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A Flexible Exponential Log-Logistic Distribution for Modeling Complex Failure Behaviors in Reliability and Engineering Data

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ABSTRACT: Parametric survival models are essential for analyzing time-to-event data in fields such as engineering and biomedicine. While the log-logistic distribution is popular for its simplicity and closed-form expressions, it often lacks the flexibility needed to capture complex hazard patterns. In this article, we propose a novel extension of the classical log-logistic distribution, termed the new exponential log-logistic (NExLL) distribution, designed to provide enhanced flexibility in modeling time-to-event data with complex failure behaviors. The NExLL model incorporates a new exponential generator to expand the shape adaptability of the baseline log-logistic distribution, allowing it to capture a wide range of hazard rate shapes, including increasing, decreasing, J-shaped, reversed J-shaped, modified bathtub, and unimodal forms. A key feature of the NExLL distribution is its formulation as a mixture of log-logistic densities, offering both symmetric and asymmetric patterns suitable for diverse real-world reliability scenarios. We establish several theoretical properties of the model, including closed-form expressions for its probability density function, cumulative distribution function, moments, hazard rate function, and quantiles. Parameter estimation is performed using seven classical estimation techniques, with extensive Monte Carlo simulations used to evaluate and compare their performance under various conditions. The practical utility and flexibility of the proposed model are illustrated using two real-world datasets from reliability and engineering applications, where the NExLL model demonstrates superior fit and predictive performance compared to existing log-logistic-based models. This contribution advances the toolbox of parametric survival models, offering a robust alternative for modeling complex aging and failure patterns in reliability, engineering, and other applied domains.

KEYWORDS: Failure rate; new exponential class; log-logistic distribution; maximum likelihood; order statistics; real-life data analysis

1 Introduction

The quality and reliability of statistical analysis fundamentally depend on the selection of an appropriate probability distribution. In recognition of this critical requirement, considerable research efforts have been devoted to developing flexible classes of probability distributions and refining associated statistical



methodologies. These advancements have found widespread application across a broad spectrum of disciplines, including insurance and actuarial science, financial risk analysis, business and economic forecasting, reliability and chemical engineering, medical and clinical research, demographic analysis, and sociological studies. The versatility of these models in capturing the complexity of real-world phenomena highlights their essential role in modern, data-driven decision-making processes.

The statistical literature has introduced several newly generated classes of univariate continuous distributions by incorporating additional shape parameters into baseline models. These extended distributions have garnered the attention of statisticians due to their flexibility and capability to model both monotonic and non-monotonic real-life data. Notable examples include the Marshall–Olkin-G [1], beta-G [2], Kumaraswamy-G [3], McDonald-G [4], Weibull-G [5], Kumaraswamy Marshall–Olkin-G [6], generalized alpha exponent power-G [7], length-biased truncated Lomax-G [8], Ramos–Louzada-G [9], Lambert-G [10], and new exponential-H (NEx-H) [11] families, among others.

The log-logistic (LL) distribution—also referred to as the Fisk distribution in economics, particularly for modeling income distributions [12–14]—is a two-parameter model defined by its scale and shape parameters. Arnold [15] extended this model by incorporating a location parameter, leading to the Pareto Type III distribution, which generalizes the LL distribution. Moreover, the LL distribution arises as a special case of two broader and more flexible families: the Burr type XII distribution [16] and the Kappa distribution [17], obtained by specifying particular values for the shape parameters within those models.

These parent distributions are especially popular in hydrological studies, where they are employed to model skewed data and extreme events, such as streamflow and precipitation. The LL distribution inherits many of these desirable characteristics while offering a more parsimonious structure, making it an appealing choice in applications where simplicity and interpretability are essential. For comprehensive discussions on the statistical properties, historical development, and wide-ranging applications of the LL distribution, refer to [18]. Recent studies have extended its applications, including detecting clinical risk shifts using LL hazard change-point models [19] and developing robust inference procedures based on minimum density power divergence estimators [20].

The LL distribution can be interpreted as the probability distribution of a random variable whose logarithm follows a logistic distribution. It is often employed as a viable alternative to the log-normal distribution, particularly due to its hazard rate function (HRF), which characteristically increases to a peak before decreasing—a feature that enhances its applicability in modeling lifetime and reliability data. To improve its flexibility and modeling capacity, numerous researchers have proposed generalized forms of the LL distribution. Among the notable extensions are the beta LL [21], the Marshall–Olkin LL [22], the McDonald LL [23], and the Zografos–Balakrishnan LL [24] distributions. Other significant contributions include the Kumaraswamy Marshall–Olkin LL [25], the extended Poisson LL [26], the odd Lomax LL [27], the alpha-power transformed LL [28], the extended LL [29], the skew LL [30], cubic transmuted LL [31], Maxwell LL [32], type-II generalized LL [33], and the recently proposed generalized Kavya–Manoharan LL distribution [34]. These developments reflect the continued interest in enhancing the LL distribution's ability to model diverse real-world data more accurately and flexibly.

In this paper, we introduce a novel extension of the LL distribution, referred to as the new exponential log-logistic (NExLL) distribution. This model is developed within the framework of the NEx-H family of distributions [11] and is designed to provide enhanced flexibility in modeling complex real-world data. Compared to existing generalizations of the LL distribution, the NExLL model offers several advantages that motivate its development including (i) The NExLL distribution is capable of capturing a wide range of HRF shapes, including increasing, decreasing, J-shaped, reversed J-shaped, modified bathtub, and unimodal forms; (ii) It is well-suited for modeling asymmetric and skewed datasets that may not be adequately

handled by traditional LL-based models; (iii) The model has potential applications across numerous fields, including survival analysis, public health, biomedical research, industrial reliability, and engineering; and (iv) Applications to two real-world datasets demonstrate that the NExLL distribution provides a superior fit compared to several well-known LL extensions. These features underscore the practical utility and theoretical contribution of the proposed distribution to the broader field of parametric survival modeling.

The final motivation of this study is to evaluate the performance of various frequentist estimation methods for the proposed NExLL distribution across different sample sizes and parameter settings. Specifically, we consider and compare the maximum likelihood estimators (MLE), least-squares estimators (LSE), weighted LS estimators (WLSE), Cramér–von Mises estimators (CRVME), Anderson–Darling estimators (ADE), right-tail AD estimators (RADE), and the maximum product of spacings estimators (MPSE). This comparative analysis aims to provide practical guidance on selecting the most suitable estimation method for the NExLL distribution, offering valuable insights for applied statisticians working with lifetime and reliability data.

While the proposed NExLL model belongs to the broader NEx-H family, it is not merely a special case of the existing NEx-H framework. The NExLL model stands out due to its complete tractability: closed-form expressions are available for all key functions, including the probability density function (PDF), cumulative distribution function (CDF), moments, quantile function, and HRF. Furthermore, the model preserves the interpretability of the baseline LL parameters, while introducing a new parameter that significantly enhances shape flexibility. This structure enables the NExLL to model complex failure rate behaviors that are not easily handled by general NEx-H models. Moreover, in practical reliability and engineering applications, the NExLL model has demonstrated superior performance, as shown in our real data illustrations.

The rest of the paper is organized into seven sections. The NExLL distribution is investigated in [Section 2](#). In [Section 3](#), some key properties of the NExLL distribution are explored. Inference about the NExLL parameters is presented in [Section 4](#). [Section 5](#) provides simulation studies. In [Section 6](#), we present two real-life data applications. [Section 7](#) gives some conclusions.

2 The NExLL Distribution

In this section, we formally define the proposed NExLL distribution and explore its flexibility graphically. Derived from the NEx-H family [\[11\]](#), the NExLL distribution extends the classical LL model by incorporating an additional shape parameter, thereby enhancing its flexibility in modeling diverse data behaviors.

The CDF of the two-parameter LL model has the form

$$G(x) = \left(1 + \frac{\lambda}{x^\beta}\right)^{-1}, \quad x > 0, \lambda, \beta > 0, \quad (1)$$

where λ and β are the scale and shape parameters, respectively. The PDF of the LL model reduces to

$$g(x) = \lambda\beta x^{-\beta-1} \left(1 + \frac{\lambda}{x^\beta}\right)^{-2}. \quad (2)$$

The NExLL distribution is constructed based on the NEx-G family, which is specified by the CDF

$$F(x) = 1 - \exp\left\{\frac{-a G(x) [2 - G(x)]}{1 - G(x)}\right\}, \quad x \in \mathbb{R}, \quad a > 0. \quad (3)$$

The corresponding PDF of the NEx-G class is expressed by

$$f(x) = \frac{a g(x)}{[1 - G(x)]^2} \left\{ 1 + [1 - G(x)]^2 \right\} \exp \left\{ \frac{-a G(x) [2 - G(x)]}{1 - G(x)} \right\}. \quad (4)$$

By incorporating the LL distribution (1) as the baseline model within the NEx-H family framework, the CDF of the NExLL distribution is defined as follows:

$$F(x) = 1 - \exp \left[\frac{-a x^\beta (x^\beta + 2\lambda)}{\lambda (x^\beta + \lambda)} \right], \quad (5)$$

where λ is the scale parameter and β and a refer to shape parameters, respectively.

The PDF of the NExLL distribution takes the form

$$f(x) = \frac{a\beta x^{\beta-1}}{\lambda} \left[1 + \left(\frac{\lambda}{x^\beta + \lambda} \right)^2 \right] \exp \left[\frac{-a x^\beta (x^\beta + 2\lambda)}{\lambda (x^\beta + \lambda)} \right]. \quad (6)$$

The HRF of the NExLL distribution reduces to

$$h(x) = \frac{a\lambda\beta x^{-\beta-1} \left(1 + \frac{\lambda}{x^\beta} \right)^{-2}}{\left[1 - \left(1 + \frac{\lambda}{x^\beta} \right)^{-1} \right]^2} \left\{ 1 + \left[1 - \left(1 + \frac{\lambda}{x^\beta} \right)^{-1} \right]^2 \right\}.$$

The quantile function (QF) of the NExLL distribution follows by inverting Eq. (5) as

$$Q(u) = \lambda^{\frac{1}{\beta}} \left[\frac{2a}{-2a - \ln(1-u) \pm \sqrt{4a^2 + \ln^2(1-u)}} - 1 \right]^{\frac{-1}{\beta}}, \quad (7)$$

where U follows the uniform (0, 1) distribution.

Figs. 1–4 illustrate the flexibility of the proposed NExLL distribution in modeling diverse data behaviors. Figs. 1 and 2 display various PDF shapes for different parameter settings while fixing $\lambda = 1$, highlighting the model's ability to capture right-skewed, left-skewed, and symmetric patterns. In contrast, Figs. 3 and 4 present a range of HRF shapes under the same condition, including increasing, decreasing, modified bathtub, and upside-down bathtub forms. These visualizations demonstrate the versatility of the NExLL model in fitting a wide spectrum of real-world reliability and survival data.

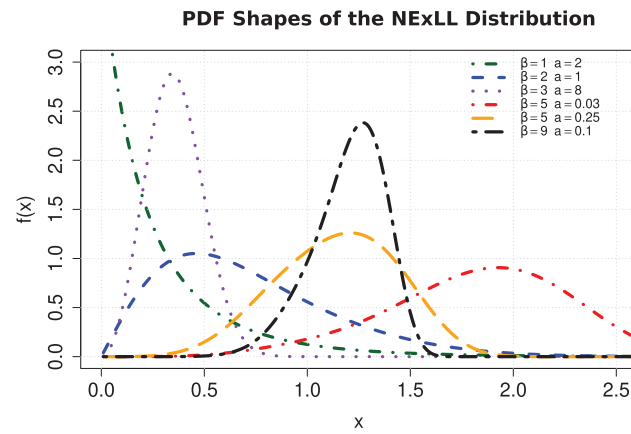


Figure 1: Some possible density shapes of the NExLL for various parametric values with $\lambda = 1$

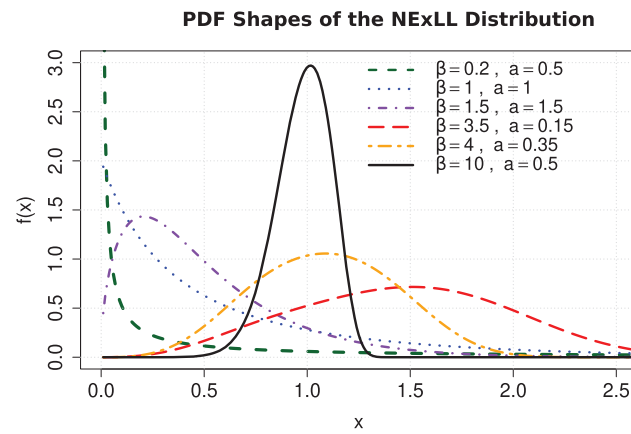


Figure 2: Some possible density shapes of the NExLL for various parametric values with $\lambda = 1$

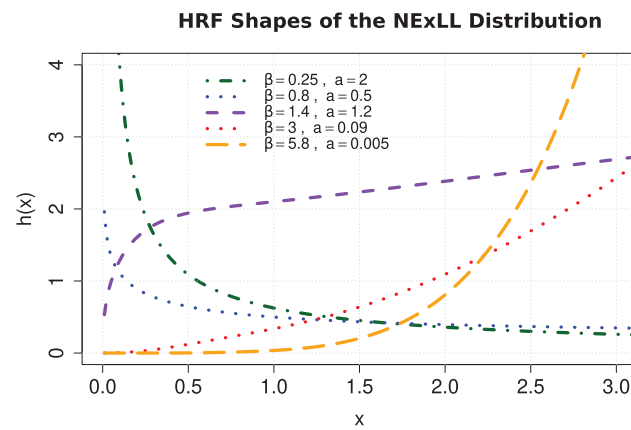


Figure 3: Some possible failure rate shapes of the NExLL for various parametric values with $\lambda = 1$

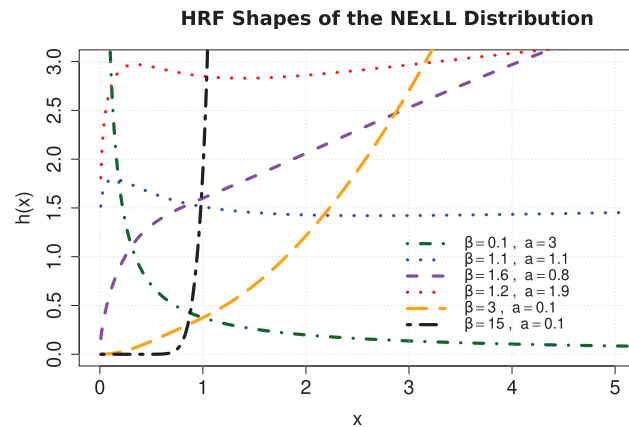


Figure 4: Some possible failure rate shapes of the NExLL for various parametric values with $\lambda = 1$

3 Properties of the NExLL Distribution

3.1 Linear Representation

To improve interpretability, we begin by deriving a series representation of the NExLL distribution. This representation expresses the density as a weighted infinite sum of simpler exponentiated-LL (ELL) components, which facilitates theoretical development and practical computation.

A useful mixture representation of the PDF of the NEx-G class was provided in [10]. According to the NEx-G density reduces to

$$f(x) = \sum_{i,j,k=0}^{\infty} \frac{(-a)^i (-1)^{k+1}}{i!} (k+i) g(x) G(x)^{i+k-1} \binom{i}{j} \binom{j-i}{k}.$$

By substituting the PDF and CDF of the LL model into the general NEx-G formulation, we derive the expanded form of the NExLL density. This expression, although complex, captures the flexible structure of the model through a combination of LL components. After some algebra, the NExLL density takes the form:

$$f(x) = \sum_{i,j,k=0}^{\infty} \frac{(-a)^i (-1)^{k+1} \lambda \beta x^{-\beta-1}}{i!} (k+i) \left(1 + \frac{\lambda}{x^\beta}\right)^{-2} \left(1 + \frac{\lambda}{x^\beta}\right)^{1-i-k} \binom{i}{j} \binom{j-i}{k}.$$

For computational efficiency and theoretical tractability, this expression can be rewritten as a more compact infinite sum involving ELL densities, each scaled by specific weight coefficients $\tau_{i,k}$, as follows:

$$f(x) = \sum_{i,k=0}^{\infty} \tau_{i,k} \zeta_{k+i}(x), \quad (8)$$

where $\zeta_{k+i}(x) = (k+i) g(x) (G(x))^{k+i-1}$ is the ELL density with power parameter and $(k+i) > 0$, and

$$\tau_{i,k} = \sum_{j=0}^{\infty} \frac{(-a)^i (-1)^{k+1}}{i!} \binom{i}{j} \binom{j-i}{k}.$$

3.2 Some Moments

Using the linear representation, we derive the moments of the NExLL distribution by evaluating the weighted integrals of ELL components. This approach simplifies the calculation of moments by leveraging known results for the ELL distribution.

