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A Unified Parametric Divergence Operator for Fermatean Fuzzy Environment and Its Applications in Machine Learning and Intelligent Decision-Making

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ABSTRACT: Uncertainty and ambiguity are pervasive in real-world intelligent systems, necessitating advanced mathematical frameworks for effective modeling and analysis. Fermatean fuzzy sets (FFSs), as a recent extension of classical fuzzy theory, provide enhanced flexibility for representing complex uncertainty. In this paper, we propose a unified parametric divergence operator for FFSs, which comprehensively captures the interplay among membership, non-membership, and hesitation degrees. The proposed operator is rigorously analyzed with respect to key mathematical properties, including non-negativity, non-degeneracy, and symmetry. Notably, several well-known divergence operators, such as Jensen-Shannon divergence, Hellinger distance, and χ^2 -divergence, are shown to be special cases within our unified framework. Extensive experiments on pattern classification, hierarchical clustering, and multiattribute decision-making tasks demonstrate the competitive performance and stability of the proposed operator. These results confirm both the theoretical significance and practical value of our method for advanced fuzzy information processing in machine learning and intelligent decision-making.

KEYWORDS: Fermatean fuzzy sets; divergence operator; pattern classification; hierarchical clustering; multiattribute decision-making

1 Introduction

Dealing with imperfect information is a common task in everyday life, as uncertainty and ambiguity inevitably permeate any real-world system, creating challenges in the decision-making process [1,2]. Whether



it is selecting the best alternative in business, diagnosing medical conditions, or making strategic decisions in engineering or finance, decision-makers are often confronted with complex situations where data is incomplete, imprecise, or ambiguous [3]. In such cases, conventional methods that rely on precise numerical values may fall short. To solve this problem, various effective theories have been established, like fuzzy sets [4,5], rough sets [6,7], Z-numbers [8,9], and Evidence theory [10,11]. These theories have a wide range of applications in various domains, such as supplier selection [12], decision-making [3], risk analysis [13], and medical diagnosis [14].

Among them, fuzzy sets, introduced by Zadeh [15], have become a foundational concept in modeling uncertainty. The notion of fuzzy sets allows for the representation of vague concepts by assigning a membership degree (MD) to elements in a set, providing a flexible way to handle ambiguous information. However, the classical fuzzy sets still face limitations when dealing with more complex uncertainty. Intuitionistic fuzzy sets (IFSs), developed by Atanassov [16], extended fuzzy sets by introducing a nonmembership degree (ND) along with the MD , providing a more comprehensive model of uncertainty. However, IFSs impose a strict constraint where the sum of the MD and ND must be less than or equal to one. This restriction can be limiting when dealing with high levels of ambiguity, as it fails to fully account for situations where both degrees exhibit substantial uncertainty. Building on IFSs, Pythagorean fuzzy sets (PFSs), introduced by Yager [17], offer a more flexible framework. PFSs allow the sum of the squares of the MD and ND to be less than or equal to one, relaxing the constraint imposed by IFSs. This relaxation provides more flexibility in modeling uncertainty and has attracted research focused on the application of PFSs [18,19]. Recently, Senapati and Yager [20] introduced Fermatean fuzzy sets (FFSs), which offer a further generalization by relaxing the constraints of IFSs and PFSs. FFSs use a more generalized relationship between the MD and ND , where the sum of the cubes of the MD and ND must be less than or equal to one. This generalized structure allows FFS to capture a broader range of uncertainty, making them highly suitable for decision-making scenarios that involve significant ambiguity and conflicting information. So far, FFS has attracted a lot of attention [21–23]. For example, Ref. [21] introduced several Fermatean fuzzy weighted average, weighted geometric, weighted power average, and weighted power geometric operators and applied them for the multiattribute decision-making (MADM) problem. Kakati et al. [22] suggested Fermatean fuzzy Archimedean Heronian mean and geometric Heronian mean operators, and developed a new MADM method for sustainable urban transport.

In practical applications, especially in fields involving pattern classification, clustering, decision-making, and artificial intelligence, measuring the discrepancy between two sets is crucial. Interestingly, distance/divergence/similarity operators for IFSs and PFSs have received great attention, and many operators have been developed. For example, some researchers have extended several widely used operators, including Hamming distance [24,25], Euclidean distance [24,26], Hausdorff distance [27,28], Jensen-Shannon divergence [29–31], Hellinger distance [32,33], cosine similarity [34,35], Jaccard similarity [36,37] and Dice similarity [38], to the environments of IFSs and PFSs, and have validated their effectiveness in scenarios such as pattern recognition, multiattribute decision-making (MADM) and medical diagnosis. Furthermore, more innovative distance and similarity operators for IFSs and PFSs can be found in the literature [39–41]. Several operators of distance and similarity have been developed for FFSs to enhance their practical utility. For example, Senapati and Yager [20] first introduced a Fermatean fuzzy operator based on Euclidean distance. Later, Onyeke and Ejegwa [42] pointed out that Senapati and Yager's work contradicts the axiomatic of distance, and developed an enhanced distance operator for FFSs. Sahoo [43] introduced several similarity operators for FFSs and applied them to group decision-making. Kirisci [44] combined cosine similarity and Euclidean distance operators for FFSs, applying these concepts within the TOPSIS framework. Subsequently, IJU [45] highlighted deficiencies in Kirisci's approach and introduced an improved cosine similarity operator. Ref. [46] designed the Hellinger distance and the triangular divergence for FFSs, respectively. Liu [14]

proposed a new triangular divergence operator for FFSs to overcome the limitations of the previous work [46]. Ref. [47] suggested some t-conorms-based distance operators for FFSs. Liu [14] presented some similarity operators between FFSs based on Tanimoto and Sørensen coefficients.

Furthermore, many MADM models have been developed to achieve successful outcomes in solving real-life problems that contain different alternatives and multiple criteria in the decision-making process [48–50]. Fuzzy logic is frequently used in the literature to overcome the uncertainties that occur in the decision-making process in MADM models [51–54]. Among them, the alternative ranking order method accounting for the two-step normalization (AROMAN) model is a novel ranking technique [55], which uses a two-step normalization mechanism to ensure that selections can be compared objectively and fairly. Currently, AROMAN has been extended to some fuzzy frameworks to solve various decision-making problems. Ref. [55] introduced the AROMAN model, which provides a new solution to the delivery decision process of cargo bicycles. Ref. [56] combined the AROMAN model and FUCOM weighting method in an interval-type 2 fuzzy environment and demonstrated its potential in practical decision-making problems. Ref. [57] integrated the MEREC weighting technique and the extended AROMAN model to assess Turkey's sustainable competitiveness level.

Despite some progress, there remain several significant research gaps:

- Some distance/divergence operators for FFSs largely ignore the hesitation degree (HD). These operators may generally fail to accurately represent the differences between FFSs, especially in real-world decision-making scenarios where HD plays a key role.
- Some existing distance/divergence operators for FFSs yield counter-intuitive results, such as producing identical values for two distinctly different fuzzy sets, undermining their reliability and practical utility.
- While the AROMAN model has shown promise in decision-making, its integration with FFSs remains underexplored. Such integration could enhance the robustness of decision-making systems by leveraging a more comprehensive representation of uncertainty.

The research objective of this paper is to fill these gaps by introducing a unified divergence operator that accounts for all three components (MD , ND , and HD) of FFSs, providing a more reliable and intuitive operator for comparing the sets. Further, by integrating the new operator with the AROMAN model, we seek to enhance the effectiveness of decision-making systems, thereby contributing to both theoretical and practical advancements in the field.

The main contributions of this paper are summarized as:

- We propose a unified parametric divergence operator for FFSs, which considers the MD , NMD , and HD .
- The mathematical properties of the proposed operator, including non-negativity, non-degeneracy, and symmetry, are rigorously analyzed.
- We demonstrate that several well-known operators are special cases of our unified operator.
- The effectiveness of the proposed operator is validated through applications on pattern classification, hierarchical clustering, and MADM tasks in various machine learning and intelligent decision-making scenarios.

The paper is organized as follows: [Section 2](#) reviews the basic concepts and the existing distance/divergence operators. [Section 3](#) proposes a unified parametric divergence operator for FFSs and offers some numerical comparisons between existing operators and the proposed operator. Applications to pattern recognition, hierarchical clustering, and MADM are provided in [Section 4](#). The conclusion is given in [Section 5](#).

2 Background

In this section, we briefly review some foundational concepts about IFSs, PFSs, and FFSs and introduce the existing distance/divergence operators.

2.1 Basic Concepts

Definition 1 ([16]): An intuitionistic fuzzy set $\mathcal{I}$ in a non-empty set $\mathbb{O} = \{o_1, \dots, o_m\}$ can be expressed as:

$$\mathcal{I} = \{\langle o_i, \mathcal{I}_{\Phi}(o_i), \mathcal{I}_{\Lambda}(o_i) \rangle | o_i \in \mathbb{O}\} \quad (1)$$

where $\mathcal{I}_{\Phi}(o_i), \mathcal{I}_{\Lambda}(o_i) : \mathbb{O} \rightarrow [0, 1]$ signify MD and ND, respectively, with the constraint that $\mathcal{I}_{\Phi}(o_i) + \mathcal{I}_{\Lambda}(o_i) \leq 1$ for $o_i \in \mathbb{O}$. The HD, denoted as $\mathcal{I}_{\Xi}(o_i) = 1 - \mathcal{I}_{\Phi}(o_i) - \mathcal{I}_{\Lambda}(o_i)$, quantifies the uncertainty of each $o_i \in \mathbb{O}$.

Definition 2 ([17]): A Pythagorean fuzzy set $\mathcal{P}$ in a non-empty set $\mathbb{O} = \{o_1, \dots, o_m\}$ can be expressed as:

$$\mathcal{P} = \{\langle o_i, \mathcal{P}_{\Phi}(o_i), \mathcal{P}_{\Lambda}(o_i) \rangle | o_i \in \mathbb{O}\} \quad (2)$$

where $\mathcal{P}_{\Phi}(o_i), \mathcal{P}_{\Lambda}(o_i) : \mathbb{O} \rightarrow [0, 1]$ signify MD and ND, respectively. It holds that $\mathcal{P}_{\Phi}^2(o_i) + \mathcal{P}_{\Lambda}^2(o_i) \leq 1$ for $o_i \in \mathbb{O}$. The HD of $\mathcal{P}$, denoted as $\mathcal{P}_{\Xi}(o_i) = \sqrt{1 - \mathcal{P}_{\Phi}^2(o_i) - \mathcal{P}_{\Lambda}^2(o_i)}$.

Definition 3 ([20]): A Fermatean fuzzy set $\mathcal{F}$ in a non-empty set $\mathbb{O} = \{o_1, \dots, o_m\}$ can be expressed as:

$$\mathcal{F} = \{\langle o_i, \mathcal{F}_{\Phi}(o_i), \mathcal{F}_{\Lambda}(o_i) \rangle | o_i \in \mathbb{O}\} \quad (3)$$

where $\mathcal{F}_{\Phi}(o_i), \mathcal{F}_{\Lambda}(o_i) : \mathbb{O} \rightarrow [0, 1]$ signify MD and ND, respectively. It holds that $\mathcal{F}_{\Phi}^3(o_i) + \mathcal{F}_{\Lambda}^3(o_i) \leq 1$ for $o_i \in \mathbb{O}$. The HD of $\mathcal{F}$, denoted as $\mathcal{F}_{\Xi}(o_i) = 3\sqrt{1 - \mathcal{F}_{\Phi}^3(o_i) - \mathcal{F}_{\Lambda}^3(o_i)}$.

Definition 4 ([58]): The score function S of the FFS is defined as follows.

$$S = \frac{(\mathcal{F}_{\Phi}(o_i))^3 - (\mathcal{F}_{\Lambda}(o_i))^3 + 1}{2} \quad (4)$$

2.2 The Existing Fermatean Fuzzy Distance/Divergence Operators

In recent years, researchers have introduced some distance/divergence operators for FFSs.

Definition 5: Sahoo [43] defined several FF distance operators:

$$\mathcal{D}_2(\mathcal{F}, \mathcal{G}) = \frac{1}{2m} \sum_{i=1}^m (|\mathcal{F}_{\Phi}(o_i) - \mathcal{G}_{\Phi}(o_i)| + |\mathcal{F}_{\Lambda}(o_i) - \mathcal{G}_{\Lambda}(o_i)| + |\mathcal{F}_{\Xi}(o_i) - \mathcal{G}_{\Xi}(o_i)|) \quad (5)$$

$$\mathcal{D}_3(\mathcal{F}, \mathcal{G}) = \frac{1}{2m} \sum_{i=1}^m (|\mathcal{F}_{\mathcal{Y}}(o_i) - \mathcal{G}_{\mathcal{Y}}(o_i)|^3)^{\frac{1}{3}} \quad (6)$$

$$\mathcal{D}_4(\mathcal{F}, \mathcal{G}) = \frac{1}{2m} \sum_{i=1}^m (|\mathcal{F}_{\mathcal{Y}}(o_i) - \mathcal{G}_{\mathcal{Y}}(o_i)|) \quad (7)$$

$$\mathcal{D}_5(\mathcal{F}, \mathcal{G}) = \frac{1}{2m} \sum_{i=1}^m \left(\frac{1}{2^i} \frac{|\mathcal{F}_{\mathcal{Y}}^3(o_i) - \mathcal{G}_{\mathcal{Y}}^3(o_i)|}{1 + |\mathcal{F}_{\mathcal{Y}}^3(o_i) - \mathcal{G}_{\mathcal{Y}}^3(o_i)|} \right) \quad (8)$$

$$\mathcal{D}_6(\mathcal{F}, \mathcal{G}) = \frac{1}{2m} \sum_{i=1}^m (|\mathcal{F}_{\mathcal{Y}}^3(o_i) - \mathcal{G}_{\mathcal{Y}}^3(o_i)|) \quad (9)$$

$$\mathcal{D}_7(\mathcal{F}, \mathcal{G}) = \frac{1}{2m} \max_i (|\mathcal{F}_Y^3(o_i) - \mathcal{G}_Y^3(o_i)|) \quad (10)$$

where

$$\mathcal{F}_Y(o_i) = \frac{1}{2}(1 + \mathcal{F}_\Phi^3(o_i) - \mathcal{F}_\Lambda^3(o_i)), \quad \mathcal{G}_Y(o_i) = \frac{1}{2}(1 + \mathcal{G}_\Phi^3(o_i) - \mathcal{G}_\Lambda^3(o_i)) \quad (11)$$

Definition 6: Liu [14] introduced Hamming distance, Euclidean distance, and Hausdorff distance operators for FFSs:

$$\mathcal{D}_8(\mathcal{F}, \mathcal{G}) = \frac{1}{2m} \sum_{i=1}^m (|\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i)| + |\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i)|) \quad (12)$$

$$\mathcal{D}_9(\mathcal{F}, \mathcal{G}) = \sqrt{\frac{1}{2m} \sum_{i=1}^m ((\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i))^2 + (\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i))^2)} \quad (13)$$

$$\mathcal{D}_{10}(\mathcal{F}, \mathcal{G}) = \frac{1}{m} \sum_{i=1}^m \max(|\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i)|, |\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i)|) \quad (14)$$

$$\mathcal{D}_{11}(\mathcal{F}, \mathcal{G}) = \frac{1}{2m} \sum_{i=1}^m (|\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i)| + |\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i)| + |\mathcal{F}_\Xi^3(o_i) - \mathcal{G}_\Xi^3(o_i)|) \quad (15)$$

$$\mathcal{D}_{12}(\mathcal{F}, \mathcal{G}) = \sqrt{\frac{1}{2m} \sum_{i=1}^m ((\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i))^2 + (\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i))^2 + (\mathcal{F}_\Xi^3(o_i) - \mathcal{G}_\Xi^3(o_i))^2)} \quad (16)$$

$$\mathcal{D}_{13}(\mathcal{F}, \mathcal{G}) = \frac{1}{m} \sum_{i=1}^m \max(|\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i)|, |\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i)|, |\mathcal{F}_\Xi^3(o_i) - \mathcal{G}_\Xi^3(o_i)|) \quad (17)$$

