

# Green's First Identity Method for Boundary-Only Solution of Self-Weight in BEM Formulation for Thick Slabs

Youssef F. Rashed<sup>1</sup>

**Abstract:** The present paper develops a new technique for treatment of self-weight for building slabs in the boundary element method (BEM). Due to the use of BEM in the analysis, all defined variables are presented on the slab boundary (mesh is defined only along the slab boundary). Self-weight, however, is usually defined over slab domain, hence domain discretisation is required, which spoils the main advantage of the BEM. In this paper a new method is presented to transform self-weight domain integrals to the boundary for such slabs. The proposed method is based on using the so-called Green's first identity. All new kernels for generalized displacements, stress-resultants, and tractions are derived and listed explicitly. The present formulation is implemented into computer code and several examples are tested. Results are compared against results obtained from other numerical method to prove the accuracy and validity of the present formulation.

**keyword:** Boundary element method, self-weight, plate bending, slabs, Green's first identity.

## 1 Introduction

Typical structural analysis of building floors involves many types of self-weight and superimposed loading. These loading are structurally modelled as uniform pressure over the overall plate domain. In finite element method (FEM) models of structures, such loading can be modelled easily as pressure over domain elements. In the boundary element method (BEM) modelling, on the other hand, the technique requires the discretisation of the floor boundary only. Therefore treatment of self-weight loading requires additional discretisation of the plate domain leading to similar mesh as that of the FEM. Hence the boundary-only discretisation advantage of the

BEM is spoiled.

Many researchers have considered transforming self-weight domain integrals to the boundary to avoid the additional domain discretisation. Vander Weeën (1982) used Green's second identity or the first term of the multiple reciprocity method to avoid such domain integrals. El-Zafrany et al. (1994) transformed such integrals using strain function representation of the displacement fundamental solution, which is equivalent to the use of the operator decoupling technique or the Helmholtz representation of a function. Wen et al. (1999) used alternative numerical integration in polar coordinates to treat domain integrals in boundary way. The formulation of Wen et al. (1999) needs transforming the origin to each collocation point on the boundary, which can be regarded as numerically inefficient. Recently Rashed (2000) treated such problem using particular integrals, which is not general method. A summary of such techniques is given by Brebbia and Rashed (2003).

The present paper is concerned of using a new method to treat self-weight domain loadings using the application of Green's first identity theory. The main advantage of using the present technique is the simplicity of the transformed kernels and it guarantees smoothness of the derived boundary integrals. All necessary kernels are derived and given in explicit form for further use. The proposed method is implemented into computer code and two examples are tested. The results are compared to traditional domain integral formulation of the BEM and also to results obtained from the FEM.

## 2 Problem definition

The boundary integral equation for an arbitrary plate governed by the Reissner theory can be written in matrix form as follows:

$$[H]\{u\} = [G]\{t\} + \{Q\} \quad (1)$$

where  $[H]$  and  $[G]$  are the well-known boundary element

<sup>1</sup> Structural Engineering Department,  
Cairo University, Giza 12613, Egypt  
Tel. 00 20 10 511 2949  
E-mail: youssef@eng.cu.edu.eg

influence matrices and  $\{u\}$  and  $\{t\}$  are the vectors of the boundary generalized displacements and tractions respectively. The vector  $\{Q\}$  is the vector of domain loading which is represented by three values for each collocation node, i.e.:

$$Q_i(\xi) = \int_{\Omega} \left[ U_{i3}(\xi, X) - \frac{\nu}{(1-\nu)\lambda^2} U_{ij,j}(\xi, X) \right] q(X) d\Omega \quad (2)$$

Where  $U_{ij}$  is the expression of the fundamental solution for the displacement (see Vander Weeën (1982)),  $i=1,2$  to represent rotational terms and  $i=3$  to represent the deflection term. The symbols  $\nu$ ,  $\lambda$  denote Poisson's ratio and the shear factor respectively. Equation 2 is written in terms of two-point notation, where  $\xi$ ,  $X$  denote the source and the field point respectively. The point  $X$  is domain point.  $q$  is the plate self-weight and assumed constant. In this paper the indicial notation is used.

In order to evaluate the integral in Eq.2, the domain  $\Omega$  has to be discretised to cells and hence it is computed numerically using certain number of Gauss points (G.PT.). This paper concerns with transforming this integral to the boundary.

