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Non-Linear Finite Element Framework for Large-Strain Wrinkling of Thin-Layer Structures

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ABSTRACT: Thin-walled lightweight structures begin to wrinkle when they are subjected to in-plane extension, and predicting this phenomenon is a challenging problem in engineering. This paper presents a study of finite-strain wrinkling of membranes using 2D strain gradient elasticity theory. Incorporating strain gradient theory instead of the classical continuum formulation is motivated by the conceptual aid it offers when dealing with size-dependent behavior or intrinsic length scale effects, to increase numerical stability, and to address the limitations of numerically solving classical wrinkling analysis. A nonlinear finite element formulation is derived for hyperelastic thin-walled membranes, which allows wrinkle patterns to occur naturally during the solution process without the need to perform a Buckingham analysis, several buckling modes, or creation of imperfections. The model proposed here has a better gradient solution and a physically consistent form than the classical understanding and is especially applicable to the study of large deformation wrinkling phenomena. The formulation is validated by comparing the membrane's response with the published results, and the accuracy and reliability of the numerical results are checked by considering residual convergence and the sensitivity of the predicted wrinkle characteristics to mesh refinement.

KEYWORDS: Strain gradient theory; finite strain wrinkling; hyperelastic membranes; nonlinear finite element method; continuum mechanics

1 Introduction

Light-walled structures, including membranes, composites, thin films, plates, and shells, are essential parts of many modern engineering systems, and the production method has been a concern. These are used in many different applications, such as inflatable systems and mechanisms, soft robotics, biomedical devices, 'flexible' and 'stretchable' electronic devices, and aerospace structures [1,2]. Due to the very small thickness, in relation to their in-plane dimensions, these structures generally have very low bending stiffness and high in-plane flexibility. Such characteristics contribute to thin-walled systems' high weight and material efficiency. In this study, the membrane is assumed to be very thin compared to the in-plane lengths of the structure, i.e., $t/L \ll 1$, where a characteristic length L is assumed. A thin-shell or thin membrane idealization is commonly used in engineering practice for structures with a thickness-to-radius or thickness-to-span ratio of 0.1 or even lower depending on geometry, loading, and boundary conditions. Given this assumption, membrane forces in the plane of the panel dominate the bending loads, and a membrane-based model is relatively efficient to use for modeling the structure [3]. Due to these properties, they are especially vulnerable to geometric

nonlinearities and structural instabilities. For this reason, instability phenomena that can occur during deformation can also be responsible for their mechanical response. Wrinkling is one of the most widely observed and practically important modes of deformation among these phenomena [4,5]. The wrinkling phenomenon is usually observed when highly compressive stresses are produced in parts of a thin structure that have insufficient bending stiffness. When compressive stresses are applied, instead of bending up and down to withstand them, the structure releases this energy in a fashion that creates a series of small out-of-plane undulations (wrinkles) throughout the structure. Wrinkles might seem justifiable at face value, but they can slightly alter the mechanical response of the structure as well. Wrinkle patterns can be formed in the stiffness distribution and change load transfer pathways, sometimes reducing the structural stability. If they are required to predict the wrinkling behavior, the design of such thin-walled systems in advanced technologies (e.g., flexible electronics, polymer films, and soft material devices) has become an important topic of research [6]. In particular, in the finite strain regime, hyperelastic material models play an important role in the prediction of deformations [7–9].

The problem becomes more complicated if the thin structure has to deal with large deformations. In many applications, thin membranes or films experience large rotations, finite strain, and large extensions. In such cases, the material nonlinearities as well as the geometric nonlinearities become very influential in the mechanical response. These effects are often inherently outside the capacity of classical small-strain theories, which assume infinitesimal deformations and linear material response. For thin material structures like elastomers, polymers, and biological tissues, however, the stress–strain relationship is typically nonlinear for large amounts of deformation. Therefore, a realistic characterization of wrinkling behavior demands formulations that are able to appropriately describe finite-strain kinematics along with the nonlinear constitutive model.

A long-standing approach to the study of wrinkling of thin membranes has been based on classical continuum mechanics and bifurcation theory. Studies have shown that, many times, linear buckling simulation is employed to predict wrinkling onset, followed by nonlinear post-buckling simulation to find the subsequent buckling patterns [10]. Another popular theory is the tension field theory, which is applied by assuming that a compressive stress state in a membrane is released by wrinkle formation, leading to a tensile stress state [11]. While these have been very helpful to the understanding of the instability of membranes, many assume rather artificial conditions, while others have to impose some imperfections into their numerical simulation to produce an instability mode [12]. This means that at times, the predicted wrinkle patterns may be very sensitive to the assumed perturbations and not to the intrinsic characteristics of the structure. Another restriction of classical elasticity theory is that there are no characteristic length scales that belong to the material itself. Experimental observation revealed that in micro- and nanoscale applications, the mechanical behavior of the structure may be dependent on its size [13,14]. Under these circumstances, a prediction by conventional continuum mechanics could differ from experimental results. The discrepancy has spurred research on generalized continuum models that can be applied to develop the application of the additional kinematic measures.

This has led to many interesting models in recent years, among which strain gradient elasticity has received a lot of interest. The latter is a theoretical framework in which the strain energy density is related not only to the classical strain tensor, but also to the spatial gradients. These higher-order strain terms introduce internal length scale parameters that can be used to model microstructural interaction in the material [15,16]. From a physical point of view, these additional terms give a way to account for size effects which are not encompassed by the traditional theory of elastic behavior. Another numerical stability benefit of gradient elasticity models has to do with the simulation of the resulting model. The higher-order terms are regularizations to smooth out localized deformation fields and to prevent out-of-range numerical

singularities that occur in instability problems. This means that, if appropriate and usually advantageous, the deformation pattern and wrinkles can occur naturally during the numerical solution process without introducing artificially growing deformations [17,18]. This characteristic makes gradient-enhanced models especially attractive for studying instability phenomena in a thin-walled structure that is subjected to large deformations [19]. In addition, constitutive laws that are able to adequately capture the nonlinear material response are needed when the thin membranes are subjected to finite strains. Often the models used for this purpose fall into the category of hyperelastic models, including the neo-Hookean, Mooney–Rivlin, and the Ogden models for modeling elastomeric and polymeric materials, respectively [4,20]. These models can trace the nonlinear stress–strain response shown by a large number of soft materials under large deformation. With strain-gradient elasticity, they offer a complete framework that encapsulates, in addition, material nonlinearity and size-dependent mechanical behavior. These formulations have recently been implemented in the study of soft-layered systems, flexible electronic substrates, and inflatable membrane structures where wrinkling patterns have an important mechanical role [21]. At the computational level, the finite element method (FEM) is still the most popular method for solving nonlinear boundary value problems (BVPs) in solid mechanics. It is flexible in its usage and extends to a wide range of complex geometries, boundary conditions, and loading scenarios in a single computational framework and new trends used 2D results to predict 3D wrinkles [22]. However, it is difficult to incorporate strain gradient elasticity into a finite element formulation due to the fact that the governing equations include higher-order spatial derivatives [23]. All of these advances enabled simulation of complex aspects of instability, such as wrinkling, localization, and post-buckling behavior, to be achieved with good numerical stability [24,25]. In recent years, the possibility of using gradient-based finite element formulations to determine the wrinkling patterns in thin films and membranes has been shown to exist under a number of loading conditions. Liu et al. [26] created a model of trans-scale surface wrinkling for stiff film-compliant substrate structures derived from the strain gradient theory and studied the scaling relationships. Other studies have investigated the wrinkling phenomenon of nano thin shells [27] and architectural fabric structures under complex loadings [2]. The extension of the continuum approach to higher-order models is underlined in the mentioned studies, which show that this approach is becoming more and more relevant for the analysis of today's modern thin-walled systems. Although significant progress has been made, modeling wrinkling of thin membranes with large strains is still a difficult problem that involves simultaneously addressing geometric nonlinearities, nonlinear material properties, and higher-order deformation gradients. The creation of comprehensive computational models known for their ability to capture these coupled effects is, therefore, of great interest, both for fundamental studies and applications to engineering.

The primary goal for the present study is to propose a nonlinear finite element approach to the analysis of large-strain wrinkling of thin-walled membranes derived from 2-dimensional strain gradient elasticity theory. The proposed approach is a bridging of finite-strain hyperelastic constitutive modeling and gradient-enhanced elasticity for capturing the nonlinearity of deformation mechanisms as well as size-dependent effects. Because the developed finite element framework includes only the altered elements, wrinkle patterns are not required in the process of solving and do not rely on classic buckling analysis and artificial errors, but are synthesized naturally during the process of solving. This gives a more stable, physically consistent modeling approach for studying wrinkling phenomena in thin-walled structures in large deformations. Indeed, there are no related numerical problems associated with the gradient model, and it is more stable than the classical model. It should be noted that the formulation proposed here is defined in the thin membrane regime; these are analyzed under the assumption provided in the subsequent continuum, constitutive sections. Hence, the claimed advantages compared to classical wrinkling methods should be read within the studied geometry, loading range, material model, and numerical discretization.