The r -th moment of X can be obtained from Eq. (8) as

$$\mu'_r = E(X^r) = \sum_{i,k=0}^{\infty} \tau_{i,k} \int_0^{\infty} x^r \zeta_{k+i}(x) dx.$$

Hence, we have

$$\mu'_r = \sum_{i,k=0}^{\infty} \tau_{i,k} \lambda \beta (k+i) \int_0^{\infty} x^{r-\beta-1} \left(1 + \frac{\lambda}{x^\beta}\right)^{-i-k-1} dx.$$

After performing the integration, we obtain a closed-form series expression for the r th moment, which is valid under the condition $\frac{r}{\beta} < 1$ to ensure convergence. Hence, we have

$$\mu'_r = \sum_{i,k=0}^{\infty} \tau_{i,k} \lambda^{\frac{r}{\beta}} (k+i) \frac{\Gamma\left(1 - \frac{r}{\beta}\right) \Gamma\left(\frac{r}{\beta} + k + i\right)}{\Gamma(k+i+1)}, \quad \frac{r}{\beta} < 1. \quad (9)$$

The mean of X , say, μ_X , follows from (9) with $r = 1$.

Similarly, the r th incomplete moment is computed by integrating the density function up to a threshold t . The resulting expression involves the hypergeometric function ${}_2F_1$, commonly encountered in survival analysis and income distribution studies.

The r th incomplete moment of the NExLL distribution has the form

$$I_r(t) = \int_0^t x^r f(x) dx.$$

Using Eq. (8), $I_r(t)$ reduces to

$$I_r(t) = \sum_{i,k=0}^{\infty} \tau_{i,k} \int_0^t x^r \zeta_{k+i}(x) dx.$$

Then, we obtain

$$I_r(t) = \frac{\tau_{i,k} \lambda \beta t^{r-\beta} (k+i)}{\beta - r} {}_2F_1\left(\frac{-r}{\beta} + 1, k+i+1; \frac{-r}{\beta} + 2; -\frac{\lambda}{t^\beta}\right),$$

where ${}_2F_1\left(\frac{-r}{\beta} + 1, k+i+1; \frac{-r}{\beta} + 2; -\frac{\lambda}{t^\beta}\right)$ is the hyper geometric function.

The mean residual life of the NExLL distribution at age t has the form

$$\psi(t) = \frac{1 - I_1(t)}{S(t) - t} = \sum_{i,k=0}^{\infty} \frac{(\beta - 1) - \tau_{i,k} \phi_{i,k} \lambda \beta t^{r-\beta} (k+i)}{(\beta - 1) \left[e^{-a\left(\frac{\omega[2-\omega]}{1-\omega}\right)} - t \right]},$$

where $\omega = \left(1 + \frac{\lambda}{t^\beta}\right)^{-1}$, $\phi_{i,k} = {}_2F_1\left(\frac{-1}{\beta} + 1, k+i+1; \frac{-1}{\beta} + 2; -\frac{\lambda}{t^\beta}\right)$ and $I_1(t)$ is the first incomplete moments.

The mean inactivity time of the NExLL distribution takes the form

$$\varphi(t) = t - \frac{I_1(t)}{F(t)} = t - \frac{\tau_{i,k} \phi_{i,k} \lambda \beta (k+i)}{(\beta-1) \left[e^{-a \left(\frac{\omega [2-\omega]}{1-\omega} \right)} - t \right]}.$$

The Lorenz (L) [35], Bonferroni (B), and Zenga (Z) curves are among the most important tools for measuring inequality, with notable applications in fields such as insurance, medicine, reliability, and economics.

The Lorenz (L) [35], Bonferroni (B) and Zenga (Z) curves are considered the most important inequality curves and have some applications in insurance, medicine, reliability, and economics.

The L curve is defined for the KAPLL distribution as follows:

$$L(p) = \frac{I_1(x_p)}{\mu'_1} = \sum_{i,k=0}^{\infty} \frac{\phi_{i,k} \lambda^{1-\frac{1}{\beta}} \beta x_p^{1-\beta} \Gamma(k+i+1)}{(\beta-1) \Gamma\left(1-\frac{1}{\beta}\right) \Gamma\left(\frac{1}{\beta} + k+i\right)},$$

where $\phi_{i,k} = {}_2F_1\left(\frac{-1}{\beta} + 1, k+i+1; \frac{-1}{\beta} + 2; -\frac{\lambda}{t^{\frac{1}{\beta}}}\right)$ and $I_1(t)$ is the first incomplete moments.

The B and Z inequality curves can be determined, through their relationship with the L curve, by the following formulae [36]

$$B(p) = \frac{L(p)}{p} \quad \text{and} \quad Z(p) = \frac{L(p) - p}{p[1 - L(p)]}.$$

The moment generating function (MGF) can also be derived using the linear representation, which leads to a nested summation form involving gamma functions. This function can be used to derive higher-order moments and cumulants.

The MGF of the NExLL model is defined by

$$M(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$

Based on Eq. (8), we can write

$$M(t) = \sum_{i,k=0}^{\infty} \tau_{i,k} \int_0^{\infty} e^{tx} \tilde{\zeta}_{k+i}(x) dx.$$

The MGF of the NExLL follows, after some algebra, as

$$M(t) = \sum_{i,k,s=0}^{\infty} \frac{\tau_{i,k} t^s \lambda^{\frac{s}{\beta}} (k+i) \Gamma\left(1-\frac{s}{\beta}\right) \Gamma\left(\frac{s}{\beta} + k+i\right)}{s! \Gamma(k+i+1)}, \quad \frac{s}{\beta} < 1.$$

Table 1 presents the computed values of μ_X using both the truncated series summation (SUM) formula and numerical integration (NI) for various combinations of β , a , and fixed $\lambda = 1$. The results indicate that as the number of terms M increases, the SUM values converge and stabilize, closely approximating the corresponding NI values. This convergence behavior confirms the accuracy and efficiency of the proposed SUM formula in estimating μ_X , particularly for $M \geq 15$, where the difference from the NI results becomes negligible across all parameter settings.

Table 1: The values of μ_x computed using the truncated SUM formula and NI for different values of β , a , and $\lambda = 1$, across various truncation levels M

β	a	M	SUM	NI
3	1.2	5	0.69830	0.71386
		7	0.69010	
		15	0.68985	
		25	0.68985	
	1.5	5	0.76047	0.65608
		7	0.73091	
		15	0.72946	
		25	0.72946	
	1.7	5	0.80670	0.62599
		7	0.74649	
		15	0.74264	
		25	0.74264	
4	1.2	5	0.67414	0.76421
		7	0.66771	
		15	0.66752	
		25	0.66752	
	1.5	5	0.73957	0.71755
		7	0.71637	
		15	0.71527	
		25	0.71527	
	1.7	5	0.78483	0.69285
		7	0.73755	
		15	0.73461	
		25	0.73461	
5	1.2	5	0.66828	0.79993
		7	0.66267	
		15	0.66250	
		25	0.66250	
	1.5	5	0.73657	0.76074
		7	0.71633	
		15	0.71538	
		25	0.71538	
	1.7	5	0.78226	0.73978
		7	0.74099	
		15	0.73846	
		25	0.73846	

Table 2 presents key descriptive measures—mean μ_x , variance (var), skew, and kur—for the NExLL distribution across various combinations of the shape parameters β , a , with λ fixed at 1. These results are

obtained using the R software, and reveal that as a increases for fixed β , both the mean and variance decrease significantly, indicating a shift toward more concentrated distributions. Additionally, the skewness and kurtosis measures decline with increasing a , suggesting a transition from highly skewed and heavy-tailed distributions (particularly at low a) to more symmetric and light-tailed forms. For higher values of β , some negative skewness appears, indicating the model's flexibility in capturing left-skewed behavior as well. These findings demonstrate the strong adaptability of the NExLL distribution to various data patterns, especially in modeling dispersion and tail behavior.

Table 2: Summary of the mean, variance, skewness, and kurtosis of the NExLL distribution for various combinations of β , a , and with $\lambda = 1$ held fixed

β	a	mean	var	skew	kur
0.5	0.5	5.56851	218.46440	7.86825	122.85770
	0.75	2.19288	36.55201	0.57377	3.92022
	1.2	0.72782	4.30826	0.02522	0.06516
	1.5	0.42919	1.52331	0.00551	0.00889
	1.7	0.31912	0.84510	0.00232	0.00286
1.5	0.5	1.17094	0.83877	1.32334	5.13052
	0.75	0.85422	0.44887	0.54578	1.56675
	1.2	0.59298	0.21226	0.18448	0.37305
	1.5	0.49945	0.14793	0.10799	0.18456
	1.7	0.45393	0.12075	0.07960	0.12370
2	0.5	1.06424	0.40627	0.84771	3.57736
	0.75	0.84048	0.25239	0.44122	1.45409
	1.2	0.64017	0.14259	0.19433	0.48715
	1.5	0.56339	0.10831	0.12880	0.28480
	1.7	0.52473	0.09282	0.10175	0.20993
3	0.5	0.99985	0.17123	0.36231	2.75153
	0.75	0.85480	0.12353	0.24587	1.48400
	1.2	0.71386	0.08337	0.14200	0.70018
	1.5	0.65608	0.06901	0.10575	0.48468
	1.7	0.62599	0.06207	0.08891	0.39347
4	0.5	0.98300	0.09794	0.10079	2.62288
	0.75	0.87427	0.07614	0.09134	1.62809
	1.2	0.76421	0.05616	0.06196	0.91256
	1.5	0.71755	0.04851	0.04733	0.68809
	1.7	0.69285	0.04469	0.03978	0.58662
5	0.5	0.97794	0.06433	-0.06791	2.66061
	0.75	0.89056	0.05230	-0.02874	1.79730
	1.2	0.79993	0.04068	-0.01704	1.11781
	1.5	0.76074	0.03605	-0.01766	0.88753
	1.7	0.73978	0.03369	-0.01868	0.77930

3.3 Rényi and m -Entropies

Measures of entropy, such as Rényi and m -entropies, quantify the uncertainty of the NExLL model. These are obtained by raising the PDF to a power m and integrating, again using the linear expansion to express the result as a tractable summation.

The Rényi entropy of a random variable X is defined by

$$P_X(m) = \frac{1}{1-m} \log \int_0^\infty f^m(x) dx, \quad m > 0, m \neq 1.$$

Using Eq. (8), the Rényi entropy of the NExLL model takes the form

$$P_X(m) = \frac{1}{1-m} \log \left[\sum_{k=0}^{\infty} \tau_{i,k}^m \frac{\beta^{m-1} \lambda^{\frac{1-m}{\beta}} (k+i)^m \Gamma\left(\frac{\beta m[k+i]-m+1}{\beta}\right) \Gamma\left(\frac{m[1+\beta]-1}{\beta}\right)}{\Gamma(m(k+i+1))} \right]. \quad (10)$$

The m -entropy, say, $L_X(m)$, is defined by

$$L_X(m) = \frac{1}{m-1} \log \left[1 - \int_0^\infty f^m(x) dx \right], \quad m > 0, m \neq 1.$$

Hence, using (10), the m -entropy of the NExLL model follows as

$$L_X(m) = \frac{1}{q-1} \log \left[1 - \sum_{k=0}^{\infty} \tau_{i,k}^m \frac{\beta^{m-1} \lambda^{\frac{1-m}{\beta}} (k+i)^m \Gamma\left(\frac{\beta m[k+i]-m+1}{\beta}\right) \Gamma\left(\frac{m[1+\beta]-1}{\beta}\right)}{\Gamma(m(k+i+1))} \right].$$

3.4 Order Statistics

The order statistics of the NExLL distribution are derived from its PDF and CDF. Given the series form of the density, the order statistic distributions also naturally take the form of infinite summations over ELL components.

Let $X_1, X_2, \dots, X_n$ be a random sample of size n and let $X_{1:n}, \dots, X_{n:n}$ be their associated order statistic. Then, the PDF of the i th order statistics, say, $X_{i:n}$, which is denoted by $f_{X_{i:n}}(x)$, reduces to

$$f_{X_{i:n}}(x) = \frac{n! f(x)}{(n-i)!(i-1)!} [F(x)]^{i-1} [1-F(x)]^{n-i}. \quad (11)$$

Substituting (5) and (6) in (11), the i th order statistic of the NExLL distribution follows as

$$f_{X_{i:n}}(x) = \frac{n!}{(n-i)!(i-1)!} \left(\sum_{k=0}^{\infty} d_{i,k} \right)^{s+n-i+1} \sum_{s=0}^{\infty} \binom{i-1}{s} (-1)^s \pi_{i+k}(x) \times [G(x)^{i+k}]^{(s+n-i)}, \quad (12)$$

where $d_{i,k} = \sum_{j=0}^{\infty} \frac{(-a)^i (-1)^k}{i!} \binom{i}{j} \binom{j-i}{k}$, $\pi_{i+k}(x) = (i+k) g(x) G(x)^{i+k-1}$, $G(x)$ and $g(x)$ are given by Eqs. (1) and (2), respectively.

After some algebra, we obtain

$$f_{X_{i:n}}(x) = \sum_{i,k,s=0}^{\infty} \frac{n! (-1)^s (i+k) [d_{i,k}]^{s+n-i+1}}{(n-i)! (i-1)!} g(x) G(x)^{(i+k)(s+n-i+1)-1} \binom{i-1}{s}.$$

The last equation can be expressed as

$$f_{i:n}(x) = \sum_{s=0}^{\infty} w_s h_{s+1}(x),$$

where $h_{s+1}(x) = (i+k)(s+n-i+1)g(x)G(x)^{(i+k)(s+n-i+1)-1}$ is the exponentiated-LL density with power parameter $(i+k)(s+n-i+1) > 0$, and

$$w_s = \sum_{i,k=0}^{\infty} \frac{n! (-1)^s [d_{i,k}]^{s+n-i+1}}{(n-i)! (i-1)! (s+n-i+1)} \binom{i-1}{s}.$$

4 Estimation Methods

In this section, we use seven methods to estimate the NExLL parameters namely: the maximum likelihood (ML), least-squares (LS), weighted LS (WLS), maximum product of spacings (MPS), Cramér-von Mises (CRVM), Anderson-Darling (AD), and right-tail AD (RAD) methods.

Let $x_1, x_2, \dots, x_n$ be a random sample from NExLL distribution then the logarithm of the likelihood function, say, (ℓ) , becomes

$$\ell = n(\ln a + \ln \beta - \ln \lambda) + (\beta - 1) \sum_{i=1}^n \ln x_i + \sum_{i=1}^n \ln \left(1 + \frac{\lambda^2}{\vartheta_i^2} \right) - a \sum_{i=1}^n \frac{x_i^\beta (\vartheta_i + \lambda)}{\lambda \vartheta_i},$$

where $\vartheta_i = x_i^\beta + \lambda$.