Definition 7: Ganie [47] defined several t -conorm-based distance operators for FFSs:

$$\mathcal{D}_{14}(\mathcal{F}, \mathcal{G}) = \frac{1}{m} \sum_{i=1}^m \left(\frac{|\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i)| + |\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i)| - 2|\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i)||\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i)|}{1 - |\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i)||\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i)|} \right) \quad (18)$$

$$\mathcal{D}_{15}(\mathcal{F}, \mathcal{G}) = \frac{1}{m} \sum_{i=1}^m (|\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i)| + |\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i)| - |\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i)||\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i)|) \quad (19)$$

$$\mathcal{D}_{16}(\mathcal{F}, \mathcal{G}) = \frac{1}{m} \sum_{i=1}^m \min(1, |\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i)| + |\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i)|) \quad (20)$$

$$\mathcal{D}_{17}(\mathcal{F}, \mathcal{G}) = \frac{1}{m} \sum_{i=1}^m \left(\frac{|\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i)| + |\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i)|}{1 + |\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i)||\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i)|} \right) \quad (21)$$

Definition 8: Deng and Wang [46] defined Hellinger distance and triangular divergence operators for FFSs:

$$\mathcal{D}_{18}(\mathcal{F}, \mathcal{G}) = \sqrt{\frac{1}{2m} \sum_{i=1}^m ((\sqrt{\mathcal{F}_\Phi^3(o_i)} - \sqrt{\mathcal{G}_\Phi^3(o_i)})^2 + (\sqrt{\mathcal{F}_\Lambda^3(o_i)} - \sqrt{\mathcal{G}_\Lambda^3(o_i)})^2)} \quad (22)$$

$$\mathcal{D}_{19}(\mathcal{F}, \mathcal{G}) = \sqrt{\frac{1}{2m} \sum_{i=1}^m \left(\frac{(\mathcal{F}_\Phi^3(o_i) - \mathcal{G}_\Phi^3(o_i))^2}{\mathcal{F}_\Phi^3(o_i) + \mathcal{G}_\Phi^3(o_i)} + \frac{(\mathcal{F}_\Lambda^3(o_i) - \mathcal{G}_\Lambda^3(o_i))^2}{\mathcal{F}_\Lambda^3(o_i) + \mathcal{G}_\Lambda^3(o_i)} \right)} \quad (23)$$

Definition 9: Mishra et al. [59] introduced an FF distance operator:

$$\mathcal{D}_{20}(\mathcal{F}, \mathcal{G}) = 1 - \frac{\sum_{i=1}^m (\min(\mathcal{F}_{\Phi}^3(o_i), \mathcal{G}_{\Phi}^3(o_i)) + \min(1 - \mathcal{F}_{\Lambda}^3(o_i), 1 - \mathcal{G}_{\Lambda}^3(o_i)))}{\sum_{i=1}^m (\max(\mathcal{F}_{\Phi}^3(o_i), \mathcal{G}_{\Phi}^3(o_i)) + \max(1 - \mathcal{F}_{\Lambda}^3(o_i), 1 - \mathcal{G}_{\Lambda}^3(o_i)))} \quad (24)$$

3 The Proposed Parametric Divergence Operator

Determining the discrepancy between FFSs remains a challenging yet critical aspect of decision-making processes. To address this, we introduce a unified parametric divergence operator for FFSs in this section. Unlike traditional divergence measures that only focus on a single aspect of membership information, our operator simultaneously incorporates the three core components of FFSs, namely the membership degree (MD), non-membership degree (NMD), and hesitation degree (HD). This design ensures that all sources of uncertainty are taken into account when comparing two fuzzy information sets. Moreover, the parameter δ provides a flexible mechanism for adjusting the sensitivity of the divergence: different values of δ emphasize different balance points between $\mathcal{F}$ and $\mathcal{G}$, allowing the operator to generalize several well-known divergences. We then explore the fundamental properties of the divergence operator, examine its connections to established operators, and analyze its efficacy. Overall, the proposed divergence operator proves more effective in differentiating FFSs and circumvents counterintuitive outcomes that may arise from using partial or incomplete measures.

Definition 10: Let $\mathcal{F} = \{\langle o_i, \mathcal{F}_{\Phi}(o_i), \mathcal{F}_{\Lambda}(o_i) \rangle | o_i \in \mathbb{O}\}$ and $\mathcal{G} = \{\langle o_i, \mathcal{G}_{\Phi}(o_i), \mathcal{G}_{\Lambda}(o_i) \rangle | o_i \in \mathbb{O}\}$ be two FFSs in $\mathbb{O}$, the parametric divergence between FFSs can be defined as:

$$\mathcal{D}^{\delta}(\mathcal{F}, \mathcal{G}) = \frac{1}{m\delta(\delta-1)} \sum_{i=1}^m \left(\sum_{\tau} ((\mathcal{F}_{\tau}^3(o_i))^{\delta} (\mathcal{G}_{\tau}^3(o_i))^{1-\delta} - \delta \mathcal{F}_{\tau}^3(o_i) + (\delta-1) \mathcal{G}_{\tau}^3(o_i)) \right) \quad (25)$$

where $\tau \in \{\Phi, \Lambda, \Xi\}$ and $\delta \in \mathbf{R} \setminus \{0, 1\}$.

This formulation highlights two important characteristics. First, the operator explicitly aggregates the divergence across all three components Φ (membership), Λ (non-membership), and Ξ (hesitation). Although Ξ is mathematically determined by Φ and Λ , its explicit inclusion does not create redundancy. On the contrary, it guarantees that the divergence takes into account the complete information of an FFS, including the degree of hesitation, which is critical in many decision-making scenarios. Moreover, including Ξ does not break the essential properties of the operator: the divergence remains non-negative, equals zero only when $\mathcal{F} = \mathcal{G}$, and is symmetric between the two sets. Thus, the operator offers a more comprehensive representation of uncertainty. Second, the parametric form with exponent δ enables a continuum of divergence behaviors. When δ approaches particular values, the operator reduces to well-known measures such as the Jensen–Shannon divergence, Hellinger distance, or χ^2 divergence. Thus, our proposal serves as a unifying generalization that bridges multiple classical operators.

The parametric divergence operator for FFSs can also be written in an expanded form as:

$$\mathcal{D}^{\delta}(\mathcal{F}, \mathcal{G}) = \frac{1}{m\delta(\delta-1)} \sum_{i=1}^m \left(\begin{aligned} &((\mathcal{F}_{\Phi}^3(o_i))^{\delta} (\mathcal{G}_{\Phi}^3(o_i))^{1-\delta} - \delta \mathcal{F}_{\Phi}^3(o_i) + (\delta-1) \mathcal{G}_{\Phi}^3(o_i)) \\ &+ ((\mathcal{F}_{\Lambda}^3(o_i))^{\delta} (\mathcal{G}_{\Lambda}^3(o_i))^{1-\delta} - \delta \mathcal{F}_{\Lambda}^3(o_i) + (\delta-1) \mathcal{G}_{\Lambda}^3(o_i)) \\ &+ ((\mathcal{F}_{\Xi}^3(o_i))^{\delta} (\mathcal{G}_{\Xi}^3(o_i))^{1-\delta} - \delta \mathcal{F}_{\Xi}^3(o_i) + (\delta-1) \mathcal{G}_{\Xi}^3(o_i)) \end{aligned} \right) \quad (26)$$

This equivalent expression shows how each component contributes to the overall divergence: the cubic powers ensure the operator respects the Fermatean structure, while the subtraction terms $-\delta \mathcal{F}_{\tau}^3(o_i)$

and $(\delta - 1)\mathcal{G}_\tau^3(o_i)$ correct for scaling and maintain non-negativity. Importantly, the explicit presence of Ξ increases the sensitivity of the measure when hesitation changes.

Example 1: To illustrate this effect, we compare divergence values computed with and without including Ξ , using (26) with $\delta = 0.5$:

Case 1: $\mathcal{F} = \{\langle 0.60, 0.20 \rangle\} \Rightarrow \mathcal{F}_\Xi = 0.9189$, $\mathcal{G} = \{\langle 0.75, 0.30 \rangle\} \Rightarrow \mathcal{G}_\Xi = 0.8199$.

$\mathcal{D}_{\text{with } \Xi}^{(0.5)} = 0.1179$, $\mathcal{D}_{\text{without } \Xi}^{(0.5)} = 0.0795$, an increase of 48.30%.

Case 2: $\mathcal{F} = \{\langle 0.50, 0.30 \rangle\} \Rightarrow \mathcal{F}_\Xi = 0.9465$, $\mathcal{G} = \{\langle 0.60, 0.20 \rangle\} \Rightarrow \mathcal{G}_\Xi = 0.9189$.

$\mathcal{D}_{\text{with } \Xi}^{(0.5)} = 0.0391$, $\mathcal{D}_{\text{without } \Xi}^{(0.5)} = 0.0360$, an increase of 8.61%.

These results confirm that including hesitation does not introduce redundancy but instead ensures that the divergence is responsive to uncertainty. When hesitation varies strongly, the divergence appropriately grows larger; when hesitation is nearly constant, the divergence remains close to the two-dimensional variant.

Example 2: Consider the following example where $\delta = 0.6$. We have two FFSs $\mathcal{F}$ and $\mathcal{G}$, given by:

$$\mathcal{F} = \{\langle 0.37, 0.45 \rangle, \langle 0.28, 0.19 \rangle\}, \quad \mathcal{G} = \{\langle 0.42, 0.38 \rangle, \langle 0.32, 0.73 \rangle\}$$

The calculations give $\mathcal{D}^{(0.6)}(\mathcal{F}, \mathcal{G}) = 0.3204$ and $\mathcal{D}^{(0.6)}(\mathcal{G}, \mathcal{F}) = 0.3885$.

These results, where $\mathcal{D}^{(0.6)}(\mathcal{F}, \mathcal{G}) \neq \mathcal{D}^{(0.6)}(\mathcal{G}, \mathcal{F})$, illustrate the asymmetry of $\mathcal{D}^{(0.6)}$. The asymmetrical nature highlights an important limitation: the measure depends on the order of the input sets. In many decision-making applications, however, symmetry is desirable, since the divergence between $\mathcal{F}$ and $\mathcal{G}$ should intuitively equal that between $\mathcal{G}$ and $\mathcal{F}$. To address this, we further extend the operator to its symmetric form, as shown below.

Definition 11: Let $\mathcal{F} = \{\langle o_i, \mathcal{F}_\Phi(o_i), \mathcal{F}_\Lambda(o_i) \rangle | o_i \in \mathbb{O}\}$ and $\mathcal{G} = \{\langle o_i, \mathcal{G}_\Phi(o_i), \mathcal{G}_\Lambda(o_i) \rangle | o_i \in \mathbb{O}\}$ be two FFSs in $\mathbb{O}$, the symmetric parametric divergence between FFSs can be defined as:

$$\mathcal{D}_L^\delta(\mathcal{F}, \mathcal{G}) = \frac{1}{2} \left[\mathcal{D}^\delta \left(\mathcal{F}, \frac{\mathcal{F} + \mathcal{G}}{2} \right) + \mathcal{D}^\delta \left(\mathcal{G}, \frac{\mathcal{F} + \mathcal{G}}{2} \right) \right] \quad (27)$$

Here, the operator computes the divergence of each set with respect to their average and then averages the results, ensuring symmetry. This construction not only eliminates the order-dependence problem but also aligns with the structure of Jensen–Shannon divergence, thereby reinforcing the interpretability and robustness of the proposed measure in practical scenarios.

The following properties are derived from the $\mathcal{D}_L^{(\delta)}(\mathcal{F}, \mathcal{G})$.

Property 1 (Non-negativity): $\mathcal{D}_L^{(\delta)}(\mathcal{F}, \mathcal{G}) \geq 0$.

Proof: For each i and τ , we consider $(\mathcal{F}_\tau^3(o_i))^\delta (\mathcal{G}_\tau^3(o_i))^{1-\delta} - \delta \mathcal{F}_\tau^3(o_i) + (\delta - 1) \mathcal{G}_\tau^3(o_i)$. We need to show that this expression is non-negative:

$$(\mathcal{F}_\tau^3(o_i))^\delta (\mathcal{G}_\tau^3(o_i))^{1-\delta} \geq \delta \mathcal{F}_\tau^3(o_i) - (\delta - 1) \mathcal{G}_\tau^3(o_i)$$

We set $x = \mathcal{F}_\tau^3(o_i)$ and $y = \mathcal{G}_\tau^3(o_i)$, where $0 \leq x, y \leq 1$:

$$f(x, y) = x^\delta y^{1-\delta} - \delta x + (\delta - 1)y$$

Case 1: $0 < \delta < 1$

In this case, the function $f(x, y)$ is convex where $0 \leq x, y \leq 1$.

By Jensen's inequality and the properties of convex functions, we have

$$x^\delta y^{1-\delta} \geq \delta x + (1-\delta)y$$

which implies that

$$x^\delta y^{1-\delta} - \delta x + (\delta-1)y \geq 0$$

Thus, the expression is non-negative for $0 < \delta < 1$.

Case 2: $\delta > 1$

In this case, the roles of x and y are reversed:

$$f(x, y) = x^\delta y^{1-\delta} - \delta x + (\delta-1)y$$

As x^δ grows faster than linear, we again leverage the fact that the convexity of power functions ensures:

$$x^\delta y^{1-\delta} \geq \delta x + (1-\delta)y$$

This ensures the term is non-negative, similarly to the case for $0 < \delta < 1$.

Summing over all i and all τ , we conclude:

$$\mathcal{D}^\delta(\mathcal{F}, \mathcal{G}) = \frac{1}{m\delta(\delta-1)} \sum_{i=1}^m \sum_{\tau} \left((\mathcal{F}_\tau^3(o_i))^\delta (\mathcal{G}_\tau^3(o_i))^{1-\delta} - \delta \mathcal{F}_\tau^3(o_i) + (\delta-1) \mathcal{G}_\tau^3(o_i) \right) \geq 0$$

This establishes that $\mathcal{D}^\delta(\mathcal{F}, \mathcal{G})$ is non-negative.

As a result,

$$\mathcal{D}_L^\delta(\mathcal{F}, \mathcal{G}) = \frac{1}{2} \left[\mathcal{D}^\delta \left(\mathcal{F}, \frac{\mathcal{F} + \mathcal{G}}{2} \right) + \mathcal{D}^\delta \left(\mathcal{G}, \frac{\mathcal{F} + \mathcal{G}}{2} \right) \right] \geq 0.$$

□

Property 2 (Non-degeneracy): $\mathcal{D}_L^{(\delta)}(\mathcal{F}, \mathcal{G}) = 0$ if and only if $\mathcal{F} = \mathcal{G}$.