The novelty and main contributions of this study are concisely summarized below:

- A non-linear finite element approach is suggested for the study of wrinkles in thin-walled membranes under large strains.
- The model is a finite-strain hyperelastic model combined with gradient-enhanced elasticity, so it takes into account material nonlinearity, geometric nonlinearity, and higher-order deformation effects at the same time.
- The formulation is different from the classical approach, which focuses on buckling or post-buckling analysis, and does not require artificial imperfections or a previous stability analysis to trigger wrinkling.
- The higher-order gradient terms give rise to a numerical regularization and lead to a more stable and physically sound simulation of wrinkle formation.

2 The Main Principles of Strain Gradient Theory

Various models of strain gradient theory have been introduced by researchers. Here, the strain gradient elasticity theory introduced by Mindlin (1964) [28], is briefly discussed. In this theory, the strain energy function is defined as follows:

$$W = W(\varepsilon, \kappa) \quad (1)$$

The constitutive response is described by a strain-energy density function that depends on the finite-deformation strain measure and its spatial gradient. The classical part of the response is associated with the deformation strain, while the higher-order contribution accounts for strain-gradient effects. The internal length scale controls the magnitude of the gradient contribution, and when this parameter tends to zero, the formulation reduces to the corresponding classical constitutive model.

The deformation is described in the reference configuration Ω_0 with material coordinates $\mathbf{X}$ and the current configuration Ω with spatial coordinates $\mathbf{x}$. The displacement field is defined as $\mathbf{u} = \mathbf{x} - \mathbf{X}$. The variation of the strain-energy functional over an arbitrary volume V can be written as:

$$\delta \int_V W dV = \int_V (\boldsymbol{\sigma} : \nabla \delta \mathbf{u} + \boldsymbol{\tau} : \nabla \nabla \delta \mathbf{u}) dV \quad (2)$$

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \varepsilon}, \quad \boldsymbol{\tau} = \frac{\partial W}{\partial \kappa} \quad (3)$$

where W is the strain-energy density, $\boldsymbol{\sigma}$ denotes the Cauchy stress tensor, $\boldsymbol{\tau}$ is the higher-order stress tensor associated with the strain-gradient term, and $\delta \mathbf{u}$ is the virtual displacement field. The operators $\nabla \delta \mathbf{u}$ and $\nabla \nabla \delta \mathbf{u}$ represent the first and second gradients of the virtual displacement, respectively. It should be mentioned that in the present study, the internal length-scale parameter is treated as a constitutive/model parameter controlling the gradient contribution. It should not be interpreted automatically as equal to the membrane thickness or wrinkle wavelength, and its value must be selected or calibrated consistently with the material and structural scale under consideration.

Using the chain derivative law and the divergence theorem, Eq. (2) becomes:

$$\delta \int_V W dV = - \int_V [\nabla \cdot (\boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\tau})] \cdot \delta \mathbf{u} dV + \int_S \mathbf{n} \cdot (\boldsymbol{\sigma} - \nabla \boldsymbol{\tau}) \cdot \delta \mathbf{u} dS \quad (4)$$

where S is the boundary curve of V and $\mathbf{n}$ is the normal vector on S . Based on the principles of variations, the gradient can be expressed in terms of the surface gradient and the normal derivative as follows:

$$\begin{aligned}\nabla \delta \mathbf{u} &= \overset{s}{\nabla} \delta \mathbf{u} + \mathbf{n} \otimes D \delta \mathbf{u} \\ \overset{s}{\nabla} &= (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \cdot \nabla \quad D = \mathbf{n} \cdot \nabla\end{aligned}\quad (5)$$

where $\mathbf{I}$ is the unity tensor. Also, if tensor Φ is a vector of order two and $\mathbf{v}$ is a vector, one can write:

$$\Phi : \nabla \mathbf{v} = \Phi : \overset{s}{\nabla} \mathbf{v} + \mathbf{n} \cdot \Phi \cdot D \mathbf{v} = \overset{s}{\nabla} \cdot (\Phi \cdot \mathbf{v}) - \left(\overset{s}{\nabla} \cdot \Phi \right) \cdot \mathbf{v} + \mathbf{n} \cdot \Phi \cdot D \mathbf{v} \quad (6)$$

Using the surface divergence theorem, the first term on the right side of Eq. (6) takes the following form:

$$\int_S \overset{s}{\nabla} \cdot (\Phi \cdot \mathbf{v}) dS = \int_S \left(\overset{s}{\nabla} \cdot \mathbf{n} \right) \mathbf{n} \cdot \Phi \cdot \mathbf{v} dS \quad (7)$$

And the integral on the left side of Eq. (7) is expressed as:

$$\begin{aligned}\int_S \Phi : \nabla \mathbf{v} dS &= \int_S (L \cdot \Phi) \cdot \mathbf{v} dS + \int_S \mathbf{n} \cdot \Phi \cdot D \mathbf{v} dS \\ L &= \mathbf{n} \overset{s}{\nabla} \cdot \mathbf{n} - \overset{s}{\nabla}\end{aligned}\quad (8)$$

Therefore, Eq. (2) becomes as follows:

$$\begin{aligned}\delta \int_V W dV &= - \int_V [\nabla \cdot (\boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\tau}) \cdot \delta \mathbf{u}] dV + \int_S \mathbf{n} \cdot (\boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\tau}) \cdot \delta \mathbf{u} dS \\ &\quad + \int_S (L \cdot \mathbf{n} \cdot \boldsymbol{\tau}) \cdot \delta \mathbf{u} dS + \int_S (\mathbf{n} \mathbf{n} : \boldsymbol{\tau}) \cdot D \delta \mathbf{u} dS\end{aligned}\quad (9)$$

As a result, based on the principle of virtual work, it is obtained that:

$$\delta \int_V W dV = \int_V \mathbf{f} \cdot \delta \mathbf{u} dV + \int_S (\mathbf{t}_1 \cdot \delta \mathbf{u} + \mathbf{t}_2 \cdot D \delta \mathbf{u}) dS \quad (10)$$

where $\mathbf{f}$ is the bulk force per unit volume and $\mathbf{t}_1$ and $\mathbf{t}_2$ are traction. Finally, the equilibrium equation is given by:

$$\nabla \cdot (\boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\tau}) + \mathbf{f} = \mathbf{0} \quad (11)$$

and the tractions are expressed as:

$$\begin{aligned}\mathbf{t}_1 &= \mathbf{n} \cdot (\boldsymbol{\sigma} - \nabla \cdot \boldsymbol{\tau}) + L \cdot \mathbf{n} \cdot \boldsymbol{\tau} \\ \mathbf{t}_2 &= \mathbf{n} \mathbf{n} : \boldsymbol{\tau}\end{aligned}\quad (12)$$

To capture size-dependent effects in a large-strain framework, we extend the classical Mindlin strain gradient theory to finite elasticity. Eq. (13) is derived by performing a Taylor expansion of the energy density around the reference configuration and retaining the lowest-order terms that maintain the stability of the gradient enhancement. Specifically:

1. The classical part follows a neo-Hookean model.
2. The gradient part is modeled as a quadratic form of $\mathbf{G}$ to introduce a single intrinsic length scale l . The material parameter l controls the strength of the strain-gradient effect and has the dimension of length.