To estimate the parameters of the NExLL distribution, we derive the score functions corresponding to the log-likelihood function. These score functions, defined as the partial derivatives of the log-likelihood with respect to the parameters a , λ and β , are solved simultaneously to obtain the MLE. The explicit expressions for the score functions are given below.

$$\begin{aligned} \frac{\partial \ell}{\partial a} &= \frac{n}{a} - \sum_{i=1}^n \frac{x_i^\beta (\vartheta_i + \lambda)}{\lambda \vartheta_i}, \\ \frac{\partial \ell}{\partial \lambda} &= \frac{-n}{\lambda} + \sum_{i=1}^n \left(1 + \frac{\lambda^2}{\vartheta_i^2} \right)^{-1} \frac{2\lambda x_i^\beta}{\vartheta_i^3} - a \sum_{i=1}^n \frac{2\lambda x_i^\beta \vartheta_i - x_i^\beta (\vartheta_i + \lambda)^2}{\lambda^2 \vartheta_i^2} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^n \ln(x_i) - \sum_{i=1}^n \left(1 + \frac{\lambda^2}{\vartheta_i^2} \right)^{-1} \frac{2\lambda^2 x_i^\beta \ln(x_i)}{\vartheta_i^3} \\ &\quad - a \sum_{i=1}^n \frac{\lambda \vartheta_i (2x_i^{2\beta} \ln(x_i) + 2\lambda x_i^\beta \ln(x_i)) - (\lambda x_i^\beta \ln(x_i) [x_i^{2\beta} + 2\lambda x_i^\beta])}{\lambda^2 \vartheta_i^2}. \end{aligned}$$

The LSE and WLSE of the NExLL parameters β , λ and a and can be obtained by minimizing

$$O(\alpha, \beta, \lambda, a, b) = \sum_{i=1}^n v_i \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2,$$

where $v_i = 1$ in case of the LSE approach and $v_i = (n+1)^2(n+2)/[i(n-i+1)]$ in case of the WLSE approach.

The LS and WLS estimators are obtained by solving the following nonlinear estimating equations:

$$\sum_{i=1}^n \left\{ 1 - \exp \left[\frac{-ax_{i:n}^\beta (\vartheta_{i:n} + \lambda)}{\lambda \vartheta_{i:n}} \right] - \frac{i}{n+1} \right\} \delta_s(x_{i:n}) = 0, \quad s = 1, 2, 3,$$

where $\delta_1(x_{i:n})$, $\delta_2(x_{i:n})$, and $\delta_3(x_{i:n})$ denote the partial derivatives of the CDF with respect to a , λ and β , respectively. These are given as:

$$\delta_1(x_{i:n}) = \frac{\partial F(x_{i:n})}{\partial a} = \frac{x_{i:n}^\beta (x_{i:n}^\beta + 2\lambda)}{\lambda (x_{i:n}^\beta + \lambda)} \exp \left\{ \frac{-ax_{i:n}^\beta (x_{i:n}^\beta + 2\lambda)}{\lambda (x_{i:n}^\beta + \lambda)} \right\}.$$

Let $\vartheta_{i:n} = x_{i:n}^\beta + \lambda$, then the above equation simplifies to:

$$\delta_1(x_{i:n}) = \frac{x_{i:n}^\beta (\vartheta_{i:n} + \lambda)}{\lambda \vartheta_{i:n}} \exp \left[\frac{-ax_{i:n}^\beta (\vartheta_{i:n} + \lambda)}{\lambda \vartheta_{i:n}} \right], \quad (13)$$

$$\delta_2(x_{i:n}) = \frac{\partial F(x_{i:n})}{\partial \lambda} = \frac{2a\lambda x_{i:n}^\beta (x_{i:n}^\beta + \lambda) - ax_{i:n}^\beta (x_{i:n}^\beta + 2\lambda)^2}{\lambda^2 (x_{i:n}^\beta + \lambda)^2} \exp \left\{ \frac{-ax_{i:n}^\beta (x_{i:n}^\beta + 2\lambda)}{\lambda (x_{i:n}^\beta + \lambda)} \right\}.$$

Using $\vartheta_{i:n}$, the last equation reduces to:

$$\delta_2(x_{i:n}) = \frac{2a\lambda x_{i:n}^\beta \vartheta_{i:n} - ax_{i:n}^\beta (x_{i:n}^\beta + 2\lambda)^2}{\lambda^2 \vartheta_{i:n}^2} \exp \left[\frac{-ax_{i:n}^\beta (\vartheta_{i:n} + \lambda)}{\lambda \vartheta_{i:n}} \right], \quad (14)$$

and

$$\begin{aligned} \delta_3(x_{i:n}) &= \frac{\partial F(x_{i:n})}{\partial \beta} = \exp \left\{ \frac{-ax_{i:n}^\beta (x_{i:n}^\beta + 2\lambda)}{\lambda (x_{i:n}^\beta + \lambda)} \right\} \\ &\times \frac{a \left(\lambda (x_{i:n}^\beta + \lambda) \left[2x_{i:n}^{2\beta} \ln(x_{i:n}) + 2\lambda x_{i:n}^\beta \ln(x_{i:n}) \right] \right) - a \left(\lambda x_{i:n}^\beta \ln(x_{i:n}) \left[x_{i:n}^{2\beta} + 2\lambda x_{i:n}^\beta \right] \right)}{\lambda^2 (x_{i:n}^\beta + \lambda)^2}. \end{aligned}$$

Using $\vartheta_{i:n}$ and letting $\eta_{i:n} = x_{i:n}^\beta \ln(x_{i:n})$ and $\vartheta_{i:n} = x_{i:n}^\beta + \lambda$, the equation above can be rewritten as:

$$\delta_3(x_{i:n}) = \frac{a \left(\lambda \vartheta_{i:n} \left[2x_{i:n}^\beta \eta_{i:n} + 2\lambda \eta_{i:n} \right] \right) - a \left(\lambda \eta_{i:n} \left[x_{i:n}^{2\beta} + 2\lambda x_{i:n}^\beta \right] \right)}{\lambda^2 \vartheta_{i:n}^2} \exp \left[\frac{-ax_{i:n}^\beta (\vartheta_{i:n} + \lambda)}{\lambda \vartheta_{i:n}} \right]. \quad (15)$$

Note: In the above expressions, $\vartheta_{i:n} = x_{i:n}^\beta + \lambda$ and $\eta_{i:n} = x_{i:n}^\beta \ln x_{i:n}$ are auxiliary terms introduced to simplify notation. These derivatives are used in solving the estimating equations for the LS, WLS, MPS, CRVM, AD, and RAD methods.

The MPSE is an alternative approach to the MLE [37,38]. The uniform spacings of a random sample of size n from the NExLL distribution are defined by

$$H(\beta, \lambda, a,) = \sum_{i=1}^{n+1} \log [D_i],$$

where $D_i = D_i(\beta, \lambda, a,) = F(x_{i:n}) - F(x_{i-1:n})$ be the uniform spacing for $(i = 1, \dots, n+1)$, where $F(x_{(0)}) = 0$, $F(x_{(n+1)}) = 1$ and $\sum_{i=1}^{n+1} D_i = 1$.

The MPSE of the NExLL parameters are determined by solving the non-linear equations

$$\sum_{i=1}^{n+1} \frac{1}{D_i} \delta_s(x_{i:n}) - \delta_s(x_{i-1:n}) = 0, \quad s = 1, 2, 3,$$

where $\delta_s(x_{i:n}) = 0$ are defined in (13)–(15) for $s = 1, 2, 3$.

The CRVM method minimizes the integrated squared difference between the theoretical and empirical CDFs. The CRVME of the NExLL parameters are obtained by minimizing the following objective function:

$$C(\beta, \lambda, a,) = \frac{1}{12n} + \sum_{i=1}^n \left\{ 1 - \exp \left[\frac{-ax_{i:n}^\beta (\vartheta_{i:n} + \lambda)}{\lambda \vartheta_{i:n}} \right] - \frac{2i-1}{2n} \right\}^2,$$

which also follow by solving the following estimating equations:

$$\sum_{i=1}^n \left\{ 1 - \exp \left[\frac{-ax_{i:n}^\beta (\vartheta_{i:n} + \lambda)}{\lambda \vartheta_{i:n}} \right] - \frac{2i-1}{2n} \right\} \delta_s(x_{i:n}) = 0, \quad s = 1, 2, 3.$$

The AD method places more weight on the tails of the distribution. The ADE of the NExLL parameters are obtained by minimizing

$$A(\beta, \lambda, a,) = -n - \frac{1}{n} + \sum_{i=1}^n (2i-1) [\log F(x_{i:n}) + \log S(x_{i:n})],$$

where $S(x_{i:n}) = 1 - F(x_{i:n})$.

The ADE can also be found as solutions of the following estimating equations:

$$\sum_{i=1}^n (2i-1) \left[\frac{\delta_s(x_{i:n})}{F(x_i)} + \frac{\delta_i(x_{n+1-i:n})}{S(x_{n+1-i:n})} \right] = 0, \quad s = 1, 2, 3.$$

The RAD method is a one-sided modification of the AD method focusing on the right tail. The RADE of the NExLL parameters β , λ and a are obtained by minimizing the following function:

$$R(\beta, \lambda, a) = \frac{1}{n} - 2 \sum_{i=1}^n F(x_{i:n}) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log S(x_{n+1-i:n}).$$

They can also be obtained from the non-linear estimating equations:

$$-2 \sum_{i=1}^n \delta_s(x_{i:n}) + \frac{1}{n} \sum_{i=1}^n (2i-1) \delta_s(x_{n+1-i:n}) / S(x_n + 1 - i : n) = 0, \quad s = 1, 2, 3,$$

where the expressions $\delta_s(x_{i:n})$ are the partial derivatives of the CDF $F(x)$ with respect to the parameters a , λ , and β , defined as:

$$\delta_1(x_{i:n}) = \frac{\partial F(x_{i:n})}{\partial a}, \quad \delta_2(x_{i:n}) = \frac{\partial F(x_{i:n})}{\partial \lambda}, \quad \delta_3(x_{i:n}) = \frac{\partial F(x_{i:n})}{\partial \beta}.$$

Each of these derivatives is given explicitly in Eqs. (13)–(15), and they are used throughout all estimating equations presented above.

In summary, the score and estimating equations for all seven estimation methods have been explicitly derived, clearly labeled, and organized for better readability. These formulations provide practitioners with the tools necessary to implement the methods and ensure reproducibility of results.

5 Numerical Simulations

This sections gives detailed simulation results to explore the behavior and performances of the introduced estimation methods in estimating the NExLL parameters based on the following three measures namely: the mean square errors (MSE), $MSE = \frac{1}{N} \sum_{i=1}^N (\hat{\theta} - \theta)^2$, average absolute biases ($|BIAS|$), $|BIAS| = \frac{1}{N} \sum_{i=1}^N |\hat{\theta} - \theta|$, and mean relative errors (MRE), $MRE = \frac{1}{N} \sum_{i=1}^N |\hat{\theta} - \theta| / \theta$, where $\theta = (\beta, \lambda, a)^T$.

Several sample sizes and parameters combinations are considered, i.e., $n = \{50, 100, 250, 500, 750\}$, $\theta = (0.5, 0.1, 0.01)^T$, $\theta = (0.5, 0.1, 0.03)^T$, $\theta = (0.5, 0.1, 0.05)^T$, $\theta = (1, 0.5, 0.01)^T$, $\theta = (1, 0.25, 0.01)^T$, $\theta = (1, 0.25, 0.02)^T$, $\theta = (2, 0.5, 0.01)^T$. We generated $N = 1000$ random samples based on the QF in (7) using the R software.

Tables 3–10 give the findings of simulations including the $|BIAS|$, MSE, and MRE for the eight estimation approaches. The results show that the values of $|BIAS|$, MSE and MRE decrease with the increase of n , showing that the given estimators are consistent. Hence, the introduced estimation methods perform very well in estimating the NExLL parameters. Generally, based on simulation results, the ordering performance of the estimators is as follows: MPSE, WLSE, ADE, MLE, LSE, CRVME, and RADE. This order confirms the superiority of the MPS estimation method with overall score of 74.

Table 3: Numerical values of the NExLL distribution for $\theta = (0.5, 0.1, 0.01)^T$

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
50	BIAS	$\hat{\beta}$	0.04490 ^{1}	0.05802 ^{5}	0.04567 ^{2}	0.06166 ^{7}	0.04700 ^{3}	0.05034 ^{4}	0.05991 ^{6}
		$\hat{\lambda}$	0.37199 ^{4}	0.54453 ^{6}	0.32292 ^{2}	0.53359 ^{5}	0.17047 ^{1}	0.36014 ^{3}	0.90240 ^{7}
		$\hat{a}$	0.02593 ^{3}	0.03727 ^{6}	0.02347 ^{2}	0.03519 ^{5}	0.01483 ^{1}	0.02700 ^{4}	0.05168 ^{7}
	MSE	$\hat{\beta}$	0.00342 ^{1}	0.00552 ^{5}	0.00352 ^{3}	0.00642 ^{7}	0.00349 ^{2}	0.00416 ^{4}	0.00618 ^{6}
		$\hat{\lambda}$	0.78690 ^{4}	0.80982 ^{5}	0.42024 ^{2}	0.85870 ^{6}	0.13081 ^{1}	0.45006 ^{3}	2.53461 ^{7}
		$\hat{a}$	0.00285 ^{4}	0.00313 ^{6}	0.00161 ^{2}	0.00291 ^{5}	0.00074 ^{1}	0.00202 ^{3}	0.00679 ^{7}
	MRE	$\hat{\beta}$	0.08980 ^{1}	0.11605 ^{5}	0.09135 ^{2}	0.12332 ^{7}	0.09401 ^{3}	0.10068 ^{4}	0.11981 ^{6}
		$\hat{\lambda}$	3.71994 ^{4}	5.44535 ^{6}	3.22919 ^{2}	5.33592 ^{5}	1.70466 ^{1}	3.60138 ^{3}	9.02403 ^{7}
		$\hat{a}$	2.59258 ^{3}	3.72740 ^{6}	2.34663 ^{2}	3.51932 ^{5}	1.48325 ^{1}	2.69963 ^{4}	5.16831 ^{7}
	$\Sigma Ranks$		25 ^{3}	50 ^{5}	19 ^{2}	52 ^{6}	14 ^{1}	32 ^{4}	60 ^{7}

(Continued)

Table 3 (continued)

