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ARTICLE

Computational Solutions of a Delay-Driven Stochastic Model for Conjunctivitis Spread

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ABSTRACT: This study investigates the transmission dynamics of conjunctivitis using stochastic delay differential equations (SDDEs). A delayed stochastic model is formulated by dividing the population into five distinct compartments: susceptible, exposed, infected, environmental irritants, and recovered individuals. The model undergoes thorough analytical examination, addressing key dynamical properties including positivity, boundedness, existence, and uniqueness of solutions. Local and global stability around the equilibrium points is studied with respect to the basic reproduction number. The existence of a unique global positive solution for the stochastic delayed model is established. In addition, a stochastic nonstandard finite difference scheme is developed, which is shown to be dynamically consistent and convergent toward the equilibrium states. The scheme preserves the essential qualitative features of the model and demonstrates improved performance when compared to existing numerical methods. Finally, the impact of time delays and stochastic fluctuations on the susceptible and infected populations is analyzed.

KEYWORDS: Conjunctivitis disease; stochastic delay differential equations (SDDE's); existence and uniqueness; unique global positivity; computational methods; results

1 Introduction

Conjunctivitis, often called pink eye, is a highly contagious disease. Infection of the conjunctiva is generally a result of viruses, bacteria, and allergens. Some viruses can cause a precise conjunctivitis known as acute hemorrhagic conjunctivitis (AHC). Bacterial conjunctivitis causes the speedy onset of conjunctiva irritation, puffiness of the eyelid, and a muggy release. Usually, signs advance mainly in a single eye and then may extend to the next eye within 2 to 5 days. Three main viruses have been studied and built as the agents responsible for acute hemorrhagic conjunctivitis (AHC): coxsackievirus A24 variation (CA24v), enterovirus 70, and adenovirus 11. Acute hemorrhagic conjunctivitis (AHC) can only occur in a human population and is transmitted through human interaction with an infected specific object, like a towel used by a diseased person. The discomfort of the eyes, sensitivity to light, sore eyes, tears, and pus discharge are some symptoms. Antibiotic eye drops, sanitized passivity, quarantine, and permitting the infection to progress in 2–3 weeks are operative ways to break the extent of conjunctivitis infection [1]. Recently, a significant outbreak occurred in

different cities in Pakistan. It began in Karachi and spread to major cities. Almost one hundred thousand cases were reported in Punjab. This massive outbreak of pink eye is a recent development in Pakistan's history [2].

In 2024, Faisal et al. [3] examined the dynamics of conjunctivitis adenovirus using a unique tool such as bifurcation. At the end, to support the dynamical results, they simulated their numerical result very comprehensively. Gabriela et al. [4] conducted a study in 2024 using a basic Susceptible-Infectious-Susceptible (SIS) model with introducing uncertainty through transmission parameters. They attained conditions of extinction and existence using threshold constraints. They considered real-world problems and proposed suitable uncertain parameters. In 2024, Essak [5] considered three types of uncertainty in their study Lévy noise, Gaussian white noise, and telegraph noise to investigate their influence on the dynamics of disease. Albani et al. [6] used the fact that model parameters evolve with time and showed the randomness of sudden change. They proposed the jump-diffusion stochastic process. They presented their dynamical analysis and compared it with the variation and jumps. They used real data for their simulation and claimed their finding are accurate. In 2024, Ali et al. [7] considered a stochastic model for influenza dynamics and also used accurate parameter values for their comprehensive understanding of influenza dynamics. They used actual data for their simulation that supports their dynamic results. In 2014, Unyong et al. [8] suggested and investigated a mathematical system for the dynamical stability of conjunctivitis with nonlinear incidence terms considered. Finally, numerical results support the analytic results. In 2012, Sangsawang et al. [9] investigated the system for the spread of Hemorrhagic Conjunctivitis. Numerical simulations of the system demonstrate that the disease-free equilibrium and endemic are stable and unique. In 2006, Chowell et al. [10] modeled the diffusion dynamics of AHC in an eruption of pink eye in the host population in Mexico. Therefore, the effects of sustenance present a public health strategy and the efficient message of the eruption, and community health evidence press statements that initiate beings on evading infection and encourage them to pursue a clinical diagnosis. They studied epidemic models with well-known computational methods based on stochastic differential equations with artificial decay terms in [11–13]. They studied well-known stochastic models such as the stochastic within-host Chikungunya (CHIKV) virus model with saturated incidence rate, Extinction, and the stationary distribution of the stochastic hepatitis B virus model, and a Zika virus as a mosquito-borne transmitted disease with environmental fluctuations as presented in [14–16]. The approach aligns with recent works such as [17], where stochastic noise is interpreted as environmental randomness impacting disease spread. Foundational contributions such as those in [18] provide detailed bifurcation analysis of delay epidemic models, highlighting the critical role of time-lag in disease dynamics. Similarly, reviews in [19,20] summarize analytical techniques for deterministic and stochastic delay systems, particularly in biological contexts. Our model structure and analytical proofs draw on these theoretical foundations, especially in establishing positivity, stability, and global existence. The authors studied the Split-Step θ-Method for Stochastic Pantograph Differential Equations, focusing on convergence and mean-square stability analysis, in [21].

Model formation and its numerical simulations are the classical origins of studying the transmission dynamics of infectious diseases because they allow us to understand the complex transmission dynamics of any contagious disease. The trajectories of the deterministic model are idealistic, but the stochastic model is closer to reality. This outcome matters for policymakers and concerned individuals. So, this study took the deterministic model and reformulated it into a stochastic delayed conjunctivitis model, then studied its stochastic behavior to understand the dynamics of the spread of disease, and an exponential delay was introduced to understand the effect of delay on the spread of disease to get more optimal results to control the disease. This work makes several novel contributions to the modeling and control of conjunctivitis epidemics:

- **Model Innovation:** The stochastic delayed compartment model for conjunctivitis is configured in a new manner such that it incorporates artificial delay effects to reflect realistic intervention strategies such as quarantine, delayed exposure, or behavioral response to symptoms.
- **Theoretical Advancement:** The model's dynamical properties, including positivity, boundedness, existence, and uniqueness, were studied rigorously. Also, the authors studied local and global stability conditions under biologically meaningful assumptions.
- **Numerical Methodology:** A completely new and original stochastic nonstandard finite difference (NSFD) method has been developed and analyzed. This method retains important dynamical properties, ensures numerical positivity and convergence, and beats standard methods under larger time steps.
- **Biological Insight:** Artificial delays significantly reduce the number of infected individuals and increase the susceptible population, with use as a control strategy. These findings are compatible with recent public health approaches and have practical relevance to the model.
- **Literature Integration:** This paper contributes to filling the gap in existing literature by making an extended union of stochastic dynamics, artificial delays, and conjunctivitis modeling under one united framework, which is justified by the most recent epidemiological events and data.
- **Public Health Interpretation of Artificial Delay:** The exponential delay term $e^{-\mu T}$ incorporated in the model captures realistic intervention strategies employed in controlling disease outbreaks. Specifically, this form of delay accounts for the time lag between exposure and infectiousness, which can be influenced by public health policies such as quarantine protocols, awareness campaigns, and social distancing mandates. For instance, if an infected individual is identified and quarantined within a specific time frame, the probability of disease transmission significantly drops, mimicking the effect of exponential decay in contact or transmission rate. Thus, the model not only reflects the underlying biology but also provides a quantitative framework for evaluating the timing and efficacy of such interventions.

The mathematical framework of this study is deeply rooted in functional analysis. This study employs key results from functional analysis, including fixed point theory, operator norms, and stability criteria in infinite-dimensional spaces, to establish existence, uniqueness, and positivity of solutions. These foundational tools ensure the rigorous formulation and analysis of the stochastic delay differential equations (SDDEs) that govern the proposed model.

The following subdivisions comprise five sections: Section 1 briefly introduces the research disease and literature review. Section 2 presents the derivation of the delayed mathematical model and its dynamical properties, equilibrium points, and local-global behavior of the delayed system. Section 3 describes the stochastic delayed model and the theorem-related unique global positive solution. Section 4 introduces the selective numerical method, and presents the derived stochastic NSFD-related theorems. Lastly, Section 5 discusses numerical simulations among selective techniques and their convergence behavior around the conjunctivitis present state with different step sizes. In addition, the discussion and conclusion are presented in different sections.

2 Derivation of the Delayed Conjunctivitis Mathematical Model

This model is present in [1] in the deterministic form. This study investigates artificial delay effects and stochastic behaviors to understand the realistic transmission dynamics. Delay-based interventions are used to control the spread of disease. Because of the exponential growth of the disease population, the exponential delay is employed to prevent disease transmission similarly. In [1], the human host population $N(t)$ is divided into five compartments $S_c(t), E_c(t), I_c(t), I_R(t), R(t)$ at any time $t \geq 0$, represent the individual susceptible, individual exposed, individual infected, number of irritants, and individual recovered, respectively. The saturated incidence in the system (1)–(5) with delayed effect represents the rate of change of susceptible

individuals to infected individuals and artificial delay $e^{-\mu T}$ that can be any artificial tactics be used to control the exponential growth of the transmission of the disease with counter artificial exponential-based interventions for all $T \geq 0$ (see Fig. 1 and Table 1).

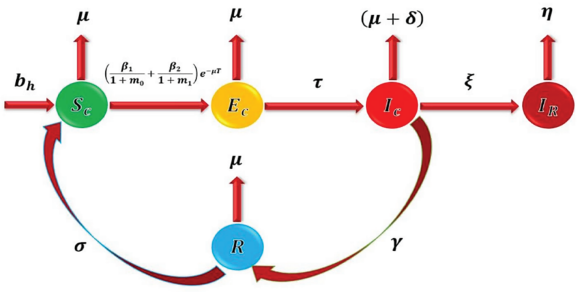

Figure 1: Flow dynamics of the Conjunctivitis disease in a population

Table 1: Physical understanding of the parameters

Parameters	Descriptions	Values/Per day	Source [1]
b_h	The rate at which susceptible enlisted into the human inhabitants.	0.05	Assumed
β_1	The rate at which susceptible procure the disease through direct transmission.	2.081 (CFE) 5.081 (CPE)	Assumed
β_2	The rate at which susceptible procure the disease through indirect transmission.	2.092 (CFE) 5.092 (CPE)	Assumed
m_0	The rate of new disease infections in humans.	0.01	Assumed
m_1	The rate of bacterial/viral infections present in irritants.	0.02	Assumed
μ	Natural death rate of human subpopulations.	0.05	Assumed
σ	The rate at which improvement becomes susceptible.	0.45	Assumed
τ	Rate of exposure to infected.	0.112	Assumed
δ	The rate at which infected lose their visualization when left untreated.	0.21	Assumed
γ	The rate at which infected become recovered.	0.142	Assumed
ξ	The amount of diseased human aid to irritants or allergens.	0.3	Assumed
η	The natural net expiry rate of bacteria/virus	0.05	Assumed

The delayed conjunctivitis mathematical model is derived under the following assumptions.

- Direct and indirect transmission paths are taken.
- Individuals who recover again become susceptible to the disease.
- Individuals who lost their vision are considered.
- The host population is equally mixed.
- The natural death rate and birth rate are the same.
- The natural death rate and natural expiry rate of bacteria/viruses are approximately equal.

Although most parameter values are directly adopted from [1], several minor modifications were made to improve computational tractability and biological interpretability. Specifically, this study assumes that the birth and natural death rates are equal (0.05/day), a common simplification in epidemic models to maintain a stable total population in the absence of disease-related deaths. This assumption allows a clearer focus on the disease-induced dynamics without the confounding effects of demographic growth or decline.

The model of the conjunctivitis delay differential equations is as follows:

$$\frac{dS_c(t)}{dt} = b_h - \frac{\beta_1 S_c(t) I_c(t-T) e^{-\mu T}}{1 + m_0 I_c(t-T)} - \frac{\beta_2 S_c(t) I_R(t)}{1 + m_1 I_R(t)} - \mu S_c(t) + \sigma R(t) \tag{1}$$

$$\frac{dE_c(t)}{dt} = \frac{\beta_1 S_c(t) I_c(t-T) e^{-\mu T}}{1 + m_0 I_c(t-T)} + \frac{\beta_2 S_c(t) I_R(t)}{1 + m_1 I_R(t)} - (\tau + \mu) E_c(t) \tag{2}$$

$$\frac{dI_c(t)}{dt} = \tau E_c(t) - (\gamma + \mu + \xi + \delta) I_c(t) \tag{3}$$

$$\frac{dI_R(t)}{dt} = \xi I_c(t) - \eta I_R(t) \tag{4}$$

$$\frac{dR(t)}{dt} = \gamma I_c(t) - (\mu + \sigma) R(t) \tag{5}$$

Given that $S_c(0) \geq 0$, $E_c(0) \geq 0$, $I_c(0) \geq 0$, $I_R(0) \geq 0$ and $R(0) \geq 0$, for all $t \geq 0$ are the initial conditions, $T < t$ where T is delay effect.

2.1 Essential Properties

Theorem 1: ***(Positivity)*** *For any given initial condition* $(S_C(0), E_c(0), I_c(0), I_R(0), R(0)) \in R_+^5$ *then the solution of system (1)–(5) for all time* $t > 0$, $(S_C(t), E_c(t), I_c(t), I_R(t), R(t)) \in R_+^5$.