The second term of the integral in Eq.2 can be directly transformed to the boundary as follows:

$$\begin{aligned} & q \frac{\nu}{(1-\nu)\lambda^2} \int_{\Omega} U_{ij,j}(\xi, X) d\Omega \\ &= q \frac{\nu}{(1-\nu)\lambda^2} \int_{\Gamma} U_{ij}(\xi, x) n_j(x) d\Gamma \end{aligned} \quad (3)$$

where  $n_j(x)$  are the component of the normal at the point  $x$ . The first term of the integral in Eq.2, can be treated using Green's first identity as follows:

$$\begin{aligned} & \int_{\Omega} U_{i3}(\xi, X) d\Omega = \int_{\Omega} G_{i3\theta,\theta}(\xi, X) d\Omega \\ &= \int_{\Gamma} G_{i3\theta}(\xi, x) n_{\theta}(x) d\Gamma \end{aligned} \quad (4)$$

Where the new kernels  $G_{i3\theta}$ , will be computed in the next section.

### 3 New equivalent boundary kernels

The main idea of this paper is to represent the fundamental solution term inside the domain integral of Eq.2 in

terms of the gradient of another functions to allow the application of the Green's first identity theory as presented in Eq. 4. In order to compute the tensor  $G_{i3\theta}$ , the following relationships can be easily proven and can be used in such representation:

$$\left[ \frac{r^2 r_{,\alpha} r_{,\theta}}{3} \right]_{,\theta} = r r_{,\alpha} \quad (5)$$

$$\left[ \frac{\lambda^2 r^3 r_{,\theta}}{4} \right]_{,\theta} = \lambda^2 r^2 \quad (6)$$

$$\left[ \frac{\ln(\lambda r) r r_{,\theta}}{2} - \frac{r r_{,\theta}}{4} \right]_{,\theta} = \ln(\lambda r) \quad (7)$$

$$\left[ \frac{\ln(\lambda r) r^2 r_{,\alpha} r_{,\theta}}{3} - \frac{r^2 r_{,\alpha} r_{,\theta}}{9} \right]_{,\theta} = r r_{,\alpha} \ln(\lambda r) \quad (8)$$

$$\left[ \frac{\ln(\lambda r) \lambda^2 r^3 r_{,\theta}}{4} - \frac{\lambda^2 r^3 r_{,\theta}}{16} \right]_{,\theta} = \lambda^2 r^2 \ln(\lambda r) \quad (9)$$

Using the former relationships, the expression of the fundamental solution can be represented as follows:

$$U_{\alpha 3} = \frac{1}{8\pi D} (2\ln(\lambda r) - 1) r r_{,\alpha} \quad (10)$$

$$= \frac{1}{8\pi D} \left( \left[ \frac{2\ln(\lambda r) r^2}{3} - \frac{5r^2}{9} \right] r_{,\alpha} r_{,\theta} \right)_{,\theta} \quad (11)$$

$$= G_{\alpha 3\theta,\theta} \quad (12)$$

and

$$\begin{aligned} U_{33} &= \frac{1}{8\pi D(1-\nu)\lambda^2} [(1-\nu)\lambda^2 r^2 (\ln(\lambda r) - 1) - 8\ln(\lambda r)] \\ & \quad (r_{,\theta} [8(1-\nu)(4\ln(\lambda r) - 5)\lambda r^3 - r(2\ln(\lambda r) - 1)])_{,\theta} \end{aligned} \quad (13)$$

$$\begin{aligned} &= \frac{1}{4\pi D(1-\nu)\lambda^2} \\ & \quad (r_{,\theta} [8(1-\nu)(4\ln(\lambda r) - 5)\lambda r^3 - r(2\ln(\lambda r) - 1)])_{,\theta} \end{aligned} \quad (14)$$

$$= G_{33\theta,\theta} \quad (15)$$

Therefore the required tensor kernels can be obtained by applying Eq.4, as follows:

$$G_{\alpha 3\gamma} = \frac{1}{8\pi D} \left[ \frac{2\ln(\lambda r) r^2}{3} - \frac{5r^2}{9} \right] r_{,\alpha} r_{,\gamma} \quad (16)$$

$$G_{33\gamma} = \frac{r,\gamma}{4\pi D(1-\nu)\lambda^2} [8(1-\nu)(4\ln(\lambda r) - 5)r^3\lambda^2 - r(2\ln(\lambda r) - 1)] \quad (17)$$