To extend this formulation to the finite-strain regime used in the wrinkling analysis, the strain energy density, W , is written as a function of the deformation gradient $\mathbf{F}$ and its gradient $\mathbf{G}$, so that the model captures both geometric nonlinearity and higher-order strain-gradient effects. In the vicinity of the reference configuration. Assuming a stress-free reference state (vanishing first-order derivatives) and material centrosymmetry (which decouples odd-order coupling between $\mathbf{F}$ and $\mathbf{G}$, the leading non-vanishing terms yield an uncoupled additive representation. To extend this representation into the finite-strain regime while ensuring material frame-indifference and energetic stability, the energy density is decomposed into a classical hyperelastic contribution and a quadratic gradient-enhancement part:

$$W(\mathbf{F}, \mathbf{G}) = W_1(\mathbf{F}) + W_2(\mathbf{F}, \mathbf{G}) \quad (13)$$

where $W_1(\mathbf{F})$ characterizes the standard macroscopic finite-strain response (modeled via a neo-Hookean formulation), and $W_2(\mathbf{F}, \mathbf{G})$ is a positive-definite quadratic form of $\mathbf{G}$ penalizing localized strain variations and introducing an intrinsic material length scale l .

$$W_1(\mathbf{F}) = \omega(\lambda_1, \lambda_2, J)$$

$$W_2(\mathbf{F}, \mathbf{G}) = \frac{1}{2}A_g(|\mathbf{g}_1|^2 + |\mathbf{g}_2|^2) + \frac{1}{2}A_\Gamma|\mathbf{\Gamma}|^2 \quad (14)$$

Here, $W_1(\mathbf{F})$ represents the classical hyperelastic strain energy density expressed as a function of the principal stretches λ_1, λ_2 and the area ratio J . The term $W_2(\mathbf{F}, \mathbf{G})$ is the higher-order (gradient-enhanced) energy contribution, defined as a quadratic function of the strain gradient tensor components to ensure energetic stability and material frame-indifference. The investigation of objectivity of Eq. (13) is done and reported in Appendix A. Physical meaning of terms in Eq. (14) is explained in following. $W_2(\mathbf{F}, \mathbf{G})$ captures the additional energy stored due to non-homogeneous strain fields and microstructural deformation gradients. It penalizes sharp spatial variations in deformation, thereby regularizing the formulation and providing an intrinsic physical length scale for natural wrinkle formation. $\mathbf{\Gamma}$ represents the higher-order strain gradient vector associated with mixed in-plane shear gradients or coupling curvature-like effects. A_g and A_Γ are positive constitutive material parameters (gradient moduli). These parameters dictate the strength of the size-dependent effect; higher values impose greater resistance against localized strain gradients and suppress spurious high-frequency wrinkling modes.

Here, $\mathbf{G} = \nabla \mathbf{F}$ is the third-order strain-gradient tensor representing the gradient of the deformation gradient. In terms of the displacement vector since the deformation gradient is given by $\mathbf{F} = \mathbf{I} + \nabla \mathbf{u}$ the components of $\mathbf{G}$ directly correspond to the second material gradient of displacement. The vector quantities $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{\Gamma}$ in Eq. (14) are defined by projecting the third-order tensor $\mathbf{G}$ onto the material principal directions $\mathbf{D}_1$ and $\mathbf{D}_2$. Physically, $\mathbf{g}_1$ and $\mathbf{g}_2$ represent the directional strain-gradient vectors along the principal material directions, while $\mathbf{\Gamma}$ captures the mixed (shear-gradient) deformation mode.

To find the response functions $\partial W / \partial F_{iA}$ and $\partial W / \partial G_{iAB}$ used in Euler's equations and natural boundary conditions, the chain derivative law is used and the resulting expression is:

$$\delta W = \frac{\partial W}{\partial F_{iA}} \delta F_{iA} + \frac{\partial W}{\partial G_{iAB}} \delta G_{iAB} = \delta W_1 + \delta W_2 \quad (15)$$

It is also obtained that:

$$\begin{aligned} J \delta J &= \lambda_2^2 \lambda_1 \delta \lambda_1 + \lambda_1^2 \lambda_2 \delta \lambda_2 - S \delta S \\ \lambda_1 \delta \lambda_1 &= \mathbf{F} \mathbf{D}_1 \cdot \delta \mathbf{F} \mathbf{D}_1 = \mathbf{d}_1 \otimes \mathbf{D}_1 \cdot \delta \mathbf{F} \end{aligned} \quad (16)$$

Hence, one can write:

$$\begin{aligned} \frac{\partial W}{\partial F_{iA}} &= (\lambda_1^{-1} \omega_1 + \lambda_2^{-2} J^{-1} \omega_J) \mathbf{d}_1 \otimes \mathbf{D}_1 + (\lambda_2^{-1} \omega_2 + \lambda_1^{-2} J^{-1} \omega_J) \mathbf{d}_2 \otimes \mathbf{D}_2 \\ &\quad - \omega_J \tan \gamma (\mathbf{d}_1 \otimes \mathbf{D}_2 + \mathbf{d}_2 \otimes \mathbf{D}_1) \end{aligned} \quad (17)$$

$$\delta W_2 = A_g (\mathbf{g}_1 \cdot \delta \mathbf{g}_1 + \mathbf{g}_2 \cdot \delta \mathbf{g}_2) + A_T \Gamma \cdot \delta \Gamma$$

$$\mathbf{g} \cdot \delta \mathbf{g} = g_i D_A D_B \delta G_{iAB}$$

$$\Gamma \cdot \delta \Gamma = \Gamma_i D_A^{(1)} D_B^{(1)} \delta G_{iAB} \quad (18)$$

Finally, using the symmetry conditions, it is obtained:

$$\frac{\partial W}{\partial G_{iAB}} = A_g (\mathbf{g}_1 \otimes \mathbf{D}_1 \otimes \mathbf{D}_1 + \mathbf{g}_2 \otimes \mathbf{D}_2 \otimes \mathbf{D}_2) + \frac{1}{2} A_T \Gamma \otimes (\mathbf{D}_1 \otimes \mathbf{D}_2 + \mathbf{D}_2 \otimes \mathbf{D}_1) \quad (19)$$

These equations are used to calculate the forces and tractions in terms of wrinkling deformation. The function W_1 which includes the first gradient tensor of deformation and its determinant can be chosen in various ways, which will be discussed further [29]. The strain energy function can be written as follows:

$$W(F) = \frac{1}{2} \tilde{\lambda} (\log J)^2 - \frac{\tilde{\mu}}{2} (\mathbf{F} : \mathbf{F} - 2 - 2 \log J) \quad (20)$$

in which:

$$\mathbf{F} : \mathbf{F} = \text{tr}(\mathbf{F}^T \mathbf{F}), J = \det \mathbf{F} \quad (21)$$

Here, $\mathbf{F} : \mathbf{F}$ denotes the inner product of the deformation gradient, whereas $\det \mathbf{F}$ represents Jacobian of the deformation gradient tensor. To relate the internal forces to the reference configuration, the first Piola-Kirchhoff stress tensor $\mathbf{P}$ is derived from the strain energy density function W with respect to the deformation gradient $\mathbf{F}$. This tensor provides a measure of force per unit of undeformed reference area and is defined as follows:

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}} = \tilde{\lambda} \log J \mathbf{F}^{-T} + \tilde{\mu} [\mathbf{F} - \mathbf{F}^{-T}] \quad (22)$$

where $\tilde{\lambda}$ and $\tilde{\mu}$ represent the effective Lamé parameters associated with the chosen hyperelastic constitutive model. The term $\mathbf{F} - \mathbf{F}^{-T}$ refers to the inverse transpose of the deformation gradient tensor, which ensures that the volumetric contribution to the stress remains objective and frame-indifferent. Furthermore, the logarithmic term $\log J$ is employed to provide a physically consistent energy growth for large compressions or expansions.

The Cauchy stress tensor is also derived from the Piola-Kirchhoff tensor by using the method:

$$\boldsymbol{\sigma} = \frac{1}{J} \mathbf{P} \cdot \mathbf{F}^T = \tilde{\lambda} \frac{\log J}{J} \tilde{\mathbf{I}} + \tilde{\mu} \frac{1}{J} [\mathbf{B} - \tilde{\mathbf{I}}] \quad (23)$$

The Cauchy stress tensor $\boldsymbol{\sigma}$ is related to the first Piola-Kirchhoff stress tensor $\mathbf{P}$ through the push-forward transformation $\boldsymbol{\sigma} = J^{-1} \mathbf{P} \cdot \mathbf{F}^T$. In this expression, $\mathbf{B}$ denotes the left Cauchy-Green deformation tensor, and $\mathbf{I}$ represents the second-order identity tensor in the spatial configuration. The presence of these tensors ensures that the stress formulation is consistently defined within the current (deformed) configuration of the membrane.