Proof: If $S_C(0) \geq 0, E_c(0) \geq 0, I_c(0) \geq 0, I_R(0) \geq 0, \mathrm{R}(0) \geq 0$ and the norm $\|\vartheta\|_\infty = Sup_{t \in D_\vartheta} |\vartheta|$. Then, Eq. (1) yields:

$$\frac{dS_c}{dt} = \left[b_h - \frac{\beta_1 S_c I_C}{1 + m_0 I_C} e^{-\mu T} - \frac{\beta_2 S_c I_R}{1 + m_1 I_R} - \mu S_c + \sigma R \right], \forall\ \mathrm{t} \geq 0$$

$$\frac{dS_c}{dt} \geq -\left[\frac{\beta_1 I_C}{1 + m_0 I_C} e^{-\mu T} + \frac{\beta_2 I_R}{1 + m_1 I_R} + \mu \right] S_c, \forall\ \mathrm{t} \geq 0$$

$$\ln S_c \geq -\left[\frac{\beta_1 \|I_c\|_\infty}{1 + m_0 \|I_c\|_\infty} e^{-\mu T} + \frac{\beta_2 \|I_R\|_\infty}{1 + m_1 \|I_R\|_\infty} + \mu \right] t + c_1, \forall\ \mathrm{t} \geq 0$$

$$S_c \geq \mathrm{S}(0)\, e^{-\left[\frac{\beta_1 \|I_c\|_\infty}{1 + m_0 \|I_c\|_\infty} e^{-\mu T} + \frac{\beta_2 \|I_R\|_\infty}{1 + m_1 \|I_R\|_\infty} + \mu \right] t} \geq 0, \forall\ \mathrm{t} \geq 0, S_c(\mathrm{t}) \geq 0, \forall\ \mathrm{t} \geq 0 \tag{6}$$

Similarly, the following relationships from system (2)–(5) can be described:

$$E_c \geq \mathrm{E}(0)\, e^{-(\tau+\mu)t} \geq 0, \forall\ \mathrm{t} \geq 0 \tag{7}$$

$$I_c \geq I_c(0)\, e^{-(\gamma+\mu+\xi+\delta)t} \geq 0, \forall\ \mathrm{t} \geq 0 \tag{8}$$

$$I_R \geq I_R(0)\, e^{-\eta t} \geq 0, \forall\ \mathrm{t} \geq 0 \tag{9}$$

$$\mathrm{R} \geq \mathrm{R}(0)\, e^{-(\mu+\sigma)t} \geq 0, \forall\ \mathrm{t} \geq 0 \tag{10}$$

Eqs. (6)–(10) are non-negative. Hence, the system (1)–(5) is positive $\forall\ \mathrm{t} \geq 0$. □

Theorem 2: ***(Boundedness)*** $\forall\, t \geq 0$, *the trajectories of the system (1)–(5) with any given initial trajectories from the set* Ω *persist around the set:*

$$\Omega = \left\{ \left(S_C\left(t\right), E_c\left(t\right), I_c\left(t\right), I_R\left(t\right), \mathrm{R}(t)\right) \epsilon\, R_+^5 : S_c\left(t\right) + E_c\left(t\right) + I_c\left(t\right) + I_R\left(t\right) + R(t) \leq \frac{b_h}{\mu} \right\}$$

if $\lim\limits_{t\to\infty} Sup\ N(t) \leq \dfrac{b_h}{\mu}$ *and* $\mu = \eta$.

Proof: After adding the system (1)–(5), the total population of the system is as follows:

$$\frac{dN}{dt} = b_h - \mu S_c - \mu E_c - \mu I_c - \delta I_c - \eta I_R - \mu R$$

$$\frac{dN}{dt} \leq b_h - \mu S_c - \mu E_c - \mu I_c - \eta I_R - \mu R$$

$$\frac{dN}{dt} \leq b_h - \mu N, \text{provided let } \mu = \eta$$

$$\frac{dN}{dt} + \mu N \leq b_h$$

The general solution

$$N\left(t\right) \leq \frac{b_h}{\mu} + \left(N\left(0\right) - \frac{b_h}{\mu}\right) e^{-\mu t} \tag{11}$$

$$\lim_{t\to\infty} Sup\ N(t) \leq \frac{b_h}{\mu}$$

Now, any given initial trajectory $N(0) \leq \frac{b_h}{\mu}$, then $\lim\limits_{t\to\infty} Sup\ N(t) \leq \frac{b_h}{\mu}$. Hence, Ω is positively invariant. Therefore, system (1)–(5) is locally Lipschitz. □

Theorem 3: ***(Existence and Uniqueness)*** *The unique solution of system (1)–(5) exists piece-wise if it satisfies linear growth and Lipschitz conditions and* $\mu = \eta$.

Proof: Define the norm $\|\vartheta\|_\infty = Sup_{t\epsilon D_\vartheta} |\vartheta|$. If $L_i, \overline{L}_i$ and ω_i be positive constants $\forall\, i = 1, 2, 3, 4, 5$ for the system (12) for uniqueness and existence, Eqs. (13) and (14) are satisfied, respectively.

Considering system (1)–(5) yields:

$$\begin{cases} S_c' = F_1\left(S_c, E_c, I_c, I_R, R, t\right) \\ E_c' = F_2\left(S_c, E_c, I_c, I_R, R, t\right) \\ I_c' = F_3\left(S_c, E_c, I_c, I_R, R, t\right), t \geq 0 \\ I_R' = F_4\left(S_c, E_c, I_c, I_R, R, t\right) \\ R' = F_5\left(S_c, E_c, I_c, I_R, R, t\right) \end{cases} \tag{12}$$

$\forall\, i = 1, 2, 3, 4, 5$ then

$$|F_i(S,t)|^2 < L_i\left(|S_i|^2 + 1\right). \tag{13}$$

$$\left|F_i(S^1,t)-F_i(S^2,t)\right|<\overline{L}_i\left|S^1-S^2\right|. \tag{14}$$

The uniqueness of Eq. (12) can be proved by satisfying Eq. (13) as follows:

$$\left|F_1\left(S_c,E_c,I_c,I_R,R,t\right)\right|^2=\left|b_h-\frac{\beta_1 S_c I_C}{1+m_0 I_C}e^{-\mu T}-\frac{\beta_2 S_c I_R}{1+m_1 I_R}-\mu S_c+\sigma R\right|^2$$

$$\left|F_1\left(S_c,E_c,I_c,I_R,R,t\right)\right|^2\le\left(\left|b_h\right|^2+\frac{\left|\beta_1\right|^2\left\|S_c\right\|_\infty\left\|I_c\right\|_\infty e^{-2\mu T}}{(1+m_0\left\|I_c\right\|_\infty)^2}+\frac{\left|\beta_2\right|^2\left\|S_c\right\|_\infty\left\|I_R\right\|_\infty}{(1+m_1\left\|I_R\right\|_\infty)^2}+\left|\mu\right|^2\left\|S_c\right\|_\infty+\left|\sigma\right|^2\left\|R\right\|_\infty\right)$$

$$\left|F_1\left(S_c,E_c,I_c,I_R,R,t\right)\right|^2\le\left(\frac{\left|\beta_1\right|^2\left\|S_c\right\|_\infty\left\|I_c\right\|_\infty e^{-2\mu T}}{(1+m_0\left\|I_c\right\|_\infty)^2}+\frac{\left|\beta_2\right|^2\left\|S_c\right\|_\infty\left\|I_R\right\|_\infty}{(1+m_1\left\|I_R\right\|_\infty)^2}+\left|\mu\right|^2\left\|S_c\right\|_\infty\right)$$
$$\left(1+\frac{\left|b_h\right|^2+\left|\sigma\right|^2\left\|R\right\|_\infty}{\left(\frac{\left|\beta_1\right|^2\left\|S_c\right\|_\infty\left\|I_c\right\|_\infty e^{-2\mu T}}{(1+m_0\left\|I_c\right\|_\infty)^2}+\frac{\left|\beta_2\right|^2\left\|S_c\right\|_\infty\left\|I_R\right\|_\infty}{(1+m_1\left\|I_R\right\|_\infty)^2}+\left|\mu\right|^2\left\|S_c\right\|_\infty\right)}\right)$$

The condition $\dfrac{\left|b_h\right|^2+\left|\sigma\right|^2\left\|R\right\|_\infty}{\left(\frac{\left|\beta_1\right|^2\left\|S_c\right\|_\infty\left\|I_c\right\|_\infty e^{-2\mu T}}{(1+m_0\left\|I_c\right\|_\infty)^2}+\frac{\left|\beta_2\right|^2\left\|S_c\right\|_\infty\left\|I_R\right\|_\infty}{(1+m_1\left\|I_R\right\|_\infty)^2}+\left|\mu\right|^2\left\|S_c\right\|_\infty\right)}<1,$

$$\left|F_1\left(S_c,E_c,I_c,I_R,R,t\right)\right|^2<L_1\left(1+\left|S_c\right|^2\right) \tag{15}$$

Similar,

$$\left|F_2\left(S_c,E_c,I_c,I_R,R,t\right)\right|^2=\left|\frac{\beta_1 S_c I_c e^{-\mu T}}{1+m_0 I_c}+\frac{\beta_2 S_c I_R}{1+m_1 I_R}-(\tau+\mu)E_c\right|^2$$

$$\left|F_2\left(S_c,E_c,I_c,I_R,R,t\right)\right|^2\le\left(\left|(\tau+\mu)\right|^2 E_{c_\infty}\right)\left(1+\frac{\frac{\left|\beta_1\right|^2\left\|S_c\right\|_\infty\left\|I_c\right\|_\infty e^{-2\mu T}}{(1+m_0\left\|I_c\right\|_\infty)^2}+\frac{\left|\beta_2\right|^2\left\|S_c\right\|_\infty\left\|I_R\right\|_\infty}{(1+m_1\left\|I_R\right\|_\infty)^2}}{\left(\left|(\tau+\mu)\right|^2\left\|E_c\right\|_\infty\right)}\right)$$

The condition $\dfrac{\frac{\left|\beta_1\right|^2\left\|S_c\right\|_\infty\left\|I_c\right\|_\infty e^{-2\mu T}}{(1+m_0\left\|I_c\right\|_\infty)^2}+\frac{\left|\beta_2\right|^2\left\|S_c\right\|_\infty\left\|I_R\right\|_\infty}{(1+m_1\left\|I_R\right\|_\infty)^2}}{\left(\left|(\tau+\mu)\right|^2\left\|E_c\right\|_\infty\right)}<1,$

$$\left|F_2\left(S_c,E_c,I_c,I_R,R,t\right)\right|^2<L_2\left(1+\left|E_c\right|^2\right) \tag{16}$$

$$\left|F_3\left(S_c,E_c,I_c,I_R,R,t\right)\right|^2=\left|\tau E_c-(\gamma+\mu+\xi+\delta)I_C\right|^2$$

$$\left|F_3\left(S_c,E_c,I_c,I_R,R,t\right)\right|^2\le\left(\left|(\gamma+\mu+\xi+\delta)\right|^2\left\|I_c\right\|_\infty\right)\left(1+\frac{\left|\tau\right|^2\left\|E_c\right\|_\infty}{\left|(\gamma+\mu+\xi+\delta)\right|^2\left\|I_c\right\|_\infty}\right)$$

The condition $\frac{|\tau|^2\|E_c\|_\infty}{|(\gamma+\mu+\xi+\delta)|^2\|I_c\|_\infty} < 1$,

$$|F_3(S_c, E_c, I_c, I_R, R, t)|^2 < L_3\left(1 + |I_c|^2\right) \tag{17}$$

$$|F_4(S_c, E_c, I_c, I_R, R, t)|^2 = |\xi I_c - \eta I_R|^2$$

$$|F_4(S_c, E_c, I_c, I_R, R, t)|^2 \le \left(|\xi|^2\|I_c\|_\infty + |\eta|^2\|I_R\|_\infty\right)$$

$$|F_4(S_c, E_c, I_c, I_R, R, t)|^2 \le |\eta|^2 I_{R_\infty}\left(1 + \frac{|\xi|^2\|I_c\|_\infty}{|\eta|^2\|I_R\|_\infty}\right)$$

The condition $\frac{|\xi|^2\|I_c\|_\infty}{|\eta|^2\|I_R\|_\infty} < 1$,

$$|F_4(S_c, E_c, I_c, I_R, R, t)|^2 < L_4\left(1 + |I_R|^2\right) \tag{18}$$

$$|F_5(S_c, E_c, I_c, I_R, R, t)|^2 = |\gamma I_C - (\mu+\sigma)R|^2$$

$$|F_5(S_c, E_c, I_c, I_R, R, t)|^2 \le \left(|\gamma|^2\|I_c\|_\infty + |(\mu+\sigma)|^2\|R\|_\infty\right)$$

$$|F_5(S_c, E_c, I_c, I_R, R, t)|^2 \le |(\mu+\sigma)|^2 R_\infty\left(1 + \frac{|\gamma|^2\|I_c\|_\infty}{|(\mu+\sigma)|^2\|R\|_\infty}\right)$$