#### 4 New kernels at internal points

The previous section treated the kernels associated with boundary values. In this section the kernels required for computation of stress resultants at internal points are derived. The integral identities for bending moments and shear forces at internal point  $\xi$  is given by Vander Weeën (1982):

$$M_{\alpha\beta}(\xi) = \int_{\Gamma(x)} U_{\alpha\beta k}(\xi, x)t_k(x)d\Gamma(x) - \int_{\Gamma(x)} T_{\alpha\beta k}(\xi, x)u_k(x)d\Gamma(x) + q \int_{\Gamma(x)} \left[ G_{\alpha\beta 3}(\xi, x) - \frac{\nu}{(1-\nu)\lambda^2} U_{\alpha\beta\theta}n_\theta \right] d\Gamma(x) + \frac{\nu}{(1-\nu)\lambda^2} q\delta_{\alpha\beta} \quad (18)$$

$$Q_{3\beta}(\xi) = \int_{\Gamma(x)} U_{3\beta k}(\xi, x)t_k(x)d\Gamma(x) - \int_{\Gamma(x)} T_{3\beta k}(\xi, x)u_k(x)d\Gamma(x) + q \int_{\Gamma(x)} \left[ G_{3\beta 3}(\xi, x) - \frac{\nu}{(1-\nu)\lambda^2} U_{3\beta\theta}n_\theta \right] d\Gamma(x) \quad (19)$$

Where  $U_{i\beta k}$ ,  $T_{i\beta k}$  are relevant kernels for the boundary displacements and tractions (Vander Weeën (1982)). The new kernels  $G_{\alpha\beta 3}$  and  $G_{3\alpha 3}$  can be computed from suitable stress-resultant-generalized displacement relationship to give (Vander Weeën (1982)):

$$G_{\alpha\beta 3} = \frac{D(1-\nu)}{2} \left( G_{\alpha 3n,\beta} + G_{\beta 3n,\alpha} + \frac{2\nu}{1-\nu} G_{\gamma\beta n,\gamma} \delta_{\alpha\beta} \right) \quad (20)$$

$$G_{3\alpha 3} = \frac{D(1-\nu)}{2} \lambda^2 (G_{\alpha 3n} + G_{33n,\alpha}) \quad (21)$$

Substituting from Eq.16 and Eq.17 into Eq.20 and Eq.21 to give:

$$G_{\alpha\beta 3} = \frac{r}{144\pi} \{ (1-\nu) [5 - 6\ln(\lambda r)] (n_\beta r_{,\alpha} + n_\alpha r_{,\beta}) + \delta_{\alpha\beta} r_{,\gamma} [23 - 5\nu - 12\ln(\lambda r)(1+\nu)] - 12r_{,\alpha} r_{,\beta} r_{,\gamma} (1-\nu) \} \quad (22)$$

$$G_{3\alpha 3} = \frac{1}{144\pi} \{ n_\alpha [18(2\ln(\lambda r) - 1) + 144(\lambda r)^2(1-\nu)(5 - 4\ln(\lambda r))] + r_{,\alpha} r_{,\gamma} [36 + (\lambda r)^2(1-\nu)(859 - 1146\ln(\lambda r))] \} \quad (23)$$

#### 5 Numerical testing (comparison to domain integral formulation)

A computer code was developed to implement the present formulation. Quadratic boundary elements are used together with constant internal cells. In this section the formulation presented in the paper is tested against domain integral formulation of the BEM. The effect of changing the number of cells and Gauss integration points are studied.

A square plate of side length 4 m is studied. The plate is clamped from all sides and has thickness of 0.2. The Young's modulus is taken equal to  $2.1 \times 10^6$  t/m<sup>2</sup> and Poisson's ratio is taken 0.16. Each plate side is discretised using 4 quadratic boundary elements. Discontinuous elements are employed at corners to cope with the discontinuity in the traction. The plate is analyzed under domain loading of  $-1$  t/m<sup>2</sup> to represent the self-weight. The analysis is carried out several times:

1. Using the traditional domain integration formulation: by changing the number of domain cells: 1, 2, 4, 8, 16 and 32. In this case the number of Gauss integration points are fixed to 4 G.PT.
2. Using the traditional domain integration formulation: by changing the number of Gauss integration points: 4, 8, 10, 20, 40 and 60. In this case only one domain cell was used to represent the domain loading.

