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An r-h Adaptive Strategy Based On Material Forces and Error Assessment

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Abstract: A new r-h adaptive scheme is proposed and formulated. It involves a combination of the configurational force based r-adaption with weighted Laplacian smoothing and mesh enrichment by h-refinement. A Zienkiewicz-Zhu best guess stress error estimator is used in the h-refinement strategy. The best sequence for combining the effectiveness of r- and h- adaption has been evolved at in this study. A further reduction in the potential energy and the relative error norm of the system is found to be achieved with combined r-adaption and mesh enrichment (in the form h-refinement). Numerical study confirms that the proposed combined r-h adaption is more efficient than a purely h-adaptive approach and more flexible than a purely r-adaptive approach with better convergence characteristics.

keyword: r-h adaptvity, Material forces, Polak-Rebiere algorithm, Discretization error.

1 Introduction

Adaptive finite element techniques seek to construct reference solutions, define error norms and in general, create a more accurate and reliable numerical solution, by using a methodology incorporating these solutions and error norms. There has been a considerable focus on theoretical and computational aspects of adaptive analysis such as those by Babuska and Rheinbold (1978), Zienkiewicz and Zhu (1987). The computed error could be based on a-priori or a-posteriori estimators. The later has gained more popularity because of its robustness, which is evident in works by Babuska and Rheinbold (1981), Zienkiewicz and Zhu (1990). In order to obtain the required accuracy with minimum cost, an optimal mesh has to be designed. Such a mesh has minimum potential energy and minimal degrees of freedom for a specified accuracy which are distributed in such a manner that error distribution is uniform indicating a flexible discretization.

Several mesh adaptive techniques such as h, p, r and s–versions that are widely reported [Zienkiewicz and Zhu (1990) Fish (1992), Fish and Guttal (1995)] are designed to optimize a spatial discretization. There have been reports on use of a combination of these methods for better performance [Madan G K and Huston (1990), Patra and Oden (1997) and Askes and Rodriguez-Farran (2000)]. Recently mesh free methods have become popular and have advantages over traditional finite element method (FEM), especially in dealing with structural problem with high stress gradient [Atluri (2004)], as they are free of post-processing in error control and adaptive analysis. True mesh free techniques such as Meshless Local Petrov-Galerkin Method (MLPG), given by Atluri and Zhu (2000) are completely devoid of background grids, and may result in more flexible adaptive grids. However, the mesh less technique is yet to make a significant inroad into commercial packages unlike the finite element methods. The focus of this paper is on refinement strategies of finite element meshes.

In the r-version of refinement strategy, henceforth called r-adaption technique, the nodes of the discretized domain are relocated iteratively in order to minimize the discretization error, while preserving the number of unknowns and order of approximation of the field variable. Typically, the mesh density increases near the regions of steeper gradients of the field variable as a result. The adaptation criteria for r-adaption as per Pierre Beal, Kokko and Touzani (2002) may be classified according to the procedure used for node relocation. Procedures based on hierarchical error estimators and based on configurational equilibrium concepts form the two broad classifications of r-adaptive procedures.

The preeminence of r-adaption procedure based on configurational equilibrium has been well established by Rajagopal, Gangadharan, Sivakumar and Thoutireddy (2004). This allows in the use of this method with suffi-

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cient poise as a whole or part of an adaptive refinement strategy. Here the mesh optimality is accomplished by minimizing the potential energy in the static case and stationarity of discrete action sum in the dynamic case Thoutireddy (2002), Thoutireddy and Ortiz (2002). The potential energy of the system is dependent on nodal coordinates in addition to the displacement field [Braun M (1997)]. Configurational or material driving forces are defined as the conjugate forces to the nodal motion with respect to the potential energy [Eshelby (1975), Maugin (1995), Gurtin and PodioGuidugli (1996)] and these forces vanish when the potential energy is a minimum. Further, the minimization of the potential energy reduces error norm due to the orthogonality of error with respect to solution space for linear problems [Thoutireddy (2002), Thoutireddy et al (2004)]. There has been considerable work on the use of material forces and their equilibrium in a computational setting such as the finite element method with notable ones are Gurtin and PodioGuidugli (1996), Shewchuk (1994), and Thoutireddy (2002). The material force imbalance is due to presence of nodes (nodes can be argued to be discrete defects as they breaks the translational symmetry of potential energy with respect to translations in reference coordinates). For a homogeneous body in continuous case linear momentum balance implies material force balance. However, introducing discretization causes non-zero material forces at the inter-element boundaries. The energy momentum tensor is computed based on strain energy though the use of energy momentum tensor based on complementary strain energy is more meaningful, especially, for nonlinear bodies. [Atluri (1982) and Atluri (1986)]. In this work, a formulation of nodal errors that is consistent with material force equilibrium has been derived. The node relocation procedure forms a vital part of r- adaption. The use of simple relaxation type iterative procedures has been reported by Muller and Maugin (2002), and Thompson et al (1999). An improved procedure by using a standard Polak-Rebiere conjugate gradient algorithm was proposed by Thoutireddy and Ortiz (2002).A modification was introduced to this algorithm for faster convergence by Rajagopal et al. (2004).

Although only an improvement of existing solution is possible through r adaption the ultimate aim of achieving a specified accuracy can be realized by successive mesh enrichment or h -refinement. Here the estimated error from the current solution is used to predict the desired element size, which may be used to subdivide the existing discretization (progressive halving) or reconstruct an entirely new discretization (remeshing). [Zienkiewicz and Zhu (1987)]. In this paper we focus on different adaptive refinement strategies. More specifically the emphasis is on global h-adaptvity and on r-adaptvity. The properties of these two schemes are complimentary, namely, expensive and sophisticated of h-adaption versus inexpensive and limited in applicability of r-adaption. This has led to the idea that the advantageous properties of the two schemes could be combined. Earlier works on r-h adaptive strategy like those of Askes and Rodriguez (2000) are based on r- or h-adaption on the parts of a domain with no refinement on the interface of these sub-domains. Although the sub-domain division is based on desired element size, these are not based on error estimators and there seems to be less mathematical vigor for refinement strategies adopted. The interfaces between domains pose topological constraints on h-adaptive remeshing as well as on the r-adaptive remeshing. There has been report on use of r-h-p refinement strategies as reported by Madan Kittur and Huston (1990), but the adaption procedure lacks physical basis and mathematical vigor.

In present work a combined r-h adaptive refinement strategy has been implemented for one-dimensional and two dimensional linear elastic plane stress and plane strain problems. The r-adaption is based on configurational force method together with modified algorithms for node relocation. The h-refinement is based on Zienkiewicz-Zhu (1987) error estimator. The present work is an extension of the earlier work by authors Rajagopal, Gangadharan, Sivakumar and Thoutireddy (2004) where preeminence of the configurational force method has been established based on qualitative and quantitative analysis in comparison to conventional techniques based on heuristic error estimators. The earlier work by Rajagopal, Gangadharan, Sivakumar and Thoutireddy (2004) differs from work by Muller and Maugin (2002) in the computation of driving force terms in the absence of body force and also in the modified node displacement procedures that have better convergence characteristics. The study indicates that combined h- and r- refinement yields useful results only if the adaption is performed in cycles of succession for every new mesh. The combined r-h strategy resolves the unhealthy mesh distortions and provides better convergence of the solution. After a brief outline of r-adaption based on configurational force approach with appropriate algorithms for node relocation and the h-refinement strategy based on best guess stress type estimator, the refinement strategy that combines the two techniques effectively is described. The last section describes and discusses on the numerical implementation of combined r-h strategy for linear elastic one and two-dimensional problems in structural mechanics.