The condition $\frac{|\gamma|^2\|I_c\|_\infty}{|(\mu+\sigma)|^2\|R\|_\infty} < 1$,

$$|F_5(S_c, E_c, I_c, I_R, R, t)|^2 < L_5\left(1 + |R|^2\right) \tag{19}$$

$$\max\left\{\begin{array}{c}\dfrac{|b_h|^2 + |\sigma|^2\|R\|_\infty}{\left(\dfrac{|\beta_1|^2\|S_c\|_\infty\|I_c\|_\infty e^{-2\mu T}}{(1+m_0\|I_c\|_\infty)^2} + \dfrac{|\beta_2|^2\|S_c\|_\infty\|I_R\|_\infty}{(1+m_1\|I_R\|_\infty)^2} + |\mu|^2\|S_c\|_\infty\right)}, \dfrac{\dfrac{|\beta_1|^2\|S_c\|_\infty\|I_c\|_\infty e^{-2\mu T}}{\left(1+m_0\|I_c\|_\infty\right)^2} + \dfrac{|\beta_2|^2\|S_c\|_\infty\|I_R\|_\infty}{\left(1+m_1\|I_R\|_\infty\right)^2}}{\left(|(\tau+\mu)|^2\|E_c\|_\infty\right)}, \\ \dfrac{|\tau|^2\|E_c\|_\infty}{|(\gamma+\mu+\xi+\delta)|^2\|I_c\|_\infty}, \dfrac{|\xi|^2\|I_c\|_\infty}{|\eta|^2\|I_R\|_\infty}, \dfrac{|\gamma|^2\|I_c\|_\infty}{|(\mu+\sigma)|^2\|R\|_\infty}\end{array}\right\} < 1$$

Now, Lipschitz's condition is verified for the existence as follows:

$$\left|F_1\left(S_c^1, t\right) - F_1\left(S_c^2, t\right)\right| = \left|\left(b_h - \frac{\beta_1 S_c^1 I_C}{1+m_0 I_C}e^{-\mu T} - \frac{\beta_2 S_c^1 I_R}{1+m_1 I_R} - \mu S_c^1 + \sigma R\right) - \left(b_h - \frac{\beta_1 S_c^2 I_C}{1+m_0 I_C}e^{-\mu T} - \frac{\beta_2 S_c^2 I_R}{1+m_1 I_R} - \mu S_c^2 + \sigma R\right)\right|$$

$$\left|F_1\left(S_c^1, t\right) - F_1\left(S_c^2, t\right)\right| \le \left(\frac{\beta_1\|I_c\|_\infty e^{-\mu T}}{1+m_0\|I_c\|_\infty} + \frac{\beta_2\|I_R\|_\infty}{1+m_1\|I_R\|_\infty} + \mu + \omega_1\right)\left|S_c^1 - S_c^2\right|$$

$$\left|F_1\left(S_c^1,t\right)-F_1\left(S_c^2t\right)\right|\le \overline{L}_1\left|S_c^1-S_c^2\right| \tag{20}$$

$$\left|F_2\left(E_c^1,t\right)-F_2\left(E_c^2,t\right)\right|=\left|\left(\frac{\beta_1 S_c I_c e^{-\mu T}}{1+m_0 I_c}+\frac{\beta_2 S_c I_R}{1+m_1 I_R}-(\tau+\mu)E_c^1\right)-\left(\frac{\beta_1 S_c I_c e^{-\mu T}}{1+m_0 I_c}+\frac{\beta_2 S_c I_R}{1+m_1 I_R}-(\tau+\mu)E_c^2\right)\right|$$

$$\left|F_2\left(E_c^1,t\right)-F_2\left(E_c^2,t\right)\right|\le\left|(\tau+\mu)+\omega_2\right|\left|\left(E_c^1-E_c^2\right)\right|$$

$$\left|F_2\left(E_c^1,t\right)-F_2\left(E_c^2,t\right)\right|\le \overline{L}_2\left|\left(E_c^1-E_c^2\right)\right| \tag{21}$$

$$\left|F_3\left(I_c^1,t\right)-F_3\left(I_c^2,t\right)\right|=\left|\left(\tau E_c-(\gamma+\mu+\xi+\delta)I_c^1\right)-\left(\tau E_c-(\gamma+\mu+\xi+\delta)I_c^2\right)\right|$$

$$\left|F_3\left(I_c^1,t\right)-F_3\left(I_c^2,t\right)\right|\le\left|(\gamma+\mu+\xi+\delta)+\omega_3\right|\left|\left(I_c^1-I_c^2\right)\right|$$

$$\left|F_3\left(I_c^1,t\right)-F_3\left(I_c^2,t\right)\right|\le \overline{L}_3\left|\left(I_c^1-I_c^2\right)\right| \tag{22}$$

$$\left|F_4\left(I_R^1,t\right)-F_4\left(I_R^2,t\right)\right|=\left|\left(\xi I_c-\eta I_R^1\right)-\left(\xi I_c-\eta I_R^2\right)\right|$$

$$\left|F_4\left(I_R^1,t\right)-F_4\left(I_R^2,t\right)\right|\le\left|\eta+\omega_4\right|\left|\left(I_R^1-I_R^2\right)\right|$$

$$\left|F_4\left(I_R^1,t\right)-F_4\left(I_R^2,t\right)\right|\le \overline{L}_4\left|\left(I_R^1-I_R^2\right)\right| \tag{23}$$

$$\left|F_5\left(R^1,t\right)-F_5\left(R^2,t\right)\right|=\left|\left(\gamma I_C-(\mu+\sigma)R^1\right)-\left(\gamma I_C-(\mu+\sigma)R^2\right)\right|$$

$$\left|F_5\left(R^1,t\right)-F_5\left(R^2,t\right)\right|\le\left|(\mu+\sigma)+\omega_5\right|\left|\left(R^1-R^2\right)\right|$$

$$\left|F_5\left(R^1,t\right)-F_5\left(R^2,t\right)\right|\le \overline{L}_5\left|\left(R^1-R^2\right)\right| \tag{24}$$

Eqs. (15)–(19) indicate that systems (1)–(5) have unique solutions and Eqs. (20)–(24) have a sure existence. □

2.2 Model Equilibrium

To obtain the condition where conjunctivitis dies out from the system (1)–(5) or becomes endemic.

$$\text{Conjunctivitis free equilibrium (CFE)} = C_1 = \left(S_c^1, E_c^1, I_c^1, I_R^1, R^1\right) = \left(\frac{b_h}{\mu}, 0, 0, 0, 0\right) \tag{25}$$

$$\text{Conjunctivitis present equilibrium (CPE)} = C_2 = \left(S_c^*, E_c^*, I_c^*, I_R^*, R^*\right) \tag{26}$$

$$S_c^* = \frac{\left(b_h+\sigma R^*\right)\left(1+m_0 I_c^*\right)\left(1+m_1 I_R^*\right)}{\beta_1 I_c^* e^{-\mu T}\left(1+m_1 I_R^*\right)+\beta_2 I_R^*\left(1+m_0 I_c^*\right)+\mu\left(1+m_0 I_c^*\right)\left(1+m_1 I_R^*\right)}$$

$$E_c^* = \frac{\beta_1 S_c^* I_c^* e^{-\mu T}\left(1 + m_1 I_R^*\right) + \beta_2 S_c^* I_R^*\left(1 + m_0 I_c^*\right)}{\left(1 + m_0 I_c^*\right)\left(1 + m_1 I_R^*\right)(\tau + \mu)}$$

$$I_c^* = \frac{R_0 \mu\,(\tau + \mu)\, E_c^*}{\beta_1 b_h e^{-\mu T}}, I_R^* = \frac{\xi I_c^*}{\eta}, R^* = \frac{\gamma I_c^*}{(\mu + \sigma)}$$

2.3 Reproduction Number

To determine the system's reproductive number [22] of (1)–(5) using the following generation matrix method.

Theorem 4: *The reproductive number of the system (1)–(5) is given by*

$$R_0 = \frac{\tau \beta_1 b_h e^{-\mu T}}{\mu\,(\tau + \mu)\,(\gamma + \mu + \xi + \delta)} \tag{27}$$

Proof: By the abovementioned method, three concerned compartments are considered, E_c, I_c, and R, under the process hypothesis and overlooked both remaining.

Now,

$$\begin{bmatrix} E_c' \\ I_c' \\ R' \end{bmatrix} = \begin{bmatrix} 0 & \frac{\beta_1 b_h e^{-\mu T}}{\mu} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_c \\ I_c \\ \mathrm{R} \end{bmatrix} - \begin{bmatrix} \tau + \mu & 0 & 0 \\ -\tau & \gamma + \mu + \xi + \delta & 0 \\ 0 & -\gamma & \mu + \sigma \end{bmatrix} \begin{bmatrix} E_c \\ I_c \\ \mathrm{R} \end{bmatrix}$$

here, $F_c = \begin{bmatrix} 0 & \frac{\beta_1 b_h e^{-\mu T}}{\mu} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and $V_c = \begin{bmatrix} \tau + \mu & 0 & 0 \\ -\tau & \gamma + \mu + \xi + \delta & 0 \\ 0 & -\gamma & \mu + \sigma \end{bmatrix}$,

where F_c = Transmission matrix at C_1, V_c = Transition matrix at C_1.

Hence, by the largest eigenvalue of $F_c {V_c}^{-1}$,

$$R_0 = \rho\left(F_c {V_c}^{-1}\right) = \frac{\tau \beta_1 b_h e^{-\mu T}}{\mu\,(\tau + \mu)\,(\gamma + \mu + \xi + \delta)}. \square$$

2.4 Stability Analysis

Now, the asymptotic behaviour of convergence for the system (1)–(5) is investigated locally and globally around defined equilibrium states with the constraints of the reproductive number.

Theorem 5: (*Local stability at C_1*) *The system (1)–(5) around the Conjunctivitis free equilibrium point is locally asymptotically stable if the reproductive number is less than one.*

Proof: To prove this result, the Jacobian matrix of the right-side system (1)–(5) at $C_1 = \left(\frac{b_h}{\mu}, 0, 0, 0, 0\right)$ is given by:

$$J(C_1) = \begin{bmatrix} -\mu & 0 & -\frac{\beta_1 b_h e^{-\mu T}}{\mu} & -\frac{\beta_2 b_h}{\mu} & \sigma \\ 0 & -(\tau+\mu) & \frac{\beta_1 b_h e^{-\mu T}}{\mu} & \frac{\beta_2 b_h}{\mu} & 0 \\ 0 & \tau & -(\gamma+\mu+\xi+\delta) & 0 & 0 \\ 0 & 0 & \xi & -\eta & 0 \\ 0 & 0 & \gamma & 0 & -(\mu+\sigma) \end{bmatrix}$$

For eigenvalue, $|J - \lambda I| = 0$,

$$\begin{vmatrix} -\mu-\lambda & 0 & -\frac{\beta_1 b_h e^{-\mu T}}{\mu} & -\frac{\beta_2 b_h}{\mu} & \sigma \\ 0 & -(\tau+\mu)-\lambda & \frac{\beta_1 b_h e^{-\mu T}}{\mu} & \frac{\beta_2 b_h}{\mu} & 0 \\ 0 & \tau & -(\gamma+\mu+\xi+\delta)-\lambda & 0 & 0 \\ 0 & 0 & \xi & -\eta-\lambda & 0 \\ 0 & 0 & \gamma & 0 & -(\mu+\sigma)-\lambda \end{vmatrix} = 0 \tag{28}$$

By expanding (28), $\lambda_1 = -\mu$ and $\lambda_2 = -(\mu+\sigma)$. The characteristic equation evaluates the remaining eigenvalues of (28):

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0 \tag{29}$$

where $A_1 = -(k_1+k_2+k_3)$, $A_2 = k_1k_3 + k_2k_3 + k_1k_2(1-R_0)$, $A_3 = -k_1k_2k_3(1-R_0) - \tau\xi k_1$ and $k_1 = -(\tau+\mu)$, $k_2 = -(\gamma+\mu+\xi+\delta)$, $k_3 = -\eta$, $k_4 = \frac{\beta_1 b_h e^{-\mu T}}{\mu}$, $k_5 = \frac{\beta_2 b_h}{\mu}$. Since $A_1A_2 > A_3$ and all $A_i > 0$ for $i = 1,2,3$ iff $R_0 < 1$. Then, Routh-Hurwitz criteria for the 3rd-order polynomial indicate that all the eigenvalues of (29) have negative real parts. Hence, the system (1)–(5) at the Conjunctivitis-free equilibrium point C_1 is locally asymptotically stable if $R_0 < 1$. □

Theorem 6: *(Local stability at C_2) The system (1)–(5) around the Conjunctivitis present equilibrium point is locally asymptotically stable if the reproductive number exceeds one.*

Proof: To prove this result, the Jacobian matrix of system (1)–(5) at $C_2 = (S_c^*, E_c^*, I_c^*, I_R^*, R^*)$ is given by:

$$J(S_c^*, E_c^*, I_c^*, I_R^*, R^*) = \begin{bmatrix} -\frac{\beta_1 I_c^* e^{-\mu T}}{1+m_0 I_c^*} - \frac{\beta_2 I_R^*}{1+m_1 I_R^*} - \mu & 0 & -\frac{\beta_1 S_c^* e^{-\mu T}}{(1+m_0 I_c^*)^2} & -\frac{\beta_2 S_c^*}{(1+m_1 I_R^*)^2} & \sigma \\ \frac{\beta_1 S_c^* e^{-\mu T}}{(1+m_0 I_c^*)^2} + \frac{\beta_2 S_c^*}{(1+m_1 I_R^*)^2} & -(\tau+\mu) & \frac{\beta_1 S_c^* e^{-\mu T}}{(1+m_0 I_c^*)^2} & \frac{\beta_2 S_c^*}{(1+m_1 I_R^*)^2} & 0 \\ 0 & \tau & -(\gamma+\mu+\xi+\delta) & 0 & 0 \\ 0 & 0 & \xi & -\eta & 0 \\ 0 & 0 & \gamma & 0 & -(\mu+\sigma) \end{bmatrix} \tag{30}$$

Routh-Hurwitz criteria for the 5th-order polynomial exhibit that all the eigenvalues of (30) through MATLAB have negative real parts. Hence, the system (1)–(5) at the conjunctivitis-present equilibrium point C_2 is locally asymptotically stable if $R_0 > 1$. □

Theorem 7: *(Global stability at C_1) The system (1)–(5) around the Conjunctivitis free equilibrium point is globally asymptotically stable if the reproductive number is less than one.*

Proof: Define Lyapunov function V: $\Omega \to \mathbb{R}$,

$$V(t) = \log\left(\frac{I_c}{I_0}\right)$$

$$\frac{dV}{dt} = \frac{1}{I_c} \times I_c'$$

$$\frac{dV}{dt} = \frac{1}{I_c} \times \left(\tau E_c - (\gamma + \mu + \xi + \delta) I_C\right)$$

$$\frac{dV}{dt} \leq (\gamma + \mu + \xi + \delta)\left(\frac{\tau \beta_1 b_h e^{-\mu T}}{\mu(\tau + \mu)(\gamma + \mu + \xi + \delta)} - 1\right)$$

$$\frac{dV}{dt} \leq (\gamma + \mu + \xi + \delta)(R_0 - 1)$$

$$\frac{dV}{dt} < 0$$

if $R_0 < 1$.