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
100	BIAS	$\hat{\beta}$	0.03478 ^{2}	0.04146 ^{5}	0.03655 ^{4}	0.04592 ^{6}	0.03221 ^{1}	0.03646 ^{3}	0.04751 ^{7}
		$\hat{\lambda}$	0.34095 ^{3}	0.49542 ^{5}	0.30492 ^{2}	0.53182 ^{6}	0.13761 ^{1}	0.34318 ^{4}	0.79555 ^{7}
		$\hat{a}$	0.02414 ^{3}	0.03594 ^{5}	0.02227 ^{2}	0.03651 ^{6}	0.01278 ^{1}	0.02609 ^{4}	0.05109 ^{7}
	MSE	$\hat{\beta}$	0.00206 ^{2}	0.00277 ^{5}	0.00223 ^{4}	0.00355 ^{6}	0.00159 ^{1}	0.00219 ^{3}	0.00359 ^{7}
		$\hat{\lambda}$	0.63561 ^{4}	0.65779 ^{5}	0.35787 ^{2}	0.81733 ^{6}	0.07817 ^{1}	0.37777 ^{3}	1.82745 ^{7}
		$\hat{a}$	0.00239 ^{4}	0.00299 ^{5}	0.00154 ^{2}	0.00324 ^{6}	0.00058 ^{1}	0.00179 ^{3}	0.00655 ^{7}
	MRE	$\hat{\beta}$	0.06956 ^{2}	0.08292 ^{5}	0.07311 ^{4}	0.09183 ^{6}	0.06442 ^{1}	0.07292 ^{3}	0.09502 ^{7}
		$\hat{\lambda}$	3.40953 ^{3}	4.95424 ^{5}	3.04924 ^{2}	5.31819 ^{6}	1.37610 ^{1}	3.43176 ^{4}	7.95547 ^{7}
		$\hat{a}$	2.41388 ^{3}	3.59427 ^{5}	2.22747 ^{2}	3.65089 ^{6}	1.27808 ^{1}	2.60925 ^{4}	5.10920 ^{7}
	$\sum Ranks$		26 ^{3}	45 ^{5}	24 ^{2}	54 ^{6}	9 ^{1}	31 ^{4}	63 ^{7}
250	BIAS	$\hat{\beta}$	0.02336 ^{2}	0.03111 ^{5}	0.02445 ^{3}	0.03261 ^{7}	0.02096 ^{1}	0.02520 ^{4}	0.03217 ^{6}
		$\hat{\lambda}$	0.25771 ^{3}	0.43312 ^{5}	0.23558 ^{2}	0.48727 ^{6}	0.12456 ^{1}	0.29627 ^{4}	0.65501 ^{7}
		$\hat{a}$	0.01946 ^{3}	0.03191 ^{5}	0.01813 ^{2}	0.03488 ^{6}	0.01124 ^{1}	0.02260 ^{4}	0.04390 ^{7}
	MSE	$\hat{\beta}$	0.00092 ^{2}	0.00156 ^{5}	0.00098 ^{3}	0.00171 ^{7}	0.00068 ^{1}	0.00105 ^{4}	0.00169 ^{6}
		$\hat{\lambda}$	0.40892 ^{4}	0.55113 ^{5}	0.24248 ^{2}	0.65357 ^{6}	0.07317 ^{1}	0.31076 ^{3}	1.43001 ^{7}
		$\hat{a}$	0.00165 ^{4}	0.00261 ^{5}	0.00118 ^{2}	0.00291 ^{6}	0.00048 ^{1}	0.00146 ^{3}	0.00556 ^{7}
	MRE	$\hat{\beta}$	0.04673 ^{2}	0.06222 ^{5}	0.04891 ^{3}	0.06523 ^{7}	0.04191 ^{1}	0.05040 ^{4}	0.06434 ^{6}
		$\hat{\lambda}$	2.57705 ^{3}	4.33117 ^{5}	2.35579 ^{2}	4.87270 ^{6}	1.24561 ^{1}	2.96271 ^{4}	6.55005 ^{7}
		$\hat{a}$	1.94616 ^{3}	3.19105 ^{5}	1.81341 ^{2}	3.48761 ^{6}	1.12428 ^{1}	2.25998 ^{4}	4.38995 ^{7}
	$\sum Ranks$		26 ^{3}	45 ^{5}	21 ^{2}	57 ^{6}	9 ^{1}	34 ^{4}	60 ^{7}
500	BIAS	$\hat{\beta}$	0.01685 ^{2}	0.02573 ^{7}	0.01934 ^{4}	0.02529 ^{6}	0.01555 ^{1}	0.01930 ^{3}	0.02475 ^{5}
		$\hat{\lambda}$	0.18437 ^{2}	0.36282 ^{5}	0.19075 ^{3}	0.36433 ^{6}	0.10048 ^{1}	0.22932 ^{4}	0.52949 ^{7}
		$\hat{a}$	0.01477 ^{2}	0.02693 ^{5}	0.01491 ^{3}	0.02726 ^{6}	0.00924 ^{1}	0.01811 ^{4}	0.03695 ^{7}
	MSE	$\hat{\beta}$	0.00051 ^{2}	0.00106 ^{7}	0.00062 ^{3,5}	0.00103 ^{5}	0.00038 ^{1}	0.00062 ^{3,5}	0.00105 ^{6}
		$\hat{\lambda}$	0.18611 ^{3}	0.40062 ^{5}	0.16553 ^{2}	0.40578 ^{6}	0.04266 ^{1}	0.21772 ^{4}	1.03062 ^{7}
		$\hat{a}$	0.00093 ^{3}	0.00189 ^{5}	0.00081 ^{2}	0.00197 ^{6}	0.00029 ^{1}	0.00109 ^{4}	0.00428 ^{7}
	MRE	$\hat{\beta}$	0.03370 ^{2}	0.05145 ^{7}	0.03867 ^{4}	0.05058 ^{6}	0.03109 ^{1}	0.03860 ^{3}	0.04951 ^{5}
		$\hat{\lambda}$	1.84375 ^{2}	3.62822 ^{5}	1.90754 ^{3}	3.64325 ^{6}	1.00482 ^{1}	2.29321 ^{4}	5.29486 ^{7}
		$\hat{a}$	1.47678 ^{2}	2.69252 ^{5}	1.49071 ^{3}	2.72561 ^{6}	0.92427 ^{1}	1.81079 ^{4}	3.69467 ^{7}
	$\sum Ranks$		20 ^{2}	51 ^{5}	27.5 ^{3}	53 ^{6}	9 ^{1}	33.5 ^{4}	58 ^{7}
750	BIAS	$\hat{\beta}$	0.01254 ^{1}	0.02157 ^{7}	0.01605 ^{4}	0.02115 ^{6}	0.01324 ^{2}	0.01596 ^{3}	0.02078 ^{5}
		$\hat{\lambda}$	0.13582 ^{2}	0.32278 ^{6}	0.18315 ^{3}	0.31330 ^{5}	0.08927 ^{1}	0.19310 ^{4}	0.39182 ^{7}
		$\hat{a}$	0.01104 ^{2}	0.02454 ^{6}	0.01453 ^{3}	0.02370 ^{5}	0.00807 ^{1}	0.01567 ^{4}	0.02795 ^{7}
	MSE	$\hat{\beta}$	0.00032 ^{2}	0.00077 ^{6}	0.00044 ^{4}	0.00077 ^{6}	0.00028 ^{1}	0.00042 ^{3}	0.00077 ^{6}
		$\hat{\lambda}$	0.11249 ^{2}	0.34525 ^{6}	0.16210 ^{4}	0.34373 ^{5}	0.03511 ^{1}	0.15328 ^{3}	0.67307 ^{7}
		$\hat{a}$	0.00059 ^{2}	0.00171 ^{6}	0.00084 ^{4}	0.00165 ^{5}	0.00022 ^{1}	0.00083 ^{3}	0.00287 ^{7}
	MRE	$\hat{\beta}$	0.02507 ^{1}	0.04315 ^{7}	0.03209 ^{4}	0.04230 ^{6}	0.02647 ^{2}	0.03191 ^{3}	0.04156 ^{5}
		$\hat{\lambda}$	1.35819 ^{2}	3.22779 ^{6}	1.83147 ^{3}	3.13295 ^{5}	0.89271 ^{1}	1.93098 ^{4}	3.91823 ^{7}
		$\hat{a}$	1.10390 ^{2}	2.45417 ^{6}	1.45270 ^{3}	2.37050 ^{5}	0.80666 ^{1}	1.56734 ^{4}	2.79530 ^{7}
	$\sum Ranks$		16 ^{2}	56 ^{6}	32 ^{4}	48 ^{5}	11 ^{1}	31 ^{3}	58 ^{7}

Table 4: Numerical values of the NExLL distribution for $\theta = (0.5, 0.1, 0.03)^T$

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
50	BIAS	$\hat{\beta}$	0.04850 ^{2}	0.05598 ^{5}	0.04784 ^{1}	0.05726 ^{7}	0.05166 ^{4}	0.04932 ^{3}	0.05703 ^{6}
		$\hat{\lambda}$	0.29981 ^{7}	0.16963 ^{5}	0.10785 ^{1}	0.16901 ^{4}	0.16406 ^{3}	0.13846 ^{2}	0.26535 ^{6}
		$\hat{a}$	0.06235 ^{7}	0.04355 ^{5}	0.02866 ^{1}	0.04182 ^{3}	0.04248 ^{4}	0.03676 ^{2}	0.05835 ^{6}
	MSE	$\hat{\beta}$	0.00401 ^{3}	0.00496 ^{5}	0.00357 ^{1}	0.00553 ^{6}	0.00402 ^{4}	0.00378 ^{2}	0.00557 ^{7}
		$\hat{\lambda}$	1.03436 ^{7}	0.07268 ^{4}	0.03654 ^{1}	0.06134 ^{3}	0.34862 ^{6}	0.04730 ^{2}	0.17315 ^{5}
		$\hat{a}$	0.03358 ^{7}	0.00637 ^{4}	0.00217 ^{1}	0.00327 ^{3}	0.01579 ^{6}	0.00302 ^{2}	0.00700 ^{5}
	MRE	$\hat{\beta}$	0.09700 ^{2}	0.11197 ^{5}	0.09568 ^{1}	0.11452 ^{7}	0.10333 ^{4}	0.09864 ^{3}	0.11407 ^{6}
		$\hat{\lambda}$	2.99813 ^{7}	1.69635 ^{5}	1.07848 ^{1}	1.69012 ^{4}	1.64063 ^{3}	1.38463 ^{2}	2.65347 ^{6}
		$\hat{a}$	2.07849 ^{7}	1.45167 ^{5}	0.95529 ^{1}	1.39384 ^{3}	1.41616 ^{4}	1.22519 ^{2}	1.94501 ^{6}
	$\Sigma Ranks$		49 ^{6}	43 ^{5}	9 ^{1}	40 ^{4}	38 ^{3}	20 ^{2}	53 ^{7}
100	BIAS	$\hat{\beta}$	0.03483 ^{2}	0.03871 ^{5}	0.03297 ^{1}	0.04221 ^{6}	0.03683 ^{4}	0.03494 ^{3}	0.04471 ^{7}
		$\hat{\lambda}$	0.28047 ^{7}	0.15064 ^{4}	0.10784 ^{1}	0.16939 ^{5}	0.14911 ^{3}	0.13291 ^{2}	0.23832 ^{6}
		$\hat{a}$	0.05943 ^{7}	0.03850 ^{4}	0.02823 ^{1}	0.04228 ^{5}	0.03840 ^{3}	0.03425 ^{2}	0.05534 ^{6}
	MSE	$\hat{\beta}$	0.00196 ^{3}	0.00240 ^{5}	0.00176 ^{1}	0.00298 ^{6}	0.00199 ^{4}	0.00195 ^{2}	0.00311 ^{7}
		$\hat{\lambda}$	1.86755 ^{7}	0.04461 ^{2}	0.03821 ^{1}	0.05869 ^{4}	0.32711 ^{6}	0.04698 ^{3}	0.13248 ^{5}
		$\hat{a}$	0.06286 ^{7}	0.00262 ^{2}	0.00221 ^{1}	0.00326 ^{4}	0.01257 ^{6}	0.00264 ^{3}	0.00605 ^{5}
	MRE	$\hat{\beta}$	0.06966 ^{2}	0.07741 ^{5}	0.06595 ^{1}	0.08442 ^{6}	0.07365 ^{4}	0.06988 ^{3}	0.08942 ^{7}
		$\hat{\lambda}$	2.80472 ^{7}	1.50636 ^{4}	1.07840 ^{1}	1.69388 ^{5}	1.49115 ^{3}	1.32905 ^{2}	2.38321 ^{6}
		$\hat{a}$	1.98109 ^{7}	1.28324 ^{4}	0.94095 ^{1}	1.40936 ^{5}	1.28008 ^{3}	1.14177 ^{2}	1.84471 ^{6}
	$\Sigma Ranks$		49 ^{6}	35 ^{3}	9 ^{1}	46 ^{5}	36 ^{4}	22 ^{2}	55 ^{7}
250	BIAS	$\hat{\beta}$	0.02267 ^{2}	0.02945 ^{6}	0.02230 ^{1}	0.02909 ^{5}	0.02402 ^{4}	0.02362 ^{3}	0.02999 ^{7}
		$\hat{\lambda}$	0.14869 ^{4}	0.15092 ^{6}	0.09054 ^{1}	0.15085 ^{5}	0.13697 ^{3}	0.11023 ^{2}	0.21008 ^{7}
		$\hat{a}$	0.03545 ^{4}	0.03842 ^{6}	0.02403 ^{1}	0.03815 ^{5}	0.03424 ^{3}	0.02885 ^{2}	0.05032 ^{7}
	MSE	$\hat{\beta}$	0.00079 ^{2}	0.00134 ^{5.5}	0.00078 ^{1}	0.00134 ^{5.5}	0.00088 ^{4}	0.00087 ^{3}	0.00140 ^{7}
		$\hat{\lambda}$	0.20722 ^{6}	0.04622 ^{4}	0.02521 ^{1}	0.04540 ^{3}	0.57043 ^{7}	0.02973 ^{2}	0.11003 ^{5}
		$\hat{a}$	0.00811 ^{6}	0.00265 ^{4}	0.00153 ^{1}	0.00256 ^{3}	0.01955 ^{7}	0.00179 ^{2}	0.00550 ^{5}
	MRE	$\hat{\beta}$	0.04533 ^{2}	0.05890 ^{6}	0.04461 ^{1}	0.05818 ^{5}	0.04805 ^{4}	0.04723 ^{3}	0.05999 ^{7}
		$\hat{\lambda}$	1.48694 ^{4}	1.50916 ^{6}	0.90544 ^{1}	1.50849 ^{5}	1.36968 ^{3}	1.10226 ^{2}	2.10084 ^{7}
		$\hat{a}$	1.18153 ^{4}	1.28079 ^{6}	0.80092 ^{1}	1.27157 ^{5}	1.14143 ^{3}	0.96156 ^{2}	1.67732 ^{7}
	$\Sigma Ranks$		34 ^{3}	49.5 ^{6}	9 ^{1}	41.5 ^{5}	38 ^{4}	21 ^{2}	59 ^{7}
500	BIAS	$\hat{\beta}$	0.01761 ^{2}	0.02382 ^{6}	0.01737 ^{1}	0.02333 ^{5}	0.01890 ^{4}	0.01832 ^{3}	0.02434 ^{7}
		$\hat{\lambda}$	0.11255 ^{4}	0.12936 ^{5}	0.07502 ^{1}	0.12974 ^{6}	0.09459 ^{3}	0.08745 ^{2}	0.18779 ^{7}
		$\hat{a}$	0.02767 ^{4}	0.03282 ^{5}	0.01999 ^{1}	0.03335 ^{6}	0.02532 ^{3}	0.02328 ^{2}	0.04507 ^{7}
	MSE	$\hat{\beta}$	0.00049 ^{2}	0.00086 ^{6}	0.00047 ^{1}	0.00083 ^{5}	0.00056 ^{4}	0.00053 ^{3}	0.00093 ^{7}
		$\hat{\lambda}$	0.07825 ^{6}	0.03607 ^{5}	0.01620 ^{1}	0.03417 ^{4}	0.02793 ^{3}	0.01963 ^{2}	0.09438 ^{7}
		$\hat{a}$	0.00348 ^{6}	0.00203 ^{5}	0.00100 ^{1}	0.00201 ^{4}	0.00168 ^{3}	0.00121 ^{2}	0.00467 ^{7}
	MRE	$\hat{\beta}$	0.03522 ^{2}	0.04764 ^{6}	0.03475 ^{1}	0.04666 ^{5}	0.03781 ^{4}	0.03664 ^{3}	0.04869 ^{7}
		$\hat{\lambda}$	1.12551 ^{4}	1.29365 ^{5}	0.75016 ^{1}	1.29736 ^{6}	0.94593 ^{3}	0.87449 ^{2}	1.87787 ^{7}
		$\hat{a}$	0.92249 ^{4}	1.09415 ^{5}	0.66649 ^{1}	1.11179 ^{6}	0.84390 ^{3}	0.77596 ^{2}	1.50243 ^{7}
	$\Sigma Ranks$		34 ^{4}	48 ^{6}	9 ^{1}	47 ^{5}	30 ^{3}	21 ^{2}	63 ^{7}

(Continued)

Table 4 (continued)

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
750	[BIAS]	$\hat{\beta}$	0.01439 ^{1}	0.02040 ^{6}	0.01461 ^{2}	0.02031 ^{5}	0.01585 ^{4}	0.01518 ^{3}	0.02155 ^{7}
		$\hat{\lambda}$	0.08967 ^{4}	0.12111 ^{6}	0.06498 ^{1}	0.11692 ^{5}	0.07574 ^{3}	0.07348 ^{2}	0.16372 ^{7}
		$\hat{a}$	0.02300 ^{4}	0.03125 ^{6}	0.01730 ^{1}	0.02998 ^{5}	0.02053 ^{3}	0.01969 ^{2}	0.03970 ^{7}
	MSE	$\hat{\beta}$	0.00035 ^{2}	0.00063 ^{5.5}	0.00034 ^{1}	0.00063 ^{5.5}	0.00039 ^{4}	0.00037 ^{3}	0.00072 ^{7}
		$\hat{\lambda}$	0.03125 ^{5}	0.03243 ^{6}	0.01464 ^{1}	0.03056 ^{4}	0.01685 ^{3}	0.01496 ^{2}	0.07741 ^{7}
		$\hat{a}$	0.00173 ^{4}	0.00193 ^{6}	0.00089 ^{1}	0.00179 ^{5}	0.00107 ^{3}	0.00092 ^{2}	0.00395 ^{7}
	MRE	$\hat{\beta}$	0.02877 ^{1}	0.04080 ^{6}	0.02922 ^{2}	0.04062 ^{5}	0.03169 ^{4}	0.03037 ^{3}	0.04310 ^{7}
		$\hat{\lambda}$	0.89669 ^{4}	1.21112 ^{6}	0.64984 ^{1}	1.16924 ^{5}	0.75736 ^{3}	0.73480 ^{2}	1.63717 ^{7}
		$\hat{a}$	0.76671 ^{4}	1.04162 ^{6}	0.57654 ^{1}	0.99924 ^{5}	0.68435 ^{3}	0.65641 ^{2}	1.32333 ^{7}
	$\Sigma Ranks$		29 ^{3}	53.5 ^{6}	11 ^{1}	44.5 ^{5}	30 ^{4}	21 ^{2}	63 ^{7}