2 r- adaption – Configurational force method

The essence of performing the r-adaption is to predict the characteristics of the optimal mesh where the number of degrees of freedom is distributed in such a manner that accuracy of solution obtained is highest possible Braun (1997), Pierre Beal, Kokko, and Touzani (2002). The errors due to discretization or approximation occurring at the nodes are typically equally distributed for better solution over the entire domain. The r adaption is based on a method of achieving material force equilibrium with the imbalance in material equilibrium being considered as a measure of error. This departure from material equilibrium is reduced by minimization of the potential with respect to nodal coordinates. This is accomplished by relocating nodes in a finite element mesh. Considering material force equilibrium results in defining energy momentum tensor in material space as given by Eshelby (1975), Maugin (1995). The components of the energy momentum tensor represent the change of total potential energy of a deformed body produced by unit material translation.

For a homogeneous body, in the continuous case, force balance implies material force balance. However, in the discrete case nodal force balance does not imply nodal material force balance due to the presence of nodes and hence element interface. Thus in a discretized form considering the material force equilibrium the non-vanishing of the divergence of energy momentum tensor at the inter element boundaries is taken as an error indicator Thoutireddy (2002), Rajagopal et al (2004). In this section the error indicator is derived by considering the material force equilibrium in a similar manner as is done in physical equilibrium. The displacement vector for a solid in the region $\Omega_0 \in \Re^3$ (See Fig. 1) in referential description is given as $u_i = x_i - X_i$ .The deformation mapping for a lagrangian description is defined as

$$x_i = x_i(X_1, X_2, X_3, t) \tag{1a}$$

$$x_i = x_i(X_I) \ or \ \Rightarrow \ \Psi(u_i, X_A) \tag{1b}$$

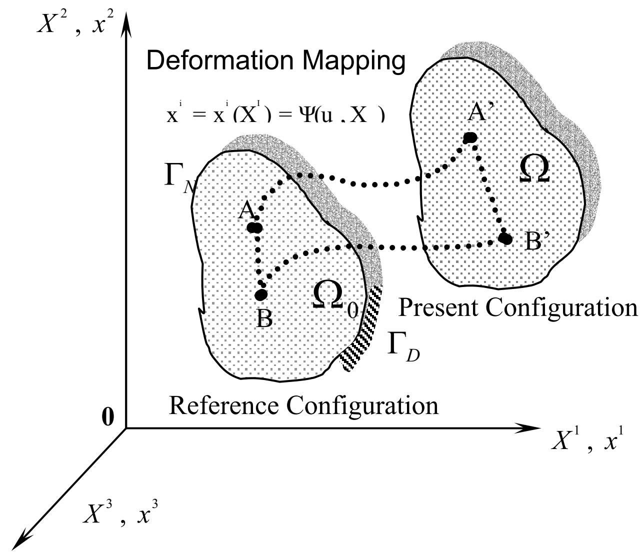


Figure 1 : Body in reference and deformed configuration

With the displacement gradient in Ω_0 given by $F_{iA} = \delta_{iA} + u_{i,A}$. The deformation mapping takes on prescribed values $\overline{\Psi}$ over the displacement part of $\Gamma(\Gamma = \Gamma_D \cup \Gamma_N)$ of the undeformed boundary. The strain energy density per unit volume of the undeformed elastic material is given by

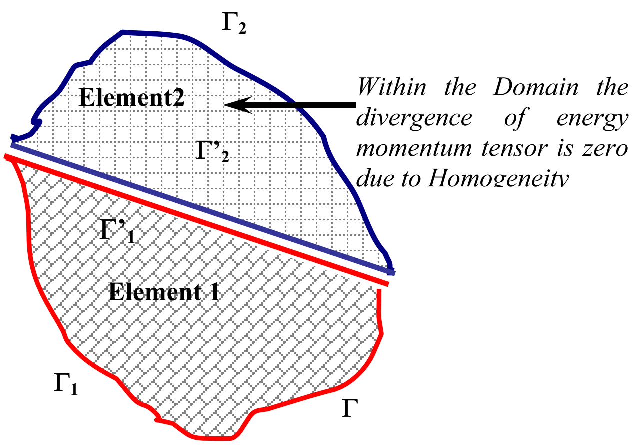
$$W = W(x^i_{,J}, X^K) \ \ or \ \ W = W(u_{i,j}, x_k) \tag{2}$$

For a linear elastic material, $W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$, where σ_{ij} is the stress tensor and ε_{ij} is the strain tensor. The physical equilibrium equation is given by $\sigma_{ij,j} + f_i = 0$, where f_i is the body force. The material gradient of strain energy results in the configurational force equilibrium and is given by

$$C_{kj,j} + g_k = 0 \tag{3}$$

Where the configurational stress tensor is given by $C_{kj} = W\delta_{kj} - \sigma_{ij}u_{i,k}$ and the Configurational force arising due to body forces are given by $g_k = -f_i u_{i,k}$. It is required to compute the discrete configurational forces arising out of discretization. In the absence of body forces the weighted residual form of the balance law equation using a vectorial test function η and integrating over the domain Ω_0 is given by

$$\int_{\Omega_0} C_{ij,j}\eta_i \, d\Omega_0 = 0 \tag{4}$$



For Discretized Domain $\Gamma = \Gamma_1 \cup \Gamma_2$

$C_{ij(1)}$ *evaluated along* $\Gamma'_2 \neq C_{ij(2)}$ *evaluated along*

Driving Force $= C_{ij(1)} - C_{ij(2)} =$ *Jump in* C_{ij}

Figure 2 : Non-satisfaction of Divergence of Energy momentum tensor resulting in a jump in driving force at Inter element Boundaries

The weak form in the absence of body forces obtained by integrating by parts can be written as

$$-\int_{\Omega_0} C_{ij}\eta_{i,j}\,d\Omega_0 + \int_{\Gamma} C_{ij}n_j\eta_i\,d\Gamma + \int_{\Gamma_e \not\subset \Gamma} C_{ij}n_j\eta_i\,d\Gamma = 0 \tag{5}$$

As a consequence of considering stationary boundaries the test function η vanishes on the boundaries of the domain Γ and hence the second term becomes zero. The divergence of the energy momentum tensor is zero for a homogeneous body without body forces. This is used to check the discrete solution obtained through finite element analysis. As finite element solutions approximate solutions the non-vanishing divergence of the energy momentum tensor provides an error indicator. The discrete jump in the energy momentum tensor (See Fig. 2) occurring at the element boundaries Γ_e (third term of the weak form equation) is the driving force used as an error measure in the node relocation process. The balance law in its weak form is analogous to the bilinear form of the governing differential equation with jump in the energy momentum tensor being similar to the traction jump occurring at the inter element boundaries [Krishnamoorthy and Mukherjee (1996)]. The discretized form of the above weak form can be written by inserting an element wise interpolation of the test function η and its gradient. Thus we can write

$$\eta_i = \sum_I N^I \eta_i^I \;\; and \;\; \eta_{i,j} = \sum_I N^I_{,j}\,\eta_i^I \tag{6}$$

Thus Eq. (5) reduces to the form

$$\sum_I \left[-\int_{\Omega e} C_{ij}N^I_{,j}\,d\Omega_0 + \int_{\Gamma_e \not\subset \Gamma} C_{ij}n_jN_i^I\,d\Gamma \right] \eta_i^I = 0 \tag{7}$$

The second term which is a traction related to the discontinuity is the configuration force of the element that needs to be numerically evaluated. Since η_I are arbitrary, each of the above in the summation over I should go to zero. Thus, the first term is equal to the negative of the second that is evaluated as the discrete configuration force, G_e^I given by

$$\int_{\Gamma_e \not\subset \Gamma} C_{ij}n_jN_i^I\,d\Gamma = \int_{\Omega e} C_{ij}N^I_{,j}\,d\Omega_0 = \left\{ \begin{array}{c} G_e{}^I{}_1 \\ G_e{}^I{}_2 \\ G_e{}^I{}_3 \end{array} \right\} = G_e^I \tag{8}$$

These configurational forces on assemblage should go to zero in the domain. This also means that the configurational forces of the elements are equidistributed for an optimal mesh. The assembled total configurational force is given by

$$G^K = \bigcup_{e=1}^{ne} G_e^I \tag{8a}$$

Where *ne* spans over the number of elements connected to a particular node.