Hence, system (1)–(5) asymptotically converges around the Conjunctivitis Free State globally if $R_0 < 1$. □

Theorem 8: *(Global stability at C_2) The system (1)–(5) around the Conjunctivitis present equilibrium point is globally asymptotically stable if the reproductive number is greater than one.*

Proof: Describe Lyapunov function V: $\Omega \to \mathbb{R}$,

$$V = \left(S_c - S_c^* - S_c^* \log\frac{S_c}{S_c^*}\right) + \left(E_c - E_c^* - E_c^* \log\frac{E_c}{E_c^*}\right) + \left(I_c - I_c^* - I_c^* \log\frac{I_c}{I_c^*}\right) + \left(I_R - I_R^* - I_R^* \log\frac{I_R}{I_R^*}\right) + \left(R - R^* - R^* \log\frac{R}{R^*}\right)$$

$$\frac{dV}{dt} = \left[\left(1 - \frac{S_c^*}{S_c}\right)\frac{dS_c}{dt}\right] + \left[\left(1 - \frac{E_c^*}{E_c}\right)\frac{dE_c}{dt}\right] + \left[\left(1 - \frac{I_c^*}{I_c}\right)\frac{dI_c}{dt}\right] + \left[\left(1 - \frac{I_R^*}{I_R}\right)\frac{dI_R}{dt}\right] + \left[\left(1 - \frac{R^*}{R}\right)\frac{dR}{dt}\right]$$

$$\frac{dV}{dt} = \left[(S_c - S_c^*)\left(\frac{b_h}{S_c} - \frac{\beta_1 I_C e^{-\mu T}}{1 + m_0 I_C} - \frac{\beta_2 I_R}{1 + m_1 I_R} - \mu + \frac{\sigma R}{S_c}\right)\right] + \left[(E_c - E_c^*)\left(\frac{\beta_1 S_c I_c e^{-\mu T}}{E_c(1 + m_0 I_c)} + \frac{\beta_2 S_c I_R}{E_c(1 + m_1 I_R)} - (\tau + \mu)\right)\right] + \left[(I_c - I_c^*)\left(\frac{\tau E_c}{I_c} - (\gamma + \mu + \xi + \delta)\right)\right] + \left[(I_R - I_R^*)\left(\frac{\xi I_c}{I_R} - \eta\right)\right] + \left[(R - R^*)\left(\frac{\gamma I_C}{R} - (\mu + \sigma)\right)\right]$$

Using the right side of the system (1)–(5):

$$\mu = \frac{b_h}{S_c^*} - \frac{\beta_1 I_c e^{-\mu T}}{1 + m_0 I_c} - \frac{\beta_2 I_R}{1 + m_1 I_R} + \frac{\sigma R}{S_c^*}$$

$$(\tau+\mu)=\frac{\beta_1 S_c I_c e^{-\mu T}}{{E_c}^*(1+m_0 I_c)}+\frac{\beta_2 S_c I_R}{{E_c}^*(1+m_1 I_R)}$$

$$(\gamma+\mu+\xi+\delta)=\frac{\tau E_c}{{I_c}^*}$$

$$\eta=\frac{\xi I_c}{{I_R}^*}$$

$$(\mu+\sigma)=\frac{\gamma I_c}{R^*}$$

$$\begin{aligned}\Rightarrow \frac{dV}{dt}&=\left[(S_c-{S_c}^*)\left(\frac{b_h}{S_c}+\frac{\sigma R}{S_c}-\left(\frac{b_h}{{S_c}^*}+\frac{\sigma R}{{S_c}^*}\right)\right)\right]\\&+\left[(E_c-{E_c}^*)\left(\frac{\beta_1 S_c I_c e^{-\mu T}}{E_c(1+m_0 I_c)}+\frac{\beta_2 S_c I_R}{E_c(1+m_1 I_R)}-\left(\frac{\beta_1 S_c I_c e^{-\mu T}}{{E_c}^*(1+m_0 I_c)}+\frac{\beta_2 S_c I_R}{{E_c}^*(1+m_1 I_R)}\right)\right)\right]\\&+\left[(I_c-{I_c}^*)\left(\frac{\tau E_c}{I_c}-\left(\frac{\tau E_c}{{I_c}^*}\right)\right)\right]+\left[(I_R-{I_R}^*)\left(\frac{\xi I_c}{I_R}-\left(\frac{\xi I_c}{{I_R}^*}\right)\right)\right]+\left[(R-R^*)\left(\frac{\gamma I_c}{R}-\left(\frac{\gamma I_c}{R^*}\right)\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{dV}{dt}&=\left[(S_c-S_c^*)\left(-\frac{b_h}{S_c^* S_c}(S_c-S_c^*)-\frac{\sigma R}{S_c^* S_c}(S_c-S_c^*)\right)\right]\\&+\left[(E_c-E_c^*)\left(-\frac{\beta_1 S_c I_c e^{-\mu T}}{E_c^* E_c(1+m_0 I_c)}(E_c-E_c^*)-\frac{\beta_2 S_c I_R}{E_c^* E_c(1+m_1 I_R)}(E_c-E_c^*)\right)\right]\\&+\left[(I_c-I_c^*)\left(-\frac{\tau E_c}{I_c^* I_c}(I_c-I_c^*)\right)\right]+\left[(I_R-I_R^*)\left(-\frac{\xi I_c}{I_R^* I_R}(I_R-I_R^*)\right)\right]\\&+\left[(R-R^*)\left(-\frac{\gamma I_C}{R^* R}(R-R^*)\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{dV}{dt}&=-\frac{b_h}{S_c^* S_c}(S_c-S_c^*)^2-\frac{\sigma R}{S_c^* S_c}(S_c-S_c^*)^2-\frac{\beta_1 S_c I_c e^{-\mu T}}{E_c^* E_c(1+m_0 I_c)}(E_c-E_c^*)^2\\&-\frac{\beta_2 S_c I_R}{E_c^* E_c(1+m_1 I_R)}(E_c-E_c^*)^2-\frac{\tau E_c}{I_c^* I_c}(I_c-I_c^*)^2-\frac{\xi I_c}{I_R^* I_R}(I_R-I_R^*)^2-\frac{\gamma I_c}{R^* R}(R-R^*)^2\end{aligned}$$

$$\frac{dV}{dt}\le 0$$

Hence, by LaSalle's invariance principle, at C_2 system is globally asymptotically stable in the feasible region Ω. □

3 Stochastic Delayed Conjunctivitis Model

Now, the stochastic behavior of the system (1)–(5) is derived by the non-parametric perturbation method for more realistic optimal outcomes. So, by non-parametric perturbation technique [23], the system (1)–(5) becomes:

$$\begin{cases} dS_c = \left[b_h - \dfrac{\beta_1 S_c I_C}{1+m_0 I_C} e^{-\mu T} - \dfrac{\beta_2 S_c I_R}{1+m_1 I_R} - \mu S_c + \sigma R \right] dt + \sigma_1 S_c dB(t) \\ dE_c = \left[\dfrac{\beta_1 S_c I_c}{1+m_0 I_c} e^{-\mu T} + \dfrac{\beta_2 S_c I_R}{1+m_1 I_R} - (\tau+\mu) E_c \right] dt + \sigma_2 E_c dB(t) \\ I_c = \left[\tau E_c - (\gamma+\mu+\xi+\delta) I_C \right] dt + \sigma_3 I_c dB(t) \\ dI_R = \left[\xi I_c - \eta I_R \right] dt + \sigma_4 I_R dB\,(t) \\ dR = \left[\gamma I_C - (\mu+\sigma) R \right] dt + \sigma_5 R dB(t) \end{cases} \tag{31}$$

Given that $S_C(0) \geq 0$, $E_C(0) \geq 0$, $I_C(0) \geq 0$, $I_R(0) \geq 0$ and $R(0) \geq 0$, for all $t > 0$ are the initial conditions, $T < t$ where T is the delay effect, $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ are the stochastic terms for each subdivision of the system, and $B(t)$ is the Brownian motion. The system (31) is non-integrable because of Brownian motion $B(t)$.

Unique Global Positivity

To prove the system (31) of SDDEs, $U(t) = (S_c(t), E_c(t), I_c(t), I_R(t), R(t))^t$ not explode to infinity in a finite time. For any given initial condition, the coefficient should satisfy both the local Lipschitz and linear growth conditions.

The coefficient of the system (31) is locally Lipschitz (Theorem 5.2.8, [24]), but it has a linear growth property. Thus, it can explode to infinity at a finite time. To prove a unique global solution, it is enough to prove $\tau_e = \infty$ almost sure.

Theorem 9: *For system (31) any given initial value* $(S_c(0), E_c(0), I_c(0), I_R(0), R(0)) \in R_+^5$, *there is a unique solution* $(S_c(t), E_c(t), I_c(t), I_R(t), R(t))$ *on* $t \geq 0$ *and will remain in* R_+^5 *almost sure.*

Proof: Since the coefficient of the system (31) is locally Lipschitz, a unique maximal local solution exists for any given initial condition.

Suppose $p_0 > 0$ be suitably outsized that any given initial condition lying within the interval $\left[\frac{1}{p_0}, p_0\right]$. Now, for each integer $p \geq p_0$, a sequence of local stopping times is defined as follows:

$$\tau_p = \inf \left\{ t \in [0, \tau_e] : \frac{1}{p} \geq U_i(t) \geq p, \text{for some } i = 1, 2, \ldots, 5 \right\}$$

$$\tau_\infty = \lim_{p \to \infty} \tau_p = \inf \left\{ t \in [0, \tau_e] : 0 \geq U_i(t) \geq \infty, \text{for some } i = 1, 2, \ldots, 5 \right\} \tag{32}$$

$inf(\{\}) = \infty$ ($\{\}$ is the empty set). τ_p is an increasing sequence as $p \to \infty$. Then, $\tau_\infty \leq \tau_e$ *a.s.* τ_e is explosion time. To complete the proof, it is enough to show $\tau_\infty = \infty$ almost sure.