The fourth-order tensor of elasticity is as follows:

$$\mathbf{A} = \frac{\partial \mathbf{P}}{\partial \mathbf{F}} = \tilde{\lambda} \mathbf{F}^{-T} \otimes \mathbf{F}^{-T} + \tilde{\mu} \tilde{\mathbf{I}} + [\tilde{\lambda} \log J - \tilde{\mu}] \tilde{\mathbf{D}} \quad (24)$$

in which:

$$\begin{aligned} \tilde{\mathbf{D}} &= \frac{\partial \mathbf{F}^{-T}}{\partial \mathbf{F}} = \mathbf{F}^{-T} \underline{\otimes} \mathbf{F} + \tilde{i} \underline{\otimes} (\mathbf{F} \cdot \mathbf{F}^{-T}) \\ \{A \underline{\otimes} B\}_{ijkl} &= A_{ik} B_{jl}, \quad \{A \underline{\otimes} B\}_{ijkl} = A_{il} B_{jk} \end{aligned} \quad (25)$$

3 Euler's Equations and Boundary Conditions

The derivation of Euler's equations and boundary conditions from the second gradient is well-known. Here, in order to achieve acceptable loads for deformation, these equations are derived. In general, according to the principle of virtual work, one can write [30]:

$$\delta E = \int_{\Omega} \delta W(\mathbf{F}, \mathbf{G}) da \quad (26)$$

In this equation δW is calculated as follows:

$$\delta W = \frac{\partial W}{\partial F_{iA}} u_{i,A} + \frac{\partial W}{\partial G_{iAB}} u_{i,AB} \quad (27)$$

The weak form leads to both classical boundary terms and higher-order boundary contributions. However, the manuscript does not explicitly prescribe the higher-order boundary conditions or corner terms for the numerical examples; these conditions must therefore be stated separately and applied consistently for each constrained, symmetry, and free boundary.

Here, the uppercase subscripts A and B (where $A, B \in [1 \text{ or } 2]$) denote the material coordinate components in the reference configuration. The comma notation followed by subscript A (or B) represents partial differentiation with respect to the A -th (or B -th) reference coordinate component. Because the strain-energy density depends on both the deformation gradient F_{iA} and its higher-order gradient G_{iAB} , the subsequent integration by parts generates the conventional equilibrium equation as well as higher-order boundary terms. Consequently, the complete boundary-value problem involves both conventional and higher-order boundary quantities.

The terms of the latter equation are derived as:

$$\frac{\partial W}{\partial F_{iA}} u_{i,A} = \left(\frac{\partial W}{\partial F_{iA}} u_i \right)_{,A} - u_i \left(\frac{\partial W}{\partial F_{iA}} \right)_{,A} \quad (28)$$

$$\frac{\partial W}{\partial G_{iAB}} u_{i,AB} = \left(\frac{\partial W}{\partial G_{iAB}} u_{i,A} \right)_{,B} - u_{i,A} \left(\frac{\partial W}{\partial G_{iAB}} \right)_{,B} \quad (29)$$

Therefore, it follows:

$$\delta E = \int_{\Omega} P_{iA} u_{i,A} da + \int_{\partial\Omega} \left(\frac{\partial W}{\partial G_{iAB}} \right) u_{i,A} N_B ds \quad (30)$$

where $\mathbf{N}$ is the unit normal vector for the boundary curve $\partial\Omega$ in the Green-Stokes theorem, and P_{iA} is:

$$P_{iA} = \frac{\partial W}{\partial F_{iA}} - \left(\frac{\partial W}{\partial G_{iAB}} \right)_{,B} \quad (31)$$

Accordingly, Eq. (26) is rewritten as:

$$\delta E = \int_{\partial\Omega} \left(u_i P_{iA} N_A + \frac{\partial W}{\partial G_{iAB}} u_{i,A} N_B \right) ds - \int_{\Omega} u_i P_{iA,A} da \quad (32)$$

When $P_{iA,A} = 0$ using the following perpendicular-tangent decomposition:

$$\nabla \mathbf{u} = \mathbf{u}' \otimes \mathbf{T} + \mathbf{u}_{,N} \otimes \mathbf{N} \quad (33)$$

where as mentioned before, $\mathbf{N}$ is the unit normal vector on $\partial\Omega$ and $\mathbf{u}' = \frac{d\mathbf{u}}{ds}$ and $\mathbf{u}_{,N}$ are the tangential and normal $\mathbf{u}$ forces on $\partial\Omega$, respectively. By applying the derivative of the product in the second term of the first integral of the Eq. (32), it is obtained that:

$$\frac{\partial W}{\partial G_{iAB}} T_A N_B u'_i = \left(\frac{\partial W}{\partial G_{iAB}} T_A N_B u_i \right)' - \left(\frac{\partial W}{\partial G_{iAB}} T_A N_B \right)' u_i \quad (34)$$

Finally, Eq. (26) results in the following relation:

$$\begin{aligned} \delta E = & \int_{\partial\Omega} u_i \left[P_{iA} N_A - \left(\frac{\partial W}{\partial G_{iAB}} T_A N_B \right)' \right] ds - \int_{\partial\Omega} u_{i,N} \left(\frac{\partial W}{\partial G_{iAB}} N_A N_B \right) ds - \\ & - \sum u_i \left(\frac{\partial W}{\partial G_{iAB}} T_A N_B \right) \end{aligned} \quad (35)$$

Also, the external power P is defined as follows:

$$P = \int_{\partial\Omega} t_i u_i ds + \int_{\partial\Omega} m_i u_{i,N} ds + \sum f_i u_i \quad (36)$$

in which:

$$\begin{aligned} t_i &= P_{iA} N_A - \left(\frac{\partial W}{\partial G_{iAB}} T_A N_B \right)' \\ m_i &= \frac{\partial W}{\partial G_{iAB}} N_A N_B \\ f_i &= - \left(\frac{\partial W}{\partial G_{iAB}} T_A N_B \right) \end{aligned} \quad (37)$$

4 Finite Element Method

For each node, the primary unknowns are the three Cartesian displacement components, and the strain and strain-gradient measures are calculated consistently from the derivatives of this interpolation. To use the finite elements method to analyze a surface, one can divide the surface into small elements given by:

$$\ell_0 = \bigcup_{i=1}^{n_c} \ell_0^c \quad (38)$$

where n_c is the number of elements on the surface. It is also possible to transfer the considered elements to natural coordinates [31]. Using the Galerkin approximation and standard interpolation functions, Eq. (39) becomes:

$$x(\xi, \eta) \approx \sum_{i=1}^{N_{node}} N^i(\xi, \eta) x^i, \quad X(\xi, \eta) \approx \sum_{i=1}^{N_{node}} N^i(\xi, \eta) X^i \quad (39)$$

where ξ and η are the natural coordinates and N^i are shape functions of the element at node i . In addition, the deformation gradient tensor on the surface is given as follows [29]:

$$\mathbf{F} = \nabla \mathbf{x} \approx \sum_{i=1}^{N_{node}} \mathbf{x}^i \otimes \nabla N^i \quad (40)$$

The weak form of the equations is as follows [29]:

$$\int_{\ell_0} \mathbf{P} \cdot \nabla N^i dA + \int_{\ell_0} [\mathbf{n} \otimes \tilde{\mathbf{S}}] \cdot N^i dA = \int_{\ell_0} b_0 N^i dA + \int_{\Omega} t_0 N^i d\Omega \quad (41)$$

In which surface-related quantity $\tilde{\mathbf{S}}$ represents the surface stress tensor in the reference configuration. $\mathbf{P}$ is the two-point tensor of the first Piola-Kirchhoff stress on the surface and b_0 is the volume force, which are given as follows:

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}, \quad b_0 = -\frac{\partial W}{\partial \mathbf{x}} \quad (42)$$

Newton-Raphson method is utilized to solve Eq. (41). For this study, the residual vector for each internal node is defined first as follows:

$$R_e^i = \int_{\ell_0} \mathbf{P} \cdot \nabla N^i dA + \int_{\ell_0} [\mathbf{n} \otimes \tilde{\mathbf{S}}] \cdot N^i dA - \int_{\ell_0} b_0 N^i dA - \int_{\Omega} t_0 N^i d\Omega \quad (43)$$

And $\mathbf{n}$ is the vertical vector output from the spatial configuration. Using the Newton-Raphson method, Eq. (45) is obtained [30]:

$$\mathbf{R}(\mathbf{d}) + \frac{\partial \mathbf{R}}{\partial \mathbf{d}} \Delta \mathbf{d} = 0, \quad d_{n+1} = d_n + \Delta d \quad (44)$$

where n is the number of iterations in each step. Moreover, $\mathbf{R}$ and $\mathbf{d}$ are the residual vectors and spatial coordinates defined as follows:

$$\mathbf{d} = [x^1, x^2, \dots, x^n]^T \quad (45)$$

$$\mathbf{R} = [R^1, R^2, \dots, R^n]^T \quad (46)$$

By solving Eq. (44), the stiffness matrix for each element is calculated as follows [29]:

$$[K_e^{ij}] = \int_{\beta_0} \{\nabla N^i\}_b \{A\}_{abcd} \{\nabla N^j\}_d dV + \int_{\ell_0} \{\tilde{\nabla} \tilde{N}^i\}_b \{\tilde{H}\}_{abcd} \{\tilde{\nabla} \tilde{N}^j\}_d dA. \quad (47)$$