Table 5: Numerical values of the NExLL distribution for $\theta = (0.5, 0.1, 0.05)^T$

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
50	[BIAS]	$\hat{\beta}$	0.04857 ^{2}	0.05481 ^{6}	0.04729 ^{1}	0.05451 ^{5}	0.05366 ^{4}	0.04858 ^{3}	0.05587 ^{7}
		$\hat{\lambda}$	0.28267 ^{7}	0.13842 ^{2}	0.15490 ^{4}	0.14076 ^{3}	0.20279 ^{6}	0.12834 ^{1}	0.16558 ^{5}
		$\hat{a}$	0.10252 ^{7}	0.06188 ^{3}	0.06792 ^{5}	0.06176 ^{2}	0.08262 ^{6}	0.05788 ^{1}	0.06783 ^{4}
	MSE	$\hat{\beta}$	0.00393 ^{3}	0.00467 ^{5}	0.00344 ^{1}	0.00494 ^{6}	0.00425 ^{4}	0.00366 ^{2}	0.00511 ^{7}
		$\hat{\lambda}$	0.59749 ^{7}	0.07696 ^{3}	0.24821 ^{5}	0.08522 ^{4}	0.52017 ^{6}	0.04914 ^{1}	0.05795 ^{2}
		$\hat{a}$	0.06084 ^{6}	0.01760 ^{4}	0.04235 ^{5}	0.01581 ^{3}	0.06142 ^{7}	0.00928 ^{2}	0.00904 ^{1}
	MRE	$\hat{\beta}$	0.09714 ^{2}	0.10961 ^{6}	0.09458 ^{1}	0.10903 ^{5}	0.10732 ^{4}	0.09716 ^{3}	0.11174 ^{7}
		$\hat{\lambda}$	2.82674 ^{7}	1.38421 ^{2}	1.54905 ^{4}	1.40756 ^{3}	2.02789 ^{6}	1.28338 ^{1}	1.65585 ^{5}
		$\hat{a}$	2.05039 ^{7}	1.23752 ^{3}	1.35846 ^{5}	1.23515 ^{2}	1.65249 ^{6}	1.15761 ^{1}	1.35656 ^{4}
	$\Sigma Ranks$		48 ^{6}	34 ^{4}	31 ^{2}	33 ^{3}	49 ^{7}	15 ^{1}	42 ^{5}
100	[BIAS]	$\hat{\beta}$	0.03666 ^{3}	0.03778 ^{5}	0.03174 ^{1}	0.04046 ^{6}	0.03740 ^{4}	0.03549 ^{2}	0.04305 ^{7}
		$\hat{\lambda}$	0.27507 ^{7}	0.10231 ^{1}	0.10500 ^{2}	0.12443 ^{4}	0.19374 ^{6}	0.12351 ^{3}	0.14452 ^{5}
		$\hat{a}$	0.10322 ^{7}	0.04574 ^{1}	0.04601 ^{2}	0.05483 ^{4}	0.07960 ^{6}	0.05362 ^{3}	0.06143 ^{5}
	MSE	$\hat{\beta}$	0.00215 ^{4}	0.00224 ^{5}	0.00157 ^{1}	0.00268 ^{6}	0.00206 ^{3}	0.00193 ^{2}	0.00283 ^{7}
		$\hat{\lambda}$	0.79904 ^{7}	0.02155 ^{1}	0.09849 ^{4}	0.05228 ^{3}	0.53248 ^{6}	0.15455 ^{5}	0.04132 ^{2}
		$\hat{a}$	0.08135 ^{7}	0.00404 ^{1}	0.01532 ^{4}	0.00916 ^{3}	0.06637 ^{6}	0.02443 ^{5}	0.00670 ^{2}
	MRE	$\hat{\beta}$	0.07332 ^{3}	0.07557 ^{5}	0.06348 ^{1}	0.08092 ^{6}	0.07480 ^{4}	0.07098 ^{2}	0.08610 ^{7}
		$\hat{\lambda}$	2.75073 ^{7}	1.02313 ^{1}	1.04997 ^{2}	1.24429 ^{4}	1.93741 ^{6}	1.23512 ^{3}	1.44518 ^{5}
		$\hat{a}$	2.06431 ^{7}	0.91482 ^{1}	0.92023 ^{2}	1.09669 ^{4}	1.59204 ^{6}	1.07235 ^{3}	1.22864 ^{5}
	$\Sigma Ranks$		52 ^{7}	21 ^{2}	19 ^{1}	40 ^{4}	47 ^{6}	28 ^{3}	45 ^{5}
250	[BIAS]	$\hat{\beta}$	0.02421 ^{3}	0.02822 ^{5}	0.02250 ^{1}	0.02846 ^{6}	0.02573 ^{4}	0.02305 ^{2}	0.02860 ^{7}
		$\hat{\lambda}$	0.19370 ^{6}	0.10009 ^{4}	0.07415 ^{1}	0.09923 ^{3}	0.20521 ^{7}	0.08684 ^{2}	0.12743 ^{5}
		$\hat{a}$	0.07353 ^{6}	0.04419 ^{4}	0.03250 ^{1}	0.04388 ^{3}	0.07705 ^{7}	0.03869 ^{2}	0.05468 ^{5}
	MSE	$\hat{\beta}$	0.00096 ^{3}	0.00121 ^{5}	0.00076 ^{1}	0.00126 ^{7}	0.00103 ^{4}	0.00085 ^{2}	0.00125 ^{6}
		$\hat{\lambda}$	0.48948 ^{6}	0.02059 ^{2}	0.02485 ^{4}	0.01867 ^{1}	1.50609 ^{7}	0.02219 ^{3}	0.03314 ^{5}
		$\hat{a}$	0.04715 ^{6}	0.00369 ^{2}	0.00404 ^{4}	0.00334 ^{1}	0.14476 ^{7}	0.00393 ^{3}	0.00546 ^{5}
	MRE	$\hat{\beta}$	0.04842 ^{3}	0.05644 ^{5}	0.04501 ^{1}	0.05692 ^{6}	0.05146 ^{4}	0.04610 ^{2}	0.05721 ^{7}
		$\hat{\lambda}$	1.93695 ^{6}	1.00088 ^{4}	0.74147 ^{1}	0.99225 ^{3}	2.05213 ^{7}	0.86838 ^{2}	1.27430 ^{5}
		$\hat{a}$	1.47056 ^{6}	0.88372 ^{4}	0.64997 ^{1}	0.87759 ^{3}	1.54093 ^{7}	0.77376 ^{2}	1.09368 ^{5}
	$\Sigma Ranks$		45 ^{5}	35 ^{4}	15 ^{1}	33 ^{3}	54 ^{7}	20 ^{2}	50 ^{6}

(Continued)

Table 5 (continued)

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
500	BIAS	$\hat{\beta}$	0.01821 ^{3}	0.02166 ^{5}	0.01727 ^{1}	0.02186 ^{6}	0.02025 ^{4}	0.01769 ^{2}	0.02249 ^{7}
		$\hat{\lambda}$	0.14943 ^{7}	0.07940 ^{3}	0.06014 ^{1}	0.08367 ^{4}	0.14040 ^{6}	0.06412 ^{2}	0.10636 ^{5}
		$\hat{a}$	0.05814 ^{7}	0.03541 ^{3}	0.02640 ^{1}	0.03747 ^{4}	0.05499 ^{6}	0.02913 ^{2}	0.04536 ^{5}
	MSE	$\hat{\beta}$	0.00053 ^{3}	0.00070 ^{5}	0.00045 ^{1}	0.00076 ^{6}	0.00062 ^{4}	0.00051 ^{2}	0.00078 ^{7}
		$\hat{\lambda}$	0.76992 ^{6}	0.01242 ^{2}	0.01529 ^{4}	0.01362 ^{3}	0.83670 ^{7}	0.01135 ^{1}	0.02612 ^{5}
		$\hat{a}$	0.07191 ^{6}	0.00228 ^{2}	0.00253 ^{4}	0.00251 ^{3}	0.07513 ^{7}	0.00208 ^{1}	0.00420 ^{5}
	MRE	$\hat{\beta}$	0.03642 ^{3}	0.04332 ^{5}	0.03455 ^{1}	0.04371 ^{6}	0.04050 ^{4}	0.03539 ^{2}	0.04498 ^{7}
		$\hat{\lambda}$	1.49430 ^{7}	0.79397 ^{3}	0.60135 ^{1}	0.83671 ^{4}	1.40397 ^{6}	0.64123 ^{2}	1.06361 ^{5}
		$\hat{a}$	1.16281 ^{7}	0.70828 ^{3}	0.52791 ^{1}	0.74930 ^{4}	1.09975 ^{6}	0.58267 ^{2}	0.90725 ^{5}
	$\Sigma Ranks$		49 ^{5}	31 ^{3}	15 ^{1}	40 ^{4}	50 ^{6}	16 ^{2}	51 ^{7}
750	BIAS	$\hat{\beta}$	0.01486 ^{3}	0.01906 ^{6}	0.01483 ^{2}	0.01885 ^{5}	0.01676 ^{4}	0.01453 ^{1}	0.02038 ^{7}
		$\hat{\lambda}$	0.11818 ^{7}	0.07027 ^{3}	0.05242 ^{1}	0.07173 ^{4}	0.07286 ^{5}	0.05581 ^{2}	0.09689 ^{6}
		$\hat{a}$	0.04617 ^{7}	0.03168 ^{3}	0.02339 ^{1}	0.03230 ^{4}	0.03266 ^{5}	0.02525 ^{2}	0.04166 ^{6}
	MSE	$\hat{\beta}$	0.00037 ^{3}	0.00055 ^{6}	0.00034 ^{1}	0.00053 ^{5}	0.00044 ^{4}	0.00035 ^{2}	0.00063 ^{7}
		$\hat{\lambda}$	0.63111 ^{7}	0.00985 ^{2}	0.01162 ^{4}	0.01000 ^{3}	0.01557 ^{5}	0.00975 ^{1}	0.02199 ^{6}
		$\hat{a}$	0.05555 ^{7}	0.00185 ^{2}	0.00194 ^{4}	0.00188 ^{3}	0.00271 ^{5}	0.00174 ^{1}	0.00366 ^{6}
	MRE	$\hat{\beta}$	0.02972 ^{3}	0.03813 ^{6}	0.02965 ^{2}	0.03771 ^{5}	0.03353 ^{4}	0.02906 ^{1}	0.04076 ^{7}
		$\hat{\lambda}$	1.18183 ^{7}	0.70265 ^{3}	0.52423 ^{1}	0.71729 ^{4}	0.72861 ^{5}	0.55810 ^{2}	0.96892 ^{6}
		$\hat{a}$	0.92331 ^{7}	0.63357 ^{3}	0.46775 ^{1}	0.64591 ^{4}	0.65310 ^{5}	0.50497 ^{2}	0.83316 ^{6}
	$\Sigma Ranks$		51 ^{6}	34 ^{3}	17 ^{2}	37 ^{4}	42 ^{5}	14 ^{1}	57 ^{7}

Table 6: Numerical values of the NExLL distribution for $\theta = (1, 0.5, 0.01)^T$

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
50	BIAS	$\hat{\beta}$	0.09487 ^{3}	0.11146 ^{5}	0.09269 ^{2}	0.12312 ^{7}	0.08800 ^{1}	0.10514 ^{4}	0.11716 ^{6}
		$\hat{\lambda}$	2.26694 ^{4}	2.54432 ^{5}	1.93898 ^{2}	3.19889 ^{6}	0.52887 ^{1}	1.98314 ^{3}	4.19138 ^{7}
		$\hat{a}$	0.02754 ^{4}	0.03243 ^{5}	0.02376 ^{2}	0.03483 ^{6}	0.01031 ^{1}	0.02586 ^{3}	0.04130 ^{7}
	MSE	$\hat{\beta}$	0.01508 ^{2}	0.02037 ^{5}	0.01548 ^{3}	0.02440 ^{7}	0.01224 ^{1}	0.01735 ^{4}	0.02283 ^{6}
		$\hat{\lambda}$	31.33821 ^{5}	20.01574 ^{4}	15.34851 ^{3}	32.46292 ^{6}	1.27203 ^{1}	14.83869 ^{2}	61.00750 ^{7}
		$\hat{a}$	0.00296 ^{6}	0.00237 ^{4}	0.00151 ^{2}	0.00282 ^{5}	0.00034 ^{1}	0.00172 ^{3}	0.00440 ^{7}
	MRE	$\hat{\beta}$	0.09487 ^{3}	0.11146 ^{5}	0.09269 ^{2}	0.12312 ^{7}	0.08800 ^{1}	0.10514 ^{4}	0.11716 ^{6}
		$\hat{\lambda}$	4.53389 ^{4}	5.08864 ^{5}	3.87796 ^{2}	6.39777 ^{6}	1.05774 ^{1}	3.96627 ^{3}	8.38277 ^{7}
		$\hat{a}$	2.75449 ^{4}	3.24273 ^{5}	2.37616 ^{2}	3.48302 ^{6}	1.03130 ^{1}	2.58561 ^{3}	4.12976 ^{7}
	$\Sigma Ranks$		35 ^{4}	43 ^{5}	20 ^{2}	56 ^{6}	9 ^{1}	29 ^{3}	60 ^{7}
100	BIAS	$\hat{\beta}$	0.07232 ^{3}	0.08977 ^{6}	0.07041 ^{2}	0.09200 ^{7}	0.06006 ^{1}	0.07243 ^{4}	0.08688 ^{5}
		$\hat{\lambda}$	2.18113 ^{4}	2.69946 ^{5}	1.85516 ^{2}	2.83437 ^{6}	0.34678 ^{1}	1.95773 ^{3}	3.96919 ^{7}
		$\hat{a}$	0.02726 ^{4}	0.03354 ^{5}	0.02337 ^{2}	0.03367 ^{6}	0.00750 ^{1}	0.02603 ^{3}	0.04340 ^{7}
	MSE	$\hat{\beta}$	0.00863 ^{3}	0.01263 ^{6}	0.00862 ^{2}	0.01405 ^{7}	0.00579 ^{1}	0.00919 ^{4}	0.01255 ^{5}
		$\hat{\lambda}$	24.33824 ^{6}	19.72583 ^{4}	13.31591 ^{2}	22.72361 ^{5}	0.58790 ^{1}	13.73306 ^{3}	46.93743 ^{7}
		$\hat{a}$	0.00288 ^{6}	0.00249 ^{4}	0.00158 ^{2}	0.00258 ^{5}	0.00018 ^{1}	0.00177 ^{3}	0.00452 ^{7}
	MRE	$\hat{\beta}$	0.07232 ^{3}	0.08977 ^{6}	0.07041 ^{2}	0.09200 ^{7}	0.06006 ^{1}	0.07243 ^{4}	0.08688 ^{5}
		$\hat{\lambda}$	4.36226 ^{4}	5.39893 ^{5}	3.71031 ^{2}	5.66875 ^{6}	0.69355 ^{1}	3.91545 ^{3}	7.93839 ^{7}
		$\hat{a}$	2.72615 ^{4}	3.35424 ^{5}	2.33743 ^{2}	3.36652 ^{6}	0.75010 ^{1}	2.60267 ^{3}	4.33978 ^{7}
	$\Sigma Ranks$		37 ^{4}	46 ^{5}	18 ^{2}	55 ^{6}	9 ^{1}	30 ^{3}	57 ^{7}