Proof: We assume two FFSs $\mathcal{F} = \mathcal{G}$:

$$\begin{aligned} \mathcal{D}_L^{(\delta)}(\mathcal{F} \parallel \mathcal{G}) &= \mathcal{D}_L^{(\delta)}(\mathcal{G} \parallel \mathcal{G}) = \mathcal{D}_L^{(\delta)}(\mathcal{F} \parallel \mathcal{F}) \\ &= \frac{1}{m\delta(\delta-1)} \sum_{i=1}^m \left(\sum_{\tau} \left((\mathcal{F}_\tau^3(o_i))^\delta (\mathcal{F}_\tau^3(o_i))^{1-\delta} - \delta \mathcal{F}_\tau^3(o_i) + (\delta-1) \mathcal{F}_\tau^3(o_i) \right) \right) \\ &= 0. \end{aligned}$$

If $\mathcal{D}_L^{(\delta)}(\mathcal{F} \parallel \mathcal{G}) = 0$, then

$$\frac{1}{2} \left[\mathcal{D}^{(\delta)} \left(\mathcal{F} \parallel \frac{\mathcal{F} + \mathcal{G}}{2} \right) + \mathcal{D}^{(\delta)} \left(\mathcal{G} \parallel \frac{\mathcal{F} + \mathcal{G}}{2} \right) \right] = 0$$

implies

$$\mathcal{D}^{(\delta)} \left(\mathcal{F} \parallel \frac{\mathcal{F} + \mathcal{G}}{2} \right) = \mathcal{D}^{(\delta)} \left(\mathcal{G} \parallel \frac{\mathcal{F} + \mathcal{G}}{2} \right) = 0.$$

Thus, we have

$$\mathcal{F} = \frac{\mathcal{F} + \mathcal{G}}{2}, \quad \mathcal{G} = \frac{\mathcal{F} + \mathcal{G}}{2}.$$

Solving these, we conclude $\mathcal{F} = \mathcal{G}$.

This completes the proof of non-degeneracy, demonstrating that $\mathcal{D}_L^{(\delta)}(\mathcal{F}||\mathcal{G}) = 0$ if and only if $\mathcal{F} = \mathcal{G}$.

□

Property 3 (Symmetry): $\mathcal{D}_L^{(\delta)}(\mathcal{F}, \mathcal{G}) = \mathcal{D}_L^{(\delta)}(\mathcal{G}, \mathcal{F})$.

Proof: We consider two FFSs $\mathcal{F}$ and $\mathcal{G}$:

$$\mathcal{D}_L^{(\delta)}(\mathcal{F}||\mathcal{G}) = \frac{1}{2} \left[\mathcal{D}^{(\delta)} \left(\mathcal{F} || \frac{\mathcal{F} + \mathcal{G}}{2} \right) + \mathcal{D}^{(\delta)} \left(\mathcal{G} || \frac{\mathcal{F} + \mathcal{G}}{2} \right) \right]$$

and

$$\mathcal{D}_L^{(\delta)}(\mathcal{G}||\mathcal{F}) = \frac{1}{2} \left[\mathcal{D}^{(\delta)} \left(\mathcal{G} || \frac{\mathcal{F} + \mathcal{G}}{2} \right) + \mathcal{D}^{(\delta)} \left(\mathcal{F} || \frac{\mathcal{F} + \mathcal{G}}{2} \right) \right].$$

By comparing these expressions, it is clear that:

$$\mathcal{D}_L^{(\delta)}(\mathcal{F}||\mathcal{G}) = \mathcal{D}_L^{(\delta)}(\mathcal{G}||\mathcal{F}).$$

This equality confirms that $\mathcal{D}_L^{(\delta)}$ is symmetric. Thus, the symmetry Property is proven. □

Property 4: When $\delta = 2$, $\mathcal{D}_L^{(\delta)}$ simplifies into the χ^2 divergence ($\mathcal{D}_{\chi^2}$):

$$\mathcal{D}_L^{(2)}(\mathcal{F}, \mathcal{G}) = \mathcal{D}_{\chi^2}(\mathcal{F}, \mathcal{G}) \quad (28)$$

where

$$\mathcal{D}_{\chi^2}(\mathcal{F}, \mathcal{G}) = \frac{1}{4m} \sum_{i=1}^m \left(\sum_{\tau} \frac{(\mathcal{F}_{\tau}^3(o_i) - \mathcal{G}_{\tau}^3(o_i))^2}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} \right) \quad (29)$$

Proof: For two FFSs $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{O}$, when $\delta = 2$, we have:

$$\begin{aligned} \mathcal{D}_L^{\delta}(\mathcal{F}, \mathcal{G}) &= \frac{1}{4m} \sum_{i=1}^m \left\{ \sum_{\tau} \left[(\mathcal{F}_{\tau}^3(o_i))^2 \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right)^{-1} - 2\mathcal{F}_{\tau}^3(o_i) + \frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right] \right. \\ &\quad \left. + \sum_{\tau} \left[(\mathcal{G}_{\tau}^3(o_i))^2 \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right)^{-1} - 2\mathcal{G}_{\tau}^3(o_i) + \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right) \right] \right\} \\ &= \frac{1}{4m} \sum_{i=1}^m \left[\sum_{\tau} \left(\frac{2(\mathcal{F}_{\tau}^3(o_i))^2}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} - 2\mathcal{F}_{\tau}^3(o_i) + \frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right) \right. \\ &\quad \left. + \sum_{\tau} \left(\frac{2(\mathcal{G}_{\tau}^3(o_i))^2}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} - 2\mathcal{G}_{\tau}^3(o_i) + \frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right) \right] \\ &= \frac{1}{4m} \sum_{i=1}^m \left[\sum_{\tau} \left(\frac{2((\mathcal{F}_{\tau}^3(o_i))^2 + (\mathcal{G}_{\tau}^3(o_i))^2) - (\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i))^2}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} \right) \right] \\ &= \frac{1}{4m} \sum_{i=1}^m \left(\sum_{\tau} \frac{(\mathcal{F}_{\tau}^3(o_i) - \mathcal{G}_{\tau}^3(o_i))^2}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} \right) \\ &= \mathcal{D}_{\chi^2}(\mathcal{F}, \mathcal{G}) \end{aligned}$$

□

Property 5: When $\delta \rightarrow 1$, $\mathcal{D}_L^{(\delta)}$ converges to the Jensen-Shannon divergence ($\mathcal{D}_{JS}$):

$$\mathcal{D}_L^{(1)}(\mathcal{F}, \mathcal{G}) = \mathcal{D}_{JS}(\mathcal{F}, \mathcal{G}) \quad (30)$$

where

$$\mathcal{D}_{JS}(\mathcal{F}, \mathcal{G}) = \frac{1}{2m} \sum_{i=1}^m \left(\sum_{\tau} \mathcal{F}_{\tau}^3(o_i) \ln \frac{2\mathcal{F}_{\tau}^3(o_i)}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} + \mathcal{G}_{\tau}^3(o_i) \ln \frac{2\mathcal{G}_{\tau}^3(o_i)}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} \right) \quad (31)$$

Proof: For two FFSs $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{O}$, when $\delta \rightarrow 1$, we have:

$$\begin{aligned} & \lim_{\delta \rightarrow 1} \mathcal{D}_L^{(\delta)}(\mathcal{F}, \mathcal{G}) \\ &= \lim_{\delta \rightarrow 1} \left\{ \frac{\frac{\partial}{\partial \delta} \left[\sum_{i=1}^m \left(\sum_{\tau} (\mathcal{F}_{\tau}^3(o_i))^{\delta} \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right)^{1-\delta} - \delta \mathcal{F}_{\tau}^3(o_i) + (\delta-1) \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right) \right) \right. \right. \\ & \quad \left. \left. + \sum_{\tau} (\mathcal{G}_{\tau}^3(o_i))^{\delta} \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right)^{1-\delta} - \delta \mathcal{G}_{\tau}^3(o_i) + (\delta-1) \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right) \right) \right]}{\frac{\partial}{\partial \delta} 2m\delta(\delta-1)} \right\} \\ &= \lim_{\delta \rightarrow 1} \left\{ \frac{\left[\sum_{i=1}^m \left(\sum_{\tau} (\mathcal{F}_{\tau}^3(o_i))^{\delta} \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right)^{1-\delta} \ln \frac{2\mathcal{F}_{\tau}^3(o_i)}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} - \mathcal{F}_{\tau}^3(o_i) + \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right) \right) \right. \right. \\ & \quad \left. \left. + \sum_{\tau} (\mathcal{G}_{\tau}^3(o_i))^{\delta} \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right)^{1-\delta} \ln \frac{2\mathcal{G}_{\tau}^3(o_i)}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} - \mathcal{G}_{\tau}^3(o_i) + \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right) \right) \right]}{2m(2\delta-1)} \right\} \\ &= \frac{1}{2m} \sum_{i=1}^m \left[\sum_{\tau} \left(\mathcal{F}_{\tau}^3(o_i) \ln \frac{2\mathcal{F}_{\tau}^3(o_i)}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} + \mathcal{G}_{\tau}^3(o_i) \ln \frac{2\mathcal{G}_{\tau}^3(o_i)}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} \right) \right] \\ &= \mathcal{D}_{JS}(\mathcal{F}, \mathcal{G}) \end{aligned}$$

□

Property 6: When $\delta = \frac{1}{2}$, $\mathcal{D}_L^{(\delta)}$ simplifies into the Hellinger distance ($\mathcal{D}_{Hel}$):

$$\mathcal{D}_L^{(\frac{1}{2})}(\mathcal{F}, \mathcal{G}) = \mathcal{D}_{Hel}(\mathcal{F}, \mathcal{G}) \quad (32)$$

where

$$\begin{aligned} \mathcal{D}_{Hel}(\mathcal{F}, \mathcal{G}) &= \frac{1}{m} \sum_{i=1}^m \left\{ \sum_{\tau} \left[\left(\sqrt{\mathcal{F}_{\tau}^3(o_i)} - \sqrt{\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2}} \right)^2 \right. \right. \\ & \quad \left. \left. + \left(\sqrt{\mathcal{G}_{\tau}^3(o_i)} - \sqrt{\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2}} \right)^2 \right] \right\} \quad (33) \end{aligned}$$

Proof: For two FFSs $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{O}$, when $\delta = \frac{1}{2}$, we have:

$$\begin{aligned} \mathcal{D}_L^\delta(\mathcal{F}, \mathcal{G}) &= -\frac{2}{m} \sum_{i=1}^m \left\{ \sum_{\tau} \left[(\mathcal{F}_{\tau}^3(o_i))^{\frac{1}{2}} \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right)^{\frac{1}{2}} - \frac{1}{2} \mathcal{F}_{\tau}^3(o_i) - \frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right] \right. \\ &\quad \left. + \sum_{\tau} \left[(\mathcal{G}_{\tau}^3(o_i))^{\frac{1}{2}} \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right)^{\frac{1}{2}} - \frac{1}{2} \mathcal{G}_{\tau}^3(o_i) - \frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right] \right\} \\ &= -\frac{2}{m} \sum_{i=1}^m \left\{ \sum_{\tau} -\frac{1}{2} \left(\sqrt{\mathcal{F}_{\tau}^3(o_i)} - \sqrt{\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2}} \right)^2 - \frac{1}{2} \sum_{\tau} \left(\sqrt{\mathcal{F}_{\tau}^3(o_i)} - \sqrt{\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2}} \right)^2 \right\} \\ &= \frac{1}{m} \sum_{i=1}^m \left\{ \sum_{\tau} \left[\left(\sqrt{\mathcal{F}_{\tau}^3(o_i)} - \sqrt{\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2}} \right)^2 + \left(\sqrt{\mathcal{F}_{\tau}^3(o_i)} - \sqrt{\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2}} \right)^2 \right] \right\} \end{aligned}$$

□

Property 7: When $\delta \rightarrow 0$, $\mathcal{D}_L^{(\delta)}$ converges to the arithmetic-geometric divergence ($\mathcal{D}_{AG}$):

$$\mathcal{D}_L^{(\frac{1}{2})}(\mathcal{F}, \mathcal{G}) = \mathcal{D}_{AG}(\mathcal{F}, \mathcal{G})$$

where

$$\mathcal{D}_{AG}(\mathcal{F}, \mathcal{G}) = \frac{1}{m} \sum_{i=1}^m \left(\sum_{\tau} \frac{(\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i))}{2} \ln \frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2\sqrt{\mathcal{F}_{\tau}^3(o_i) * \mathcal{G}_{\tau}^3(o_i)}} \right) \quad (34)$$

Proof: For two FFSs $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{O}$, when $\delta \rightarrow 0$, we have:

$$\begin{aligned} &\lim_{\delta \rightarrow 0} \mathcal{D}_L^{(\delta)}(\mathcal{F}, \mathcal{G}) \\ &= \lim_{\delta \rightarrow 0} \left\{ \frac{\frac{\partial}{\partial \delta} \left[\sum_{i=1}^m \left(\sum_{\tau} (\mathcal{F}_{\tau}^3(o_i))^{\delta} \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right)^{1-\delta} - \delta \mathcal{F}_{\tau}^3(o_i) + (\delta - 1) \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right) \right) \right.}{\frac{\partial}{\partial \delta} 2m\delta(\delta - 1)} \right. \\ &\quad \left. + \sum_{\tau} (\mathcal{G}_{\tau}^3(o_i))^{\delta} \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right)^{1-\delta} - \delta \mathcal{G}_{\tau}^3(o_i) + (\delta - 1) \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right) \right] \right\} \\ &= \lim_{\delta \rightarrow 0} \left\{ \frac{\left[\sum_{i=1}^m \left(\sum_{\tau} (\mathcal{F}_{\tau}^3(o_i))^{\delta} \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right)^{1-\delta} \ln \frac{2\mathcal{F}_{\tau}^3(o_i)}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} - \mathcal{F}_{\tau}^3(o_i) + \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right) \right) \right.}{2m(2\delta - 1)} \right. \\ &\quad \left. + \sum_{\tau} (\mathcal{G}_{\tau}^3(o_i))^{\delta} \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right)^{1-\delta} \ln \frac{2\mathcal{G}_{\tau}^3(o_i)}{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)} - \mathcal{G}_{\tau}^3(o_i) + \left(\frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \right) \right] \right\} \\ &= \frac{1}{m} \sum_{i=1}^m \left[\sum_{\tau} \frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2} \ln \frac{\mathcal{F}_{\tau}^3(o_i) + \mathcal{G}_{\tau}^3(o_i)}{2\sqrt{\mathcal{F}_{\tau}^3(o_i) \mathcal{G}_{\tau}^3(o_i)}} \right] \end{aligned}$$

$$= \mathcal{D}_{AG}(\mathcal{F}, \mathcal{G})$$

□

Next, we use several numerical analyses to report the efficiency of the proposed parametric divergence operator.