2.1 Node relocation procedure

The interior nodes are updated by an iterative rule such as $X^K = X^K - cG^k$ [Muller and Maugin (2002)]. The constant c is chosen sufficiently small to achieve convergence (to avoid unhealthy mesh distortions). For better convergence a nonlinear conjugate gradient method, known as Polak-Rebiere method [Thoutireddy and Ortiz (2002), Rajagopal et al (2004)] for minimization of energy function has been incorporated. The algorithm has

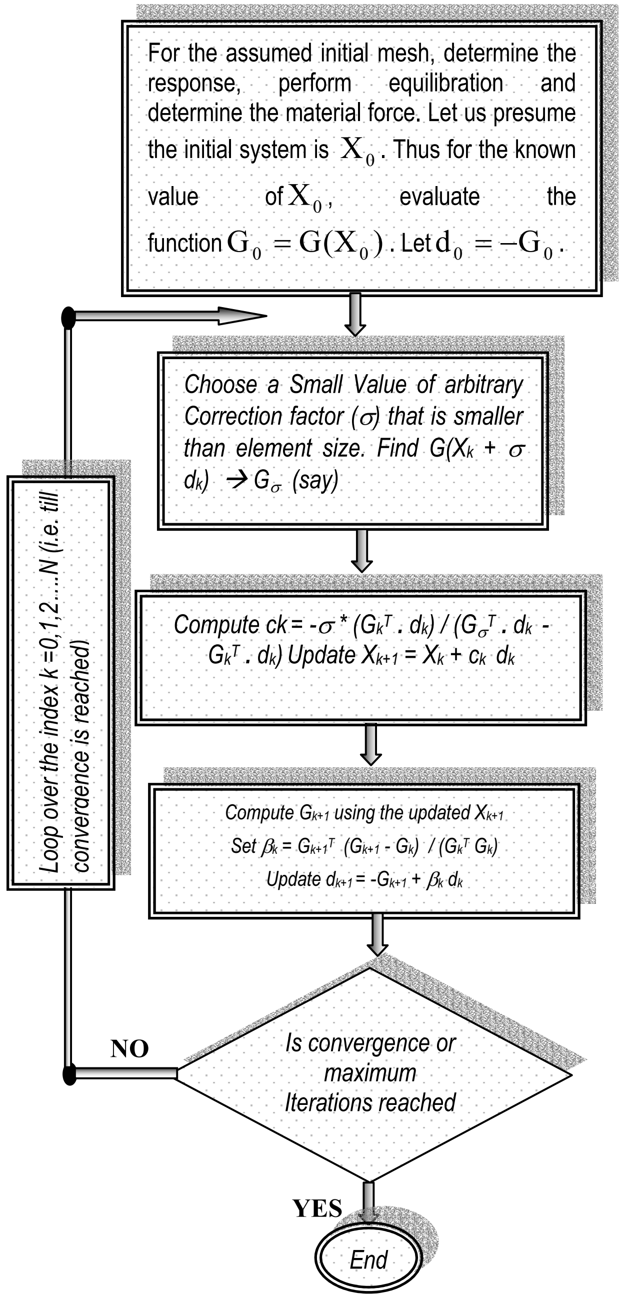


Figure 3 : Steps in Node displacement Procedure - Polak Rebrie Algorithm

been explained in Fig. 3. The method has two levels of iteration. The outer loop is the undeformed coordinate iterative update or nodal coordinate update. Nodal coordinate iterative loop contains solution for equilibrium solution for deformed coordinate for a fixed mesh. This ensures that the configurational forces for undeformed coordinate update correspond to equilibrium solution. A linear projection method, which is similar to the successive over relaxation iterative technique, shows better convergence [Rajagopal, Gangadharan, Sivakumar and Thoutireddy (2004)].

3 h-adaptive strategies

The h adaptive strategy is based on best guess type or Zienkiewicz-zhu (1987) error estimator. The general form the estimator is as given below

$$\|e\| = \left(\int_{\Omega} \{e_\sigma\}^T |C|^{-1} \{e_\sigma\} \, d\Omega \right)^{1/2} \tag{9}$$

Where $\{e_\sigma\} = \{\sigma^*\} - \{\sigma_h\}$.

$\{\sigma\}^*$ and $\{\sigma_h\}$ are the best guess stress and finite element stress respectively. The best guess stress is obtained through a simple projection technique as given by Zienkiewicz-zhu (1990). The absolute value of the error over domain is calculated. The global percentage error η is given by

$$\eta = \frac{\|e\|}{\|u\|} * 100 \tag{10}$$

Where, $\|e\|$ is a suitable error norm and $\|u\|$ is the displacement norm. Since the exact displacement norm is not known, we use an approximate norm $\|u^*\|$. Specifying the global error in the energy norm $\overline{\eta}$ in the form of a percentage of total energy norm one can compute the permissible error in the energy norm. Thus for a uniform distribution of error one can compute the permissible error and refinement index for each element. The mesh density can thus be computed based on present size of the element, order of approximation and refinement index. A subdivision technique for h- refinement has been incorporated making sure that the initial topology is maintained to an extent.

4 Combined r-h adaptive strategy

It is observed that there is no change in the topology of the domain when corrections are made for configurational forces. There is only an increase or decrease in the element size h_i . The aim of adaptive post processing technique is to obtain softer discretization, along with stationary value of potential and to get better displacement or stress solution across element boundaries with a good mesh. The criteria for goodness of mesh are based upon strain energy, displacement and stress values at selected critical points of a structure. An adaption based on Material forces tends to result in bad shape elements and approximation. This is from the understanding that the displacement polynomial approximation made within the element assumes extreme values at the nodes. To get better Finite element solution we need to change the topology of the domain once the stationary value of the potential is reached after completion of mesh adaption iterations. Furthermore an optimal mesh is one in which the number of degrees of freedom is minimal for a specified accuracy. This can be achieved only through mesh enrichment. The process of adaption and enrichment may follow one another as one single cycle or may be repeated in cycles.

4.1 Convergence characteristics

The criterion for convergence criterion decides the number of iterations and reflects the quality of estimated driving force during iteration of node relocation process. A modified criterion is required for the following reasons. A tolerance criterion based on configurational forces or potential energy is stringent and results in large number of iterations. Further smoothing or any such procedure during node relocation is likely to affect the potential energy and the configurational forces. A measure to account for variability in initial meshes for a given problem is thus required. A possible efficient way is to prescribe the relative percentage change in configurational force given by

$$Relative\,G\,norm\,(\%age)\,\eta_G = \frac{\|\Delta G_i\|}{\|G_i + \Delta G_i\|} \times 100\% \tag{11}$$

$\|G_i\|$ is the elemental value of the configurational force at every iteration. Similarly we can find the global value by adding up all the elemental values and can be denoted as $\|G_i\|_g$. The variation of η_G reduces with iterations, indicating that the system reaches a stationary value of the potential. If $\overline{\eta}_G = specified\ value\ of\ \eta_G$, we get a measure of tolerance in considering the stationary value of the potential. We can thus define $\overline{\eta}_G = specified\ value\ of\ \eta_G$ for specifying the shift in equilibrium from view point of configurational force mesh adaption.