Prove this result by contradiction, then there exists $J > 0$ and $\vartheta \in (0,1)$ such that:

$$P\{\tau_\infty \leq J\} > \vartheta \tag{33}$$

This, there is an integer $p_1 > p_0$ such that:

$$P\left\{\tau_p \leq J\right\} \geq \vartheta \forall p \geq p_1 \tag{34}$$

For any K such that $K - 1 - \log K > 0$, a positive definite function can be defined as follows:

C^5- function $V: R_+^5 \to R_+$ can be defined by

$$V = (S_c - 1 - \log S_c) + (E_c - 1 - \log E_c) + (I_c - 1 - \log I_c) + (I_R - 1 - \log I_R) + (R - 1 - \log R)$$

The following can be calculated using Ito's formula:

$$dV = \left(1-\frac{1}{S_c}\right)dS_c + \left(1-\frac{1}{E_c}\right)dE_c + \left(1-\frac{1}{I_c}\right)dI_c + \left(1-\frac{1}{I_R}\right)dI_R + \left(1-\frac{1}{R}\right)dR + \frac{5}{2}\sigma^2 dt$$

$$\begin{aligned}
\Rightarrow dV = {} & \left(1-\frac{1}{S_c}\right)\left[\left(b_h - \frac{\beta_1 S_c I_C}{1+m_0 I_C}e^{-\mu T} - \frac{\beta_2 S_c I_R}{1+m_1 I_R} - \mu S_c + \sigma R\right)dt + \sigma_1 S_c dB(t)\right] \\
& + \left(1-\frac{1}{E_c}\right)\left[\left(\frac{\beta_1 S_c I_c}{1+m_0 I_c}e^{-\mu T} + \frac{\beta_2 S_c I_R}{1+m_1 I_R} - (\tau+\mu)E_c\right)dt + \sigma_2 E_c dB(t)\right] \\
& + \left(1-\frac{1}{I_c}\right)\left[(\tau E_c - (\gamma+\mu+\xi+\delta)I_C)dt + \sigma_3 I_c dB(t)\right] \\
& + \left(1-\frac{1}{I_R}\right)\left[(\xi I_c - \eta I_R)dt + \sigma_4 I_R dB(t)\right] \\
& + \left(1-\frac{1}{R}\right)\left[(\gamma I_C - (\mu+\sigma)R)dt + \sigma_5 R dB(t)\right] + \frac{5}{2}\sigma^2 dt \\
\Longrightarrow dV \le {} & \left(b_h + 4\mu + \tau + \gamma + \xi + \delta + \eta + \sigma + \frac{5}{2}\sigma^2\right)dt + \sigma_1 S_c dB(t) + \sigma_2 E_c dB(t) + \sigma_3 I_c dB(t) \\
& + \sigma_4 I_R dB(t) + \sigma_5 R dB(t)
\end{aligned}$$

If $P = b_h + 4\mu + \tau + \gamma + \xi + \delta + \eta + \sigma + 5/2\sigma^2$, then the equation can be written as follows:

$$dV \le Pdt + \sigma_1 S_c dB(t) + \sigma_2 E_c dB(t) + \sigma_3 I_c dB(t) + \sigma_4 I_R dB(t) + \sigma_5 R dB(t)$$

After integrating from 0 to $\tau_p \wedge j$, where p is a positive constant, yields

$$\begin{aligned}
& \int_0^{\tau_p \wedge j} dV(S_c(s), E_c(s), I_c(s), I_R(s), R(s)) \\
& \le \int_0^{\tau_p \wedge j} P ds \\
& + \int_0^{\tau_p \wedge j} (\sigma_1 S_c dB(s) + \sigma_2 E_c dB(s) + \sigma_3 I_c dB(s) + \sigma_4 I_R dB(s) + \sigma_5 R dB(s))
\end{aligned} \tag{35}$$

Using the fact $\mathrm{E}\left(\int_0^t dB(s)\right) = 0$, where $\tau_p \wedge j = \min(\tau_p, \mathrm{J})$, integrating Eq. (35) and taking the expectations, yields

$$EV\left(S_c(\tau_p \wedge j), E_c(\tau_p \wedge j), I_c(\tau_p \wedge j), I_R(\tau_p \wedge j), R(\tau_p \wedge j)\right) - V(S_c(0), E_c(0), I_c(0), I_R(0), R(0)) \le PT$$
$$EV\left(S_c(\tau_p \wedge j), E_c(\tau_p \wedge j), I_c(\tau_p \wedge j), I_R(\tau_p \wedge j), R(\tau_p \wedge j)\right) \le V(S_c(0), E_c(0), I_c(0), I_R(0), R(0)) + PT$$

Set $\upsilon_p = \{\tau_p \le J\}$ for $p > p_1$, and from (34), then $P(\upsilon_p) \ge \vartheta$, for every $h \in \upsilon_p$ there are some i such that $u_i(\upsilon_p, h)$ equals either p or $\frac{1}{p}$ for $i = 1, 2, 3, 4, 5$.

Hence, $V\left(S_c(\tau_p \wedge j), E_c(\tau_p \wedge j), I_c(\tau_p \wedge j), I_R(\tau_p \wedge j), R(\tau_p \wedge j)\right)$ is less than $\min\left\{p - 1 - \ln p, \frac{1}{p} - 1 - \ln\frac{1}{p}\right\}$,

$$\begin{aligned}
& \Rightarrow V(S_c(0), E_c(0), I_c(0), I_R(0), R(0)) + PT \ge E\left(I_{\upsilon_p(h)} V(S_c(\tau_p), E_c(\tau_p), I_c(\tau_p), I_R(\tau_p), R(\tau_p))\right) \\
& \ge \left\{\min\left(p - 1 - \ln p, \frac{1}{p} - 1 - \ln\frac{1}{p}\right)\right\}
\end{aligned} \tag{36}$$

The indicator function is represented by $I_{\upsilon_p(h)}$ of υ_p. As, $m \to \infty$

$$V\left(S_c(0), E_c(0), I_c(0), I_R(0), R(0)\right) + PT = \infty$$

It tends to be a contradiction because

$$V\left(S_c(0), E_c(0), I_c(0), I_R(0), R(0)\right) + PT < \infty$$

here, $\tau_\infty = inf\left(\{\}\right) = \infty$. So, $\tau_\infty = \infty$almost sure. Therefore, the system (31) will explode to infinity in infinite time. □

4 Numerical Procedure

This section presents the numerical competitive analysis for the stochastic delayed conjunctivitis model (31) among three standard and one nonstandard method. In which for all $n \in N$ and $H > 0$ uniform partition of the interval $[0, H]$ of N sub-intervals, let us define step size $h = \frac{H-0}{N}$ and $t_n = nh$ for all $n \in I_N$, where $I_N = \{0, 1, 2, \ldots, N\}$. Initial condition $S_c(0) = 0.2, E_c(0) = 0.4, I_c(0) = 0.1, I_R(0) = 0.1, R(0) = 0.2$.

4.1 Transition Probabilities of the Model

If $U(t) = (S_c, E_c, I_c, I_R, R)^{\mathrm{t}}$, and the number of chances of an event is listed in Table 2:

Table 2: Transition probabilities of the model

T_i = Transition	P_i = Probabilities
$T_1 = \begin{bmatrix}1 & 0 & 00 & 0\end{bmatrix}^{\mathrm{t}}$	$P_1 = b_h \Delta t$
$T_2 = \begin{bmatrix}-1 & 1 & 00 & 0\end{bmatrix}^{t}$	$P_2 = \left(\frac{\beta_1 S_c I_C e^{-\mu T}}{1+m_0 I_C} + \frac{\beta_2 S_c I_R}{1+m_1 I_R}\right)\Delta t$
$T_3 = \begin{bmatrix}-1 & 0 & 00 & 0\end{bmatrix}^{\mathrm{t}}$	$P_3 = \mu S_c \Delta t$
$T_4 = \begin{bmatrix}1 & 0 & 00 & -1\end{bmatrix}^{\mathrm{t}}$	$P_4 = \sigma R \Delta t$
$T_5 = \begin{bmatrix}0 & -1 & 10 & 0\end{bmatrix}^{\mathrm{t}}$	$P_5 = \tau E_c \Delta t$
$T_6 = \begin{bmatrix}0 & -1 & 00 & 0\end{bmatrix}^{\mathrm{t}}$	$P_6 = \mu E_c \Delta t$
$T_7 = \begin{bmatrix}0 & 0 & -10 & 1\end{bmatrix}^{\mathrm{t}}$	$P_7 = \gamma I_C \Delta t$
$T_8 = \begin{bmatrix}0 & 0 & -10 & 0\end{bmatrix}^{\mathrm{t}}$	$P_8 = \mu I_C \Delta t$
$T_9 = \begin{bmatrix}0 & 0 & -11 & 0\end{bmatrix}^{\mathrm{t}}$	$P_9 = \xi I_C \Delta t$
$T_{10} = \begin{bmatrix}0 & 0 & -10 & 0\end{bmatrix}^{\mathrm{t}}$	$P_{10} = \delta I_C \Delta t$
$T_{11} = \begin{bmatrix}0 & 0 & 0-1 & 0\end{bmatrix}^{\mathrm{t}}$	$P_{11} = \eta I_R \Delta t$
$T_{12} = \begin{bmatrix}0 & 0 & 00 & -1\end{bmatrix}^{\mathrm{t}}$	$P_{12} = \mu R \Delta t$

The expectation and variance of the system (31) can be described as follows:

$$\text{Expectation} = \mathrm{E}^{*}\left[\Delta \mathrm{U}\right] = \sum_{\mathrm{i}=1}^{12} \mathrm{P_i T_i} = \begin{bmatrix} b_h - \dfrac{\beta_1 S_c I_C e^{-\mu T}}{1+m_0 I_C} - \dfrac{\beta_2 S_c I_R}{1+m_1 I_R} - \mu S_c + \sigma R \\ \dfrac{\beta_1 S_c I_C e^{-\mu T}}{1+m_0 I_C} + \dfrac{\beta_2 S_c I_R}{1+m_1 I_R} - (\tau+\mu) E_c \\ \tau E_c - (\gamma+\mu+\xi+\delta) I_C \\ \xi I_c - \eta I_R \\ \gamma I_C - (\mu+\sigma) R \end{bmatrix} \Delta t$$

$$\text{Var} = \mathrm{E}^{*}\left[\Delta \mathrm{U} \Delta \mathrm{U}^{\mathrm{T}}\right] = \sum_{\mathrm{i}=1}^{12} \mathrm{P_i}[\mathrm{T_i}][\mathrm{T_i}]^{\mathrm{T}}$$

$$= \begin{bmatrix} I_{11} & I_{12} & 0 & 0 & I_{15} \\ I_{21} & I_{22} & I_{23} & 0 & 0 \\ 0 & I_{32} & I_{33} & I_{34} & I_{35} \\ 0 & 0 & I_{43} & I_{44} & 0 \\ I_{51} & 0 & I_{53} & 0 & I_{55} \end{bmatrix} \Delta t$$

$I_{11} = \mathrm{P_1} + \mathrm{P_2} + \mathrm{P_3} + \mathrm{P_4}$, $I_{12} = -\mathrm{P_2}$, $I_{15} = -\mathrm{P_4}$, $I_{21} = -\mathrm{P_2}$, $I_{22} = \mathrm{P_2} + \mathrm{P_5} + \mathrm{P_6}$, $I_{23} = -\mathrm{P_5}$, $I_{32} = -\mathrm{P_5}$, $I_{33} = \mathrm{P_5} + \mathrm{P_7} + \mathrm{P_8} + \mathrm{P_9} + \mathrm{P_{10}}$, $I_{34} = -\mathrm{P_9}$, $I_{35} = -\mathrm{P_7}$, $I_{43} = -\mathrm{P_9}$, $I_{44} = \mathrm{P_9} + \mathrm{P_{11}}$, $I_{51} = -\mathrm{P_4}$, $I_{53} = -\mathrm{P_7}$, $I_{55} = \mathrm{P_4} + \mathrm{P_7} + \mathrm{P_{12}}$.

If $\text{Drift} = \mathrm{H}\left(\mathrm{U}\left(\mathrm{t}\right), \mathrm{t}\right) = \frac{\mathrm{E}^{*}[\Delta \mathrm{U}]}{\Delta \mathrm{t}}$, $\text{Diffusion} = \mathrm{K}\left(\mathrm{U}(\mathrm{t}), \mathrm{t}\right) = \sqrt{\frac{\mathrm{E}^{*}[\Delta \mathrm{U} \Delta \mathrm{U}^{\mathrm{T}}]}{\Delta \mathrm{t}}}$, then

$$\frac{\mathrm{dU}\left(\mathrm{t}\right)}{dt} = \mathrm{H}\left(\mathrm{U}\left(\mathrm{t}\right), \mathrm{t}\right) + \mathrm{K}\left(\mathrm{U}\left(\mathrm{t}\right), \mathrm{t}\right) \frac{\mathrm{dB(t)}}{dt} \tag{37}$$

Eq. (37) is the stochastic differential equation with initial condition $S_c\left(0\right) = 0.2, E_c\left(0\right) = 0.4, I_c\left(0\right) = 0.1, I_R\left(0\right) = 0.3$, and non-integrable $\mathrm{B(t)}$ is the Brownian motion.

4.2 Euler-Maruyama Technique

In this technique, the weak order of convergence with weak order one under sufficient smoothness property [25] is higher than the firm order of convergence with substantial order 0.5 under Lipschitz and bounded growth properties on H and K of equation [26,27].

$$\Delta U = \mathrm{H}\left(\mathrm{U(t)}, \mathrm{t}\right) \Delta t + \mathrm{K}\left(\mathrm{U(t)}, \mathrm{t}\right) \Delta B_n \tag{38}$$

With initial condition $\mathrm{U}\left(t_0\right) = U_0$. The random process ΔB_n is independent of distinct, discrete time intervals with $0 = t_0 < t_1 < t_2 \ldots < t_N = H$ with the standard normally distributed random process, i.e., $\Delta B_n \sim \sqrt{t_{n+1} - t_n} N(0,1)$. Where, $\Delta t = t_{n+1} - t_n$ and $\Delta B_n = B_{n+1} - B_n \; \forall n \in I_N$.

4.3 Stochastic Euler Method

The stochastic Euler method can be formulated from the system (31) with given initial conditions $S_c\left(0\right) = 0.2, E_c\left(0\right) = 0.4, I_c\left(0\right) = 0.1, I_R\left(0\right) = 0.1, R\left(0\right) = 0.2$ as follows:

$$\begin{cases} S_c^{n+1} = S_c^n + h\left[b_h - \dfrac{\beta_1 S_c^n I_c^n e^{-\mu T}}{1+m_0 I_c^n} - \dfrac{\beta_2 S_c^n I_R^n}{1+m_1 I_R^n} - \mu S_c^n + \sigma R^n + \sigma_1 S_c^n \Delta B_n\right] \\ E_c^{n+1} = E_c^n + h\left[\dfrac{\beta_1 S_c^n I_c^n e^{-\mu T}}{1+m_0 I_c^n} + \dfrac{\beta_2 S_c^n I_R^n}{1+m_1 I_R^n} - (\tau+\mu)E_c^n + \sigma_2 E_c^n \Delta B_n\right] \\ I_c^{n+1} = I_c^n + h\left[\tau E_c^n - (\gamma+\mu+\xi+\delta)I_c^n + \sigma_3 I_c^n \Delta B_n\right] \\ I_R^{n+1} = I_R^n + h\left[\xi I_c^n - \eta I_R^n + \sigma_4 I_R^n \Delta B_n\right] \\ R^{n+1} = R^n + h\left[\gamma I_c^n - (\mu+\sigma)R^n + \sigma_5 R^n \Delta B_n\right] \end{cases} \tag{39}$$

where, $n = 0, 1, 2, \ldots, N$ and $\Delta B_n = B_{n+1} - B_n \forall n \in I_N$ is a standard normally distributed random process, i.e., $\Delta B_n \sim \sqrt{t_{n+1} - t_n} N(0,1)$.