(Continued)

Table 6 (continued)

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
250	BIAS	$\hat{\beta}$	0.04565 ^{2}	0.06092 ^{5}	0.04815 ^{3}	0.06277 ^{6}	0.03854 ^{1}	0.05340 ^{4}	0.06359 ^{7}
		$\hat{\lambda}$	1.41499 ^{2}	2.28926 ^{6}	1.43402 ^{3}	2.23696 ^{5}	0.25679 ^{1}	1.74010 ^{4}	3.27585 ^{7}
		$\hat{a}$	0.01917 ^{2}	0.03083 ^{6}	0.01987 ^{3}	0.02911 ^{5}	0.00564 ^{1}	0.02359 ^{4}	0.03966 ^{7}
	MSE	$\hat{\beta}$	0.00353 ^{2}	0.00593 ^{5}	0.00367 ^{3}	0.00616 ^{6}	0.00234 ^{1}	0.00468 ^{4}	0.00639 ^{7}
		$\hat{\lambda}$	12.75668 ^{4}	13.75515 ^{5}	8.70761 ^{2}	14.23246 ^{6}	0.52116 ^{1}	10.73943 ^{3}	32.01686 ^{7}
		$\hat{a}$	0.00159 ^{4}	0.00213 ^{6}	0.00129 ^{2}	0.00200 ^{5}	0.00015 ^{1}	0.00148 ^{3}	0.00393 ^{7}
	MRE	$\hat{\beta}$	0.04565 ^{2}	0.06092 ^{5}	0.04815 ^{3}	0.06277 ^{6}	0.03854 ^{1}	0.05340 ^{4}	0.06359 ^{7}
		$\hat{\lambda}$	2.82998 ^{2}	4.57851 ^{6}	2.86804 ^{3}	4.47392 ^{5}	0.51357 ^{1}	3.48020 ^{4}	6.55170 ^{7}
		$\hat{a}$	1.91650 ^{2}	3.08262 ^{6}	1.98689 ^{3}	2.91137 ^{5}	0.56422 ^{1}	2.35903 ^{4}	3.96612 ^{7}
	$\sum Ranks$		22 ^{2}	50 ^{6}	25 ^{3}	49 ^{5}	9 ^{1}	34 ^{4}	63 ^{7}
500	BIAS	$\hat{\beta}$	0.03382 ^{2}	0.05002 ^{6}	0.03637 ^{3}	0.05012 ^{7}	0.02849 ^{1}	0.04019 ^{4}	0.04947 ^{5}
		$\hat{\lambda}$	1.08204 ^{2}	1.94429 ^{5}	1.09341 ^{3}	1.99082 ^{6}	0.14460 ^{1}	1.41529 ^{4}	2.66194 ^{7}
		$\hat{a}$	0.01542 ^{3}	0.02630 ^{5}	0.01538 ^{2}	0.02697 ^{6}	0.00343 ^{1}	0.02031 ^{4}	0.03407 ^{7}
	MSE	$\hat{\beta}$	0.00206 ^{2}	0.00395 ^{7}	0.00228 ^{3}	0.00388 ^{6}	0.00125 ^{1}	0.00263 ^{4}	0.00387 ^{5}
		$\hat{\lambda}$	6.39441 ^{3}	10.24676 ^{5}	5.53983 ^{2}	10.97119 ^{6}	0.28733 ^{1}	7.31199 ^{4}	21.82101 ^{7}
		$\hat{a}$	0.00096 ^{3}	0.00159 ^{5}	0.00081 ^{2}	0.00171 ^{6}	0.00007 ^{1}	0.00120 ^{4}	0.00298 ^{7}
	MRE	$\hat{\beta}$	0.03382 ^{2}	0.05002 ^{6}	0.03637 ^{3}	0.05012 ^{7}	0.02849 ^{1}	0.04019 ^{4}	0.04947 ^{5}
		$\hat{\lambda}$	2.16407 ^{2}	3.88857 ^{5}	2.18683 ^{3}	3.98164 ^{6}	0.28920 ^{1}	2.83058 ^{4}	5.32387 ^{7}
		$\hat{a}$	1.54171 ^{3}	2.62955 ^{5}	1.53832 ^{2}	2.69676 ^{6}	0.34258 ^{1}	2.03096 ^{4}	3.40700 ^{7}
	$\sum Ranks$		22 ^{2}	49 ^{5}	23 ^{3}	56 ^{6}	9 ^{1}	36 ^{4}	57 ^{7}
750	BIAS	$\hat{\beta}$	0.02411 ^{2}	0.04274 ^{6}	0.02962 ^{3}	0.04442 ^{7}	0.02168 ^{1}	0.03642 ^{4}	0.04091 ^{5}
		$\hat{\lambda}$	0.80015 ^{2}	1.63590 ^{5}	0.90079 ^{3}	1.74803 ^{6}	0.07398 ^{1}	1.32147 ^{4}	2.02815 ^{7}
		$\hat{a}$	0.01163 ^{2}	0.02282 ^{5}	0.01316 ^{3}	0.02395 ^{6}	0.00216 ^{1}	0.01892 ^{4}	0.02684 ^{7}
	MSE	$\hat{\beta}$	0.00120 ^{2}	0.00286 ^{6}	0.00151 ^{3}	0.00312 ^{7}	0.00076 ^{1}	0.00217 ^{4}	0.00285 ^{5}
		$\hat{\lambda}$	4.68162 ^{3}	8.03852 ^{5}	4.12171 ^{2}	8.75820 ^{6}	0.08618 ^{1}	6.17373 ^{4}	14.52214 ^{7}
		$\hat{a}$	0.00069 ^{3}	0.00128 ^{5}	0.00067 ^{2}	0.00137 ^{6}	0.00003 ^{1}	0.00100 ^{4}	0.00204 ^{7}
	MRE	$\hat{\beta}$	0.02411 ^{2}	0.04274 ^{6}	0.02962 ^{3}	0.04442 ^{7}	0.02168 ^{1}	0.03642 ^{4}	0.04091 ^{5}
		$\hat{\lambda}$	1.60031 ^{2}	3.27180 ^{5}	1.80158 ^{3}	3.49605 ^{6}	0.14796 ^{1}	2.64295 ^{4}	4.05631 ^{7}
		$\hat{a}$	1.16263 ^{2}	2.28176 ^{5}	1.31649 ^{3}	2.39549 ^{6}	0.21602 ^{1}	1.89217 ^{4}	2.68442 ^{7}
	$\sum Ranks$		20 ^{2}	48 ^{5}	25 ^{3}	57 ^{6.5}	9 ^{1}	36 ^{4}	57 ^{6.5}

Table 7: Numerical values of the NExLL distribution for $\theta = (1, 0.25, 0.01)^T$

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
50	BIAS	$\hat{\beta}$	0.09540 ^{3}	0.11356 ^{5}	0.09461 ^{2}	0.12388 ^{7}	0.08925 ^{1}	0.10574 ^{4}	0.11932 ^{6}
		$\hat{\lambda}$	1.08117 ^{4}	1.29510 ^{5}	0.95246 ^{2}	1.56126 ^{6}	0.33386 ^{1}	0.99041 ^{3}	2.16544 ^{7}
		$\hat{a}$	0.02780 ^{4}	0.03355 ^{5}	0.02506 ^{2}	0.03681 ^{6}	0.01207 ^{1}	0.02673 ^{3}	0.04525 ^{7}
	MSE	$\hat{\beta}$	0.01570 ^{2}	0.02153 ^{5}	0.01576 ^{3}	0.02491 ^{7}	0.01261 ^{1}	0.01776 ^{4}	0.02384 ^{6}
		$\hat{\lambda}$	6.69085 ^{5}	5.20787 ^{4}	3.46496 ^{2}	7.34425 ^{6}	0.47494 ^{1}	3.88305 ^{3}	15.48404 ^{7}
		$\hat{a}$	0.00294 ^{5}	0.00250 ^{4}	0.00163 ^{2}	0.00313 ^{6}	0.00048 ^{1}	0.00184 ^{3}	0.00521 ^{7}
	MRE	$\hat{\beta}$	0.09540 ^{3}	0.11356 ^{5}	0.09461 ^{2}	0.12388 ^{7}	0.08925 ^{1}	0.10574 ^{4}	0.11932 ^{6}
		$\hat{\lambda}$	4.32466 ^{4}	5.18039 ^{5}	3.80986 ^{2}	6.24503 ^{6}	1.33546 ^{1}	3.96163 ^{3}	8.66175 ^{7}
		$\hat{a}$	2.77983 ^{4}	3.35528 ^{5}	2.50599 ^{2}	3.68141 ^{6}	1.20663 ^{1}	2.67293 ^{3}	4.52521 ^{7}
	$\sum Ranks$		34 ^{4}	43 ^{5}	19 ^{2}	57 ^{6}	9 ^{1}	30 ^{3}	60 ^{7}

(Continued)

Table 7 (continued)

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
100	BIAS	$\hat{\beta}$	0.07378 ^{4}	0.09027 ^{6}	0.07265 ^{2}	0.09260 ^{7}	0.06199 ^{1}	0.07283 ^{3}	0.08877 ^{5}
		$\hat{\lambda}$	1.06611 ^{4}	1.33046 ^{5}	0.88942 ^{2}	1.40160 ^{6}	0.27978 ^{1}	0.93225 ^{3}	2.03651 ^{7}
		$\hat{a}$	0.02833 ^{4}	0.03515 ^{5}	0.02421 ^{2}	0.03552 ^{6}	0.01032 ^{1}	0.02624 ^{3}	0.04698 ^{7}
	MSE	$\hat{\beta}$	0.00889 ^{3}	0.01279 ^{5}	0.00888 ^{2}	0.01430 ^{7}	0.00609 ^{1}	0.00932 ^{4}	0.01316 ^{6}
		$\hat{\lambda}$	5.71159 ^{6}	4.67331 ^{4}	2.79704 ^{2}	5.52604 ^{5}	0.37822 ^{1}	3.01583 ^{3}	12.01410 ^{7}
		$\hat{a}$	0.00314 ^{6}	0.00277 ^{4}	0.00164 ^{2}	0.00293 ^{5}	0.00035 ^{1}	0.00181 ^{3}	0.00537 ^{7}
	MRE	$\hat{\beta}$	0.07378 ^{4}	0.09027 ^{6}	0.07265 ^{2}	0.09260 ^{7}	0.06199 ^{1}	0.07283 ^{3}	0.08877 ^{5}
		$\hat{\lambda}$	4.26443 ^{4}	5.32184 ^{5}	3.55767 ^{2}	5.60641 ^{6}	1.11911 ^{1}	3.72900 ^{3}	8.14605 ^{7}
		$\hat{a}$	2.83308 ^{4}	3.51458 ^{5}	2.42142 ^{2}	3.55193 ^{6}	1.03191 ^{1}	2.62358 ^{3}	4.69849 ^{7}
	$\sum Ranks$		39 ^{4}	45 ^{5}	18 ^{2}	55 ^{6}	9 ^{1}	28 ^{3}	58 ^{7}
250	BIAS	$\hat{\beta}$	0.04651 ^{2}	0.06192 ^{5}	0.05001 ^{3}	0.06368 ^{6}	0.04019 ^{1}	0.05363 ^{4}	0.06518 ^{7}
		$\hat{\lambda}$	0.68189 ^{2}	1.16733 ^{6}	0.74237 ^{3}	1.12680 ^{5}	0.23958 ^{1}	0.85324 ^{4}	1.70951 ^{7}
		$\hat{a}$	0.01946 ^{2}	0.03258 ^{6}	0.02161 ^{3}	0.03080 ^{5}	0.00888 ^{1}	0.02436 ^{4}	0.04323 ^{7}
	MSE	$\hat{\beta}$	0.00363 ^{2}	0.00612 ^{5}	0.00393 ^{3}	0.00633 ^{6}	0.00252 ^{1}	0.00472 ^{4}	0.00674 ^{7}
		$\hat{\lambda}$	2.74323 ^{4}	3.70332 ^{6}	2.14536 ^{2}	3.62712 ^{5}	0.32942 ^{1}	2.50081 ^{3}	8.78422 ^{7}
		$\hat{a}$	0.00160 ^{3,5}	0.00249 ^{6}	0.00148 ^{2}	0.00231 ^{5}	0.00031 ^{1}	0.00160 ^{3,5}	0.00482 ^{7}
	MRE	$\hat{\beta}$	0.04651 ^{2}	0.06192 ^{5}	0.05001 ^{3}	0.06368 ^{6}	0.04019 ^{1}	0.05363 ^{4}	0.06518 ^{7}
		$\hat{\lambda}$	2.72756 ^{2}	4.66930 ^{6}	2.96947 ^{3}	4.50720 ^{5}	0.95832 ^{1}	3.41297 ^{4}	6.83802 ^{7}
		$\hat{a}$	1.94640 ^{2}	3.25787 ^{6}	2.16119 ^{3}	3.07995 ^{5}	0.88836 ^{1}	2.43621 ^{4}	4.32284 ^{7}
	$\sum Ranks$		21.5 ^{2}	51 ^{6}	25 ^{3}	48 ^{5}	9 ^{1}	34.5 ^{4}	63 ^{7}
500	BIAS	$\hat{\beta}$	0.03561 ^{2}	0.05104 ^{5}	0.03797 ^{3}	0.05107 ^{6}	0.03031 ^{1}	0.04064 ^{4}	0.05119 ^{7}
		$\hat{\lambda}$	0.55914 ^{2}	0.99457 ^{5}	0.58626 ^{3}	1.01913 ^{6}	0.17455 ^{1}	0.70468 ^{4}	1.42063 ^{7}
		$\hat{a}$	0.01653 ^{2}	0.02793 ^{5}	0.01716 ^{3}	0.02856 ^{6}	0.00658 ^{1}	0.02098 ^{4}	0.03748 ^{7}
	MSE	$\hat{\beta}$	0.00223 ^{2}	0.00412 ^{6}	0.00255 ^{3}	0.00406 ^{5}	0.00140 ^{1}	0.00270 ^{4}	0.00418 ^{7}
		$\hat{\lambda}$	1.55525 ^{3}	2.77392 ^{5}	1.47275 ^{2}	3.02239 ^{6}	0.17318 ^{1}	1.79430 ^{4}	6.38873 ^{7}
		$\hat{a}$	0.00104 ^{3}	0.00188 ^{5}	0.00097 ^{2}	0.00202 ^{6}	0.00017 ^{1}	0.00130 ^{4}	0.00379 ^{7}
	MRE	$\hat{\beta}$	0.03561 ^{2}	0.05104 ^{5}	0.03797 ^{3}	0.05107 ^{6}	0.03031 ^{1}	0.04064 ^{4}	0.05119 ^{7}
		$\hat{\lambda}$	2.23654 ^{2}	3.97829 ^{5}	2.34503 ^{3}	4.07653 ^{6}	0.69818 ^{1}	2.81870 ^{4}	5.68254 ^{7}
		$\hat{a}$	1.65337 ^{2}	2.79271 ^{5}	1.71646 ^{3}	2.85618 ^{6}	0.65799 ^{1}	2.09818 ^{4}	3.74807 ^{7}
	$\sum Ranks$		20 ^{2}	46 ^{5}	25 ^{3}	53 ^{6}	9 ^{1}	36 ^{4}	63 ^{7}
750	BIAS	$\hat{\beta}$	0.02447 ^{2}	0.04357 ^{6}	0.03142 ^{3}	0.04556 ^{7}	0.02387 ^{1}	0.03702 ^{4}	0.04206 ^{5}
		$\hat{\lambda}$	0.38507 ^{2}	0.83913 ^{5}	0.48552 ^{3}	0.89920 ^{6}	0.12427 ^{1}	0.66220 ^{4}	1.06361 ^{7}
		$\hat{a}$	0.01169 ^{2}	0.02417 ^{5}	0.01459 ^{3}	0.02549 ^{6}	0.00491 ^{1}	0.01961 ^{4}	0.02893 ^{7}
	MSE	$\hat{\beta}$	0.00121 ^{2}	0.00301 ^{5}	0.00176 ^{3}	0.00330 ^{7}	0.00091 ^{1}	0.00224 ^{4}	0.00306 ^{6}
		$\hat{\lambda}$	1.03263 ^{2}	2.19452 ^{5}	1.10076 ^{3}	2.42171 ^{6}	0.08960 ^{1}	1.58097 ^{4}	4.14078 ^{7}
		$\hat{a}$	0.00069 ^{2}	0.00152 ^{5}	0.00077 ^{3}	0.00164 ^{6}	0.00010 ^{1}	0.00110 ^{4}	0.00252 ^{7}
	MRE	$\hat{\beta}$	0.02447 ^{2}	0.04357 ^{6}	0.03142 ^{3}	0.04556 ^{7}	0.02387 ^{1}	0.03702 ^{4}	0.04206 ^{5}
		$\hat{\lambda}$	1.54027 ^{2}	3.35653 ^{5}	1.94210 ^{3}	3.59681 ^{6}	0.49709 ^{1}	2.64880 ^{4}	4.25445 ^{7}
		$\hat{a}$	1.16930 ^{2}	2.41713 ^{5}	1.45903 ^{3}	2.54876 ^{6}	0.49109 ^{1}	1.96087 ^{4}	2.89327 ^{7}
	$\sum Ranks$		18 ^{2}	47 ^{5}	27 ^{3}	57 ^{6}	9 ^{1}	36 ^{4}	58 ^{7}