The variation of relative error norm percentage and global G norm percentage (Eq. 10 and Eq. 11) over the mesh adaption iterations for the structured mesh are of importance. The relative error norm percentage decreases initially with adaption owing to the increase in the flexibility of the system. With adaption there is progressive distortion of the element and with topology being preserved, the approximations of the field variable and hence the recovery based error estimator tend to be bad. This is reflected through the increase in the value of

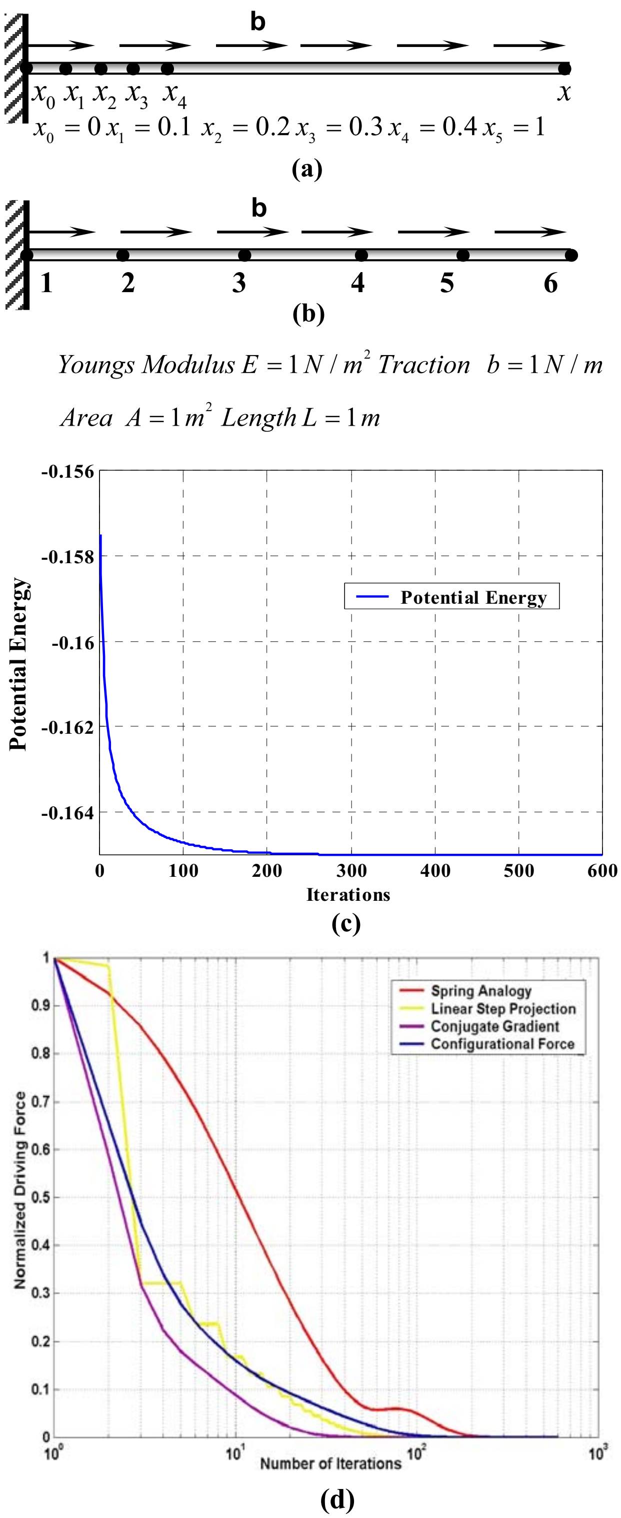


Figure 4 : (a) Initial Mesh of Linear elements. (b) Optimal uniform adapted Mesh of Linear elements (c) Variation of Potential energy during Node relocation process (d) Normalized Driving force variation for Various Node displacement procedures.

the error norm percentage at later iterations. The effect of smoothing mesh is reflected in the plots of potential and relative error norm percentage. The smoothing tends to reduce the error norm percentage and potential. It is thus expected that η would increase with the iterations made for iteration. This is likely only in many elements of the initial mesh. It is also likely that in some elements the relative error norm percentage reduces.

The efficiency of any r-adaption technique is measured in terms of the extent of element distortion that occurs in the relocation procedure. Since r-adaption is used as a part of the combined r-h refinement strategy, which incorporates mesh enrichment through h-refinement, the estimation of distortion metric for elements is not considered important. An accurate analysis would incorporate the error in the energy norm arising due to element distortion. This forms the scope of future work.

5 Results and Discussions

In this section we report the results of numerical tests that establish effectiveness of combined r-h adaptive strategy. Numerical studies have been made on implementing in the first case node relocation followed by mesh enrichment as one single cycle and in the second case node relocation followed by enrichment in succession.

5.1 One Dimensional example:

A linear elastic axial rod fixed at one end and free at other end which is under a uniform body force b as shown in Fig.4 (a). was considered for the implementation and comparison of the configurational force and the spring analogy method in one dimension. The displacement solution for this axial rod is given by $u(x) = \frac{b}{E}\left(x - \frac{x^2}{2}\right)$. Where E = Young's modulus of material, here we chose E=1N/mm^2, b=1N/m. Corresponding to this displacement the strain and stress are linear in$x$. This suggests that uniform mesh is the optimal mesh corresponding to this solution. To authenticate this by mesh adaption, the elastic rod is discretized using linear elements as shown in Fig. 4(b). A set of six nodes were considered with one node at free end to define geometry and other nodes clustered near to fixed end. Mesh adaption based on configurational forces was performed with appropriate choice of correction factor. Fig. 4(c) shows the final adapted optimal uniform mesh. Convergence characteristics of various node relocation techniques were studied during

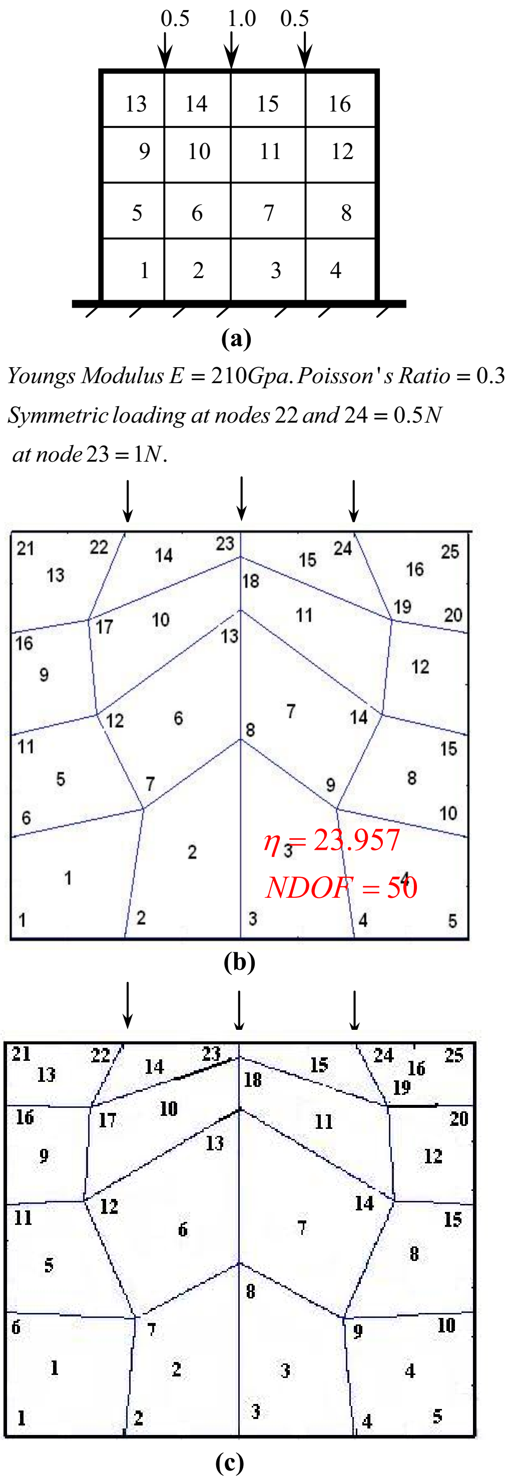


Figure 5 : (a) Block under pressure Initial Mesh (b) Mesh adaption using Polak rebrie algorithm with boundary nodes Fixed (c) Mesh adaption using Polak Rebrie algorithm –boundary nodes Moving.