4.4 Stochastic Runge-Kutta Method

The stochastic Runge-Kutta method of order four can be derived from the system (31) with given initial condition $S_c(0) = 0.2, E_c(0) = 0.4, I_c(0) = 0.1, I_R(0) = 0.1, R(0) = 0.2$ as follows:

Stage 1

$$\begin{cases} k_1 = h\left[b_h - \dfrac{\beta_1 S_c^n I_c^n e^{-\mu T}}{1+m_0 I_c^n} - \dfrac{\beta_2 S_c^n I_R^n}{1+m_1 I_R^n} - \mu S_c^n + \sigma R^n + \sigma_1 S_c^n \Delta B_n\right] \\ l_1 = h\left[\dfrac{\beta_1 S_c^n I_c^n e^{-\mu T}}{1+m_0 I_c^n} + \dfrac{\beta_2 S_c^n I_R^n}{1+m_1 I_R^n} - (\tau+\mu)E_c^n + \sigma_2 E_c^n \Delta B_n\right] \\ m_1 = h\left[\tau E_c^n - (\gamma+\mu+\xi+\delta)I_c^n + \sigma_3 I_c^n \Delta B_n\right] \\ n_1 = h\left[\xi I_c^n - \eta I_R^n + \sigma_4 I_R^n \Delta B_n\right] \\ p_1 = h\left[\gamma I_c^n - (\mu+\sigma)R^n + \sigma_5 R^n \Delta B_n\right] \end{cases}$$

Stage 2

$$\begin{cases} k_2 = h\left[b_h - \dfrac{\beta_1\left(S_c^n + \frac{k_1}{2}\right)\left(I_c^n + \frac{m_1}{2}\right)e^{-\mu T}}{1+m_0\left(I_c^n + \frac{m_1}{2}\right)} - \dfrac{\beta_2\left(S_c^n + \frac{k_1}{2}\right)\left(I_R^n + \frac{n_1}{2}\right)}{1+m_1\left(I_R^n + \frac{n_1}{2}\right)} - \mu\left(S_c^n + \frac{k_1}{2}\right) + \sigma\left(R^n + \frac{p_1}{2}\right) + \sigma_1\left(S_c^n + \frac{k_1}{2}\right)\Delta B_n\right] \\ l_2 = h\left[\dfrac{\beta_1\left(S_c^n + \frac{k_1}{2}\right)\left(I_c^n + \frac{m_1}{2}\right)e^{-\mu T}}{1+m_0\left(I_c^n + \frac{m_1}{2}\right)} + \dfrac{\beta_2\left(S_c^n + \frac{k_1}{2}\right)\left(I_R^n + \frac{n_1}{2}\right)}{1+m_1\left(I_R^n + \frac{n_1}{2}\right)} - (\tau+\mu)\left(E_c^n + \frac{l_1}{2}\right) + \sigma_2\left(E_c^n + \frac{l_1}{2}\right)\Delta B_n\right] \\ m_2 = h\left[\tau\left(E_c^n + \frac{l_1}{2}\right) - (\gamma+\mu+\xi+\delta)\left(I_c^n + \frac{m_1}{2}\right) + \sigma_3\left(I_c^n + \frac{m_1}{2}\right)\Delta B_n\right] \\ n_2 = h\left[\xi\left(I_c^n + \frac{m_1}{2}\right) - \eta\left(I_R^n + \frac{n_1}{2}\right) + \sigma_4\left(I_R^n + \frac{n_1}{2}\right)\Delta B_n\right] \\ p_2 = h\left[\gamma\left(I_c^n + \frac{m_1}{2}\right) - (\mu+\sigma)\left(R^n + \frac{p_1}{2}\right) + \sigma_5\left(R^n + \frac{p_1}{2}\right)\Delta B_n\right] \end{cases}$$

Stage 3

$$\begin{cases} k_3 = h\left[b_h - \dfrac{\beta_1\left(S_c^n + \frac{k_2}{2}\right)\left(I_c^n + \frac{m_2}{2}\right)e^{-\mu T}}{1 + m_0\left(I_c^n + \frac{m_2}{2}\right)} - \dfrac{\beta_2\left(S_c^n + \frac{k_2}{2}\right)\left(I_R^n + \frac{n_2}{2}\right)}{1 + m_1\left(I_R^n + \frac{n_2}{2}\right)} - \mu\left(S_c^n + \frac{k_2}{2}\right) + \sigma\left(R^n + \frac{p_2}{2}\right) + \sigma_1\left(S_c^n + \frac{k_2}{2}\right)\Delta B_n\right] \\ l_3 = h\left[\dfrac{\beta_1\left(S_c^n + \frac{k_2}{2}\right)\left(I_c^n + \frac{m_2}{2}\right)e^{-\mu T}}{1 + m_0\left(I_c^n + \frac{m_2}{2}\right)} + \dfrac{\beta_2\left(S_c^n + \frac{k_2}{2}\right)\left(I_R^n + \frac{n_2}{2}\right)}{1 + m_1\left(I_R^n + \frac{n_2}{2}\right)} - (\tau + \mu)\left(E_c^n + \frac{l_2}{2}\right) + \sigma_2\left(E_c^n + \frac{l_2}{2}\right)\Delta B_n\right] \\ m_3 = h\left[\tau\left(E_c^n + \frac{l_2}{2}\right) - (\gamma + \mu + \xi + \delta)\left(I_c^n + \frac{m_2}{2}\right) + \sigma_3\left(I_c^n + \frac{m_2}{2}\right)\Delta B_n\right] \\ n_3 = h\left[\xi\left(I_c^n + \frac{m_2}{2}\right) - \eta\left(I_R^n + \frac{n_2}{2}\right) + \sigma_4\left(I_R^n + \frac{n_2}{2}\right)\Delta B_n\right] \\ p_3 = h\left[\gamma\left(I_c^n + \frac{m_2}{2}\right) - (\mu + \sigma)\left(R^n + \frac{p_2}{2}\right) + \sigma_5\left(R^n + \frac{p_2}{2}\right)\Delta B_n\right] \end{cases}$$

Stage 4

$$\begin{cases} k_4 = h\left[b_h - \dfrac{\beta_1\left(S_c^n + k_3\right)\left(I_c^n + m_3\right)e^{-\mu T}}{1 + m_0\left(I_c^n + m_3\right)} - \dfrac{\beta_2(S_c^n + k_3)(I_R^n + n_3)}{1 + m_1\left(I_R^n + n_3\right)} - \mu(S_c^n + k_3) + \sigma(R^n + p_3) + \sigma_1(S_c^n + k_3)\Delta B_n\right] \\ l_4 = h\left[\dfrac{\beta_1(S_c^n + k_3)(I_c^n + m_3)e^{-\mu T}}{1 + m_0(I_c^n + m_3)} + \dfrac{\beta_2(S_c^n + k_3)(I_R^n + n_3)}{1 + m_1(I_R^n + n_3)} - (\tau + \mu)(E_c^n + l_3) + \sigma_2(E_c^n + l_3)\Delta B_n\right] \\ m_4 = h\left[\tau(E_c^n + l_3) - (\gamma + \mu + \xi + \delta)(I_c^n + m_3) + \sigma_3(I_c^n + m_3)\Delta B_n\right] \\ n_4 = h\left[\xi(I_c^n + m_3) - \eta(I_R^n + n_3) + \sigma_4(I_R^n + n_3)\Delta B_n\right] \\ p_4 = h\left[\gamma(I_c^n + m_3) - (\mu + \sigma)\left(R^n + p_3\right) + \sigma_5(R^n + p_3)\Delta B_n\right] \end{cases}$$

Final stage

$$\begin{cases} S_c^{n+1} = S_c^n + \frac{1}{6}(k_1 + 2k_2 + 2k_2 + k_4) \\ E_c^{n+1} = E_c^n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \\ I_c^{n+1} = I_c^n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \\ I_R^{n+1} = I_R^n + \frac{1}{6}(n_1 + 2n_2 + 2n_3 + n_4) \\ R^{n+1} = R^n + \frac{1}{6}(p_1 + 2p_2 + 2p_3 + p_4) \end{cases} \tag{40}$$

where, $n = 0, 1, 2, \ldots, N$ and $\Delta B_n = B_{n+1} - B_n \ \forall n \in I_N$ is a standard normally distributed random process, i.e., $\Delta B_n \sim \sqrt{t_{n+1} - t_n}N(0,1)$.

4.5 Stochastic Nonstandard Finite Difference Method

The proposed stochastic NSFD method [28,29] of the non-parametric perturbation stochastic system (31) can be expressed in the following manner, with given initial condition $S_c(0) = 0.2, E_c(0) = 0.4, I_c(0) = 0.1, I_R(0) = 0.1, R(0) = 0.2$.

$$\frac{S_c^{n+1} - S_c^n}{h} = b_h - \frac{\beta_1 S_c^{n+1} I_c^n e^{-\mu T}}{1 + m_0 I_c^n} - \frac{\beta_2 S_c^{n+1} I_R^n}{1 + m_1 I_R^n} - \mu S_c^{n+1} + \sigma R^n + \sigma_1 S_c^n \Delta B_n$$

$$S_c^{n+1} = \frac{S_c^n + \phi(h)\left(b_h + \sigma R^n + \sigma_1 S_c^n \Delta B_n\right)}{1 + \phi(h)\left(\frac{\beta_1 I_c^n e^{-\mu T}}{1 + m_0 I_c^n} + \frac{\beta_2 I_R^n}{1 + m_1 I_R^n} + \mu\right)} \tag{41}$$

Similarly,

$$E_c^{n+1} = \frac{E_c^n + \phi(h)\left(\frac{\beta_1 S_c^n I_c^n e^{-\mu T}}{1 + m_0 I_c^n} + \frac{\beta_2 S_c^n I_R^n}{1 + m_1 I_R^n} + \sigma_2 E_c^n \Delta B_n\right)}{1 + \phi(h)(\tau + \mu)} \tag{42}$$

$$I_c^{n+1} = \frac{I_c^n + \phi(h)\left(\tau E_c^n + \sigma_3 I_c^n \Delta B_n\right)}{1 + \phi(h)(\gamma + \mu + \xi + \delta)} \tag{43}$$

$$I_R^{n+1} = \frac{I_R^n + \phi(h)\left(\xi I_c^n + \sigma_4 I_R^n \Delta B_n\right)}{1 + \phi(h)\eta} \tag{44}$$

$$R^{n+1} = \frac{R^n + \phi(h)\left(\gamma I_c^n + \sigma_5 R^n \Delta B_n\right)}{1 + \phi(h)(\mu + \sigma)} \tag{45}$$

where, $\phi(h) = 1 - e^{-h}$ and "h" is any time step size and $n = 0, 1, 2, \ldots, N$ and $\Delta B_n = B_{n+1} - B_n \ \forall n \in I_N$ is a standard normally distributed random process, i.e., $\Delta B_n \sim \sqrt{t_{n+1} - t_n} N(0,1)$.

4.6 Essential Properties for the Stochastic NSFD

It is important to check the essential properties of the proposed discrete system derived from the continuous stochastic delayed conjunctivitis system (31).

Theorem 10: *For any given initial value* $(S_c^n(0), E_c^n(0), I_c^n(0), I_R^n(0), R^n(0)) \in R_+^5$, *Eqs. (41)–(45) have a unique positive solution* $(S_c^n(t), E_c^n(t), I_c^n(t), I_R^n(t), R^n(t)) \in R_+^5$ *on* $n \geq 0$.

Proof: The proof is straightforward because the constraint of biological problems is non-negative. □

Theorem 11: *The region* $\Gamma = \left\{(S_c^n(t), E_c^n(t), I_c^n(t), I_R^n(t), R^n(t)) \in R_+^5 : S_c^n \geq 0, E_c^n \geq 0, I_c^n \geq 0, I_R^n \geq 0, R^n \geq 0, S_c^n + E_c^n + I_c^n + I_R^n + R^n \leq \frac{b_h}{\mu}\right\}$ *for all,* $n \geq 0$ *is a positive invariant feasible region for Eqs. (41)–(45).*

Proof: Prove this result by mathematical induction on 'n'. Case I is obvious.