Here, $\tilde{N}_b$ denotes the interpolation function in the reference configuration, and $\tilde{\nabla} \tilde{N}_b$ denotes its gradient with respect to the reference coordinates X . In the discrete formulation, Eq. (50) is obtained from the weak form by substituting the finite-element interpolation of the displacement field into the virtual work expression and collecting the element contributions in matrix form. Therefore, Eq. (48) represents the discretized counterpart of the continuum weak form introduced in Section 3. $\mathbf{A}$ and $\tilde{\mathbf{H}}$ are fourth-order two-point elasticity tensors defined with respect to the deformation gradient $\mathbf{F}$:

$$\mathbf{A} = \frac{\partial \mathbf{P}}{\partial \mathbf{F}}, \quad \tilde{\mathbf{H}} = \frac{\partial [\mathbf{n} \otimes \tilde{\mathbf{S}}]}{\partial \tilde{\mathbf{F}}} \quad (48)$$

In Eqs. (49) and (50), the tilde notation (e.g., in $\tilde{H}$, $\tilde{S}$, and $\tilde{\nabla}$) is employed to signify that these operators and tensors are defined with respect to the reference (material) configuration. For instance, $\tilde{\nabla}$ denotes the gradient operator in the material coordinates, and $\tilde{S}$ represents the surface stress tensor in the reference state. This notation ensures a clear distinction from spatial quantities.

The rectangular element of the thin-walled membrane is considered in the xy plane. In each node such as i , the displacement vector variables a_i are introduced, which has three basic components, for example, if the displacement in the x -direction is represented by u , these three components are displacement in the x -direction, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. The nodal displacement vector a_i is defined as follows. The displacement element, as usual, is expressed as the following nodal displacements with 12 components for each element [32]:

$$\mathbf{a}^e = \left\{ a_i \quad a_j \quad a_k \quad a_l \right\}^T, \quad a_i = \left\{ u \quad \frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \right\}^T. \quad (49)$$

It is easy to use a polynomial expression to define shape functions with 12 parameters. Specific terms must be removed from the 4th-order polynomial of the desired term. A polynomial expression in the following form is presented:

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x^3 y + \alpha_{12} xy^3 \equiv \tilde{\mathbf{P}}\boldsymbol{\alpha} \quad (50)$$

The above polynomial has certain advantages. For example, along any line $x = \text{const}$ or $y = \text{const}$ the displacement vector u will change as a third order polynomial. The borders of the element or its common faces consist of such lines. Because some of the final values are common in adjacent elements, the continuity condition of u will be satisfied along every common face. The constants α_1 to α_{12} are obtained by writing 12 simultaneous equations that relate the values of u and their derivatives at the nodes:

$$u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \dots, \left(\frac{\partial u}{\partial x} \right)_i = \alpha_2 + \alpha_5 y_i + \dots, \left(\frac{\partial u}{\partial y} \right)_i = \alpha_3 + \alpha_5 x_i + \dots \quad (51)$$

By listing all 12 equations, they can be converted into the following matrix form:

$$\mathbf{a}^e = \mathbf{C}\boldsymbol{\alpha} \quad (52)$$

where $\mathbf{C}$ is a 12×12 matrix and depends on nodal coordinates, and $\boldsymbol{\alpha}$ is the vector contains 12 unknown constants. By inverting the above relationship, one can obtain:

$$\boldsymbol{\alpha} = \mathbf{C}^{-1}\mathbf{a}^e \quad (53)$$

Now one can write an expression to transfer the element in the standard form as follows:

$$\mathbf{u} \equiv \mathbf{N}\mathbf{a}^e = \mathbf{P}\mathbf{C}^{-1}\mathbf{a}^e \quad (54)$$

Finally, the shape functions are obtained from the following relation:

$$\mathbf{N} = \mathbf{P}\mathbf{C}^{-1} \quad (55)$$

where N has 12 values, corresponding to the 12 degrees of freedom, for the shape function, and these functions were calculated for the rectangular element using MAPLE software:

$$\begin{aligned}
 N_1 &= \frac{1}{8} [-(\xi - 1)(\eta - 1)(\eta^2 + \eta + \xi^2 + \xi - 2)], & N_2 &= \frac{1}{8} [-a(\xi + 1)(\xi - 1)^2(\eta - 1)], \\
 N_3 &= \frac{1}{8} [-b(\eta + 1)(\eta - 1)^2(\xi - 1)], & N_4 &= \frac{1}{8} [(\xi + 1)(\eta - 1)(\eta^2 + \eta + \xi^2 - \xi - 2)], \\
 N_5 &= \frac{1}{8} [-a(\xi - 1)(\xi + 1)^2(\eta - 1)], & N_6 &= \frac{1}{8} [b(\eta + 1)(\eta - 1)^2(\xi + 1)], \\
 N_7 &= \frac{1}{8} [-(\xi + 1)(\eta + 1)(\eta^2 - \eta + \xi^2 - \xi - 2)], & N_8 &= \frac{1}{8} [a(\xi - 1)(\xi + 1)^2(\eta + 1)], \\
 N_9 &= \frac{1}{8} [b(-1 + \eta)(\eta + 1)^2(\xi + 1)], & N_{10} &= \frac{1}{8} [(\xi - 1)(\eta + 1)(\eta^2 - \eta + \xi^2 + \xi - 2)], \\
 N_{11} &= \frac{1}{8} [a(\xi + 1)(\xi - 1)^2(\eta + 1)], & N_{12} &= \frac{1}{8} [-b(\eta - 1)(\eta + 1)^2(\xi - 1)].
 \end{aligned} \tag{56}$$

Using these functions the stiffness matrix can be calculated. As a result, the final equation is as follows:

$$\mathbf{Kd} = \mathbf{R} \tag{57}$$

It is worth noting that the present gradient-enhanced formulation introduces additional degrees of freedom, which increases the size of the global system and the computational cost of each nonlinear iteration relative to classical FEM. Nevertheless, unlike the classical wrinkling analysis, which typically requires a preliminary buckling analysis followed by a post-buckling computation, the present method directly predicts the wrinkled state within a single nonlinear procedure. As a result, the additional local cost associated with the extra degree of freedoms (DOFs) is partly offset by the elimination of the separate buckling stage, making the overall computational effort competitive.

Also, it is important to note that the rectangular element employed in this framework is a C^0 -continuous element and does not strictly satisfy the C^1 continuity requirement usually associated with second-gradient theories. However, in the context of the present non-linear large-strain analysis, this non-conforming approximation is utilized to maintain computational efficiency. The effectiveness of this choice in capturing the physical wrinkling patterns is demonstrated through the convergence and validation cases presented in the numerical examples section. To trace the post-bifurcation response, the incremental-iterative Newton–Raphson scheme is employed. Since the flat configuration represents a trivial symmetric equilibrium path, a small symmetry-breaking perturbation (in the form of a minor geometry imperfection or out-of-plane force perturbation) is introduced to trigger the wrinkled post-bifurcation branch. While the gradient regularization provides numerical stabilization and dictates the characteristic wrinkle length scale, the actual branch selection is governed by this imposed perturbation.

5 Numerical Simulations

In the previous sections, the principles of membrane deformation based on strain gradient elasticity theory and discretization of equations based on the finite element method were presented. In this section, the performance of the presented formulation is demonstrated using examples. For this purpose, a finite element code was written in MATLAB R2025 for the formulation presented here. The nonlinear equilibrium equations are solved incrementally using the Newton–Raphson method. At each load or displacement increment, convergence is declared when both the normalized residual and the relative displacement correction satisfy prescribed tolerances. Note on Higher-Order Boundary Conditions: In the finite element

implementation of the current framework, while the theoretical derivation in Section 3 accounts for higher-order tractions and corner forces, these terms are set to zero in the following numerical simulations. This is justified as the considered thin-layer structures are assumed to be either smooth (no sharp edges) or have free-edge conditions where higher-order external prescriptions are absent. Thus, the external work contribution is reduced to standard traction and body forces, as shown in the weak form Eq. (42). In the first example, the deformation of a square membrane under the application of out-of-plane displacement to its center is analyzed. In the second example, the issue of wrinkling under uniaxial extension is examined and investigated in detail.