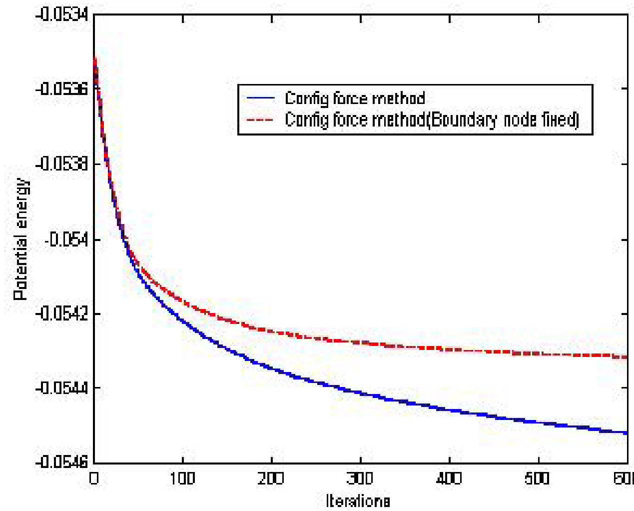


Figure 6 : Plot of Potential Energy for Various Iterations for Mesh with boundary nodes fixed and moving

the adaption process. Fig. 4(d) shows a plot of the normalized driving force versus number of iterations. It is seen that the conjugate gradient algorithm together with a linear step projection considerably improves the convergence rates.

5.2 Two Dimensional examples:

5.2.1 Block under pressure:

A homogeneous square block of linear elastic isotropic material with nondimensionalized length of four units with a symmetric loading is considered. The vertical displacements on the bottom edge are fixed. The block is discretized using four noded bilinear elements.

The initial mesh is shown in Fig. 5(a). A plane strain state is assumed. For the given loading and boundary conditions mesh adaption is performed by Polak-Rebiere conjugate gradient node relocation algorithm to get adapted mesh as shown in Fig. 5(b). The boundary nodes are generally fixed during the adaption process Fig. 5(b). Some times the boundary nodes can also be made to move in one direction Fig. 5(c). This process causes a further reduction in the potential of the system as shown in Fig. 6. The gradient of the strain energy gets reduced with adaption this is evident from Fig. 7(a) and Fig. 7(b).

Once the convergence criterion based on potential energy is reached, the mesh is enriched by h- refinement.

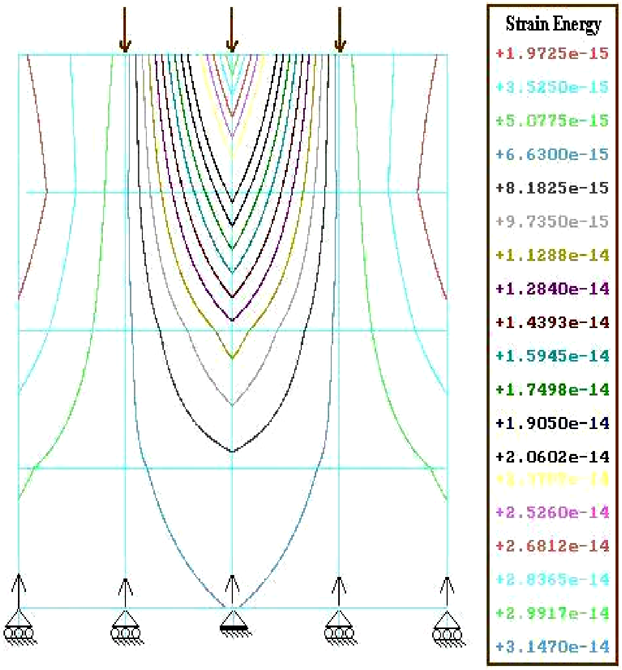


Figure 7a : Contours of Strain energy Distribution (Isoenenergetics) Before Adaption

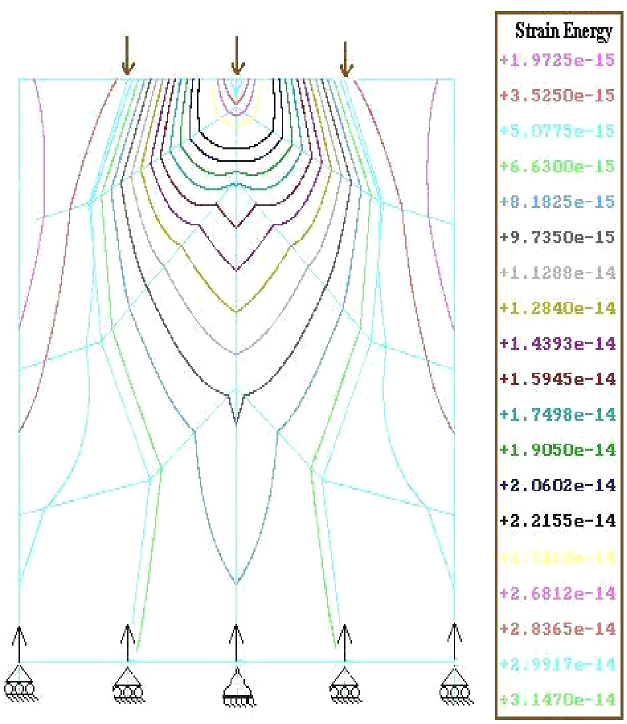


Figure 7b : Contours of Strain energy Distribution (Isoenenergetics) after Adaption