Now for case-II

If the result is true for any $n \in I_{N-1}$

$$\Rightarrow N^n = S_c^n + E_c^n + I_c^n + I_R^n + R^n \leq \frac{b_h}{\mu} \tag{i}$$

The system (41)–(45) can be decomposed as follows:

$$\frac{S_c^{n+1} - S_c^n}{h} = b_h - \frac{\beta_1 S_c^n I_c^n e^{-\mu T}}{1 + m_0 I_c^n} - \frac{\beta_2 S_c^n I_R^n}{1 + m_1 I_R^n} - \mu S_c^n + \sigma R^n + \sigma_1 S_c^n \Delta B_n$$

$$\frac{E_c^{n+1} - E_c^n}{h} = \frac{\beta_1 S_c^n I_c^n e^{-\mu T}}{1 + m_0 I_c^n} + \frac{\beta_2 S_c^n I_R^n}{1 + m_1 I_R^n} - (\tau + \mu) E_c^n + \sigma_2 E_c^n \Delta B_n$$

$$\frac{I_c^{n+1} - I_c^n}{h} = \tau E_c^n - (\gamma + \mu + \xi + \delta) I_c^n + \sigma_3 I_c^n \Delta B_n$$

$$\frac{I_R^{n+1} - I_R^n}{h} = \xi I_c^n - \eta I_R^n + \sigma_4 I_R^n \Delta B_n$$

$$\frac{R^{n+1} - R^n}{h} = \gamma I_c^n - (\mu + \sigma) R^n + \sigma_5 R^n \Delta B_n$$

These equations are added by supposing non-perturbation is zero.

$$\frac{\left(S_c^{n+1}+E_c^{n+1}+I_c^{n+1}+I_R^{n+1}+R^{n+1}\right)-\left(S_c^n+E_c^n+I_c^n+I_R^n+R^n\right)}{h}$$
$$= b_h - \mu S_c^n - \mu E_c^n - \mu I_c^n - \delta I_c^n - \eta I_R^n - \mu R^n$$

$$\frac{N^{n+1} - N^n}{h} \le b_h - \mu S_c^n - \mu E_c^n - \mu I_c^n - \eta I_R^n - \mu R^n$$

By Eq. (i) and suppose $\eta = \mu$,

$$\frac{N^{n+1} - N^n}{h} \le b_h - \mu \left(S_c^n + E_c^n + I_c^n + I_R^n + R^n\right)$$

$$N^{n+1} \le h b_h - h\mu N^n + N^n$$

$$N^{n+1} \le h b_h - h\mu \frac{b_h}{\mu} + \frac{b_h}{\mu}$$

$$N^{n+1} \le \frac{b_h}{\mu}$$

By mathematical induction, the result is valid for the proposed NSFD method and hence is bounded for all $n \ge 0$. □

4.7 Consistency Analysis of Stochastic Nonstandard Finite Difference (NSFD)

Theorem 12: *The equilibria for the stochastic proposed nonstandard system (41)–(45) are the same for any $n \ge 0$ in discrete and continuous dynamical systems.*

Proof: By solving the system (41)–(45), and $\Delta B_n = 0$.

Conjunctivitis free equilibrium (CFE) = $C_1{}^n = \left(S_c^{n1}, E_c^{n1}, I_c^{n1}, I_R^{n1}, R^{n1}\right) = \left(\frac{b_h}{\mu}, 0, 0, 0, 0\right)$,

Conjunctivitis present equilibrium (CPE) = $C_2{}^n = \left(S_c^{n2}, E_c^{n2}, I_c^{n2}, I_R^{n2}, R^{n2}\right)$,

$$S_c^{n2} = \frac{\left(b_h + \sigma R^{n2}\right)\left(1 + m_0 I_c^{n2}\right)\left(1 + m_1 I_R^{n2}\right)}{\beta_1 I_c^{n2} e^{-\mu T}\left(1 + m_1 I_R^{n2}\right) + \beta_2 I_R^{n2}\left(1 + m_0 I_c^{n2}\right) + \mu\left(1 + m_0 I_c^{n2}\right)\left(1 + m_1 I_R^{n2}\right)}$$

$$E_c^{n2} = \frac{\beta_1 S_c^{n2} I_c^{n2} e^{-\mu T}\left(1 + m_1 I_R^{n2}\right) + \beta_2 S_c^{n2} I_R^{n2}\left(1 + m_0 I_c^{n2}\right)}{\left(1 + m_0 I_c^{n2}\right)\left(1 + m_1 I_R^{n2}\right)(\tau + \mu)}$$

$I_c^{n2} = \frac{R_0 \mu (\tau+\mu) E_c^{n2}}{\beta_1 b_h e^{-\mu T}}$, $I_R^{n2} = \frac{\xi I_c^{n2}}{\eta}$, $R^{n2} = \frac{\gamma I_c^{n2}}{(\mu+\sigma)}$. □

Theorem 13: *The deterministic form of the stochastic nonstandard computational system (41)–(45) is stable if the eigenvalue lies in the unit circle $\forall n \ge 0$ [30].*

Proof: The system (41)–(45), and $\Delta B_n = 0$, is as follows:

$$A = \frac{S_c + h(b_h + \sigma R)}{1 + \dfrac{h\beta_1 I_c e^{-\mu T}}{1 + m_0 I_c} + \dfrac{h\beta_2 I_R}{1 + m_1 I_R} + h\mu},\ B = \frac{E_c + h\left(\dfrac{\beta_1 S_c I_c e^{-\mu T}}{1 + m_0 I_c} + \dfrac{\beta_2 S_c I_R}{1 + m_1 I_R}\right)}{1 + h(\tau + \mu)},\ C = \frac{I_c + h\tau E_c}{1 + h(\gamma + \mu + \xi + \delta)},\ D = \frac{I_R + h\xi I_c}{1 + h\eta},\ F = \frac{R + h\gamma I_c}{1 + h(\mu + \sigma)}$$

The Jacobian of the above equations:

$$J = \begin{bmatrix} \frac{\partial A}{\partial S_c} & \frac{\partial A}{\partial E_c} & \frac{\partial A}{\partial I_c} & \frac{\partial A}{\partial I_R} & \frac{\partial A}{\partial R} \\ \frac{\partial B}{\partial S_c} & \frac{\partial B}{\partial E_c} & \frac{\partial B}{\partial I_c} & \frac{\partial B}{\partial I_R} & \frac{\partial B}{\partial R} \\ \frac{\partial C}{\partial S_c} & \frac{\partial C}{\partial E_c} & \frac{\partial C}{\partial I_c} & \frac{\partial C}{\partial I_R} & \frac{\partial C}{\partial R} \\ \frac{\partial D}{\partial S_c} & \frac{\partial D}{\partial E_c} & \frac{\partial D}{\partial I_c} & \frac{\partial D}{\partial I_R} & \frac{\partial D}{\partial R} \\ \frac{\partial F}{\partial S_c} & \frac{\partial F}{\partial E_c} & \frac{\partial F}{\partial I_c} & \frac{\partial F}{\partial I_R} & \frac{\partial F}{\partial R} \end{bmatrix}$$

where

$$\frac{\partial A}{\partial S_c} = \frac{1}{1 + \dfrac{h\beta_1 I_c e^{-\mu T}}{1 + m_0 I_c} + \dfrac{h\beta_2 I_R}{1 + m_1 I_R} + h\mu},\ \frac{\partial B}{\partial S_c} = \frac{h\left(\dfrac{\beta_1 I_c e^{-\mu T}}{1 + m_0 I_c} + \dfrac{\beta_2 I_R}{1 + m_1 I_R}\right)}{1 + h(\tau + \mu)}$$

$$\frac{\partial A}{\partial E_c} = 0,\ \frac{\partial B}{\partial E_c} = \frac{1}{1 + h(\tau + \mu)}$$

$$\frac{\partial A}{\partial I_c} = \frac{-(S_c + hb_h + h\sigma R)h\beta_1 e^{-\mu T}}{(1 + m_0 I_c)^2\left(1 + \dfrac{h\beta_1 I_c e^{-\mu T}}{1 + m_0 I_c} + \dfrac{h\beta_2 I_R}{1 + m_1 I_R} + h\mu\right)^2},\ \frac{\partial B}{\partial I_c} = \frac{h\beta_1 S_c e^{-\mu T}}{(1 + h\tau + h\mu)(1 + m_0 I_c)^2}$$

$$\frac{\partial A}{\partial I_R} = \frac{-(S_c + hb_h + h\sigma R)h\beta_2}{(1 + m_1 I_R)^2\left(1 + \dfrac{h\beta_1 I_c e^{-\mu T}}{1 + m_0 I_c} + \dfrac{h\beta_2 I_R}{1 + m_1 I_R} + h\mu\right)^2},\ \frac{\partial B}{\partial I_R} = \frac{h\beta_2 S_c}{(1 + h\tau + h\mu)(1 + m_1 I_R)^2}$$

$$\frac{\partial A}{\partial R} = \frac{h\sigma}{1 + \dfrac{h\beta_1 I_c e^{-\mu T}}{1 + m_0 I_c} + \dfrac{h\beta_2 I_R}{1 + m_1 I_R} + h\mu},\ \frac{\partial B}{\partial R} = 0,\ \frac{\partial F}{\partial S_c} = 0$$

$$\frac{\partial C}{\partial S_c} = 0,\ \frac{\partial D}{\partial S_c} = 0,\ \frac{\partial F}{\partial E_c} = 0$$

$$\frac{\partial C}{\partial E_c} = \frac{h\tau}{1 + h(\gamma + \mu + \xi + \delta)},\ \frac{\partial D}{\partial E_c} = 0,\ \frac{\partial F}{\partial I_c} = \frac{h\gamma}{1 + h(\mu + \sigma)}$$

$$\frac{\partial C}{\partial I_c} = \frac{1}{1 + h(\gamma + \mu + \xi + \delta)},\ \frac{\partial D}{\partial I_c} = \frac{h\xi}{1 + h\eta},\ \frac{\partial F}{\partial I_R} = 0$$

$$\frac{\partial C}{\partial I_R} = 0,\ \frac{\partial D}{\partial I_R} = \frac{1}{1 + h\eta},\ \frac{\partial F}{\partial R} = \frac{1}{1 + h(\mu + \sigma)}$$

$$\frac{\partial C}{\partial R} = 0, \frac{\partial D}{\partial R} = 0$$

At Conjunctivitis free equilibrium (CFE) = $C_1{}^n = \left(S_c^{n1}, E_c^{n1}, I_c^{n1}, I_R^{n1}, R^{n1}\right) = \left(\frac{b_h}{\mu}, 0, 0, 0, 0\right)$ the given Jacobian is

$$J(E_0^*) = \begin{bmatrix} \frac{1}{1+h\mu} & 0 & \frac{-(b_h+\mu h b_h)h\beta_1 e^{-\mu T}}{\mu(1+h\mu)^2} & \frac{-(b_h+\mu h b_h)h\beta_2}{\mu(1+h\mu)^2} & \frac{h\sigma}{1+h\mu} \\ 0 & \frac{1}{1+h(\tau+\mu)} & \frac{h\beta_1 b_h e^{-\mu T}}{(1+h\tau+h\mu)\mu} & \frac{h\beta_2 b_h}{(1+h\tau+h\mu)\mu} & 0 \\ 0 & \frac{h\tau}{1+h(\gamma+\mu+\xi+\delta)} & \frac{1}{1+h(\gamma+\mu+\xi+\delta)} & 0 & 0 \\ 0 & 0 & \frac{h\xi}{1+h\eta} & \frac{1}{1+h\eta} & 0 \\ 0 & 0 & \frac{h\gamma}{1+h(\mu+\sigma)} & 0 & \frac{1}{1+h(\mu+\sigma)} \end{bmatrix} \tag{46}$$

$\lambda_1 = \frac{1}{1+h\mu} < 1, \lambda_2 = \frac{1}{1+h(\mu+\sigma)} < 1$. The remaining eigenvalues of (46) can be obtained by the characteristic equation.

$$\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0 \tag{47}$$

All remaining eigenvalues lie within the unit circle as depicted in Fig. 2 by representing Eq. (47) using the parameter values provided in Table 1. Since the largest eigenvalue is shown to be less than one, it follows that the rest are also less than one, which confirms the system's stability. □

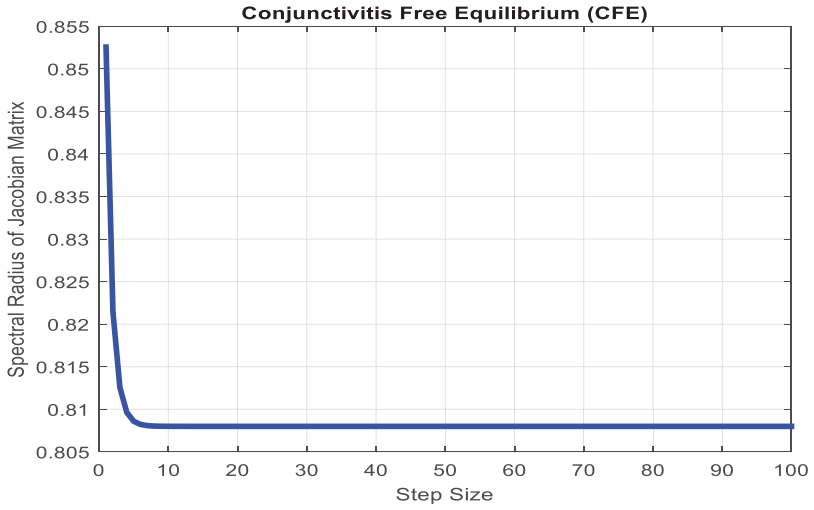

Figure 2: The spectral radius of the Jacobian matrix of the proposed nonstandard computational method at the conjunctivitis-free equilibrium (CFE) is less than one for a discrete time step size [0, 100]

Finally, the delayed nonstandard finite difference system driven by the system (41)–(45) is unconditionally asymptotically convergent to its equilibrium states for any step size.