Example 3: Consider two FFSs $\mathcal{F}, \mathcal{G}$ in $\mathbb{O}$ under different cases, where

$$\text{Case 1 : } \mathcal{F} = \{\langle 0.48, 0.25 \rangle, \langle 0.37, 0.57 \rangle\}, \mathcal{G} = \{\langle 0.42, 0.38 \rangle, \langle 0.32, 0.73 \rangle\}$$

$$\text{Case 2 : } \mathcal{F} = \{\langle 0.26, 0.67 \rangle, \langle 0.79, 0.61 \rangle\}, \mathcal{G} = \{\langle 0.73, 0.63 \rangle, \langle 0.73, 0.77 \rangle\}$$

To evaluate the difference between these fuzzy sets, we calculate the divergence operators $\mathcal{D}_L^{(\delta)}(\mathcal{F}, \mathcal{G})$ with different values of δ .

$$\mathcal{D}_L^{(2)}(\mathcal{F}, \mathcal{G}) = 0.0168, \mathcal{D}_L^{(1)}(\mathcal{F}, \mathcal{G}) = 0.0172, \mathcal{D}_L^{(\frac{1}{2})}(\mathcal{F}, \mathcal{G}) = 0.0176, \mathcal{D}_L^{(0)}(\mathcal{F}, \mathcal{G}) = 0.0181.$$

$$\mathcal{D}_L^{(2)}(\mathcal{F}, \mathcal{G}) = 0.0702, \mathcal{D}_L^{(1)}(\mathcal{F}, \mathcal{G}) = 0.0803, \mathcal{D}_L^{(\frac{1}{2})}(\mathcal{F}, \mathcal{G}) = 0.0934, \mathcal{D}_L^{(0)}(\mathcal{F}, \mathcal{G}) = 0.1184.$$

These numerical results illustrate that the proposed divergence can capture fine differences between FFSs under varying δ . More importantly, the results confirm that our proposed operators satisfy the property of non-negativity: $\mathcal{D}_L^\delta(\mathcal{F}, \mathcal{G}) \geq 0$. In real-world applications such as risk evaluation or decision analysis, this ensures that the computed divergence values are always meaningful and interpretable as distances.

Example 4: Consider the FFSs $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{O}$ under different cases, where

$$\text{Case 1 : } \mathcal{F} = \{\langle 0.58, 0.68 \rangle, \langle 0.73, 0.49 \rangle\}, \mathcal{G} = \{\langle 0.58, 0.68 \rangle, \langle 0.73, 0.49 \rangle\}$$

$$\text{Case 2 : } \mathcal{F} = \{\langle 0.72, 0.77 \rangle, \langle 0.67, 0.82 \rangle\}, \mathcal{G} = \{\langle 0.72, 0.77 \rangle, \langle 0.67, 0.82 \rangle\}$$

For both cases, the calculated distances are:

$$\mathcal{D}_L^{(2)}(\mathcal{F}, \mathcal{G}) = 0.0000, \mathcal{D}_L^{(1)}(\mathcal{F}, \mathcal{G}) = 0.0000, \mathcal{D}_L^{(\frac{1}{2})}(\mathcal{F}, \mathcal{G}) = 0.0000, \mathcal{D}_L^{(0)}(\mathcal{F}, \mathcal{G}) = 0.0000.$$

The results confirm that our proposed operators satisfy the property of non-degeneracy: $\mathcal{D}_L^\delta(\mathcal{F}, \mathcal{G}) = 0$ if and only if $\mathcal{F} = \mathcal{G}$. This property is crucial in classification and clustering tasks: when two fuzzy objects are identical, the divergence must be zero to avoid misleading conclusions.

Example 5: Consider the FFSs $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{O}$ under different cases, where

$$\text{Case 1 : } \mathcal{F} = \{\langle 0.32, 0.82 \rangle, \langle 0.89, 0.12 \rangle\}, \mathcal{G} = \{\langle 0.63, 0.63 \rangle, \langle 0.73, 0.70 \rangle\}$$

$$\text{Case 2 : } \mathcal{F} = \{\langle 0.44, 0.60 \rangle, \langle 0.73, 0.37 \rangle\}, \mathcal{G} = \{\langle 0.62, 0.60 \rangle, \langle 0.80, 0.70 \rangle\}$$

The distances for each case demonstrate the property of symmetry, as shown below:

$$\mathcal{D}_L^{(2)}(\mathcal{F}, \mathcal{G}) = 0.0892, \dots, \mathcal{D}_L^{(0)}(\mathcal{F}, \mathcal{G}) = 0.2288.$$

$$\mathcal{D}_L^{(2)}(\mathcal{G}, \mathcal{F}) = 0.0892, \dots, \mathcal{D}_L^{(0)}(\mathcal{G}, \mathcal{F}) = 0.2288.$$

The results confirm that our proposed operators satisfy the property of symmetry: $\mathcal{D}_L^\delta(\mathcal{F}, \mathcal{G}) = \mathcal{D}_L^\delta(\mathcal{G}, \mathcal{F})$. Symmetry is highly desirable in applications like clustering and pattern recognition, as it ensures that the

divergence between two fuzzy patterns does not depend on their order, thereby avoiding bias in the learning process.

Example 6: Consider the FFSs $\mathcal{F}$ and $\mathcal{G}$ in $\mathbb{O}$ under four different cases:

Case 1 : $\mathcal{F} = \{\langle 0.67, 0.58 \rangle, \langle 0.74, 0.67 \rangle\}$, $\mathcal{G} = \{\langle 0.74, 0.67 \rangle, \langle 0.70, 0.79 \rangle\}$

Case 2 : $\mathcal{F} = \{\langle 0.79, 0.74 \rangle, \langle 0.74, 0.67 \rangle\}$, $\mathcal{G} = \{\langle 0.74, 0.67 \rangle, \langle 0.70, 0.79 \rangle\}$

Case 3 : $\mathcal{F} = \{\langle 0.74, 0.67 \rangle, \langle 0.67, 0.58 \rangle\}$, $\mathcal{G} = \{\langle 0.79, 0.74 \rangle, \langle 0.63, 0.63 \rangle\}$

Case 4 : $\mathcal{F} = \{\langle 0.74, 0.67 \rangle, \langle 0.79, 0.74 \rangle\}$, $\mathcal{G} = \{\langle 0.67, 0.58 \rangle, \langle 0.82, 0.70 \rangle\}$

To compare the performance of the existing operators, we analyzed the results presented in Table 1. This table summarizes the effectiveness of different distance operators across four distinct cases.

Table 1: The results of different distance operators

	1	2	3	4
Case	$\mathcal{F} = \{\langle \mathbf{0.67}, \mathbf{0.58} \rangle, \langle \mathbf{0.74}, \mathbf{0.67} \rangle\}$ $\mathcal{G} = \{\langle \mathbf{0.74}, \mathbf{0.67} \rangle, \langle \mathbf{0.70}, \mathbf{0.79} \rangle\}$	$\mathcal{F} = \{\langle \mathbf{0.79}, \mathbf{0.74} \rangle, \langle \mathbf{0.74}, \mathbf{0.67} \rangle\}$ $\mathcal{G} = \{\langle \mathbf{0.74}, \mathbf{0.67} \rangle, \langle \mathbf{0.70}, \mathbf{0.79} \rangle\}$	$\mathcal{F} = \{\langle \mathbf{0.74}, \mathbf{0.67} \rangle, \langle \mathbf{0.67}, \mathbf{0.58} \rangle\}$ $\mathcal{G} = \{\langle \mathbf{0.79}, \mathbf{0.74} \rangle, \langle \mathbf{0.63}, \mathbf{0.63} \rangle\}$	$\mathcal{F} = \{\langle \mathbf{0.74}, \mathbf{0.67} \rangle, \langle \mathbf{0.79}, \mathbf{0.74} \rangle\}$ $\mathcal{G} = \{\langle \mathbf{0.67}, \mathbf{0.58} \rangle, \langle \mathbf{0.82}, \mathbf{0.70} \rangle\}$
$D_1(\mathcal{F}, \mathcal{G})$	0.1103	0.1103	0.0696	0.0696
$D_2(\mathcal{F}, \mathcal{G})$	0.1427	0.1560	0.1036	0.0835
$D_3(\mathcal{F}, \mathcal{G})$	0.0313	0.0313	0.0125	0.0125
$D_4(\mathcal{F}, \mathcal{G})$	0.0313	0.0313	0.0125	0.0125
$D_5(\mathcal{F}, \mathcal{G})$	0.0051	0.0051	0.0025	0.0030
$D_6(\mathcal{F}, \mathcal{G})$	0.0224	0.0224	0.0103	0.0124
$D_7(\mathcal{F}, \mathcal{G})$	0.0224	0.0224	0.0103	0.0124
$D_8(\mathcal{F}, \mathcal{G})$	0.1125	0.1125	0.0750	0.0750
$D_9(\mathcal{F}, \mathcal{G})$	0.1250	0.1250	0.0791	0.0791
$D_{10}(\mathcal{F}, \mathcal{G})$	0.1500	0.1500	0.0750	0.0750
$D_{11}(\mathcal{F}, \mathcal{G})$	0.2000	0.2000	0.1250	0.1250
$D_{12}(\mathcal{F}, \mathcal{G})$	0.1768	0.1768	0.1275	0.1275
$D_{13}(\mathcal{F}, \mathcal{G})$	0.1000	0.1000	0.1000	0.1000
$D_{14}(\mathcal{F}, \mathcal{G})$	0.2071	0.2071	0.1385	0.1385
$D_{15}(\mathcal{F}, \mathcal{G})$	0.2151	0.2151	0.1438	0.1438
$D_{16}(\mathcal{F}, \mathcal{G})$	0.2251	0.2251	0.1500	0.1500
$D_{17}(\mathcal{F}, \mathcal{G})$	0.2228	0.2228	0.1489	0.1489
$D_{18}(\mathcal{F}, \mathcal{G})$	0.1053	0.0998	0.0667	0.0710
$D_{19}(\mathcal{F}, \mathcal{G})$	0.1480	0.1403	0.0942	0.1000
$D_{20}(\mathcal{F}, \mathcal{G})$	0.1957	0.1957	0.1304	0.1250
$D_L^{(2)}(\mathcal{F}, \mathcal{G})$	0.0235	0.0286	0.0169	0.0112
$D_L^{(1)}(\mathcal{F}, \mathcal{G})$	0.0237	0.0294	0.0175	0.0113
$D_L^{(\frac{1}{2})}(\mathcal{F}, \mathcal{G})$	0.0240	0.0301	0.0181	0.0114
$D_L^{(0)}(\mathcal{F}, \mathcal{G})$	0.0236	0.0303	0.0104	0.0117

Note: Bold numbers indicate that the calculation results are the same.

The existing operators show consistent results in several cases, indicating their limited ability to distinguish subtle differences between FFSs. For example, $\mathcal{D}_1, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6, \mathcal{D}_7, \dots, \mathcal{D}_{17}$ produced identical values for Case 1 and Case 2, as well as for Case 3 and Case 4. This consistency reveals a lack of sensitivity to small variations, which may lead to misjudgments in practical decision-making.

In contrast, the proposed operators $\mathcal{D}_L^{(2)}(\mathcal{F}, \mathcal{G})$, $\mathcal{D}_L^{(1)}(\mathcal{F}, \mathcal{G})$, $\mathcal{D}_L^{(\frac{1}{2})}(\mathcal{F}, \mathcal{G})$, and $\mathcal{D}_L^{(0)}(\mathcal{F}, \mathcal{G})$ exhibit distinct results across all cases. For instance, $\mathcal{D}_L^{(2)}(\mathcal{F}, \mathcal{G})$ shows results of 0.0235 in Case 1, 0.0286 in Case 2, 0.0169 in Case 3, and 0.0112 in Case 4. These variations demonstrate that our operator is capable of detecting subtle differences between fuzzy objects that traditional operators fail to capture.

4 Applications

This section demonstrates the effectiveness of the proposed parametric divergence operator through three representative applications: pattern classification and hierarchical clustering (as typical machine learning tasks), and multiattribute decision-making (as an intelligent decision-making task).

Across the three applications, we aim to establish feasibility, interpretability, and breadth of applicability. Application 1 shows that our operator can be used for pattern recognition with results consistent with most methods. Applications 2 and 3 extend to hierarchical clustering of vehicle buyers and MADM of blockchain platforms selection, respectively, illustrating that the operator integrates well with both machine learning and decision analytics and yields actionable insights.

4.1 Pattern Classification

Let $\mathbb{O} = \{o_1, \dots, o_m\}$ denote a non-empty set containing m elements. Given n reference patterns characterized by FFSs $\{\mathbb{M}_j\}_{j=1}^n$. We present a systematic classification method for an unknown pattern represented by the FFS $\mathbb{S}$ through the following operational procedure.

Step 1: Compute the divergence operator between each reference pattern $\mathbb{M}_j (j = 1, \dots, n)$ and the target pattern $\mathbb{S}$ using the operator defined in (27).

Step 2: Determine the minimum divergence value through comparative analysis:

$$\mathcal{D}_L^\delta(\mathbb{M}_j', \mathbb{S}) = \min_{1 \leq j \leq n} \mathcal{D}_L^\delta(\mathbb{M}_j, \mathbb{S}) \quad (35)$$

Step 3: Find the classification result of $\mathbb{S}$ as:

$$j' = \arg \min_{1 \leq j \leq n} \mathcal{D}_L^\delta(\mathbb{M}_j, \mathbb{S}), \mathbb{M}_{j'} \rightarrow \mathbb{S} \quad (36)$$

Step 4: Quantify the discriminative capability using the degree of confidence (Doc) [60]:

$$Doc = \sum_{j=1, j \neq j'}^n |\tilde{\mathcal{D}}_L^\delta(\mathbb{M}_j, \mathbb{S}) - \tilde{\mathcal{D}}_L^\delta(\mathbb{M}_{j'}, \mathbb{S})| \quad (37)$$

where $\tilde{\mathcal{D}}_L^\delta$ is a normalized divergence, and $\mathbb{M}_{j'}$ is the classified result with $\mathbb{S}$. A higher Doc indicates a better result. It is important to note that DoC is introduced as a deterministic confidence margin rather than a probabilistic test statistic. Specifically, DoC quantifies the separation between the best-matched alternative and its competitors, thereby reflecting robustness against ties or instability. It does not itself imply statistical significance.

Application 1: This application involves a pattern classification task for classifying building materials into four predefined classes using ten attributes [61]. The task is to assign an unknown building material, denoted as $\mathbb{S}$,

to one of the four known types, $\mathbb{M}_j$ (where $j = 1, 2, 3, 4$), based on attribute set $\mathbb{Z}_i$ ($i = 1, 2, \dots, 10$), as detailed in Table 2. The classification is determined through both existing and proposed divergence operators on FFSs to establish the closest match among the classes.