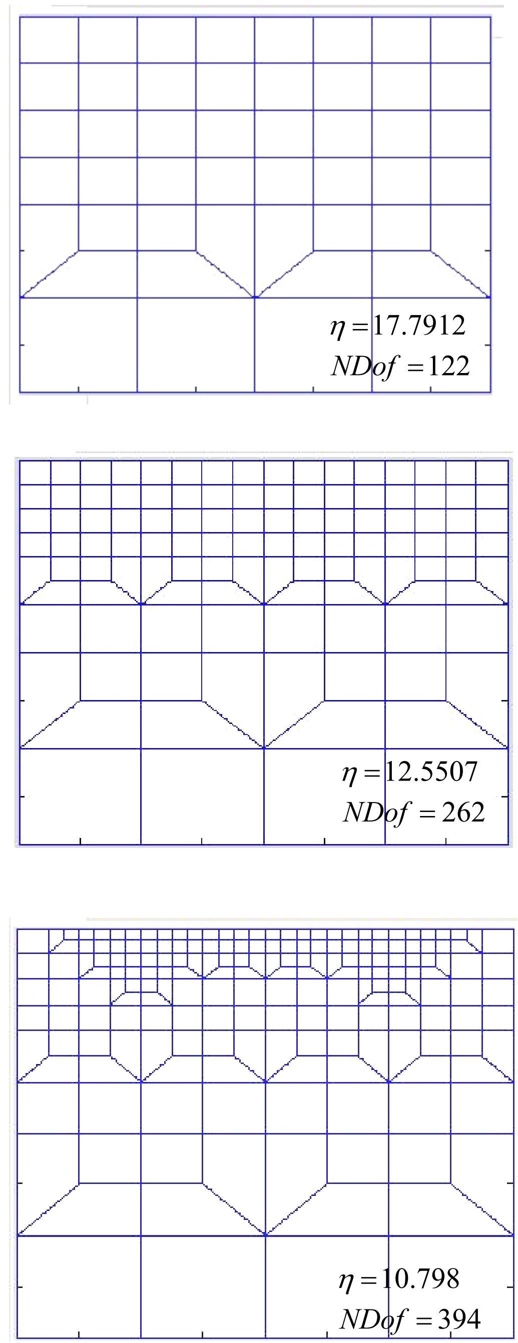


Figure 8a : Mesh enrichment by h refinement without r-Adaption

The topology of the mesh is thus changed. The current mesh is not optimal. Thus r-adaption iterations are continued for this enriched mesh. This process of successive adaption followed by enrichment is continued till a specified degree of accuracy is achieved. Fig. 8(a) shows the

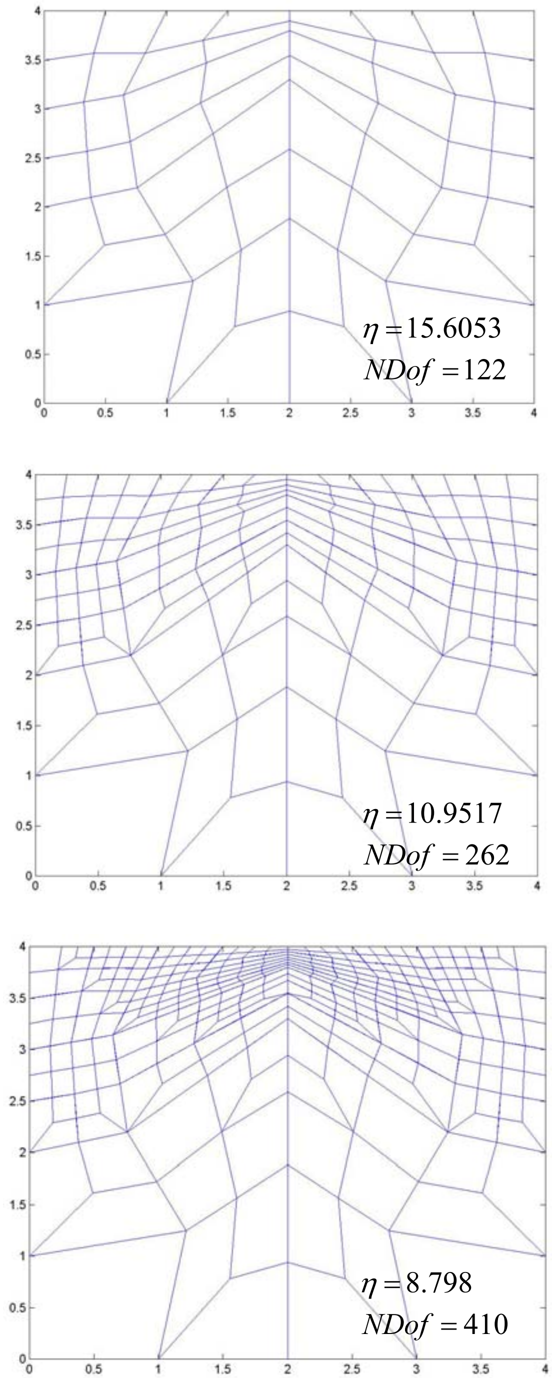
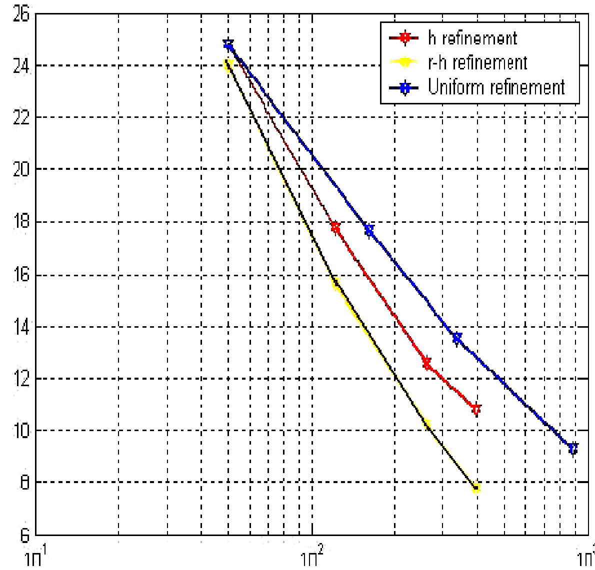
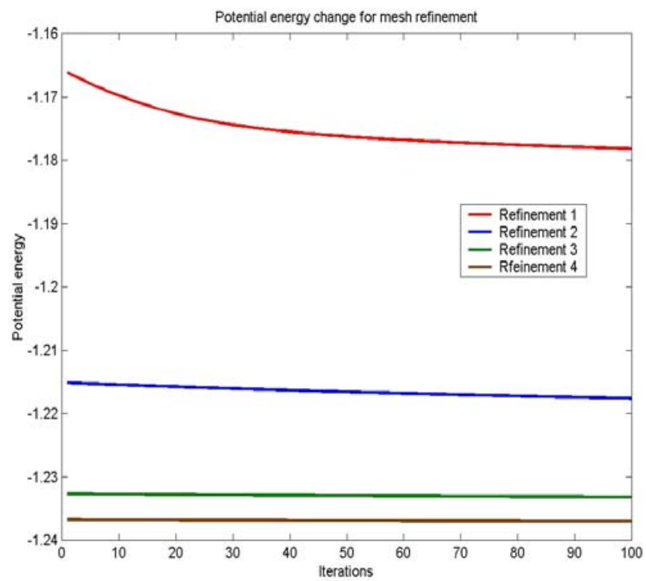


Figure 8b : Combined r-adaption and mesh enrichment by h refinement



(a)

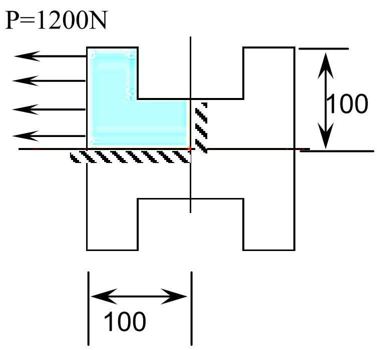


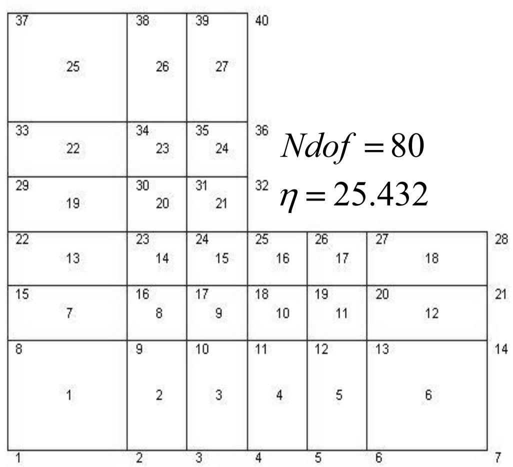
(b)

Figure 9 : (a) Relative Error norm percentage versus Number of Degrees of freedom. (b) Plot of Potential energy versus the number r- adaption iterations at various refinement levels.