5.1 Deformation of a Square Membrane Subjected to Point Force

In this example, a 10×10 membrane with fixed edges is considered. The out-of-plane force is applied at the center. The Lamé constants for this problem are considered as $\lambda = 1.3$ GPa and $\mu = 0.4$ GPa. Using the symmetry, it is possible to consider one-fourth of the fabric. The reaction force curve in terms of the vertical displacement of the center is shown in Fig. 1. Different curves for various values of A_g and A_Γ have been produced in the form $A_g = A_\Gamma = \mu l^2$. In this regard, length l is a material characteristic and it is considered as a multiple of the thickness before deformation, i.e., h . Therefore, it can be seen that the curve obtained based on the gradient formulation is very close to the curve reported in the article by Nadler et al. [33], provided that the values $A_g = A_\Gamma = \mu (h/4)^2$ are used in the calculations. In other words, with a condition $l = h/4$, the gradient formulation used in this paper is able to produce the sheet response based on the two-scale formulation that starts from the primary structure of the structure and formulates the macro behavior of it based on its micro structure. The characteristic length l controls the influence of the higher-order gradient terms and therefore affects the regularization capability of the present formulation. In this study, the value $l = h/4$ is chosen as a representative constitutive length scale proportional to the membrane thickness. This choice provides a physically reasonable relation between the internal material length and the structural dimension, while remaining sufficiently small compared with the in-plane dimensions of the sheet. It should be noted, however, that the quantitative response may depend on the selected value of l . Therefore, although $l = h/4$ is adopted here for demonstration purposes, a systematic sensitivity analysis and experimental calibration of this parameter would be desirable in future work.

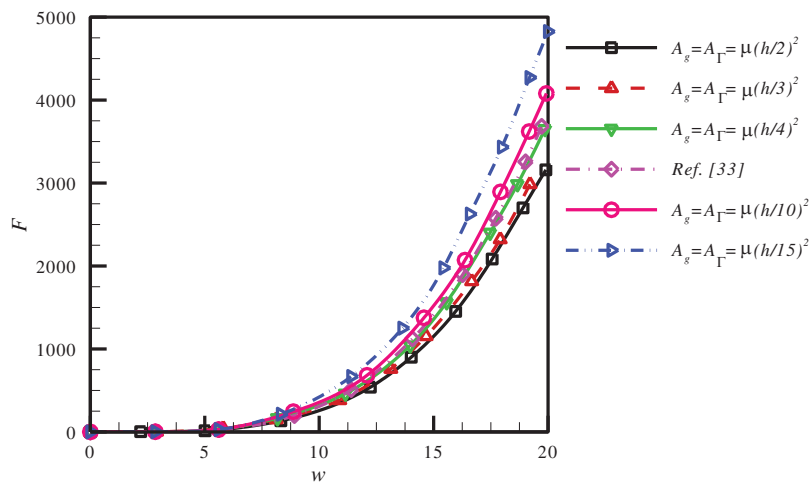


Figure 1: Force-displacement curve for a square fabric under loading in the center [33].

The contours of $\partial w/\partial x$, $\partial w/\partial x$ and $\partial w/\partial y$ can be seen in Figs. 2 and 3. Due to the symmetry of the sheet and loading, these two contours are practically equal and each one is obtained by rotating the other by 90 degrees.

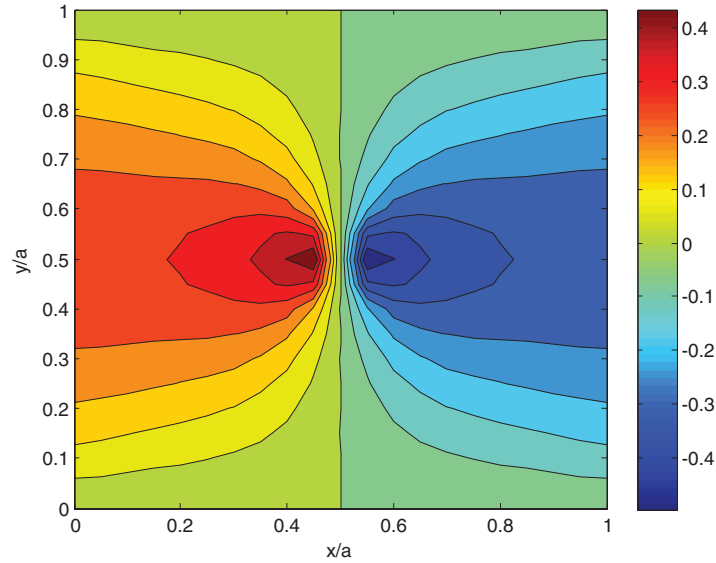


Figure 2: Contour of $\partial w/\partial x$ for the square fabric.

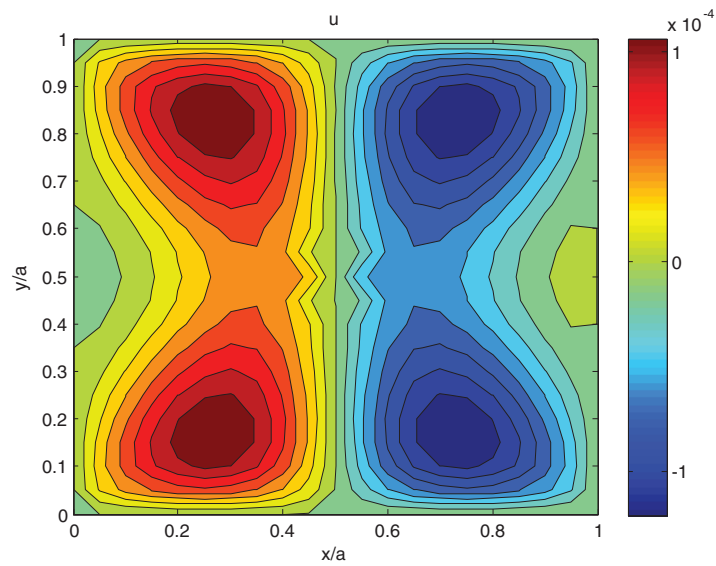


Figure 3: Contour of the displacement component u for the fabric.

The contour plots for the displacement derivatives and components are presented in Figs. 2 and 3. Specifically, Fig. 2 illustrates the distribution of $\partial w/\partial x$, while Fig. 3 shows the displacement component u . It should be noted that due to the symmetry of the square geometry and the boundary conditions, the contours for $\partial w/\partial y$ and the displacement component v are identical to those of $\partial w/\partial x$ and u , respectively, subject to a 90-degree rotation. Therefore, for the sake of brevity, only the x -direction results are displayed.

5.2 Membrane Wrinkling under Uniaxial Extension

In this problem, we focus on wrinkling in the thin-walled membranes under uniaxial extension. We consider a membrane with dimensions $L \times b \times h$ where L is its length in the direction of load, b is the width, and h is its thickness. As seen in Fig. 4, two Cartesian coordinate systems are considered on the fabric (one in the lower left corner and the other in the center) where the X and x axes are along the length of the sheet and the Y and y axes are along its width. The direction of the third axis is also in the direction of the thickness and perpendicular to the plane. The $x = 0$ edge of the membrane is completely fixed and has no movement in the three directions of the coordinate system. Also, the end of the membrane (due to being connected to the jaws of the extension device) is only able to move in the x -direction, and its movement in other directions is zero. The other two edges of the sheet are also free.

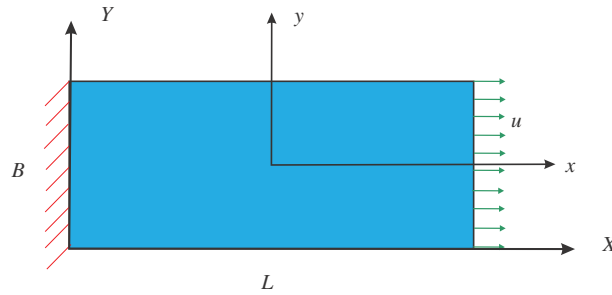


Figure 4: Geometry of a rectangular membrane under uniaxial extension.

It is assumed that the woven material has properties close to a polymeric material with Young's modulus $E = 1$ MPa and Poisson's ratio $\nu = 0.49$. According to Ref. [34], length, width, and thickness of the sheet are $L = 0.254$ m, $B = 0.102$ m, and $h = 0.1$ mm, respectively. The comparison with [34] is performed using the same specimen dimensions, material properties, loading level, and boundary conditions. The wrinkle wavelength and amplitude are extracted using the same geometric definitions, and the relative differences are reported for the selected strain levels. Due to the symmetry property of deformation, only one-quarter of the geometry of the membrane can be considered. To discretize the model, the one-quarter of membrane is meshed with 50×50 elements. A complete mesh-sensitivity study is required before the predicted wrinkle wavelength and amplitude can be regarded as fully mesh-independent. Therefore, the present results are interpreted as verification results for the selected discretization rather than as a complete convergence demonstration. A ten percent axial strain was applied to the fabric, which is equivalent to applying a displacement of $u = 2.54$ cm to the right edge of the sheet. The transverse bending curve of the sheet along the axis of symmetry $x = L/2$ is shown in Fig. 5. It should be noted that the curve is shown on the entire width in the figure, while in the finite element analysis, only one-quarter model is considered. Different curves have been produced for different values of A_g and A_Γ . Therefore, it can be seen that the curve obtained based on the gradient formulation is very close to the curve reported in [34], provided that the values $A_g = A_\Gamma = \mu (h/4)^2$ are used in the calculations. We note that the main advantage of the formulation used in this thesis is that it does not have the problems of classical formulation. In fact, in the classical formulation used in the treatise of [34], a buckling analysis must first be applied to the sheet (or fabric). Then a multiple of the buckling modes should be applied to the sheet as the primary task and finally the main tension should be applied to the sheet (or tissue). The gradient model presented here eliminates the multiple steps of analysis. And the gradient model is more stable than the classical model and numerically more efficient.