Table 2: Classes of building materials on application 1

	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_5$
$\mathbb{M}_1$	$\langle 0.173, 0.524 \rangle$	$\langle 0.102, 0.818 \rangle$	$\langle 0.530, 0.326 \rangle$	$\langle 0.965, 0.008 \rangle$	$\langle 0.420, 0.351 \rangle$
$\mathbb{M}_2$	$\langle 0.510, 0.365 \rangle$	$\langle 0.627, 0.125 \rangle$	$\langle 1.000, 0.000 \rangle$	$\langle 0.125, 0.648 \rangle$	$\langle 0.026, 0.823 \rangle$
$\mathbb{M}_3$	$\langle 0.495, 0.387 \rangle$	$\langle 0.603, 0.298 \rangle$	$\langle 0.987, 0.006 \rangle$	$\langle 0.073, 0.849 \rangle$	$\langle 0.037, 0.923 \rangle$
$\mathbb{M}_4$	$\langle 1.000, 0.000 \rangle$	$\langle 1.000, 0.000 \rangle$	$\langle 0.857, 0.123 \rangle$	$\langle 0.734, 0.158 \rangle$	$\langle 0.021, 0.896 \rangle$
$\mathbb{S}$	$\langle 0.978, 0.003 \rangle$	$\langle 0.980, 0.012 \rangle$	$\langle 0.798, 0.132 \rangle$	$\langle 0.693, 0.213 \rangle$	$\langle 0.051, 0.876 \rangle$
	$\mathbb{Z}_6$	$\mathbb{Z}_7$	$\mathbb{Z}_8$	$\mathbb{Z}_9$	$\mathbb{Z}_{10}$
$\mathbb{M}_1$	$\langle 0.008, 0.956 \rangle$	$\langle 0.331, 0.512 \rangle$	$\langle 1.000, 0.000 \rangle$	$\langle 0.215, 0.625 \rangle$	$\langle 0.432, 0.534 \rangle$
$\mathbb{M}_2$	$\langle 0.732, 0.153 \rangle$	$\langle 0.556, 0.303 \rangle$	$\langle 0.650, 0.267 \rangle$	$\langle 1.000, 0.000 \rangle$	$\langle 0.145, 0.762 \rangle$
$\mathbb{M}_3$	$\langle 0.690, 0.268 \rangle$	$\langle 0.147, 0.812 \rangle$	$\langle 0.213, 0.653 \rangle$	$\langle 0.501, 0.284 \rangle$	$\langle 1.000, 0.000 \rangle$
$\mathbb{M}_4$	$\langle 0.076, 0.912 \rangle$	$\langle 0.152, 0.712 \rangle$	$\langle 0.113, 0.756 \rangle$	$\langle 0.489, 0.389 \rangle$	$\langle 1.000, 0.000 \rangle$
$\mathbb{S}$	$\langle 0.123, 0.756 \rangle$	$\langle 0.152, 0.721 \rangle$	$\langle 0.113, 0.732 \rangle$	$\langle 0.494, 0.368 \rangle$	$\langle 0.987, 0.000 \rangle$

The results are summarized in the Table 3 and Fig. 1. We can learn that across nearly all operators, $\mathbb{M}_4$ is consistently identified as the best alternative, suggesting that it is particularly well-suited for identifying and classifying the unknown building material $\mathbb{S}$. The proposed operator produced the highest Doc (3.1446 versus values between 1.3968 and 2.8333 for other operators), which reflects a clearer separation between the true class and competing alternatives. This suggests that the method may provide more reliable decision support in cases where ambiguity exists.

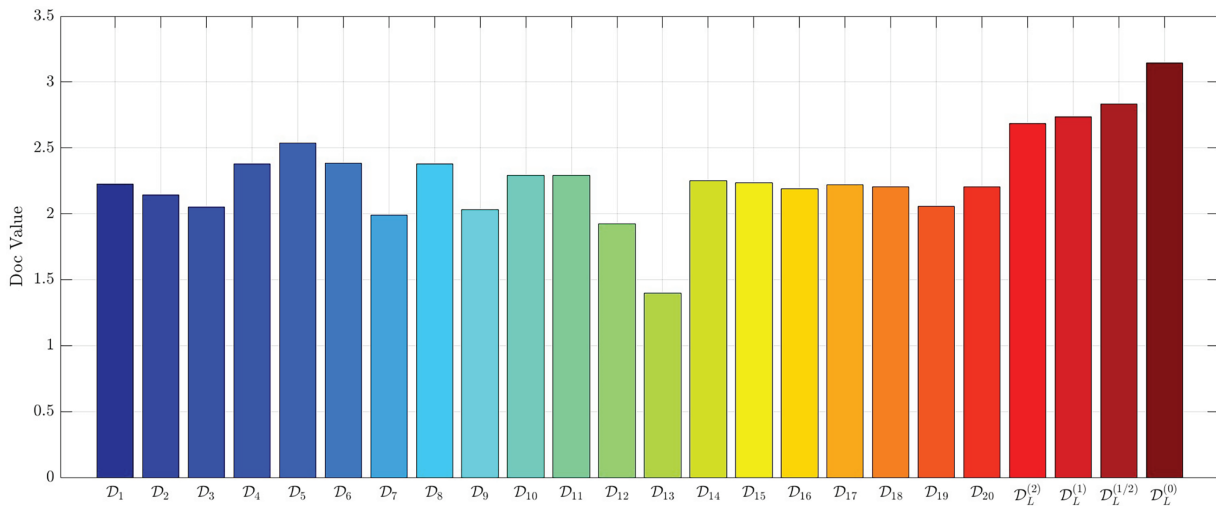
Table 3: Pattern recognition results on application 1

Operator	$(\mathbb{M}_1, \mathbb{S})$	$(\mathbb{M}_2, \mathbb{S})$	$(\mathbb{M}_3, \mathbb{S})$	$(\mathbb{M}_4, \mathbb{S})$	Result	Doc
$\mathcal{D}_1$	0.3578	0.3209	0.2070	0.0488	$\mathbb{M}_3$	2.2277
$\mathcal{D}_2$	0.6202	0.5865	0.3424	0.1089	$\mathbb{M}_4$	2.1448
$\mathcal{D}_3$	0.0521	0.0449	0.0335	0.0084	$\mathbb{M}_4$	2.0518
$\mathcal{D}_4$	0.1919	0.1765	0.1055	0.0199	$\mathbb{M}_4$	2.3775
$\mathcal{D}_5$	0.0107	0.0079	0.0046	0.0014	$\mathbb{M}_4$	2.5366
$\mathcal{D}_6$	0.2339	0.2348	0.1314	0.0217	$\mathbb{M}_4$	2.3860
$\mathcal{D}_7$	0.0486	0.0461	0.0378	0.0056	$\mathbb{M}_4$	1.9884
$\mathcal{D}_8$	0.3839	0.3531	0.2109	0.0399	$\mathbb{M}_4$	2.3774
$\mathcal{D}_9$	0.5087	0.4556	0.3315	0.0832	$\mathbb{M}_4$	2.0313
$\mathcal{D}_{10}$	0.6180	0.5422	0.3458	0.0782	$\mathbb{M}_4$	2.2924
$\mathcal{D}_{11}$	0.6180	0.5422	0.3458	0.0782	$\mathbb{M}_4$	2.2924
$\mathcal{D}_{12}$	0.6251	0.5681	0.4236	0.1172	$\mathbb{M}_4$	1.9241
$\mathcal{D}_{13}$	0.0999	0.0958	0.0814	0.0326	$\mathbb{M}_4$	1.3968
$\mathcal{D}_{14}$	0.6307	0.5904	0.3671	0.0795	$\mathbb{M}_4$	2.2509
$\mathcal{D}_{15}$	0.6409	0.6137	0.3815	0.0796	$\mathbb{M}_4$	2.2336

(Continued)

Table 3 (continued)

Operator	(M_1, S)	(M_2, S)	(M_3, S)	(M_4, S)	Result	Doc
$\mathcal{D}_{16}$	0.6409	0.6660	0.4218	0.0797	M_4	2.1876
$\mathcal{D}_{17}$	0.6484	0.6318	0.3920	0.0797	M_4	2.2215
$\mathcal{D}_{18}$	0.5163	0.4724	0.3285	0.0589	M_4	2.2035
$\mathcal{D}_{19}$	0.5687	0.5393	0.3868	0.0799	M_4	2.0561
$\mathcal{D}_{20}$	0.5182	0.4587	0.3181	0.0651	M_4	2.2056
$\mathcal{D}_L^{(2)}$	0.2442	0.2218	0.1197	0.0111	M_4	2.6857
$\mathcal{D}_L^{(1)}$	0.3066	0.2796	0.1441	0.0129	M_4	2.7357
$\mathcal{D}_L^{(\frac{1}{2})}$	0.4257	0.3861	0.1811	0.0171	M_4	2.8333
$\mathcal{D}_L^{(0)}$	0.1760	0.1224	0.0605	0.0056	M_4	3.1446

**Figure 1:** Doc for different operators on application 1

All operators correctly identified M_4 as the classification result, showing consistency across approaches. The proposed operator produced the highest Degree of Confidence (3.1446 versus values between 1.3968 and 2.8333 for other methods), which reflects a clearer separation between the true class and competing alternatives. This suggests that the method may provide more reliable decision support in cases where ambiguity exists.

4.2 Hierarchical Clustering

In this subsection, we use hierarchical clustering to show the performance of the proposed operator. Here we employ the clustering method proposed by [14].

Application 2: In the automotive industry, it is essential to understand consumer preferences to design vehicles that meet different needs and increase customer satisfaction. Automakers can now segment their market based on specific buying patterns and preferences.

We propose a hierarchical clustering method for FFSs to deal with the ambiguity and uncertainty in consumer preferences and behaviors. It will categorize vehicle buyers and provide insights into different buyers. This will help automakers understand the major factors influencing consumer choices and the types of buyers

in the market, enabling more targeted product development and marketing strategies. We consider five vehicle buyer types ($\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5$) and five attributes: Budget ($\mathbb{F}_1$), Performance ($\mathbb{F}_2$), Fuel Efficiency ($\mathbb{F}_3$), Brand ($\mathbb{F}_4$), Aesthetic Preferences ($\mathbb{F}_5$). Table 4 provides data for five different vehicle buyer types in the form of FFSs. Let us make $\mathcal{D}_L^{(2)}$ as an example.

Table 4: FFSs of the five vehicle buyer types on application 2

	$\mathbb{F}_1$	$\mathbb{F}_2$	$\mathbb{F}_3$	$\mathbb{F}_4$	$\mathbb{F}_5$
$\mathbb{B}_1$	(0.499, 0.602)	(0.633, 0.418)	(0.577, 0.271)	(0.465, 0.417)	(0.261, 0.514)
$\mathbb{B}_2$	(0.369, 0.772)	(0.413, 0.261)	(0.317, 0.418)	(0.213, 0.651)	(0.419, 0.799)
$\mathbb{B}_3$	(0.478, 0.621)	(0.591, 0.346)	(0.479, 0.321)	(0.394, 0.541)	(0.301, 0.617)
$\mathbb{B}_4$	(0.761, 0.314)	(0.713, 0.113)	(0.879, 0.193)	(0.807, 0.201)	(0.811, 0.102)
$\mathbb{B}_5$	(0.314, 0.761)	(0.473, 0.281)	(0.369, 0.491)	(0.286, 0.661)	(0.411, 0.776)

Step 1: Establish the divergence matrix $\mathbb{T} = [t_{pq}]_{5 \times 5}$ using (27) as:

$$\mathbb{T} = \begin{pmatrix} 0.0000 & 0.1715 & 0.1315 & 0.1863 & 0.1613 \\ 0.1715 & 0.0000 & 0.1025 & 0.2820 & 0.0246 \\ 0.1315 & 0.1025 & 0.0000 & 0.2297 & 0.0921 \\ 0.1863 & 0.2820 & 0.2297 & 0.0000 & 0.2685 \\ 0.1613 & 0.0246 & 0.0921 & 0.2685 & 0.0000 \end{pmatrix}$$

where $t_{pq} = \mathcal{D}_L^2(\mathbb{B}_i, \mathbb{B}_{i'})$.

Step 2: Calculate $\mathbb{T}^2$, for $1 \leq p, q \leq 5$, where

$$\mathbb{T}^2 = \mathbb{T} \circ \mathbb{T} = [\tilde{t}_{pq}]_{5 \times 5} = \max_r \left\{ \min \{t_{pr}, t_{rq}\} \right\}$$

Then $\mathbb{T}$ is an equivalent correlation matrix if $\mathbb{T}^2 \subseteq \mathbb{T}$. If $\mathbb{T}^2 \notin \mathbb{T}$, then the equivalent correlation matrix is calculated as

$$\mathbb{T}^{2^r} : \mathbb{T} \rightarrow \mathbb{T}^2 \rightarrow \mathbb{T}^4 \rightarrow \dots \rightarrow \mathbb{T}^{2^r} \rightarrow \dots$$

until

$$\mathbb{T}^{2^r} = \mathbb{T}^{2^{r+1}}$$

$$\mathbb{T}^2 = \mathbb{T} \circ \mathbb{T}$$

$$= \begin{pmatrix} 0.1035 & 0.1035 & 0.1035 & 0.0533 & 0.1035 \\ 0.1035 & 0.1765 & 0.1238 & 0.0533 & 0.1685 \\ 0.1035 & 0.1238 & 0.1238 & 0.0275 & 0.1238 \\ 0.0533 & 0.0533 & 0.0275 & 0.1765 & 0.0464 \\ 0.1035 & 0.1685 & 0.1238 & 0.0464 & 0.1685 \end{pmatrix}$$

It is clear that $\mathbb{T}^2 \notin \mathbb{T}$, so we continue compute $\mathbb{T}^4$.

$$\mathbb{T}^4 = \mathbb{T}^2 \circ \mathbb{T}^2$$

$$= \begin{pmatrix} 0.1035 & 0.1035 & 0.1035 & 0.0533 & 0.1035 \\ 0.1035 & 0.1765 & 0.1238 & 0.0533 & 0.1685 \\ 0.1035 & 0.1238 & 0.1238 & 0.0533 & 0.1238 \\ 0.0533 & 0.0533 & 0.0533 & 0.1765 & 0.0533 \\ 0.1035 & 0.1685 & 0.1238 & 0.0533 & 0.1685 \end{pmatrix}$$

It is evident that $\mathbb{T}^4 \notin \mathbb{T}^2$, so we continue compute $\mathbb{T}^8$.

$$\mathbb{T}^8 = \mathbb{T}^4 \circ \mathbb{T}^4$$

$$= \begin{pmatrix} 0.1035 & 0.1035 & 0.1035 & 0.0533 & 0.1035 \\ 0.1035 & 0.1765 & 0.1238 & 0.0533 & 0.1685 \\ 0.1035 & 0.1238 & 0.1238 & 0.0533 & 0.1238 \\ 0.0533 & 0.0533 & 0.0533 & 0.1765 & 0.0533 \\ 0.1035 & 0.1685 & 0.1238 & 0.0533 & 0.1685 \end{pmatrix}$$

$\mathbb{T}^8 \subset \mathbb{T}^4$, so $\mathbb{T}^8$ is an equivalent association matrix.

Step 3: Compute the η -cutting matrix $\mathbb{T}_\eta = [t_{pq}^\eta]_{5 \times 5}$, the potential classes for the five vehicle buyer types in $\mathcal{D}_L^{(2)}$ are delineated in Table 5. In addition, we present a summary of other potential clusters using both existing and proposed distance operators in Tables 6–8. This overview thoroughly examines the possible classes and clusters for vehicle buyer types based on different operators.