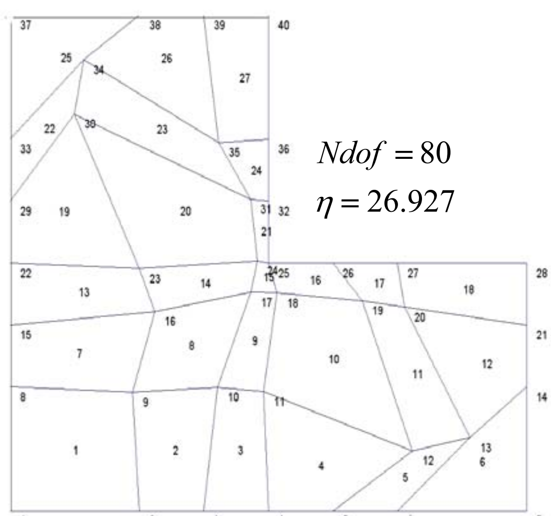
meshes obtained by adaptive mesh enrichment (h refinement) alone. Fig. 8(b) shows combined r and h refinement.

A comparison of the convergence characteristics of uniform h refinement, adaptive h refinement and combined r-h adaptive refinement is made. It is seen that for the same number of elements a combined r-h strategy results in an optimal mesh with reduced errors and the method shows faster convergence. This is evident from the plot of relative error norm percentage for various refinements as shown in Fig. 9 (a). The reduction in potential energy





(a)



(b)

Figure 10 : (a) L shaped Domain –Initial Mesh (b) Final Adapted mesh

of the system with progressive refinement is very evident as shown in Fig. 9(b).

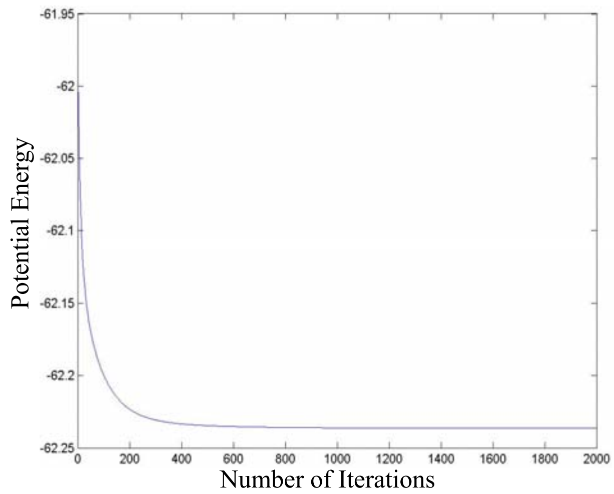
5.2.2 L shaped Domain:

A homogeneous L-shaped domain of linear elastic, isotropic material (Young's modulus=70 Gpa and Poisson ratio=0.33), is considered with specified dimensions and loading. A plane stress state is assumed. The domain is discretized using four noded bilinear elements. For the given loading and boundary conditions mesh adaption based on configurational force is performed using Polak-Rebiere conjugate gradient algorithm The initial structured discretization is as shown in Fig. 10(a). The adapted mesh after certain number of iterations is shown Fig. 10 (b). It is reported in literature and observed here that the mesh adaption results in distorted and degenerate elements. It is required to continue the adaption procedure for further minimization of potential at the same time it is required to avoid degeneracy. In this context a weighted laplacian smoothing is performed to smooth the elements. The error norm percentage initially reduces and owing to distortion increases and reaches a constant value during the mesh adaption process. The variation of the potential energy with iterations with an intermediate smoothing is shown in Fig. 11(a). The variation of the relative G norm percentage during node relocation is as shown in Fig. 11(b).

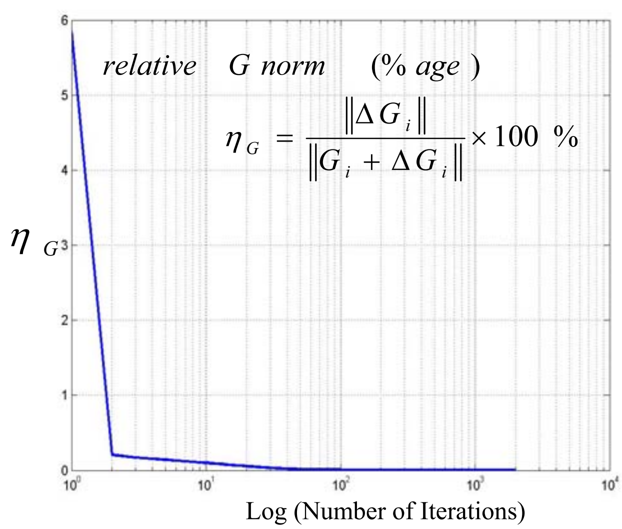
Mesh enrichment is carried out after adaption. In the present case the combined procedure is not successive. In the sense r adaption is performed only once at the beginning followed by h refinement. Fig. 12 and Fig. 13 shows the meshes obtained by h refinement alone and by combined r-h refinement respectively. The convergence rates are improved (See Fig. 14) but are not that pronounced as in successive r followed by h refinement made for previous example.

6 Conclusions

In this study a new r-h adaptive scheme has been proposed and formulated. The r-adaption is based on a modified configurational force methods together with a weighted Laplacian smoothing...Mesh enrichment by h -refinement is based on Zienkiewicz-Zhu best guess stress error estimator. The present scheme results in an optimal initial mesh because the numbers of degrees of freedom are distributed in such a manner that error distribution



(a)



(b)

Figure 11 : (a) Variation of Potential Energy during relocation (b) Relative G norm percentage during Iterations