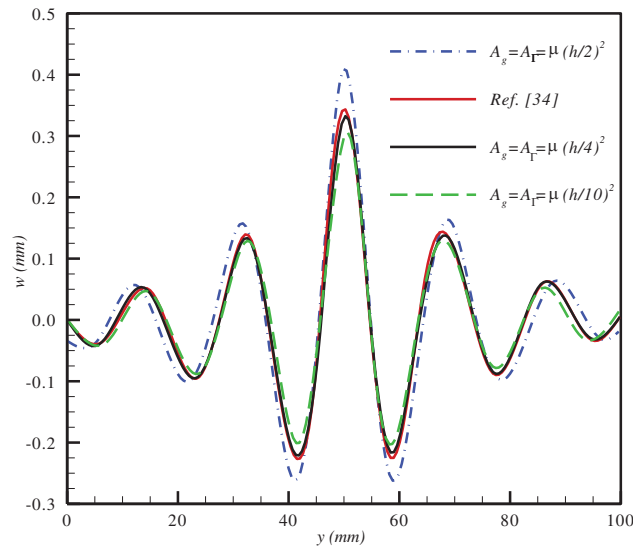


Figure 5: Transverse bending curve of the sheet along the $x = L/2$ line [34].

The deformation contour for $A_g = A_T = \mu (h/4)^2$ is given in Fig. 6. Moreover, in Fig. 7, the crosswise rise of the tissue in the one-quarter model is shown in different stages of applying horizontal stretching. In fact, this figure shows the profile of wrinkling in different stages. Each curve represents the application of one tenth of the total horizontal displacement in the x -direction. This figure indicates that with the increase in the amount of horizontal elongation, the transverse amplitude of the sheet starts to increase. The value of the wrinkle wavelength (that is, the distance between two consecutive peaks) also decreases with the increase in elongation.

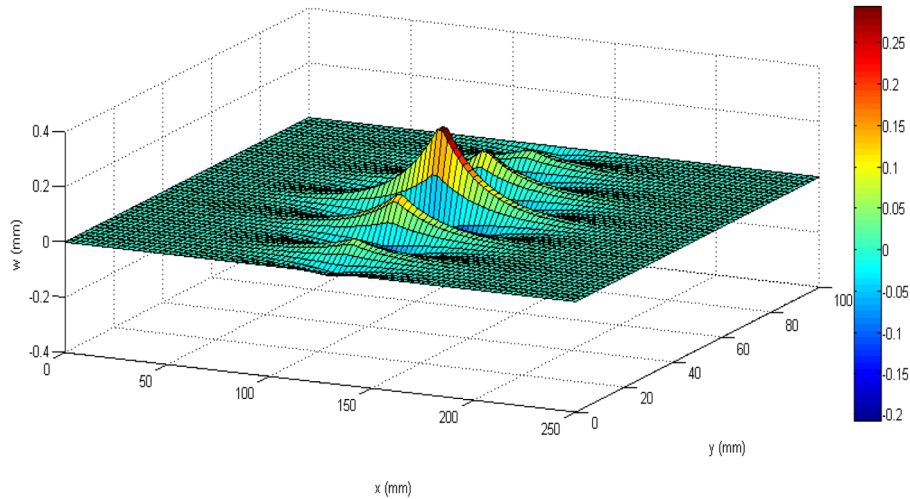


Figure 6: The deformed geometry of the membrane under uniaxial extension.

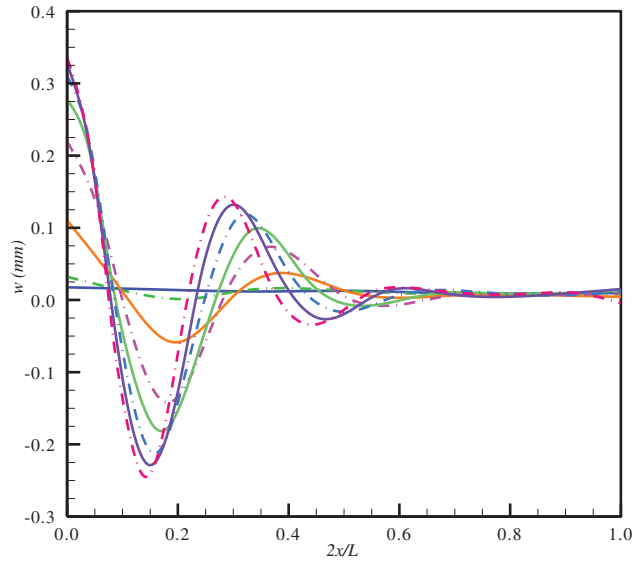


Figure 7: Transverse bending curve of the sheet along the $x = L/2$ line in seven equal stages of loading.

The horizontal (u) and vertical (v) displacement contours of the membrane in the quarter model are shown in Figs. 8 and 9. It should be noted that the horizontal displacement is displayed as dimensionless $2u/L$, which is the same as small strain ε_{11} . We also note that the displacement component v in the quarter model is negative and its value is zero on the horizontal symmetry line of the fabric. Also, as mentioned in the loading method, the value of v on the edge of applying horizontal tension is equal to zero.

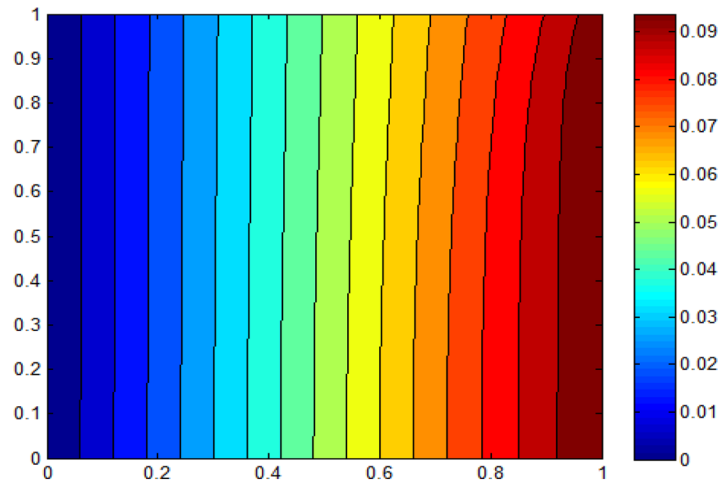


Figure 8: Contour of the displacement component u on the model of a quarter of the membrane.

Fig. 10 shows the wavelength of wrinkles as a function of the axial strain applied to the membrane for different thicknesses. In all the curves, values have been used based on the strain gradient theory $A_g = A_\Gamma = \mu (h/4)^2$. From the figure, it can be seen that for the thickness $h = 0.1$ mm, the response obtained from the gradient theory in this work is close to the response reported in [34]. Different curves in the figure indicate that the wavelength of the wrinkles increases with the increase in woven thickness. Also, with the increase

of applied strain, the wavelength of wrinkles follows a decreasing trend. In this case, the number of wrinkles increases, but their amplitude and wavelength decrease.

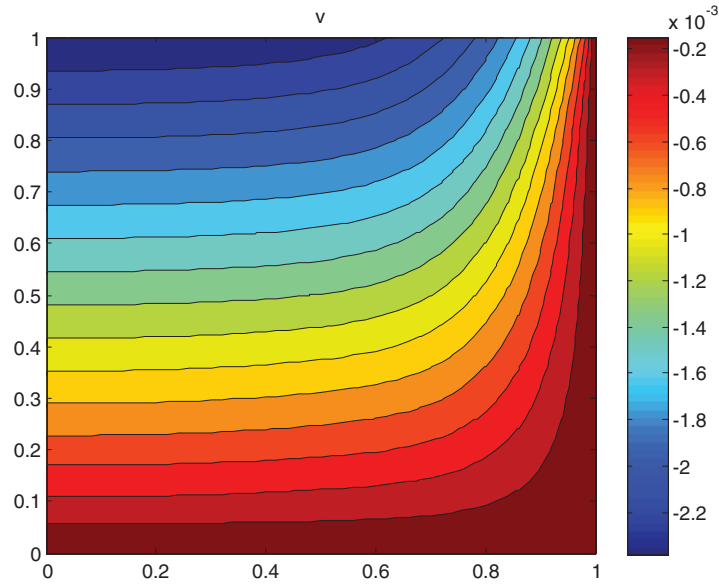


Figure 9: Contour of the displacement component v on the model of a quarter of the membrane.

