +\left[\frac{\gamma_\beta\tilde{Q}_{11}}{\gamma_\alpha R_x^2} + \frac{\tilde{Q}_{12}+\tilde{Q}_{21}}{R_xR_y} + \frac{\gamma_\alpha\tilde{Q}_{22}}{\gamma_\beta R_y^2}\right]\tilde{w}^{(k)} \\ -\left(\frac{\gamma_\beta b_{13}}{R_x} + \frac{\gamma_\alpha b_{23}}{R_y}\right)\frac{e}{Q}\tilde{D}_z^{(k)} + \tilde{m}\tilde{\tau}_{xz}^{(k)} \\ +\tilde{n}\tilde{\tau}_{yz}^{(k)} + \frac{\tilde{m}z}{R_y}\tilde{\tau}_{xz}^{(k-1)} + \frac{\tilde{n}z}{R_x}\tilde{\tau}_{yz}^{(k-1)} \\ +\left[\begin{array}{l}\frac{\gamma_\beta}{R_x}\frac{e_{33}e_{31}+\eta_{33}c_{31}}{e_{33}^2+\eta_{33}c_{33}} \\ +\frac{\gamma_\alpha}{R_y}\frac{e_{33}e_{32}+\eta_{33}c_{32}}{e_{33}^2+\eta_{33}c_{33}}\end{array}\right]\tilde{\sigma}_z^{(k-1)}\end{array}\right\}dz,\end{aligned}$$

$$\tilde{\phi}^{(k)} = \int_{-1}^{z}\left\{\begin{array}{l}\frac{e}{Q}\frac{\tilde{m}}{\gamma_\alpha}\frac{e_{33}c_{13}-e_{31}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\tilde{u}^{(k)} \\ +\frac{e}{Q}\frac{\tilde{n}}{\gamma_\beta}\frac{e_{33}c_{23}-e_{32}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\tilde{v}^{(k)} \\ -\frac{e}{Q\gamma_\alpha R_x}\frac{e_{33}c_{13}-e_{31}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\tilde{w}^{(k)} \\ -\frac{e}{Q\gamma_\beta R_y}\frac{e_{33}c_{23}-e_{32}c_{33}}{e_{33}^2+c_{33}\eta_{33}}\tilde{w}^{(k)} \\ -\frac{c_{33}}{Q}\frac{e^2}{e_{33}^2+c_{33}\eta_{33}}\tilde{D}_z^{(k)} \\ +\frac{e_{33}e}{e_{33}^2+c_{33}\eta_{33}}\tilde{\sigma}_z^{(k-1)}\end{array}\right\}dz.$$