3. Using the present formulation without domain cells and with only 4 G.PT.

The deflection, bending moments, and shear forces at points A (center point X=0, Y=0) and B (located at the center of the first quadrant of the plate, X=3 m, Y=3 m) are plotted in figs. (1) to (5). It can be seen that the present formulation produces accurate results with only 4 G.PT to compute boundary integrals. The accuracy of the present formulation is similar to using at least 40-60 G.PT with one domain cell, or using at least 16 to 32 domain cells.

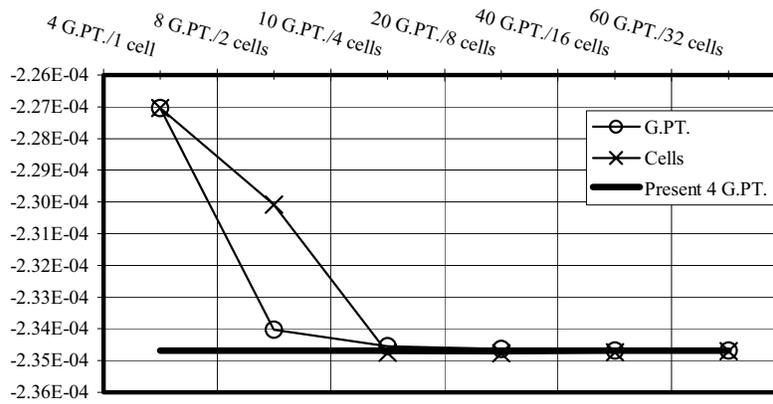


Figure 1 : Deflection (m) at point A.

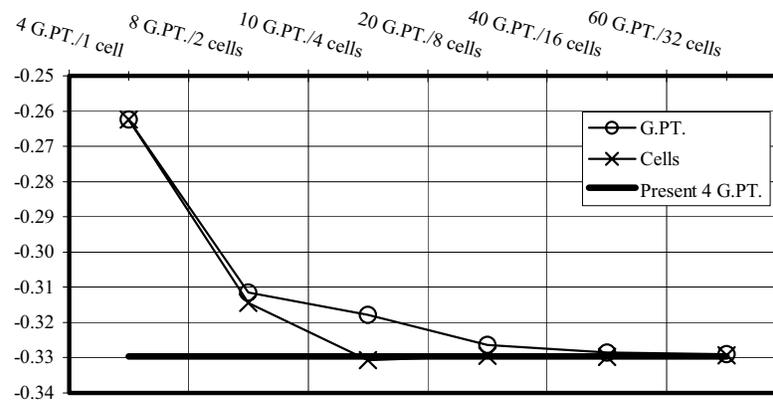


Figure 2 : Bending moment (m.t/m) at point A.

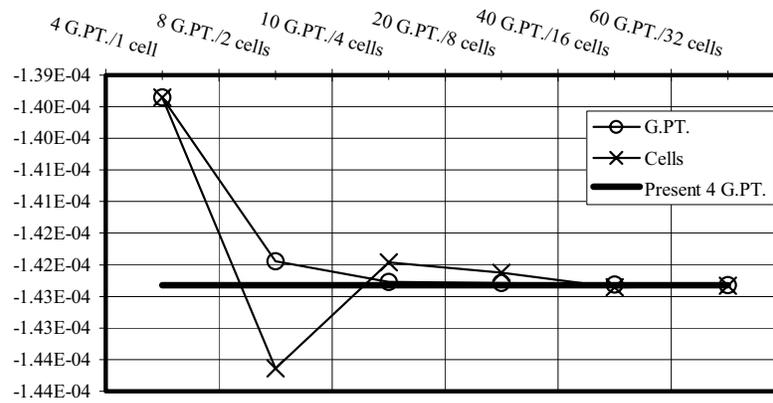


Figure 3 : Deflection (m) at point B.