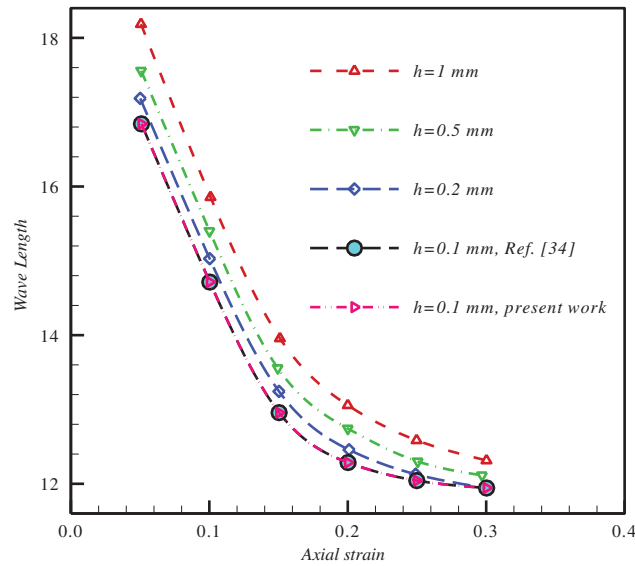


Figure 10: Wrinkle wavelength according to the applied axial strain for different thicknesses [34].

6 Conclusion

In this work, large deformations of woven fabrics were investigated based on the two-dimensional strain gradient theory. First, the theory of strain gradient and its application to membranes were developed. Second, the finite element method was used to obtain the general formulation, and finally, the application of the presented theory was shown with two examples. According to the governing equations and the solved examples, the following results can be presented:

- (1) The gradient formulation used in this work can predict membrane behavior after deformation and is in full agreement with previous studies.
- (2) The main advantage of the gradient formulation used in this paper is numerical efficiency. The gradient theory does not have the problems of the classical formulation. In other words, in the classical formulation, a buckling analysis should be applied first, then the buckling modes should be applied to the fabric as the initial rise and finally the answers will be obtained; however, in the current gradient formulation, buckling analysis will not be needed and only applying a very small deformation can lead to the correct results.
- (3) In the problem of controlling the displacement of the square membrane, the behavior of the fabric is predicted using the two-scale formulation.
- (4) In the problem of membrane wrinkling under uniaxial extension, with the increase in the amount of elongation, the transverse amplitude of the membrane increases and the wrinkling wavelength decreases. Moreover, as thickness increases, the wavelength of wrinkles increases, which is also expected from a physical point of view.
- (5) Once wrinkling occurs, the wrinkle amplitude initially increases with increasing axial strain in the woven layer and reaches a maximum. In the present simulations, when the applied axial strain exceeds approximately 10%, the wrinkle amplitude exhibits a decreasing trend, accompanied by a reduction in the number of visible wrinkles.

The present results demonstrate consistency with the selected published benchmark studies; however, they should be regarded as numerical verification rather than experimental validation. The conclusions are limited to the investigated geometries, constitutive assumptions, internal length-scale values, mesh discretizations, and loading conditions.

Despite the efficiency of the developed non-linear finite element framework, it should be noted that the current model primarily assumes isotropic material behavior and simplified interface conditions, which may not fully represent the complexity of heterogeneous or multi-layered systems. Future research could enhance this approach by incorporating anisotropic material laws and considering the influence of dynamic loading conditions on the stability of wrinkling patterns.

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Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Appendix A.1 Material Frame-Indifference (Objectivity) of the Formulation

To demonstrate that the proposed energy density in Eq. (13) satisfies the principle of material frame-indifference, consider a superposed rigid-body motion on the current spatial configuration:

$$\mathbf{x}^*(X, t) = \mathbf{Q}(t)\mathbf{x}(X, t) + \mathbf{c}(t), \quad (\text{A1})$$

where $\mathbf{Q}(t)$ is a time-dependent proper orthogonal rotation tensor satisfying $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$ and $\det(\mathbf{Q}) = 1$, and $\mathbf{c}(t)$ is an arbitrary rigid translation vector. Under this transformation, the material coordinates X and the material principal directions remain unchanged. The deformation gradient and its second gradient transform as:

$$\mathbf{F}^* = \nabla_X \mathbf{x}^* = \mathbf{Q} \nabla_X \mathbf{x} = \mathbf{Q} \mathbf{F}, \quad (\text{A2})$$

$$\mathbf{G}^* = \nabla_X \mathbf{F}^* = \mathbf{Q} \nabla_X \mathbf{F} = \mathbf{Q} \mathbf{G}. \quad (\text{A3})$$

Appendix A.2 Invariance of the Classical Part $W_1(\mathbf{F})$

The classical hyperelastic part depends on the deformation through the right Cauchy–Green tensor and the Jacobian. Under the rotation $\mathbf{Q}$:

$$\mathbf{C}^* = \mathbf{F}^{*T} \mathbf{F}^* = (\mathbf{Q} \mathbf{F})^T (\mathbf{Q} \mathbf{F}) = \mathbf{F}^T \mathbf{Q}^T \mathbf{Q} \mathbf{F} = \mathbf{F}^T \mathbf{F} = \mathbf{C}, \quad (\text{A4})$$

$$J^* = \det(\mathbf{Q} \mathbf{F}) = \det(\mathbf{Q}) \det(\mathbf{F}) = J, \quad (\text{A5})$$

hence, $W_1(\mathbf{F}^*) = W_1(\mathbf{F})$.

Appendix A.3 Invariance of the Higher-Order Part $W_2(\mathbf{F}, \mathbf{G})$

The transformed higher-order vectors $\mathbf{g}_1^*$, $\mathbf{g}_2^*$, and Γ^* in Eq. (14) become:

$$\mathbf{g}_1^* = \mathbf{G}^* : (\mathbf{D}_1 \otimes \mathbf{D}_1) = (\mathbf{Q} \mathbf{G}) : (\mathbf{D}_1 \otimes \mathbf{D}_1) = \mathbf{Q} \mathbf{g}_1, \quad (\text{A6})$$

$$\mathbf{g}_2^* = \mathbf{G}^* : (\mathbf{D}_2 \otimes \mathbf{D}_2) = (\mathbf{Q} \mathbf{G}) : (\mathbf{D}_2 \otimes \mathbf{D}_2) = \mathbf{Q} \mathbf{g}_2, \quad (\text{A7})$$

$$\Gamma^* = \mathbf{G}^* : (\mathbf{D}_1 \otimes \mathbf{D}_2 + \mathbf{D}_2 \otimes \mathbf{D}_1) = \mathbf{Q} \Gamma. \quad (\text{A8})$$

Consequently, the scalar squared norms appearing in W_2 transform as:

$$|\mathbf{g}_\omega^*|^2 = \mathbf{g}_\omega^{*T} \mathbf{g}_\omega^* = (\mathbf{Q} \mathbf{g}_\omega)^T (\mathbf{Q} \mathbf{g}_\omega) = \mathbf{g}_\omega^T \mathbf{Q}^T \mathbf{Q} \mathbf{g}_\omega = |\mathbf{g}_\omega|^2, \quad \omega = 1, 2, \quad (\text{A9})$$

$$|\Gamma^*|^2 = \Gamma^{*T} \Gamma^* = (\mathbf{Q} \Gamma)^T (\mathbf{Q} \Gamma) = \Gamma^T \mathbf{Q}^T \mathbf{Q} \Gamma = |\Gamma|^2. \quad (\text{A10})$$

Substituting these invariants into Eq. (14) yields:

$$W_2(\mathbf{F}^*, \mathbf{G}^*) = \frac{1}{2} A_g \left(|\mathbf{g}_1^*|^2 + |\mathbf{g}_2^*|^2 \right) + \frac{1}{2} A_\Gamma |\Gamma^*|^2 = W_2(\mathbf{F}, \mathbf{G}). \quad (\text{A11})$$

Therefore, the total strain energy density satisfies $W_2(\mathbf{F}^*, \mathbf{G}^*) = W_2(\mathbf{F}, \mathbf{G})$, proving that Eq. (13) is rigorously objective and material frame-indifferent.

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