5 Results

This section presents the comparative numerical analysis of the considered standard schemes with a proposed nonstandard finite difference scheme and deterministic simulations by MATLAB. For a conjunctivitis-free state, $b_h = 0.05$, $\mu = 0.05$, $\beta_1 = 2.081$, $\beta_2 = 2.092$, $\sigma = 0.45$, $\tau = 0.112$, $\xi = 0.3$, $\gamma = 0.142$,$\delta = 0.21$, $\eta = 0.05$, $m_0 = 0.01$, $m_1 = 0.02$, $0 < \sigma_i < 1$, $i = 1, 2, 3, 4, 5$ and they give $R_0 < 1$. For the conjunctivitis present state, $b_h = 0.05$, $\mu = 0.05$, $\beta_1 = 5.081$; $\beta_2 = 5.092$, $\sigma = 0.45$, $\tau = 0.112$, $\xi = 0.3$,$\gamma = 0.142$, $\delta = 0.21$, $\eta = 0.05$, $m_0 = 0.01$, $m_1 = 0.02$, $0 < \sigma_i < 1$, $i = 1, 2, 3, 4, 5$ gives $R_0 > 1$ and these parameter values taken from [1], and some minor changes in some values, and use one of the assumptions that the death and birth rates are the same for these simulations. Only the proposed nonstandard finite difference scheme satisfies the theoretical dynamic and continuous study.

6 Discussion

In the present study, the necessary dynamical properties like positivity, boundedness, uniqueness, and the existence of a conjunctivitis delayed model are studied. For small step size, Fig. 3a,c,e shows infected individuals converge to the actual endemic state, and after an increase in step size, Fig. 3b,d,f, all considered standard schemes, is inconsistent and refuses to comply with fundamental properties. In addition, the deterministic solution is the expected solution of the stochastic proposed nonstandard finite difference, which preserves all dynamic properties and a consistent scheme at any time step size. As the delay in the model increases, susceptibility exponentially rises and leads to the disease dying out from the system. Fig. 4a indicates that as the artificial delay in the model increases, the number of infected individuals decreases exponentially and, after some delay, converges to zero, and the disease dies out of the system. In contrast, Fig. 4b, in the same manner, presents an increase in the number of susceptible individuals and *vice versa*. Fig. 5 reveals that if $T > 0.3$, the reproductive number (R_0) becomes less than one, the system changes its behavior and becomes a Conjunctivitis Free Equilibrium (CFE). The asymptotic convergence behavior of the model around the derived equilibrium states is proved under the constraints on the reproductive number. The stochastic delayed model is derived from the non-parametric perturbation technique and proves its unique global positivity. For the proposed stochastic nonstandard finite difference scheme, dynamical and consistency analyses are done because the proposed discrete model gives approximated solutions, and the remaining considered schemes are classical and standard, and they depend on the time step size and have limitations on the dynamical properties. After numerical analysis of the stochastic conjunctivitis delayed model, only the stochastic proposed nonstandard finite difference (NSFD) scheme is consistent. It follows all dynamical properties, and all other considered standard schemes are stable and convergent around their defined equilibrium states, conditionally. Hence, the authors recommend the stochastic nonstandard finite difference scheme for any stochastic delayed epidemic model. Lastly, delaying strategies are an efficient way to control the strain of diseases. **Positivity Violation in Standard Schemes:** The loss of positivity under standard stochastic numerical schemes is investigated by running 100 Monte Carlo simulations for each method and recording the proportion of simulations in which any of the state variables $(S.E.I.Z.R)$ became negative. For $\boldsymbol{h}$ = **0.05**, then: **Euler-Maruyama:** 27% of runs violated positivity, **Stochastic Euler:** 31% of runs violated positivity, **Stochastic Runge-Kutta:** 19% of runs violated positivity and **Proposed NSFD:** 0% violation (fully positive trajectories).

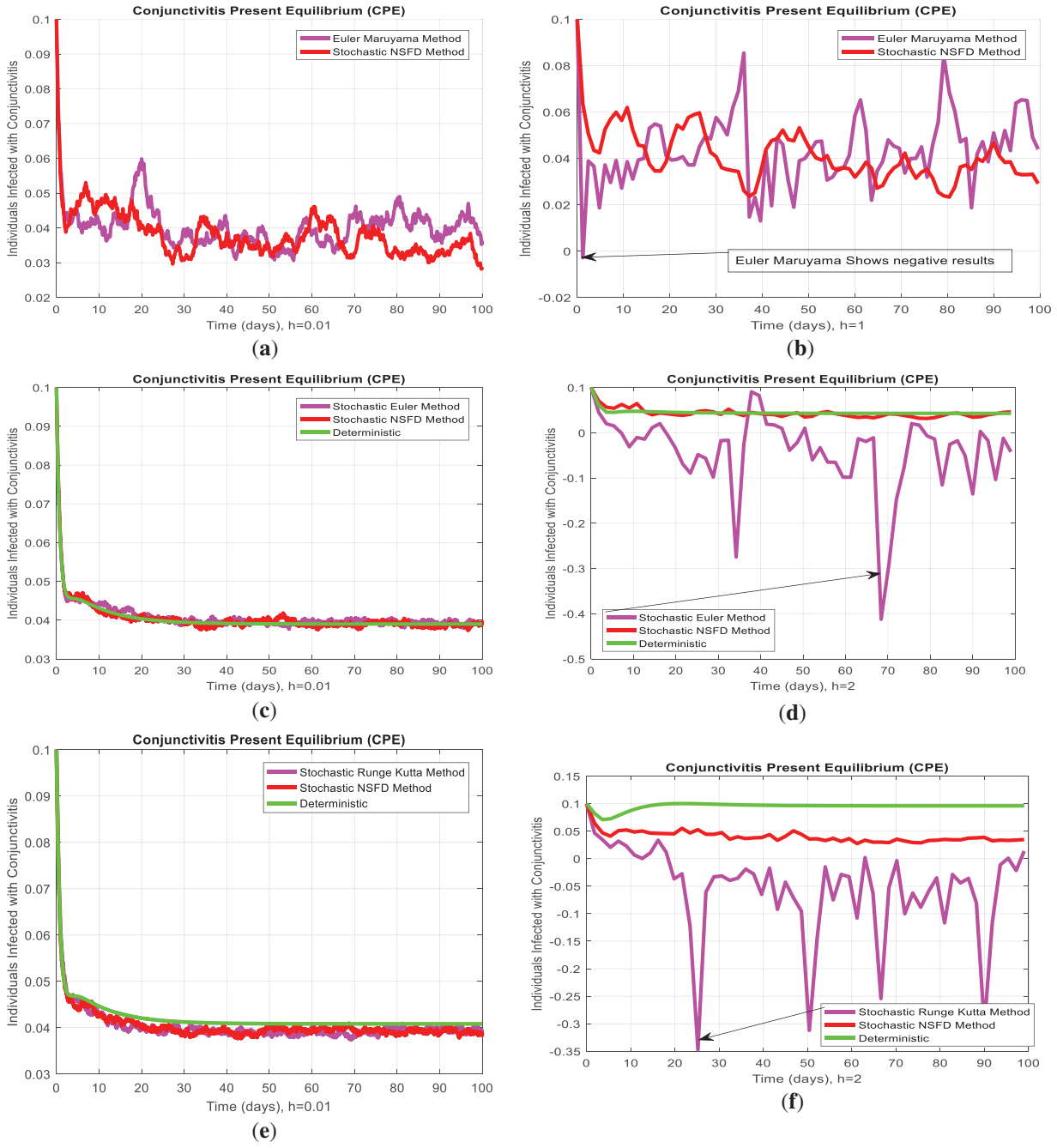

Figure 3: (**a**) Euler Maruyama scheme for infected individuals converges to the true endemic state for small step size $h = 0.01$. (**b**) Euler Maruyama scheme disobeys positivity for large step size $h = 1$. (**c**) Stochastic Euler scheme for infected individuals converges to the true CPE for small time step size $h = 0.01$. (**d**) For larger time step sizes like $h = 2$, Stochastic Euler loses positivity of the model. (**e**) Stochastic Runge Kutta method converges at $h = 0.01$ (**f**) Stochastic Runge Kutta produces negativity at $h = 2$

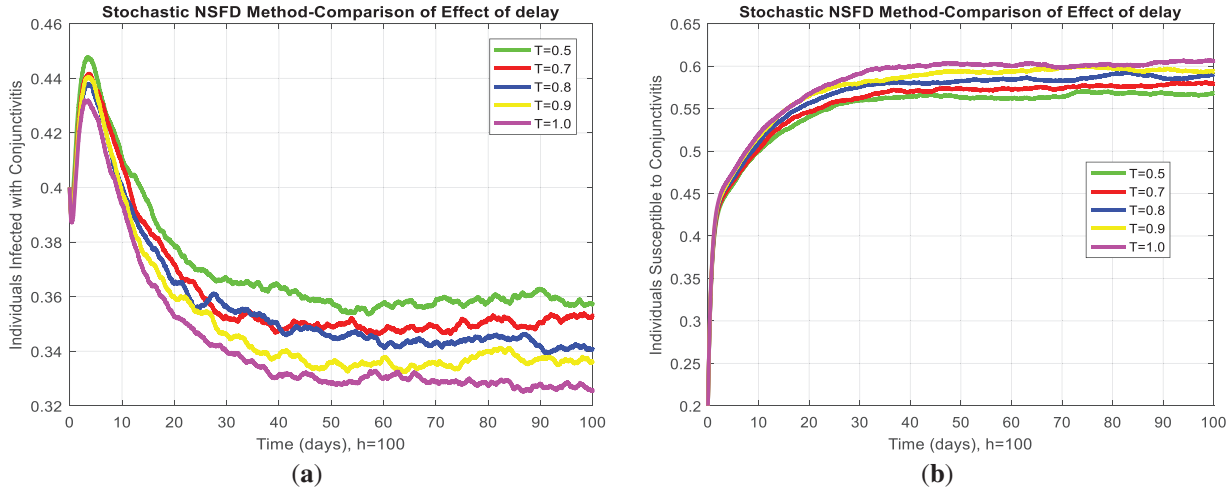

Figure 4: (**a**) Infected individuals decrease by increasing the delay for any step size. (**b**) For any step size, susceptible individuals increase whenever the delay increases

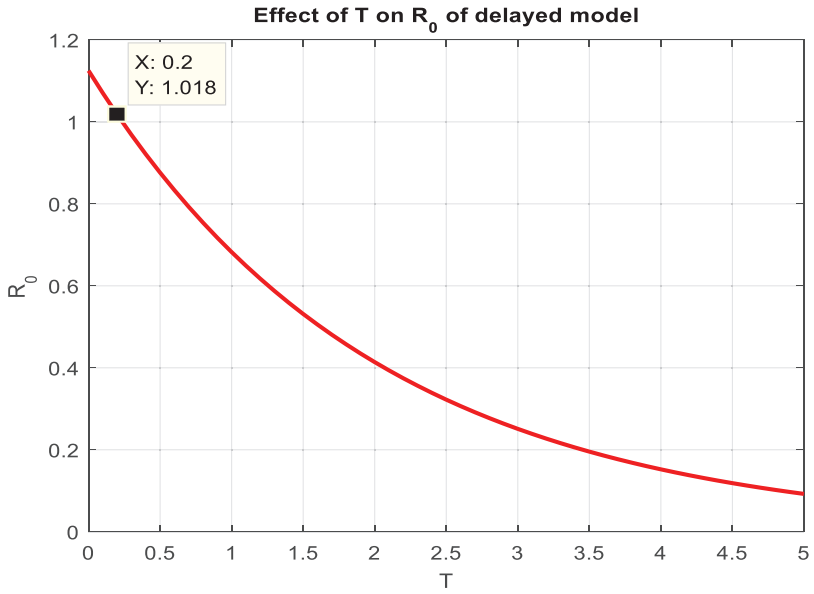

Figure 5: Almost after a three-day delay, the system changes its behavior and becomes a Conjunctivitis Free Equilibrium (CFE)

7 Conclusion

This study develops and analyzes a stochastic delay differential equation model to investigate the transmission dynamics of conjunctivitis. The model incorporates both direct and indirect transmission pathways and includes an artificial exponential delay to assess the impact of intervention strategies on disease progression. This research establishes key mathematical properties of the model, including positivity, boundedness, and the existence and uniqueness of global solutions. Stability analysis is performed around both disease-free and endemic equilibrium states under the constraint of the basic reproduction number. A novel stochastic nonstandard finite difference (NSFD) scheme is proposed, which demonstrates strong consistency with theoretical results and preserves the system's dynamic properties across a range of time

step sizes. Extensive numerical simulations indicate that artificial delay effectively reduces the number of infected individuals and promotes disease eradication, highlighting its potential as a viable public health intervention strategy. Comparison to standard numerical schemes reveals that the proposed NSFD method offers improved stability, positivity, and convergence. This study contributes a mathematically rigorous and computationally reliable framework for understanding and controlling conjunctivitis outbreaks using stochastic modeling with delay effects. The findings support the inclusion of artificial delays in epidemic models to reflect real-world disease mitigation efforts. In the future, this work could be improved by using quantum computing methods to make epidemic simulations faster and more efficient. This could allow real-time analysis and better optimization of complex disease models, especially when working with large amounts of data or uncertain situations.

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