Table 8: Numerical values of the NExLL distribution for $\theta = (1, 0.25, 0.02)^T$

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
50	BIAS	$\hat{\beta}$	0.09764 ^{3}	0.10891 ^{5}	0.08783 ^{1}	0.12015 ^{7}	0.09211 ^{2}	0.10274 ^{4}	0.11836 ^{6}
		$\hat{\lambda}$	0.89701 ^{6}	0.60787 ^{4}	0.40951 ^{2}	0.75408 ^{5}	0.30302 ^{1}	0.52984 ^{3}	1.15463 ^{7}
		$\hat{a}$	0.04802 ^{6}	0.03567 ^{4}	0.02561 ^{2}	0.04093 ^{5}	0.02157 ^{1}	0.03267 ^{3}	0.05357 ^{7}
	MSE	$\hat{\beta}$	0.01620 ^{3}	0.01988 ^{5}	0.01322 ^{2}	0.02354 ^{6}	0.01304 ^{1}	0.01686 ^{4}	0.02423 ^{7}
		$\hat{\lambda}$	3.73899 ^{6}	0.99525 ^{4}	0.61429 ^{2}	1.43177 ^{5}	0.37487 ^{1}	0.84527 ^{3}	4.01775 ^{7}
		$\hat{a}$	0.00834 ^{7}	0.00251 ^{4}	0.00168 ^{2}	0.00331 ^{5}	0.00136 ^{1}	0.00234 ^{3}	0.00621 ^{6}
	MRE	$\hat{\beta}$	0.09764 ^{3}	0.10891 ^{5}	0.08783 ^{1}	0.12015 ^{7}	0.09211 ^{2}	0.10274 ^{4}	0.11836 ^{6}
		$\hat{\lambda}$	3.58804 ^{6}	2.43148 ^{4}	1.63802 ^{2}	3.01633 ^{5}	1.21207 ^{1}	2.11934 ^{3}	4.61851 ^{7}
		$\hat{a}$	2.40107 ^{6}	1.78365 ^{4}	1.28040 ^{2}	2.04660 ^{5}	1.07862 ^{1}	1.63367 ^{3}	2.67833 ^{7}
	$\sum Ranks$		46 ^{5}	39 ^{4}	16 ^{2}	50 ^{6}	11 ^{1}	30 ^{3}	60 ^{7}
100	BIAS	$\hat{\beta}$	0.07042 ^{3}	0.08613 ^{5}	0.06716 ^{2}	0.08847 ^{7}	0.06577 ^{1}	0.07161 ^{4}	0.08671 ^{6}
		$\hat{\lambda}$	0.69457 ^{5}	0.61638 ^{4}	0.39204 ^{2}	0.69592 ^{6}	0.28792 ^{1}	0.48029 ^{3}	1.07603 ^{7}
		$\hat{a}$	0.04003 ^{6}	0.03698 ^{4}	0.02458 ^{2}	0.03939 ^{5}	0.02044 ^{1}	0.03003 ^{3}	0.05503 ^{7}
	MSE	$\hat{\beta}$	0.00802 ^{3}	0.01173 ^{5}	0.00762 ^{2}	0.01329 ^{7}	0.00676 ^{1}	0.00879 ^{4}	0.01249 ^{6}
		$\hat{\lambda}$	2.11811 ^{6}	0.84696 ^{4}	0.46254 ^{2}	1.19588 ^{5}	0.31141 ^{1}	0.64984 ^{3}	3.13197 ^{7}
		$\hat{a}$	0.00570 ^{6}	0.00262 ^{4}	0.00150 ^{2}	0.00308 ^{5}	0.00119 ^{1}	0.00197 ^{3}	0.00662 ^{7}
	MRE	$\hat{\beta}$	0.07042 ^{3}	0.08613 ^{5}	0.06716 ^{2}	0.08847 ^{7}	0.06577 ^{1}	0.07161 ^{4}	0.08671 ^{6}
		$\hat{\lambda}$	2.77827 ^{5}	2.46552 ^{4}	1.56816 ^{2}	2.78369 ^{6}	1.15167 ^{1}	1.92115 ^{3}	4.30412 ^{7}
		$\hat{a}$	2.00134 ^{6}	1.84878 ^{4}	1.22887 ^{2}	1.96943 ^{5}	1.02205 ^{1}	1.50162 ^{3}	2.75144 ^{7}
	$\sum Ranks$		43 ^{5}	39 ^{4}	18 ^{2}	53 ^{6}	9 ^{1}	30 ^{3}	60 ^{7}
250	BIAS	$\hat{\beta}$	0.04774 ^{3}	0.05819 ^{5}	0.04705 ^{2}	0.06033 ^{6}	0.04239 ^{1}	0.05063 ^{4}	0.06469 ^{7}
		$\hat{\lambda}$	0.51009 ^{4}	0.58143 ^{5}	0.36156 ^{2}	0.58769 ^{6}	0.19907 ^{1}	0.46441 ^{3}	0.91010 ^{7}
		$\hat{a}$	0.03049 ^{4}	0.03575 ^{6}	0.02305 ^{2}	0.03547 ^{5}	0.01494 ^{1}	0.02921 ^{3}	0.05035 ^{7}
	MSE	$\hat{\beta}$	0.00359 ^{3}	0.00551 ^{5}	0.00349 ^{2}	0.00578 ^{6}	0.00281 ^{1}	0.00426 ^{4}	0.00659 ^{7}
		$\hat{\lambda}$	1.26439 ^{6}	0.78719 ^{4}	0.47623 ^{2}	0.83061 ^{5}	0.15262 ^{1}	0.60583 ^{3}	2.20367 ^{7}
		$\hat{a}$	0.00337 ^{6}	0.00254 ^{4}	0.00153 ^{2}	0.00257 ^{5}	0.00062 ^{1}	0.00192 ^{3}	0.00574 ^{7}
	MRE	$\hat{\beta}$	0.04774 ^{3}	0.05819 ^{5}	0.04705 ^{2}	0.06033 ^{6}	0.04239 ^{1}	0.05063 ^{4}	0.06469 ^{7}
		$\hat{\lambda}$	2.04035 ^{4}	2.32572 ^{5}	1.44624 ^{2}	2.35078 ^{6}	0.79628 ^{1}	1.85764 ^{3}	3.64040 ^{7}
		$\hat{a}$	1.52435 ^{4}	1.78730 ^{6}	1.15248 ^{2}	1.77360 ^{5}	0.74702 ^{1}	1.46061 ^{3}	2.51755 ^{7}
	$\sum Ranks$		37 ^{4}	45 ^{5}	18 ^{2}	50 ^{6}	9 ^{1}	30 ^{3}	63 ^{7}
500	BIAS	$\hat{\beta}$	0.03686 ^{3}	0.05064 ^{5}	0.03593 ^{2}	0.05088 ^{6}	0.03132 ^{1}	0.03820 ^{4}	0.05296 ^{7}
		$\hat{\lambda}$	0.40417 ^{4}	0.55413 ^{5}	0.28536 ^{2}	0.57150 ^{6}	0.14829 ^{1}	0.37771 ^{3}	0.83839 ^{7}
		$\hat{a}$	0.02486 ^{4}	0.03337 ^{5}	0.01845 ^{2}	0.03421 ^{6}	0.01119 ^{1}	0.02455 ^{3}	0.04717 ^{7}
	MSE	$\hat{\beta}$	0.00218 ^{3}	0.00394 ^{5}	0.00217 ^{2}	0.00397 ^{6}	0.00155 ^{1}	0.00234 ^{4}	0.00427 ^{7}
		$\hat{\lambda}$	0.84839 ^{6}	0.74622 ^{4}	0.30063 ^{2}	0.80382 ^{5}	0.08482 ^{1}	0.39798 ^{3}	1.96202 ^{7}
		$\hat{a}$	0.00235 ^{5}	0.00230 ^{4}	0.00102 ^{2}	0.00239 ^{6}	0.00036 ^{1}	0.00140 ^{3}	0.00529 ^{7}
	MRE	$\hat{\beta}$	0.03686 ^{3}	0.05064 ^{5}	0.03593 ^{2}	0.05088 ^{6}	0.03132 ^{1}	0.03820 ^{4}	0.05296 ^{7}
		$\hat{\lambda}$	1.61668 ^{4}	2.21653 ^{5}	1.14143 ^{2}	2.28600 ^{6}	0.59315 ^{1}	1.51085 ^{3}	3.35355 ^{7}
		$\hat{a}$	1.24324 ^{4}	1.66828 ^{5}	0.92252 ^{2}	1.71068 ^{6}	0.55941 ^{1}	1.22765 ^{3}	2.35843 ^{7}
	$\sum Ranks$		36 ^{4}	43 ^{5}	18 ^{2}	53 ^{6}	9 ^{1}	30 ^{3}	63 ^{7}

(Continued)

Table 8 (continued)

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
750	BIAS	$\hat{\beta}$	0.02896 ^{2}	0.04342 ^{5}	0.02995 ^{3}	0.04582 ^{7}	0.02605 ^{1}	0.03547 ^{4}	0.04492 ^{6}
		$\hat{\lambda}$	0.31776 ^{3}	0.48366 ^{5}	0.22128 ^{2}	0.51361 ^{6}	0.13514 ^{1}	0.34694 ^{4}	0.68369 ^{7}
		$\hat{a}$	0.02024 ^{3}	0.02990 ^{5}	0.01469 ^{2}	0.03123 ^{6}	0.01013 ^{1}	0.02257 ^{4}	0.03909 ^{7}
	MSE	$\hat{\beta}$	0.00143 ^{2}	0.00284 ^{5}	0.00148 ^{3}	0.00319 ^{6}	0.00111 ^{1}	0.00198 ^{4}	0.00332 ^{7}
		$\hat{\lambda}$	0.44322 ^{4}	0.58498 ^{5}	0.18056 ^{2}	0.66764 ^{6}	0.07875 ^{1}	0.31063 ^{3}	1.45108 ^{7}
		$\hat{a}$	0.00140 ^{4}	0.00191 ^{5}	0.00064 ^{2}	0.00208 ^{6}	0.00033 ^{1}	0.00112 ^{3}	0.00393 ^{7}
	MRE	$\hat{\beta}$	0.02896 ^{2}	0.04342 ^{5}	0.02995 ^{3}	0.04582 ^{7}	0.02605 ^{1}	0.03547 ^{4}	0.04492 ^{6}
		$\hat{\lambda}$	1.27104 ^{3}	1.93465 ^{5}	0.88510 ^{2}	2.05445 ^{6}	0.54056 ^{1}	1.38774 ^{4}	2.73478 ^{7}
		$\hat{a}$	1.01199 ^{3}	1.49487 ^{5}	0.73443 ^{2}	1.56141 ^{6}	0.50636 ^{1}	1.12836 ^{4}	1.95457 ^{7}
	$\sum Ranks$		26 ^{3}	45 ^{5}	21 ^{2}	56 ^{6}	9 ^{1}	34 ^{4}	61 ^{7}

Table 9: Numerical values of the NExLL distribution for $\theta = (2, 0.5, 0.01)^T$

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
50	BIAS	$\hat{\beta}$	0.19825 ^{2}	0.22666 ^{5}	0.19872 ^{3}	0.24657 ^{7}	0.18304 ^{1}	0.21261 ^{4}	0.23541 ^{6}
		$\hat{\lambda}$	2.38206 ^{4}	2.56407 ^{5}	2.18176 ^{3}	3.20910 ^{6}	0.55474 ^{1}	2.01825 ^{2}	4.11218 ^{7}
		$\hat{a}$	0.02836 ^{4}	0.03264 ^{5}	0.02749 ^{3}	0.03457 ^{6}	0.01082 ^{1}	0.02646 ^{2}	0.04195 ^{7}
	MSE	$\hat{\beta}$	0.06581 ^{2}	0.08395 ^{5}	0.06902 ^{3}	0.09825 ^{7}	0.05118 ^{1}	0.07060 ^{4}	0.09046 ^{6}
		$\hat{\lambda}$	34.34063 ^{6}	19.75405 ^{4}	16.51153 ^{3}	32.11487 ^{5}	1.35945 ^{1}	14.96315 ^{2}	57.41411 ^{7}
		$\hat{a}$	0.00309 ^{6}	0.00237 ^{4}	0.00178 ^{2}	0.00275 ^{5}	0.00036 ^{1}	0.00180 ^{3}	0.00450 ^{7}
	MRE	$\hat{\beta}$	0.09913 ^{2}	0.11333 ^{5}	0.09936 ^{3}	0.12329 ^{7}	0.09152 ^{1}	0.10631 ^{4}	0.11770 ^{6}
		$\hat{\lambda}$	4.76412 ^{4}	5.12813 ^{5}	4.36352 ^{3}	6.41821 ^{6}	1.10949 ^{1}	4.03649 ^{2}	8.22437 ^{7}
		$\hat{a}$	2.83565 ^{4}	3.26401 ^{5}	2.74934 ^{3}	3.45688 ^{6}	1.08240 ^{1}	2.64600 ^{2}	4.19467 ^{7}
	$\sum Ranks$		34 ^{4}	43 ^{5}	26 ^{3}	55 ^{6}	9 ^{1}	25 ^{2}	60 ^{7}
100	BIAS	$\hat{\beta}$	0.14995 ^{3}	0.17978 ^{5}	0.14119 ^{2}	0.18397 ^{7}	0.12688 ^{1}	0.15588 ^{4}	0.18251 ^{6}
		$\hat{\lambda}$	2.33838 ^{4}	2.71704 ^{5}	2.03826 ^{3}	2.82490 ^{6}	0.37229 ^{1}	1.92769 ^{2}	3.96646 ^{7}
		$\hat{a}$	0.02877 ^{4}	0.03391 ^{6}	0.02672 ^{3}	0.03370 ^{5}	0.00807 ^{1}	0.02587 ^{2}	0.04344 ^{7}
	MSE	$\hat{\beta}$	0.03650 ^{3}	0.05059 ^{5}	0.03457 ^{2}	0.05628 ^{7}	0.02523 ^{1}	0.03965 ^{4}	0.05390 ^{6}
		$\hat{\lambda}$	28.22428 ^{6}	19.44160 ^{4}	14.16505 ^{3}	22.16041 ^{5}	0.73371 ^{1}	12.71886 ^{2}	45.10800 ^{7}
		$\hat{a}$	0.00332 ^{6}	0.00250 ^{4}	0.00187 ^{3}	0.00258 ^{5}	0.00022 ^{1}	0.00176 ^{2}	0.00451 ^{7}
	MRE	$\hat{\beta}$	0.07497 ^{3}	0.08989 ^{5}	0.07059 ^{2}	0.09199 ^{7}	0.06344 ^{1}	0.07794 ^{4}	0.09125 ^{6}
		$\hat{\lambda}$	4.67675 ^{4}	5.43407 ^{5}	4.07653 ^{3}	5.64980 ^{6}	0.74459 ^{1}	3.85537 ^{2}	7.93291 ^{7}
		$\hat{a}$	2.87707 ^{4}	3.39145 ^{6}	2.67192 ^{3}	3.37005 ^{5}	0.80707 ^{1}	2.58744 ^{2}	4.34393 ^{7}
	$\sum Ranks$		37 ^{4}	45 ^{5}	24 ^{2.5}	53 ^{6}	9 ^{1}	24 ^{2.5}	60 ^{7}
250	BIAS	$\hat{\beta}$	0.09336 ^{2}	0.12240 ^{5}	0.09898 ^{3}	0.12619 ^{6}	0.08064 ^{1}	0.10780 ^{4}	0.12963 ^{7}
		$\hat{\lambda}$	1.51852 ^{2}	2.29132 ^{6}	1.65418 ^{3}	2.25320 ^{5}	0.29168 ^{1}	1.71260 ^{4}	3.11924 ^{7}
		$\hat{a}$	0.02054 ^{2}	0.03083 ^{6}	0.02280 ^{3}	0.02940 ^{5}	0.00627 ^{1}	0.02356 ^{4}	0.03752 ^{7}
	MSE	$\hat{\beta}$	0.01470 ^{2}	0.02379 ^{5}	0.01554 ^{3}	0.02471 ^{6}	0.01013 ^{1}	0.01864 ^{4}	0.02643 ^{7}
		$\hat{\lambda}$	13.92248 ^{5}	13.69033 ^{4}	10.44752 ^{3}	14.34206 ^{6}	0.67837 ^{1}	10.15974 ^{2}	28.91119 ^{7}
		$\hat{a}$	0.00180 ^{4}	0.00212 ^{6}	0.00156 ^{3}	0.00203 ^{5}	0.00019 ^{1}	0.00144 ^{2}	0.00349 ^{7}
	MRE	$\hat{\beta}$	0.04668 ^{2}	0.06120 ^{5}	0.04949 ^{3}	0.06309 ^{6}	0.04032 ^{1}	0.05390 ^{4}	0.06482 ^{7}
		$\hat{\lambda}$	3.03704 ^{2}	4.58264 ^{6}	3.30837 ^{3}	4.50641 ^{5}	0.58336 ^{1}	3.42521 ^{4}	6.23847 ^{7}
		$\hat{a}$	2.05371 ^{2}	3.08334 ^{6}	2.28007 ^{3}	2.93977 ^{5}	0.62703 ^{1}	2.35563 ^{4}	3.75241 ^{7}
	$\sum Ranks$		23 ^{2}	49 ^{5.5}	27 ^{3}	49 ^{5.5}	9 ^{1}	32 ^{4}	63 ^{7}