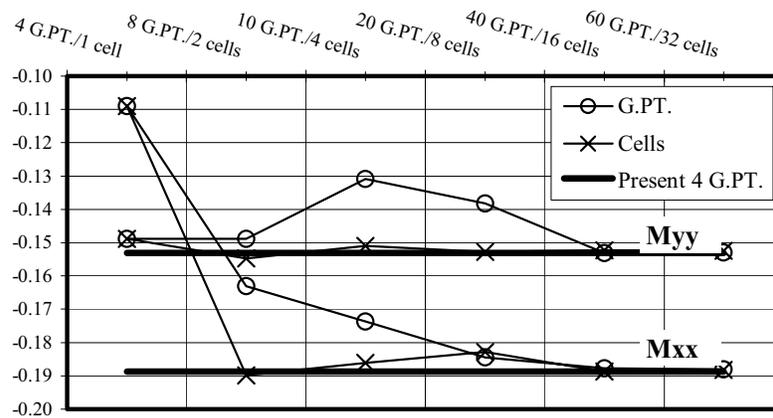


Figure 4 : Bending moments (m.t./m) at point B.

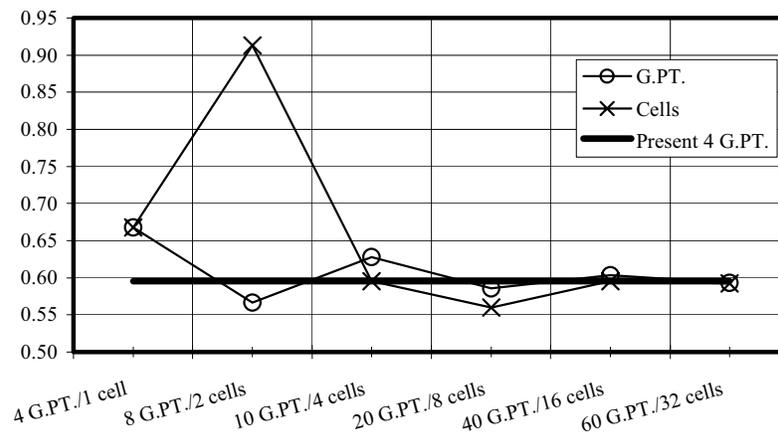


Figure 5 : Shear force (t/m) at point B.

## 6 Curved slab application (comparison to the finite element method)

The former example demonstrated the accuracy of the present formulation against traditional domain integral computation. In this example the accuracy of the present formulation is tested against the results of the FEM.

The shown slab in fig. (6) is analyzed using the BEM based on the present formulation and analyzed another time using the FEM. The slab is of 0.25 m thickness and clamped along GH, HI, IJ, JD, whereas it has free edge on the circular boundary DEFG. The slab is supported also on two columns (0.4×0.4m) inside the domain. Columns are of length 3m and 3m at the levels below and above the slab and are fixed at their ends. Young's modulus was taken  $2.1 \times 10^6$  t/m<sup>2</sup> and Poisson's ratio is taken 0.2. The slab is analyzed under domain loading of 1.1 t/m<sup>2</sup> to represent both self-weight and live load.

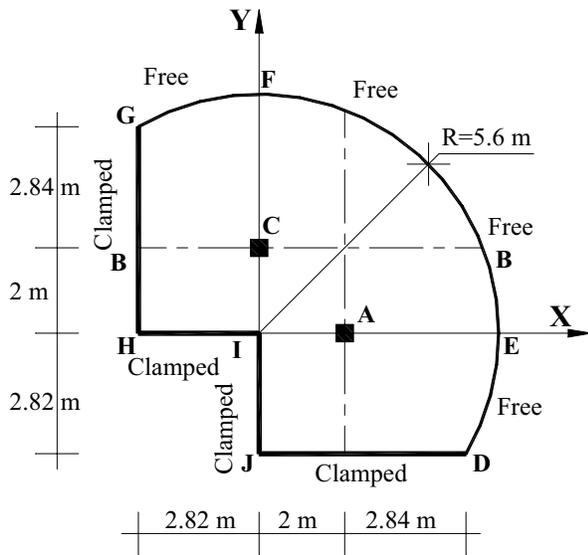
In the FEM analysis, the 4-node rectangular thick plate element was used. Curved boundary of the slab is approximated using straight lines. The element size adopted is about 0.4×0.4 m. Triangular elements are used to fit along the circular boundary. Columns are modeled using skeletal frame element connected to the slab at single node.