Table 5: Clustering results of divergence operator $\mathcal{D}_L^{(2)}$ on application 2

	Class	Confidence level	Clusters
$\mathcal{D}_L^{(2)}$	1	$0 < \eta \leq 0.0533$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.0533 < \eta \leq 0.1035$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.1035 < \eta \leq 0.1238$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.1238 < \eta \leq 0.1685$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.1685 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$

Table 6: Clustering results of proposed divergence operator $\mathcal{D}_L$ on application 2

	Class	Confidence level	Clusters
$\mathcal{D}_L^{(2)}$	1	$0 < \eta \leq 0.0533$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.0533 < \eta \leq 0.1035$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.1035 < \eta \leq 0.1238$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.1238 < \eta \leq 0.1685$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.1685 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_L^{(1)}$	1	$0 < \eta \leq 0.0566$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.0566 < \eta \leq 0.1175$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.1175 < \eta \leq 0.1424$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.1424 < \eta \leq 0.2026$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.2026 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$

(Continued)

Table 6 (continued)

	Class	Confidence level	Clusters
$\mathcal{D}_L^{(\frac{1}{2})}$	1	$0 < \eta \leq 0.0560$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.0560 < \eta \leq 0.1175$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.1175 < \eta \leq 0.1424$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.1424 < \eta \leq 0.2026$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.2026 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_L^{(0)}$	1	$0 < \eta \leq 0.0560$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.0560 < \eta \leq 0.1142$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.1142 < \eta \leq 0.1601$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.1601 < \eta \leq 0.2859$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.2859 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$

Table 7: Clustering results of existing operator $\mathcal{D}_i$ ($i = 1, 2, \dots, 10$) on application 2

	Class	Confidence level	Clusters
$\mathcal{D}_1$	1	$0 < \eta \leq 0.1145$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.1145 < \eta \leq 0.2147$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.2147 < \eta \leq 0.2297$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.2297 < \eta \leq 0.2685$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.2685 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_2$	1	$0 < \eta \leq 0.2472$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.2472 < \eta \leq 0.3491$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.3491 < \eta \leq 0.3903$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.3903 < \eta \leq 0.4779$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.4779 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_3$	1	$0 < \eta \leq 0.023$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.0233 < \eta \leq 0.0433$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.0433 < \eta \leq 0.0502$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.0502 < \eta \leq 0.0656$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.0656 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_4$	1	$0 < \eta \leq 0.0635$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.0635 < \eta \leq 0.1162$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.1162 < \eta \leq 0.1343$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.1343 < \eta \leq 0.1775$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.1775 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_5$	1	$0 < \eta \leq 0.0001$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.0001 < \eta \leq 0.0008$	$\{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_5\}$
	3	$0.0008 < \eta \leq 0.0009$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	4	$0.0009 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$

(Continued)

Table 7 (continued)

	Class	Confidence level	Clusters
$\mathcal{D}_6$	1	$0 < \eta \leq 0.0368$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.0368 < \eta \leq 0.1415$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.1415 < \eta \leq 0.1546$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.1546 < \eta \leq 0.1773$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.1773 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_7$	1	$0 < \eta \leq 0.0100$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.0100 < \eta \leq 0.0383$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.0383 < \eta \leq 0.0428$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.0428 < \eta \leq 0.0474$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.0474 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_8$	1	$0 < \eta \leq 0.1492$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.1492 < \eta \leq 0.2324$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.2324 < \eta \leq 0.2689$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.2689 < \eta \leq 0.3550$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.3550 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_9$	1	$0 < \eta \leq 0.1794$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.1794 < \eta \leq 0.2930$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.2930 < \eta \leq 0.3245$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.3245 < \eta \leq 0.3967$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.3967 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_{10}$	1	$0 < \eta \leq 0.2326$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.2326 < \eta \leq 0.3706$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.3706 < \eta \leq 0.4055$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.4055 < \eta \leq 0.4527$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.4527 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$

Table 8: Clustering results of existing operator $\mathcal{D}_i$ ($i = 11, 12, \dots, 20$) on application 2

	Class	Confidence level	Clusters
$\mathcal{D}_{11}$	1	$0 < \eta \leq 0.2548$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.2548 < \eta \leq 0.3706$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.3706 < \eta \leq 0.4055$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.4055 < \eta \leq 0.4527$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.4527 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_{12}$	1	$0 < \eta \leq 0.2482$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.2482 < \eta \leq 0.3710$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.3710 < \eta \leq 0.3937$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.3937 < \eta \leq 0.4412$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.4412 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$

(Continued)

Table 8 (continued)

	Class	Confidence level	Clusters
$\mathcal{D}_{13}$	1	$0 < \eta \leq 0.0860$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.0860 < \eta \leq 0.1031$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.1031 < \eta \leq 0.1138$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.1138 < \eta \leq 0.1258$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.1258 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_{14}$	1	$0 < \eta \leq 0.2725$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.2725 < \eta \leq 0.4125$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.4125 < \eta \leq 0.4544$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.4544 < \eta \leq 0.5426$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.5426 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_{15}$	1	$0 < \eta \leq 0.2832$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.2832 < \eta \leq 0.4307$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.4307 < \eta \leq 0.4815$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.4815 < \eta \leq 0.5899$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.5899 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_{16}$	1	$0 < \eta \leq 0.2984$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.2984 < \eta \leq 0.4647$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.4647 < \eta \leq 0.5372$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.5372 < \eta \leq 0.7187$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.7187 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_{17}$	1	$0 < \eta \leq 0.2936$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.2936 < \eta \leq 0.4471$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.4471 < \eta \leq 0.5045$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.5045 < \eta \leq 0.6333$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.6333 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_{18}$	1	$0 < \eta \leq 0.2174$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.2174 < \eta \leq 0.3264$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.3264 < \eta \leq 0.4649$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.4649 < \eta \leq 0.4772$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.4772 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$
$\mathcal{D}_{19}$	1	$0 < \eta \leq 0.2906$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.2906 < \eta \leq 0.4049$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.4049 < \eta \leq 0.4538$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.4538 < \eta \leq 0.5529$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.5529 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$

(Continued)

Table 8 (continued)

	Class	Confidence level	Clusters
$\mathcal{D}_{20}$	1	$0 < \eta \leq 0.2826$	$\{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_4, \mathbb{B}_5\}$
	2	$0.2826 < \eta \leq 0.3101$	$\{\mathbb{B}_4\} \{\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	3	$0.3101 < \eta \leq 0.3585$	$\{\mathbb{B}_1\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_3, \mathbb{B}_5\}$
	4	$0.3585 < \eta \leq 0.4738$	$\{\mathbb{B}_1\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_2, \mathbb{B}_5\}$
	5	$0.4738 < \eta \leq 1.0000$	$\{\mathbb{B}_1\} \{\mathbb{B}_2\} \{\mathbb{B}_3\} \{\mathbb{B}_4\} \{\mathbb{B}_5\}$

Result Analysis: The clustering results provide automakers with a nuanced understanding of the vehicle buyer market. By identifying the critical factors that influence buyer decisions, manufacturers can tailor their product development and marketing strategies to better meet the specific needs of each buyer segment. For example, targeted marketing campaigns can be designed to appeal to $\mathbb{B}_2$ eco-conscious buyers by emphasizing fuel efficiency and low emissions, while luxury features and brand prestige can be emphasized when targeting $\mathbb{B}_5$ luxury buyers.

Significance of Clustering: The proposed operator produces a clustering structure that is both interpretable and actionable for segmenting vehicle buyers. Qualitatively, the early separation of $\mathbb{B}_4$ across a broad range of η (Tables 5–8) is consistent with its profile: high emphasis on performance/brand and low hesitation, which matches the characteristics of a “premium performance” segment. Meanwhile, $\{\mathbb{B}_2, \mathbb{B}_5\}$ tend to co-locate at intermediate η thresholds, reflecting their shared prioritization of fuel efficiency and cost considerations, characteristic of “eco-/budget-oriented” buyers. The remaining $\mathbb{B}_1$ and $\mathbb{B}_3$ form a middle segment with moderate preferences, which aligns with a “mainstream” cluster. The results imply distinct product/marketing levers: premium features and brand positioning for $\mathbb{B}_4$, efficiency-based pricing/communication for $\mathbb{B}_2/\mathbb{B}_5$, and balanced offerings for $\mathbb{B}_1/\mathbb{B}_3$. This connects the clustering outcome to tangible actions, addressing the stated goal of segmenting vehicle buyers.

4.3 Multiattribute Decision Making

4.3.1 Proposed MADM Model

- **Construct the decision matrix.**

Step 1: Given a set of n alternatives $\mathbb{A} = \{\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_n\}$, and a collection of m evaluation attributes $\mathbb{C} = \{\mathbb{C}_1, \mathbb{C}_2, \dots, \mathbb{C}_m\}$, the goal is to assess each alternative for the defined attribute. Then, we construct the decision matrix $\mathcal{P} = (p_{ij})_{m \times n}$ as:

$$\mathcal{P} = (p_{ij})_{m \times n} = \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1n} \\ p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \dots & p_{mn} \end{pmatrix} \quad (38)$$

where $p_{ij} = (\mathcal{F}_{\Phi}(o_{ij}), \mathcal{F}_{\Lambda}(o_{ij}))$.

- **Compute the criteria weights using proposed divergence operator.**

Step 2: It is assumed that each criterion has a different level of importance and that these weights are independent. Let $\Omega = (\omega_1, \dots, \omega_n)^T$ be the vector of attribute weights, where each ω_i is in the closed interval $[0, 1]$, and the sum of all weights satisfies $\sum_{i=1}^n \omega_i = 1$. The procedure for calculating the attribute weights is described as:

$$\omega_i = \frac{\frac{1}{n-1} \sum_{j=1}^n \sum_{k=1}^n \mathcal{D}_L^{(\delta)}(p_{ij}, p_{ik})}{\sum_{i=1}^m \left(\frac{1}{n-1} \sum_{j=1}^n \sum_{k=1}^n \mathcal{D}_L^{(\delta)}(p_{ij}, p_{ik}) \right)} \quad (39)$$

- **Calculate the alternative rankings using AROMAN model.**

Step 3: Construct score decision matrix $\mathcal{DM}(s)$ based on (4):

$$\mathcal{DM}(s) = \begin{bmatrix} \mathbb{A}_1 & \mathbb{A}_2 & \cdots & \mathbb{A}_n \\ \mathbb{C}_1 & S(\mathbb{A}_1 \mathbb{C}_1) & S(\mathbb{A}_2 \mathbb{C}_1) & \cdots & S(\mathbb{A}_n \mathbb{C}_1) \\ \mathbb{C}_2 & S(\mathbb{A}_1 \mathbb{C}_2) & S(\mathbb{A}_2 \mathbb{C}_2) & \cdots & S(\mathbb{A}_n \mathbb{C}_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbb{C}_m & S(\mathbb{A}_1 \mathbb{C}_m) & S(\mathbb{A}_2 \mathbb{C}_m) & \cdots & S(\mathbb{A}_n \mathbb{C}_m) \end{bmatrix} \quad (40)$$

Step 4: We first use linear normalization to standardize the matrix.

$$F_{ij}^L = \frac{\mathcal{DM}(s_{ij}) - \min_j \mathcal{DM}(s_{ij})}{\max_j \mathcal{DM}(s_{ij}) - \min_j \mathcal{DM}(s_{ij})} \quad (41)$$

Step 5: Then, we use vector normalization to standardize the matrix.

$$F_{ij}^V = \frac{\mathcal{DM}(s_{ij})}{\sqrt{\sum_{j=1}^n \mathcal{DM}(s_{ij})^2}} \quad (42)$$

Step 6: The two normalized matrices are then combined to obtain an aggregated normalization matrix.

$$F_{ij}^{\text{Norm}} = \frac{\gamma F_{ij}^L + (1 - \gamma) F_{ij}^V}{2} \quad (43)$$

where $\gamma \in [0, 1]$ serves as a balancing coefficient between the linear and vector normalizations. Normally, we set $\gamma = 0.5$ when two normalizations are considered equally important.

Step 7: The aggregated normalization matrix is multiplied by the criterion weights to produce a weighted decision matrix.

$$\hat{F}_{ij} = w_i \cdot F_{ij}^{\text{Norm}} \quad (44)$$

- **Step 8:** The benefit attributes are summed using (45), and the cost attributes are summed using (46) to get the normalized weighted values:

$$\mathcal{P}_j = \sum_{i=1}^m \widehat{F}_{ij}^{(\max)} \quad (45)$$

$$\mathcal{Q}_j = \sum_{i=1}^m \widehat{F}_{ij}^{(\min)} \quad (46)$$

Step 9: The performance score for each alternative is calculated by integrating the benefit and cost components.

$$\mathcal{R}_j = \mathcal{P}_j^\zeta + \mathcal{Q}_j^{(1-\zeta)} \quad (47)$$

Here, $\zeta \in [0, 1]$ denotes the relative importance of benefit versus cost criteria. Normally, we set $\zeta = 0.5$ when both criteria types are considered equally important. The alternatives are then ranked in descending order according to $\mathcal{R}_j$ values, where a higher score indicates a more favorable alternative.

4.3.2 Application on Blockchain Platforms Selection

Application 3: In recent years, blockchain technology has emerged as a transformative force in industries such as finance, supply chain, healthcare, and digital identity [62]. As organisations consider adopting blockchain into their operations, selecting the most appropriate platform becomes a critical decision. Given the variety of blockchain platforms available—each with different strengths, trade-offs, and technical specifications—a structured MADM approach is required to ensure an optimal choice that meets strategic objectives.

A technology consultancy has been tasked with recommending the most suitable blockchain platform for a consortium of companies that aims to adopt a decentralized application (dApp) for secure, transparent data sharing. The decision involves evaluating four prominent blockchain platforms based on six key criteria.

Blockchain Platforms:

- $\mathbb{A}_1$ (Ethereum)—A widely adopted open-source platform known for its robust smart contract capabilities.
- $\mathbb{A}_2$ (Hyperledger Fabric)—A permissioned blockchain built for enterprise use with a modular architecture.
- $\mathbb{A}_3$ (Polkadot)—A next-generation blockchain focused on interoperability and scalability through its parachain model.
- $\mathbb{A}_4$ (Solana)—A high-performance blockchain that emphasises speed and low transaction costs.

Criteria for Evaluation:

- Scalability (Benefit)—The platform's ability to handle increasing numbers of transactions without performance degradation.
- Operational Expense (Cost)—The total cost of deploying and maintaining applications on the platform.
- Security (Benefit)—The robustness of the platform's security features, including consensus mechanisms and protection against attacks.
- Interoperability (Benefit)—The ease with which the platform can interact with other blockchain networks and systems.
- Development Maturity (Benefit)—The availability of developer tools, community support, and documentation.
- Governance Complexity (Cost)—The level of difficulty and friction involved in participating in or influencing platform governance decisions.

The consultancy will use a multiattribute decision-making method to rank these platforms and provide a data-driven recommendation.

Our proposed MADM model results are shown in the following part.

Step 1: Construct the blockchain platforms decision matrix $\mathcal{P} = (p_{ij})_{6 \times 4}$, as shown in Table 9.

Table 9: Transposed evaluation of blockchain platforms based on multiple criteria

	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
$\mathbb{C}_1$	$\langle 0.64, 0.42 \rangle$	$\langle 0.93, 0.41 \rangle$	$\langle 0.76, 0.68 \rangle$	$\langle 0.51, 0.45 \rangle$
$\mathbb{C}_2$	$\langle 0.46, 0.88 \rangle$	$\langle 0.53, 0.67 \rangle$	$\langle 0.65, 0.62 \rangle$	$\langle 0.74, 0.63 \rangle$
$\mathbb{C}_3$	$\langle 0.75, 0.81 \rangle$	$\langle 0.73, 0.76 \rangle$	$\langle 0.49, 0.69 \rangle$	$\langle 0.94, 0.35 \rangle$
$\mathbb{C}_4$	$\langle 0.82, 0.65 \rangle$	$\langle 0.54, 0.62 \rangle$	$\langle 0.72, 0.45 \rangle$	$\langle 0.61, 0.79 \rangle$
$\mathbb{C}_5$	$\langle 0.82, 0.53 \rangle$	$\langle 0.68, 0.65 \rangle$	$\langle 0.88, 0.67 \rangle$	$\langle 0.76, 0.43 \rangle$
$\mathbb{C}_6$	$\langle 0.70, 0.74 \rangle$	$\langle 0.57, 0.83 \rangle$	$\langle 0.60, 0.80 \rangle$	$\langle 0.58, 0.61 \rangle$

Step 2: The divergence operator $\mathcal{D}_L^{(2)}$ is applied to (39), and the weight of each attribute is computed as shown in Fig. 2.