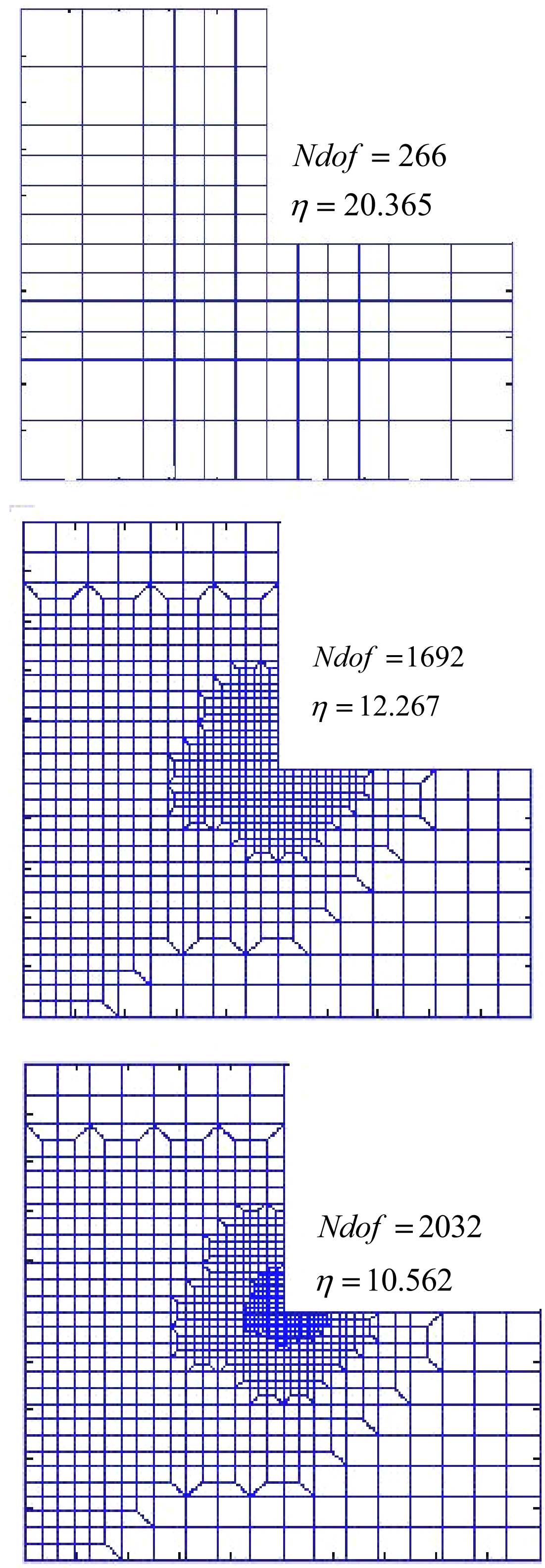


Figure 12 : L shape domain –h refinement without adaption

is uniform and potential energy of the system is a minimum indicating a flexible discretization. Furthermore the present method also results in an optimal adapted mesh in which the number of degrees of freedom is minimal for a specified accuracy .The best sequence for combining the effectiveness of r- and h- adaption has been evolved at in this study.

It is seen that a combined r-h strategy with r-adaption followed by h-adaption in cyclic sequence performs well

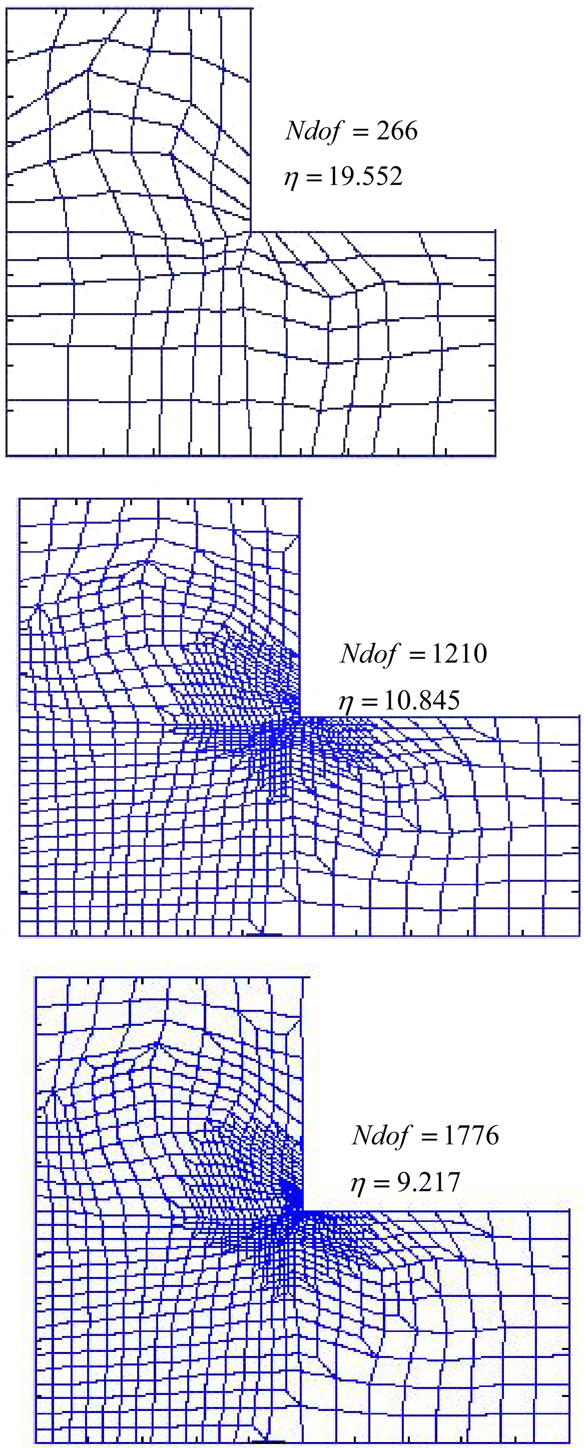


Figure 13 : L shaped Domain Combined r-h refinement

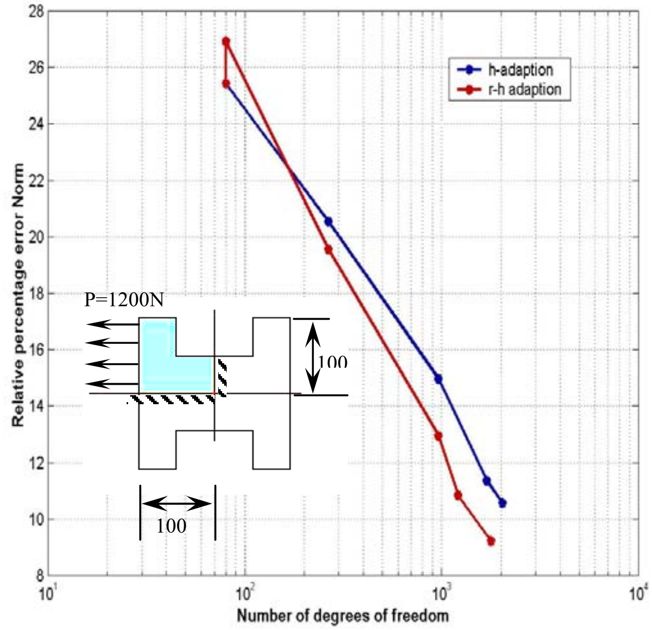


Figure 14 : Plot of relative error norm percentage before and after combined r-h adaption

than a single cycle of r- followed by h –adaption. A further reduction in the potential energy and the relative error norm of the system is found to be achieved with combined r-adaption and mesh enrichment (in the form h-refinement). Numerical study confirms that the proposed combined r-h adaption is more efficient than a purely h-adaptive approach and more flexible than a purely r-adaptive approach with better convergence characteristics. The present work also gives a scope for study of optimality of combined r-h strategy by considering r and h parameters to appear explicitly in the formulation. The smoothing algorithm can be improved to include a combined laplacian and optimization based approach to improve mesh quality at lower computational cost. A step to improvement over the present method would be to consider the stationarity (maximization) of potential incorporating the energy arising due to distortion of elements. Furthermore the recent trends in adaptivity such as the use of meshless techniques can be resorted to. The notion of conservation based on complimentary energy can be worked out for achieving mesh adaption in nonlinear elastic problems and these form the scope of further study.

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Appendix A: Nomenclature

$e(x)$, $\|e(x)\|$	Error function, Norm of error function
$u(x)$	Exact solution for displacement
$u_h(x)$	Finite element solution
N	Interpolation function used for FE approximation
$\hat{u}_h(x)$	Smoothed or recovered finite element solution
h_i	Element size
c	Relaxation and Correction factors
f, q	Body force and traction terms
Ω, Ω_i	Domain and elemental volumes
$\Gamma = \Gamma_D \cup \Gamma_N$	Domain Boundary union of Drichlet and Von Neumann boundaries
$\Psi(u_i, X_A)$	Deformation mapping function
X_A, x_a	Referential and Present Coordinates
F_{iA}	Displacement gradient
$W(u_{i,j}, x_k)$	Strain energy density
σ_{ij}, C_{ij}	Cauchy stress tensor and Energy momentum tensor
g_k	Configurational body forces
G_e^I, G^k	Elemental and assembled nodal configurational force
G_{x1}, G_{x2}	Configurational force at successive present configurations
$\|\Delta G_i\|$	Norm of change in configurational force
η_G, η	Relative G norm and Relative error norm percentage