In the present BEM analysis, the plate is discretised using 26 boundary elements. Discontinuous elements are used only at corners: H, I, J to model discontinuous tractions at these points. Columns are modeled using two internal cells with suitable axial and rotational stiffness. The self-weight and live load on the slab is treated using the derived boundary integrals in the present work.

Table 1 demonstrates the column reaction bending moments and axial forces obtained from the FEM and the present BEM models. Results for the deflections, ro-

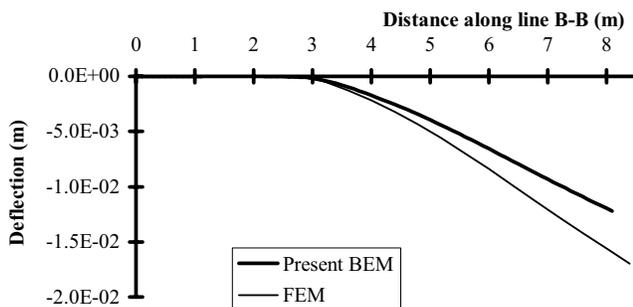
**Table 1** : Column reactions for the curved slab

	Present BEM	FEM
Bending moment $M_x$ (m.t.)	5.68	5.54
Bending moment $M_y$ (m.t.)	6.40	6.28
Axial force (t)	25.81	27.24

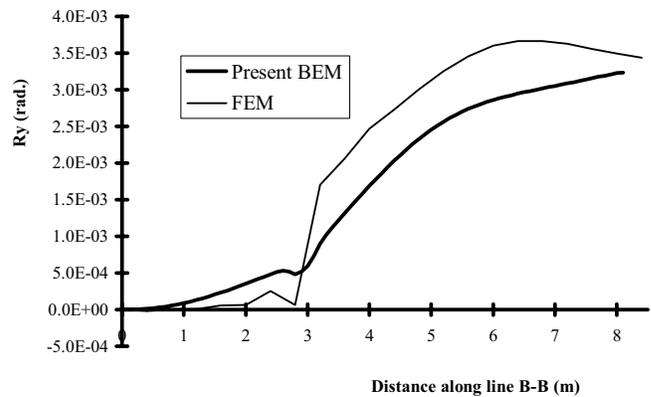


**Figure 6** : The considered curved slab.

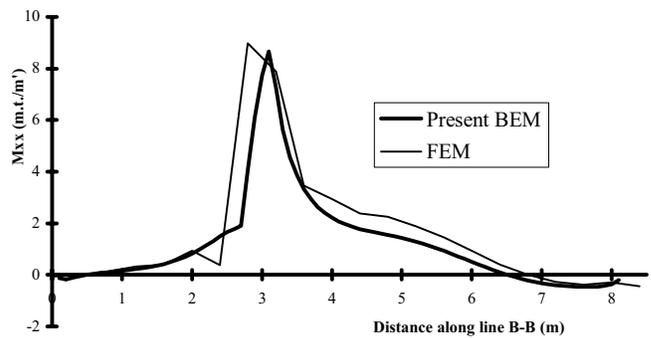
tations, bending moments, twisting moments and shear forces in X and Y direction along the line B-B are plotted as shown in figs. (7) to (12) respectively. It can be seen that the results of the present formulation are in good agreements with those obtained from the FEM. The observed difference in the deflection, rotation and shear results is due to:



**Figure 7** : Deflections of the curved slab.



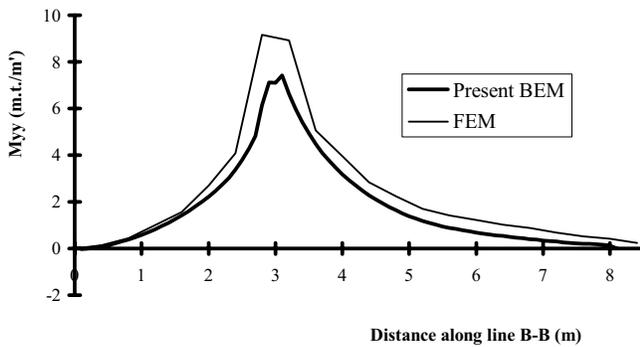
**Figure 8** : Rotation of the curved slab.