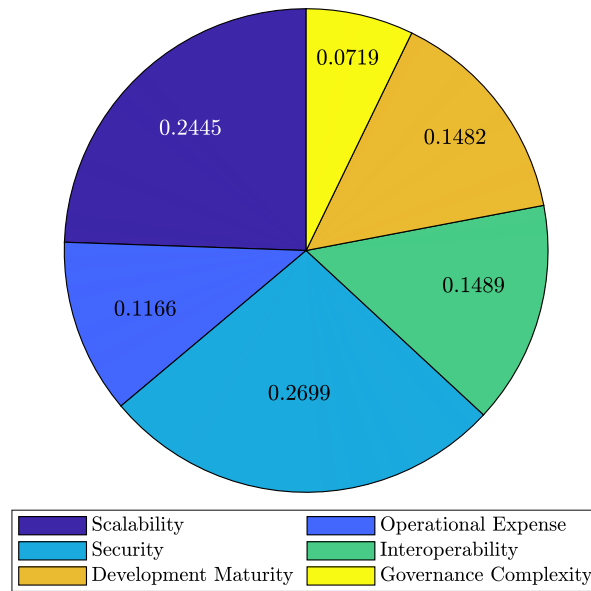


Figure 2: The weight of each attribute

Step 3: The score decision matrix $\mathcal{DM}(s)$ is computed using (4) and shown in Table 10.

Table 10: The score decision matrix

	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
$\mathbb{C}_1$	0.5940	0.8677	0.5623	0.5208
$\mathbb{C}_2$	0.2079	0.4241	0.5181	0.5776
$\mathbb{C}_3$	0.4452	0.4750	0.3946	0.8939
$\mathbb{C}_4$	0.6384	0.4596	0.6411	0.3670
$\mathbb{C}_5$	0.7012	0.5199	0.6904	0.6797
$\mathbb{C}_6$	0.4689	0.3067	0.3520	0.4841

Step 4: Calculate the linear normalization matrix using (41), shown in Table 11.

Table 11: The linear normalization matrix

	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
$\mathbb{C}_1$	0.2112	1.0000	0.1196	0.0000
$\mathbb{C}_2$	0.0000	0.5847	0.8392	1.0000
$\mathbb{C}_3$	0.1014	0.1611	0.0000	1.0000
$\mathbb{C}_4$	0.9902	0.3378	1.0000	0.0000
$\mathbb{C}_5$	1.0000	0.0000	0.9399	0.8814
$\mathbb{C}_6$	0.9144	0.0000	0.2554	1.0000

Step 5: Calculate the vector normalization matrix using (42), shown in Table 12.

Table 12: The vector normalization matrix

	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
$\mathbb{C}_1$	0.4565	0.6669	0.4321	0.4002
$\mathbb{C}_2$	0.2289	0.4668	0.5704	0.6358
$\mathbb{C}_3$	0.3792	0.4046	0.3361	0.7613
$\mathbb{C}_4$	0.5916	0.4259	0.5941	0.3401
$\mathbb{C}_5$	0.5377	0.3987	0.5294	0.5212
$\mathbb{C}_6$	0.5719	0.3741	0.4293	0.5904

Step 6: Through (43), the aggregated averaged normalization matrix is obtained and displayed in Table 13.

Table 13: The aggregated averaged normalization matrix

	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
$\mathbb{C}_1$	0.1669	0.4167	0.1379	0.1001
$\mathbb{C}_2$	0.0572	0.2629	0.3524	0.4090
$\mathbb{C}_3$	0.1202	0.1414	0.0840	0.4403
$\mathbb{C}_4$	0.3954	0.1909	0.3985	0.0850
$\mathbb{C}_5$	0.3844	0.0997	0.3673	0.3507
$\mathbb{C}_6$	0.3716	0.0935	0.1712	0.3976

Step 7: Using (44), the weighted aggregated averaged normalization matrix is obtained and displayed in Table 14.

Table 14: The weighted aggregated averaged normalization matrix

	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$
$\mathbb{C}_1$	0.0408	0.1019	0.0337	0.0245
$\mathbb{C}_2$	0.0067	0.0307	0.0411	0.0477
$\mathbb{C}_3$	0.0324	0.0382	0.0227	0.1188
$\mathbb{C}_4$	0.0589	0.0284	0.0593	0.0127
$\mathbb{C}_5$	0.0570	0.0148	0.0544	0.0520
$\mathbb{C}_6$	0.0267	0.0067	0.0123	0.0286

Step 8: The normalized weighted values are computed as: $\mathcal{P}_1 = 0.1891$, $\mathcal{P}_2 = 0.1833$, $\mathcal{P}_3 = 0.1702$, $\mathcal{P}_4 = 0.2079$ and $\mathcal{Q}_1 = 0.0334$, $\mathcal{Q}_2 = 0.0374$, $\mathcal{Q}_3 = 0.0534$, $\mathcal{Q}_4 = 0.0763$.

Step 9: The final ranking of alternatives are shown as: $\mathcal{R}_1 = 0.6176$, $\mathcal{R}_2 = 0.6215$, $\mathcal{R}_3 = 0.6436$, $\mathcal{R}_4 = 0.7322$. Therefore, $\mathbb{A}_4$ is the best alternative.

To evaluate the robustness and validity of the proposed model, a comparative analysis was conducted against several existing decision-making approaches, including the FFS-TOPSIS model [20,44], the FFS-VIKOR method [63]. In the TOPSIS model, a higher final score indicates better performance of the

alternatives. In contrast, the VIKOR model follows an opposite convention, where a lower score signifies a more favorable alternative.

The comparative analysis is shown in Table 15 and Fig. 3. The FFS-TOPSIS and the FFS-VIKOR model used the Euclidean distance to determine weight based on (39) $w = (0.1992, 0.1318, 0.2301, 0.1609, 0.1803, 0.0976)$. The proposed model ranks the $\mathbb{A}_4$ (Solana) with the highest score (0.7322), followed by $\mathbb{A}_3$ (Polkadot), $\mathbb{A}_2$ (Hyperledger Fabric), and $\mathbb{A}_1$ (Ethereum). This ranking indicates that $\mathbb{A}_4$ (Solana) is the most favorable option. Moreover, the FFS-TOPSIS model also selects $\mathbb{A}_4$ (Solana) as the best alternative. According to the VIKOR decision rules, we can learn that based on two VIKOR conditions, “acceptable advantage” is not satisfied, but “acceptable stability” is satisfied. By calculating and analyzing, the optimal alternative is $\mathbb{A}_4$ (Solana) and $\mathbb{A}_2$ (Hyperledger Fabric). The comparative results support the validity and stability of the proposed model in MADM scenario and further confirm its potential as a reliable alternative to existing models based on FFSs.

Table 15: The final ranking of different models

Model	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$	Ranking	Best alternative
Proposed model	0.6175	0.6214	0.6436	0.7322	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$	$\mathbb{A}_4$
FFS-TOPSIS	0.4844	0.4010	0.5123	0.5712	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_1 > \mathbb{A}_2$	$\mathbb{A}_4$
FFS-VIKOR (S_j)	0.7408	0.6524	0.7039	0.3061	$\mathbb{A}_1 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_4$	$\mathbb{A}_4$
FFS-VIKOR (R_j)	0.2031	0.1879	0.2301	0.1992	$\mathbb{A}_3 > \mathbb{A}_1 > \mathbb{A}_4 > \mathbb{A}_2$	$\mathbb{A}_2$
FFS-VIKOR (Q_j)	0.6792	0.3839	0.9516	0.1336	$\mathbb{A}_3 > \mathbb{A}_1 > \mathbb{A}_2 > \mathbb{A}_4$	$\mathbb{A}_4, \mathbb{A}_2$

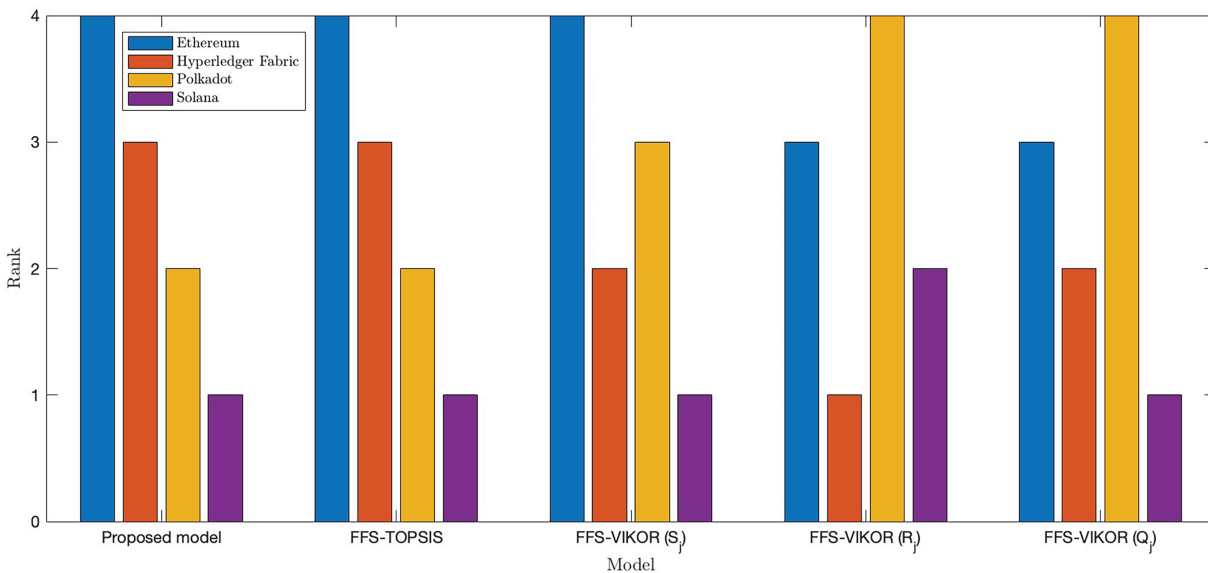


Figure 3: The result of the comparative analysis

4.3.3 Parameter Analysis

To evaluate the stability of the proposed model, this section conducts a sensitivity analysis on three parameters: δ , γ , and ζ . The initial configuration sets the baseline values as $\delta = 2$, $\gamma = \zeta = 0.5$. Subsequently, δ is varied within the range (0, 5), while γ and ζ are each independently varied with the interval [0, 1], keeping

the remaining parameters constant during each variation. This analysis enables an assessment of how changes in each parameter individually affect the utility scores and the ranking of alternatives. A comprehensive summary of the results is provided in [Table 16](#).

Table 16: The sensitivity analysis of parameter δ , γ and ζ

Parameter	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$	Ranking
$\delta \rightarrow 0$	0.6924	0.6181	0.7047	0.7720	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_1 > \mathbb{A}_2$
$\delta = 0.5$	0.6127	0.6116	0.6370	0.7313	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_1 > \mathbb{A}_2$
$\delta \rightarrow 1.0$	0.6148	0.6170	0.6402	0.7316	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\delta = 1.5$	0.6166	0.6200	0.6424	0.7320	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\delta = 2.0$	0.6175	0.6214	0.6436	0.7322	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\delta = 2.5$	0.6178	0.6218	0.6440	0.7322	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\delta = 3.0$	0.6175	0.6214	0.6436	0.7322	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\delta = 3.5$	0.6167	0.6205	0.6427	0.7320	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\delta = 4.0$	0.6154	0.6191	0.6411	0.7316	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\delta = 4.5$	0.6136	0.6172	0.6390	0.7309	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\delta = 5.0$	0.6116	0.6149	0.6366	0.7302	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
Parameter	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$	Ranking
$\gamma = 0.0$	0.6210	0.6459	0.6468	0.7058	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\gamma = 0.1$	0.6203	0.6411	0.6463	0.7115	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\gamma = 0.2$	0.6196	0.6362	0.6457	0.7169	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\gamma = 0.3$	0.6189	0.6313	0.6450	0.7222	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\gamma = 0.4$	0.6182	0.6264	0.6443	0.7273	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\gamma = 0.5$	0.6175	0.6214	0.6436	0.7322	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\gamma = 0.6$	0.6169	0.6164	0.6429	0.7369	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_1 > \mathbb{A}_2$
$\gamma = 0.7$	0.6162	0.6113	0.6421	0.7415	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_1 > \mathbb{A}_2$
$\gamma = 0.8$	0.6155	0.6062	0.6412	0.7460	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_1 > \mathbb{A}_2$
$\gamma = 0.9$	0.6148	0.6011	0.6404	0.7503	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_1 > \mathbb{A}_2$
$\gamma = 1.0$	0.6141	0.5959	0.6395	0.7545	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_1 > \mathbb{A}_2$
Parameter	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{A}_4$	Ranking
$\zeta = 0.0$	1.1891	1.1832	1.1702	1.2079	$\mathbb{A}_4 > \mathbb{A}_1 > \mathbb{A}_2 > \mathbb{A}_3$
$\zeta = 0.1$	0.9351	0.9370	0.9492	1.0164	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\zeta = 0.2$	0.7705	0.7755	0.7991	0.8824	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\zeta = 0.3$	0.6723	0.6780	0.7047	0.7952	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\zeta = 0.4$	0.6248	0.6298	0.6553	0.7470	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\zeta = 0.5$	0.6175	0.6214	0.6436	0.7322	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\zeta = 0.6$	0.6437	0.6464	0.6649	0.7471	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\zeta = 0.7$	0.6993	0.7012	0.7165	0.7893	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\zeta = 0.8$	0.6993	0.7012	0.7165	0.7893	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\zeta = 0.9$	0.8935	0.8959	0.9093	0.9533	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$
$\zeta = 1.0$	1.0334	1.0374	1.0534	1.0763	$\mathbb{A}_4 > \mathbb{A}_3 > \mathbb{A}_2 > \mathbb{A}_1$

In all three cases, while the utility scores of alternatives show sensitivity to parameter variations, the ranking results remain largely stable. Notably, alternative $\mathbb{A}_4$ consistently emerges as the top-ranked option, highlighting the model's resilience and reliability across differing parameter settings.

5 Conclusion

This paper introduced a unified Fermatean fuzzy parametric divergence operator that fully accounts for membership, non-membership, and hesitation degrees, addressing key shortcomings in existing divergence operators for FFSs. The proposed operator satisfies fundamental mathematical properties, including non-negativity, non-degeneracy, and symmetry. It also generalizes several well-known divergence operators such as Jensen-Shannon, Hellinger, χ^2 -divergence, and arithmetic-geometric divergence. The effectiveness of the proposed operator is validated through three practical applications: pattern classification, hierarchical clustering, and MADM for blockchain platform selection via the AROMAN model. The results confirm the superior sensitivity, robustness, and versatility of our operator in various machine learning and intelligent decision-making scenarios.

Despite the promising outcomes, some limitations should be acknowledged. First, the current divergence operator is defined for precise FFSs, which limits its direct applicability to more complex environments such as interval-valued, complex, or hesitant FFSs. Second, the divergence operator has not yet been integrated into learning-based frameworks, making it less adaptive in dynamic or data-driven decision contexts. Future research will focus on several directions to overcome these limitations. These include generalizing the operator to interval-valued, complex, or hesitant FFSs and embedding the divergence operator within machine learning models such as fuzzy neural networks. Such extensions aim to enhance both the theoretical flexibility and practical utility of the proposed operator in increasingly complex and large-scale decision-making environments.

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References

1. Gligorić Z, Gligorić M, Miljanović I, Lutovac S, Milutinović A. Assessing criteria weights by the symmetry point of criterion (novel spc method)—application in the efficiency evaluation of the mineral deposit multi-criteria partitioning algorithm. *Comput Model Eng Sci.* 2023;136(1):955–79. doi:10.32604/cmesci.2023.025021.
2. Abdalla MEM, Uzair A, Ishtiaq A, Tahir M, Kamran M. Algebraic structures and practical implications of interval-valued Fermatean neutrosophic super HyperSoft sets in healthcare. *Spec Oper Res.* 2025;2(1):199–218.

3. Liu Z, Letchmunan S. Enhanced fuzzy clustering for incomplete instance with evidence combination. *ACM Trans Knowl Discov Data*. 2024;18(3):1–20. doi:10.1145/3638061.
4. Tešić D, Marinković D. Application of fermatean fuzzy weight operators and MCDM model DIBR-DIBR II-NWBM-BM for efficiency-based selection of a complex combat system. *J Decis Anal Intell Comput*. 2023;3(1):243–56.
5. Majd SS, Maleki A, Basirat S, Golkarfard A. Fermatean fuzzy TOPSIS method and its application in ranking business intelligence-based strategies in smart city context. *J Oper Intell*. 2025;3(1):1–16.
6. Gokasar I, Pamucar D, Deveci M, Ding W. A novel rough numbers based extended MACBETH method for the prioritization of the connected autonomous vehicles in real-time traffic management. *Expert Syst Appl*. 2023;211(1):118445. doi:10.1016/j.eswa.2022.118445.
7. Pamucar D, Deveci M, Gokasar I, Brito-Parada PR, Martínez L. Evaluation of process technologies for sustainable mining using interval rough number based heronian and power averaging functions. *Knowl Based Syst*. 2024;289(4):111494. doi:10.1016/j.knosys.2024.111494.
8. Qi J, Hu J, Huang H, Peng Y. New customer-oriented design concept evaluation by using improved Z-number-based multi-criteria decision-making method. *Adv Eng Inform*. 2022;53(2):101683. doi:10.1016/j.aei.2022.101683.
9. Nazari-Shirkouhi S, Tavakoli M, Govindan K, Mousakhani S. A hybrid approach using Z-number DEA model and artificial neural network for resilient supplier selection. *Expert Syst Appl*. 2023;222(04):119746. doi:10.1016/j.eswa.2023.119746.
10. Lyu S, Liu Z. A belief sharma-mittal divergence with its application in multi-sensor information fusion. *Comput Appl Math*. 2024;43(1):34. doi:10.1007/s40314-023-02542-0.
11. Liu Z, Huang H, Letchmunan S, Deveci M. Adaptive weighted multi-view evidential clustering with feature preference. *Knowl Based Syst*. 2024;294(3):111770. doi:10.1016/j.knosys.2024.111770.
12. Singh RR, Zindani D, Maity SR. A novel fuzzy-prospect theory approach for hydrogen fuel cell component supplier selection for automotive industry. *Expert Syst Appl*. 2024;246(3):123142. doi:10.1016/j.eswa.2024.123142.
13. Rong Y, Yu L, Liu Y, Simic V, Pamucar D, Garg H. A novel failure mode and effect analysis model based on extended interval-valued q-rung orthopair fuzzy approach for risk analysis. *Eng Appl Artif Intell*. 2024;136(1):108892. doi:10.1016/j.engappai.2024.108892.
14. Liu Z. Fermatean fuzzy similarity measures based on Tanimoto and Sørensen coefficients with applications to pattern classification, medical diagnosis and clustering analysis. *Eng Appl Artif Intell*. 2024;132(1):107878. doi:10.1016/j.engappai.2024.107878.
15. Zadeh LA. Fuzzy sets. *Inf Control*. 1965;8(3):338–53.
16. Atanassov KT. Intuitionistic fuzzy sets. *Fuzzy Sets Syst*. 1986;20(1):87–96. doi:10.1016/s0165-0114(86)80034-3.
17. Yager RR. Pythagorean membership grades in multicriteria decision making. *IEEE Trans Fuzzy Syst*. 2014;22(4):958–65. doi:10.1109/tfuzz.2013.2278989.
18. Ejegwa PA, Feng Y, Tang S, Agbetayo JM, Dai X. New Pythagorean fuzzy-based distance operators and their applications in pattern classification and disease diagnostic analysis. *Neural Comput Appl*. 2023;35(14):10083–95. doi:10.1007/s00521-022-07679-3.
19. Zulqarnain RM, Ma WX, Siddique I, Ahmad H, Askar S. An intelligent MCGDM model in green suppliers selection using interactional aggregation operators for interval-valued pythagorean fuzzy soft sets. *Comput Model Eng Sci*. 2024;139(2):1829–62. doi:10.32604/cmes.2023.030687.
20. Senapati T, Yager RR. Fermatean fuzzy sets. *J Ambient Intell Humaniz Comput*. 2020;11(2):663–74. doi:10.1007/s12652-019-01377-0.
21. Senapati T, Yager RR. Fermatean fuzzy weighted averaging/geometric operators and its application in multi-criteria decision-making methods. *Eng Appl Artif Intell*. 2019;85(S1):112–21. doi:10.1016/j.engappai.2019.05.012.
22. Kakati P, Senapati T, Moslem S, Pilla F. Fermatean fuzzy archimedean heronian mean-based model for estimating sustainable urban transport solutions. *Eng Appl Artif Intell*. 2024;127(1):107349. doi:10.1016/j.engappai.2023.107349.

23. Saha A, Dabic-Miletic S, Senapati T, Simic V, Pamucar D, Ala A, et al. Fermatean fuzzy dombi generalized maclaurin symmetric mean operators for prioritizing bulk material handling technologies. *Cognit Comput*. 2024;16(6):3096–121. doi:10.1007/s12559-024-10323-y.
24. Szmidt E, Kacprzyk J. Distances between intuitionistic fuzzy sets. *Fuzzy Sets Syst*. 2000;114(3):505–18. doi:10.1016/s0165-0114(98)00244-9.
25. Ejegwa PA. Distance and similarity measures for Pythagorean fuzzy sets. *Granul Comput*. 2020;5(2):225–38. doi:10.1007/s41066-018-00149-z.
26. Ejegwa PA. Modified Zhang and Xu's distance measure for Pythagorean fuzzy sets and its application to pattern recognition problems. *Neural Comput Appl*. 2020;32(14):10199–208. doi:10.1007/s00521-019-04554-6.
27. Grzegorzewski P. Distances between intuitionistic fuzzy sets and/or interval-valued fuzzy sets based on the Hausdorff metric. *Fuzzy Sets Syst*. 2004;148(2):319–28. doi:10.1016/j.fss.2003.08.005.
28. Yang Y, Chiclana F. Consistency of 2D and 3D distances of intuitionistic fuzzy sets. *Expert Syst Appl*. 2012;39(10):8665–70. doi:10.1016/j.eswa.2012.01.199.
29. Xiao F, Ding W. Divergence measure of Pythagorean fuzzy sets and its application in medical diagnosis. *Applied Soft Comput*. 2019;79(2):254–67. doi:10.1016/j.asoc.2019.03.043.
30. Wu X, Zhu Z, Chen G, Pedrycz W, Liu L, Aggarwal M. Generalized TODIM method based on symmetric intuitionistic fuzzy Jensen-Shannon divergence. *Expert Syst Appl*. 2024;237(11):121554. doi:10.1016/j.eswa.2023.121554.
31. Wu X, Zhu Z, Chen SM. Strict intuitionistic fuzzy distance/similarity measures based on Jensen-Shannon divergence. *Inf Sci*. 2024;661(6):120144. doi:10.1016/j.ins.2024.120144.
32. Li X, Liu Z, Han X, Liu N, Yuan W. An intuitionistic fuzzy version of hellinger distance measure and its application to decision-making process. *Symmetry*. 2023;15(2):500. doi:10.3390/sym15020500.
33. Liu Z. Hellinger distance measures on pythagorean fuzzy environment via their applications. *Int J Knowl-Based Intell Eng Syst*. 2024;28(2):211–29. doi:10.3233/kes-230150.
34. Ye J. Cosine similarity measures for intuitionistic fuzzy sets and their applications. *Math Comput Model*. 2011;53(1–2):91–7. doi:10.1016/j.mcm.2010.07.022.
35. Zhang Q, Hu J, Feng J, Liu A, Li Y. New similarity measures of Pythagorean fuzzy sets and their applications. *IEEE Access*. 2019;7:138192–202. doi:10.1109/access.2019.2942766.
36. Hwang CM, Yang MS, Hung WL. New similarity measures of intuitionistic fuzzy sets based on the Jaccard index with its application to clustering. *Int J Intell Syst*. 2018;33(8):1672–88. doi:10.1002/int.21990.
37. Hussain Z, Alam S, Hussain R, ur Rahman S. New similarity measure of Pythagorean fuzzy sets based on the Jaccard index with its application to clustering. *Ain Shams Eng J*. 2024;15(1):102294. doi:10.1016/j.asej.2023.102294.
38. Tang Y, Wen L, Wei G. Approaches to multiple attribute group decision making based on the generalized Dice similarity measures with intuitionistic fuzzy information. *Int J Knowl-Based Intell Eng Syst*. 2017;21(2):85–95. doi:10.3233/kes-170354.
39. Patel A, Jana S, Mahanta J. Intuitionistic fuzzy EM-SWARA-TOPSIS approach based on new distance measure to assess the medical waste treatment techniques. *Appl Soft Comput*. 2023;144(3):110521. doi:10.1016/j.asoc.2023.110521.
40. Kumar R, Kumar S. An extended combined compromise solution framework based on novel intuitionistic fuzzy distance measure and score function with applications in sustainable biomass crop selection. *Expert Syst Appl*. 2024;239(10):122345. doi:10.1016/j.eswa.2023.122345.
41. Sun G, Wang M. Pythagorean fuzzy information processing based on centroid distance measure and its applications. *Expert Syst Appl*. 2024;236(4):121295. doi:10.1016/j.eswa.2023.121295.
42. Onyeke IC, Ejegwa PA. Modified Senapati and Yager's Fermatean fuzzy distance and its application in students'course placement in tertiary institution. In: *Real life applications of multiple criteria decision making techniques in fuzzy domain*. Cham, Switzerland: Springer; 2022. p. 237–53 doi: 10.1007/978-981-19-4929-6_11.
43. Sahoo L. Similarity measures for Fermatean fuzzy sets and its applications in group decision-making. *Decis Sci Letters*. 2022;11(2):167–80.

44. Kirişçi M. New cosine similarity and distance measures for Fermatean fuzzy sets and TOPSIS approach. *Knowl Inf Syst.* 2023;65(2):855–68. doi:10.1007/s10115-022-01776-4.
45. Liu Z, Huang H. Comment on “new cosine similarity and distance measures for fermatean fuzzy sets and topsis approach”. *Knowl Inf Syst.* 2023;65(12):5151–7. doi:10.1007/s10115-023-01926-2.
46. Deng Z, Wang J. New distance measure for Fermatean fuzzy sets and its application. *Int J Intell Syst.* 2022;37(3):1903–30. doi:10.1002/int.22760.
47. Ganie AH. Multicriteria decision-making based on distance measures and knowledge measures of Fermatean fuzzy sets. *Granul Comput.* 2022;7(4):979–98. doi:10.1007/s41066-021-00309-8.
48. Behzadian M, Otaghsara SK, Yazdani M, Ignatius J. A state-of-the-art survey of TOPSIS applications. *Expert Syst Appl.* 2012;39(17):13051–69. doi:10.1016/j.eswa.2012.05.056.
49. Gul M, Celik E, Aydin N, Gumus AT, Guneri AF. A state of the art literature review of VIKOR and its fuzzy extensions on applications. *Appl Soft Comput.* 2016;46:60–89. doi:10.1016/j.asoc.2016.04.040.
50. Torkayesh AE, Deveci M, Karagoz S, Antucheviciene J. A state-of-the-art survey of evaluation based on distance from average solution (EDAS): developments and applications. *Expert Syst Appl.* 2023;221(1):119724. doi:10.1016/j.eswa.2023.119724.
51. Chang TH. Fuzzy VIKOR method: a case study of the hospital service evaluation in Taiwan. *Inf Sci.* 2014;271(3):196–212. doi:10.1016/j.ins.2014.02.118.
52. Ghorabae MK, Amiri M, Zavadskas EK, Hooshmand R, Antuchevičienė J. Fuzzy extension of the CODAS method for multi-criteria market segment evaluation. *J Bus Econ Manag.* 2017;18(1):1–19. doi:10.3846/16111699.2016.1278559.
53. Salih MM, Zaidan B, Zaidan A, Ahmed MA. Survey on fuzzy TOPSIS state-of-the-art between 2007 and 2017. *Comput Operat Res.* 2019;104:207–27.
54. Ogundoyin SO, Kamil IA. An integrated fuzzy-BWM, fuzzy-LBWA and V-fuzzy-CoCoSo-LD model for gateway selection in fog-bolstered Internet of Things. *Appl Soft Comput.* 2023;143(5):110393. doi:10.1016/j.asoc.2023.110393.
55. Bošković S, Švadlenka L, Jovčić S, Dobrodolac M, Simić V, Bacanin N. An alternative ranking order method accounting for two-step normalization (AROMAN)—a case study of the electric vehicle selection problem. *IEEE Access.* 2023;11(3):39496–507. doi:10.1109/access.2023.3265818.
56. Nikolić I, Milutinović J, Božanić D, Dobrodolac M. Using an interval type-2 fuzzy AROMAN decision-making method to improve the sustainability of the postal network in rural areas. *Mathematics.* 2023;11(14):3105. doi:10.3390/math11143105.
57. Kara K, Yalçın GC, Acar AZ, Simic V, Konya S, Pamucar D. The MEREC-AROMAN method for determining sustainable competitiveness levels: a case study for Turkey. *Socioecon Plann Sci.* 2024;91(48):101762. doi:10.1016/j.seps.2023.101762.
58. Deveci M, Varouchakis EA, Brito-Parada PR, Mishra AR, Rani P, Bolgkoranou M, et al. Evaluation of risks impeding sustainable mining using Fermatean fuzzy score function based SWARA method. *Appl Soft Comput.* 2023;139(1):110220. doi:10.1016/j.asoc.2023.110220.
59. Mishra AR, Rani P, Saha A, Senapati T, Hezam IM, Yager RR. Fermatean fuzzy copula aggregation operators and similarity measures-based complex proportional assessment approach for renewable energy source selection. *Complex Intell Syst.* 2022;8(6):5223–48. doi:10.1007/s40747-022-00743-4.
60. Zhu S, Liu Z, Ulutagay G, Deveci M, Pamučar D. Novel α -divergence measures on picture fuzzy sets and interval-valued picture fuzzy sets with diverse applications. *Eng Appl Artif Intell.* 2024;136(12):109041. doi:10.1016/j.engappai.2024.109041.
61. Bozyiugit MC, Olgun M, Ünver M, Söylemez D. Parametric picture fuzzy cross-entropy measures based on d-Choquet integral for building material recognition. *Appl Soft Comput.* 2024;166(3):112167. doi:10.1016/j.asoc.2024.112167.
62. Nanayakkara S, Rodrigo M, Perera S, Weerasuriya GT, Hijazi AA. A methodology for selection of a Blockchain platform to develop an enterprise system. *J Ind Inf Integr.* 2021;23(04):100215. doi:10.1016/j.jii.2021.100215.
63. Gül S. Fermatean fuzzy set extensions of SAW, ARAS, and VIKOR with applications in COVID-19 testing laboratory selection problem. *Expert Syst.* 2021;38(8):e12769. doi:10.1111/exsy.12769.