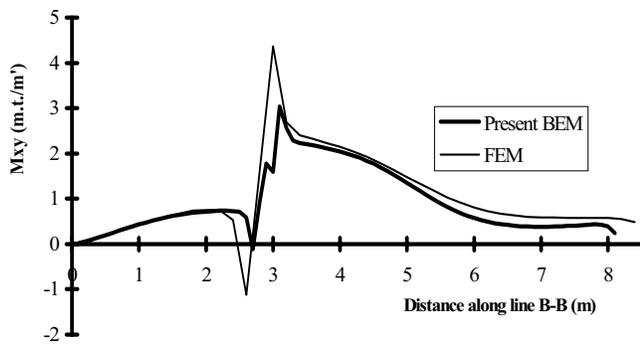
**Figure 9** : Bending moment  $M_{xx}$  of the curved slab.

1. The FEM treats the plate as discretised structure, whereas BEM models the plate as continuum. This affects the results of the deflection and rotation.
2. The difference in modeling the column: in FEM the column is treated as skeletal frame connected to the plate at single node, whereas in the BEM the column is modeled using its actual cross section. This affects the result of the shear, especially in the vicinity of the column.

These modeling differences lead to more accurate results obtained from the BEM.

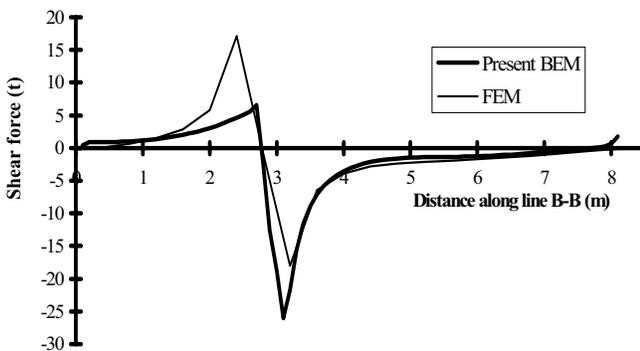


**Figure 10** : Bending moment  $M_{yy}$  of the curved slab.



**Fig. (11):** Twisting moment  $M_{xy}$  of the curved slab.

**Figure 11** : Twisting moment  $M_{xy}$  of the curved slab.



**Figure 12** : Shear force of the curved slab.

## 7 Conclusions

The present work develops a technique to treat self-weight in the BEM for plates using boundary integrals. This new development allows the analysis of slabs using the BEM with boundary-only discretisation. Green's

first identity was used in the developments of the new equivalent boundary integrals. Kernels necessary for the present transformation were derived and given in explicit form. The developed formulation was tested using two examples: clamped square plates, and curved cantilever slab. Results are compared to traditional BEM formulation using domain cells, and FEM. It was shown that the accuracy of the present formulation is very accurate even with few number of Gauss integration points. Also it can easily model curved boundaries without any approximations. It has been also demonstrated that the BEM using the developed formulation can accurately models structures as continuum leading to more accurate and realistic results.

## References

- Brebbia, C. A.; Rashed, Y. F.** (2003): On the treatment of domain integrals in BEM, In Transformation of Domain Effects to the Boundary", Y.F. Rashed, C.A. Brebbia (ed), WIT Press, pp.1-22.
- El-Zafrany, A.; Debbih, M.; Fadhil, S.** (1994): Boundary Element Analysis of Thick Reissner Plates in Bending, *Engineering Analysis with Boundary Elements*, vol.14, pp. 159-169.
- Rashed, Y. F.** (2000): A new efficient method for treatment of uniform loading in thick plates, In 8<sup>th</sup> Arab structural engineering conf., Egypt, pp.37-51.
- Vander Weeën, F.** (1982): Application of the boundary integral equation method to Reissner's plate model, *Int. J. Numerical Methods in Engineering*, vol. 18, pp. 1-10.
- Wen, P. H.; Aliabadi, M. H.; Young, A.** (1999): Transformation of Domain Integrals to Boundary Integrals in BEM Analysis of Shear Deformable Plate Bending Problems, *Computational Mechanics*, vol. 24, pp 304-309.