(Continued)

Table 9 (continued)

n	Est.	Param.	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
500	MRE	$\hat{\beta}$	0.07182 ^{2}	0.10041 ^{6}	0.07636 ^{3}	0.10051 ^{7}	0.05695 ^{1}	0.08374 ^{4}	0.09836 ^{5}
		$\hat{\lambda}$	1.18972 ^{2}	1.95005 ^{5}	1.34320 ^{3}	1.98623 ^{6}	0.16503 ^{1}	1.53265 ^{4}	2.66659 ^{7}
		$\hat{a}$	0.01672 ^{2}	0.02641 ^{5}	0.01909 ^{3}	0.02694 ^{6}	0.00377 ^{1}	0.02176 ^{4}	0.03442 ^{7}
	MSE	$\hat{\beta}$	0.00912 ^{2}	0.01583 ^{7}	0.01005 ^{3}	0.01559 ^{6}	0.00501 ^{1}	0.01114 ^{4}	0.01537 ^{5}
		$\hat{\lambda}$	7.05808 ^{3}	10.22967 ^{5}	6.77288 ^{2}	10.84626 ^{6}	0.34392 ^{1}	7.85339 ^{4}	21.85827 ^{7}
		$\hat{a}$	0.00103 ^{2}	0.00160 ^{5}	0.00108 ^{3}	0.00170 ^{6}	0.00009 ^{1}	0.00129 ^{4}	0.00306 ^{7}
	MRE	$\hat{\beta}$	0.03591 ^{2}	0.05020 ^{6}	0.03818 ^{3}	0.05025 ^{7}	0.02848 ^{1}	0.04187 ^{4}	0.04918 ^{5}
		$\hat{\lambda}$	2.37945 ^{2}	3.90011 ^{5}	2.68641 ^{3}	3.97246 ^{6}	0.33005 ^{1}	3.06531 ^{4}	5.33318 ^{7}
		$\hat{a}$	1.67151 ^{2}	2.64109 ^{5}	1.90858 ^{3}	2.69407 ^{6}	0.37706 ^{1}	2.17620 ^{4}	3.44234 ^{7}
	$\Sigma Ranks$		19 ^{2}	49 ^{5}	26 ^{3}	56 ^{6}	9 ^{1}	36 ^{4}	57 ^{7}
750	[BIAS]	$\hat{\beta}$	0.05076 ^{2}	0.08604 ^{6}	0.06181 ^{3}	0.08910 ^{7}	0.04418 ^{1}	0.07483 ^{4}	0.08057 ^{5}
		$\hat{\lambda}$	0.84665 ^{2}	1.65137 ^{5}	1.01170 ^{3}	1.74966 ^{6}	0.07959 ^{1}	1.35406 ^{4}	1.94966 ^{7}
		$\hat{a}$	0.01230 ^{2}	0.02303 ^{5}	0.01465 ^{3}	0.02401 ^{6}	0.00226 ^{1}	0.01949 ^{4}	0.02612 ^{7}
	MSE	$\hat{\beta}$	0.00504 ^{2}	0.01156 ^{6}	0.00685 ^{3}	0.01251 ^{7}	0.00317 ^{1}	0.00885 ^{4}	0.01094 ^{5}
		$\hat{\lambda}$	4.94224 ^{3}	8.15856 ^{5}	4.58136 ^{2}	8.73776 ^{6}	0.11018 ^{1}	6.00786 ^{4}	13.22241 ^{7}
		$\hat{a}$	0.00073 ^{3}	0.00130 ^{5}	0.00072 ^{2}	0.00137 ^{6}	0.00003 ^{1}	0.00099 ^{4}	0.00189 ^{7}
	MRE	$\hat{\beta}$	0.02538 ^{2}	0.04302 ^{6}	0.03091 ^{3}	0.04455 ^{7}	0.02209 ^{1}	0.03741 ^{4}	0.04028 ^{5}
		$\hat{\lambda}$	1.69331 ^{2}	3.30274 ^{5}	2.02339 ^{3}	3.49931 ^{6}	0.15917 ^{1}	2.70811 ^{4}	3.89932 ^{7}
		$\hat{a}$	1.22956 ^{2}	2.30289 ^{5}	1.46466 ^{3}	2.40122 ^{6}	0.22576 ^{1}	1.94910 ^{4}	2.61167 ^{7}
	$\Sigma Ranks$		20 ^{2}	48 ^{5}	25 ^{3}	57 ^{6.5}	9 ^{1}	36 ^{4}	57 ^{6.5}

Table 10: The ordering performance of all estimators for combinations of parameters

θ^T	n	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
$\beta = 0.5, \lambda = 0.1, a = 0.01$	50	3	5	2	6	1	4	7
	100	3	5	2	6	1	4	7
	250	3	5	2	6	1	4	7
	500	2	5	3	6	1	4	7
	700	2	6	4	5	1	3	7
$\beta = 0.5, \lambda = 0.1, a = 0.03$	50	6	5	1	4	3	2	7
	100	6	5	1	5	4	2	7
	250	3	6	1	5	4	2	7
	500	4	6	1	5	3	2	7
	700	3	6	1	5	4	2	7
$\beta = 0.5, \lambda = 0.1, a = 0.05$	50	6	4	2	3	7	1	5
	100	7	2	1	4	6	3	5
	250	5	4	1	3	7	2	6
	500	5	3	1	4	6	2	7
	700	6	3	2	4	5	1	7

(Continued)

Table 10 (continued)

θ^T	n	MLE	LSE	WLSE	CRVME	MPSE	ADE	RADE
$\beta = 1, \lambda = 0.5, a = 0.01$	50	4	5	2	6	1	3	7
	100	4	5	2	6	1	3	7
	250	2	6	3	5	1	4	7
	500	2	5	3	6	1	4	7
	700	2	5	3	6.5	1	4	6.5
$\beta = 1, \lambda = 0.25, a = 0.01$	50	4	5	2	6	1	3	7
	100	4	5	2	6	1	3	7
	250	2	6	3	5	1	4	7
	500	2	5	3	6	1	4	7
	700	2	5	3	6	1	4	7
$\beta = 1, \lambda = 0.25, a = 0.02$	50	5	4	2	6	1	3	7
	100	5	4	2	6	1	3	7
	250	4	5	2	6	1	3	7
	500	4	5	2	6	1	3	7
	700	3	5	2	6	1	4	7
$\beta = 2, \lambda = 0.5, a = 0.01$	50	4	5	3	6	1	2	7
	100	4	5	2.5	6	1	2.5	7
	250	2	5.5	3	5.5	1	4	7
	500	2	5	3	6	1	4	7
	700	2	5	3	6.5	1	4	6.5
$\sum Ranks$		127	170.5	75.5	189.5	74	106.5	239
Overall Rank		4	5	2	6	1	3	7

6 Real-World Data Applications

This section illustrates the importance and applicability of the NExLL distribution by fitting two real-life data applications from Reliability and engineering sciences. The first data set is addressed by [39], and it contains 40 observations of time to failure ($10^3 h$) of turbocharger of one type of engine. The data are: 3.5, 1.6, 5.4, 4.8, 6.0, 7.0, 6.5, 7.3, 8.0, 7.7, 8.4, 3.9, 2.0, 5.0, 6.1, 5.6, 8.1, 6.5, 7.3, 7.1, 7.8, 2.6, 8.4, 4.5, 5.8, 5.1, 6.3, 7.3, 6.7, 7.7, 8.3, 7.9, 8.5, 4.6, 3.0, 5.3, 8.7, 6.0, 9.0, 8.8.

The second data set refers to breaking stress of carbon fibers in (Gba) for 100 observations, and it is studied by [40]. The data are: 0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.80, 1.84, 1.84, 1.87, 1.89, 1.92, 2.0, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.60, 3.65, 3.68, 3.68, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90, 4.91, 5.08, 5.56.

The fits of the NExLL distributions is compared with some existing LL distributions namely: the alpha power transformed LL (APTLL) [28], Poisson Burr type X LL (PBXLL) [26], beta LL (BLL) [21], Weibull generalized LL (WGLL) by [41], McDonald LL (McLL) [23], additive Weibull IL (AWLL) [42], and LL distributions.

The performance of the fitted distributions is explored using some measures including the Akaike information criterion (AIC), consistent Akaike IC (CAIC), Hannan–Quinn IC (HQIC), Bayesian IC (BIC), $(-\ell)$, where ℓ the maximized log-likelihood, as well as some statistics include the Cramér–Von Mises (W^*), Anderson–Darling (A^*), Kolmogorov–Smirnov (KS) and the KS p -value (PV).

All numerical results in this section are calculated using the R software. Tables 11 and 12 present the goodness-of-fit (GOF) results for the proposed NExLL distribution alongside several competing LL-based models, applied to two real datasets. The evaluation is conducted using a comprehensive set of model selection criteria and goodness-of-fit statistics to ensure robust and meaningful comparisons.

Table 11: GOF measures of the NExLL distribution and other competing LL models for Dataset 1

Distribution	AIC	CAIC	BIC	HQIC	W^*	A^*	$-\ell$	KS	PV
NExLL	169.2820	169.9487	174.3487	171.1140	0.0602	0.4599	81.6410	0.1028	0.7922
LL	181.4133	181.7376	184.7910	182.6345	0.2142	1.4072	88.7066	0.1437	0.3807
APTLL	183.3957	184.0623	188.4623	185.2276	0.2133	1.4019	88.6978	0.1441	0.3774
PBXLL	175.9246	176.5913	180.9912	177.7565	0.1380	0.9618	84.9623	0.1160	0.6542
BLL	182.6718	183.8147	189.4274	185.1144	0.2028	1.3474	87.3359	0.1261	0.5480
WGLL	172.9608	174.1037	179.7164	175.4034	0.0771	0.5740	82.4804	0.1078	0.7418
McLL	183.8549	185.6196	192.2993	186.9081	0.1901	1.2735	86.9274	0.1244	0.5663
AWLL	176.9510	179.4965	187.0843	180.6149	0.0770	0.5730	82.4755	0.1077	0.7423

Table 12: GOF measures of the NExLL distribution and other competing LL models for Dataset 2

Distribution	AIC	CAIC	BIC	HQIC	W^*	A^*	$-\ell$	KS	PV
NExLL	289.0275	289.2775	296.8430	292.1906	0.0622	0.4140	141.5138	0.0602	0.8612
LL	296.5591	296.6828	301.7694	298.6678	0.2390	1.2409	146.2795	0.0903	0.3889
APTLL	298.5591	298.8091	306.3746	301.7221	0.2390	1.2409	146.2795	0.0903	0.3889
PBXLL	289.4567	289.7067	297.2722	292.6198	0.0945	0.4968	141.7284	0.0776	0.5836
BLL	295.2779	295.6989	305.6985	299.4953	0.1595	0.8179	143.6389	0.0946	0.3328
WGLL	291.0523	291.4734	301.4730	295.2698	0.0623	0.4157	141.5262	0.0605	0.8573
McLL	295.2981	295.9364	308.3240	300.5699	0.1123	0.5953	142.6491	0.0860	0.4502
AWLL	295.0583	295.9615	310.6893	301.3845	0.0623	0.4159	141.5292	0.0605	0.8578

Specifically, the model selection criteria include the AIC, CAIC, HQIC, and BIC. These criteria assess the trade-off between model fit and complexity, where lower values generally indicate better-fitting models with fewer penalties for parameter count. Additionally, the statistic $-\ell$ reflects how well the model fits the data—lower values imply a better fit.

To evaluate the agreement between the empirical and theoretical distributions, we also report GOF statistics including W^* , A^* , and the KS statistic along with its corresponding PV. These statistics test the discrepancy between the empirical distribution function and the cumulative distribution function of the fitted model. Smaller values of W^* , A^* , and KS, and higher p -values, suggest a closer agreement between the model and observed data.

From Tables 11 and 12, it is evident that the NExLL distribution consistently achieves the lowest values for all considered information criteria and test statistics across both datasets. It also records the highest KS p -values, indicating no significant deviation from the empirical data. These results strongly support the

superior flexibility and fitting capability of the NExLL model compared to other well-known and recently proposed LL-based distributions.

The ML estimates and standard errors (SEs) for the models' parameters are reported in [Table 13](#), for the two datasets. Overall, the NExLL distribution have the lowest values for goodness-of-fit criteria among all LL fitted models. Then, it could be chosen as the most adequate model for fitting the two datasets.

Table 13: The ML estimates (first line) and SEs (second line) for the two datasets

Distribution		Estimates				
Dataset 1						
NExLL	4.2518	203.3933	0.0504			
(β, λ, a)	0.6650	549.4226	0.0884			
LL	4.8416	6996.4772				
(β, λ)	0.7012	9446.9359				
APTLL	1.9329	5.8212	4.8134			
(α, a, b)	5.6222	1.7283	0.6953			
PBXLL	0.1657	30.3102	144.4757			
(α, β, λ)	0.0246	3.7432	159.3419			
BLL	1693.8481	1.0372	6.9900	2322.0832		
(α, β, a, b)	1292.9657	0.2790	3.9446	1582.0007		
WGLL	84.4278	0.9009	93.1452	0.0474		
(α, β, a, b)	23.1604	0.1025	580.5337	0.0178		
McLL	5.8200	77.1080	1.1030	79.9610	0.1821	
$(\alpha, \beta, \lambda, a, b)$	3.4211	3.0119	0.8265	50.6929	0.1453	
AWLL	1.8938	41.1786	0.0167	4.7330	998.9035	2.0448
$(\alpha, \lambda, a, b, c, d)$	26.2972	346.1821	75.6685	76.7921	32,508.3351	28.3936
Dataset 2						
NExLL	2.8003	5971.7872	145.5678			
(β, λ, a)	0.2197	29,123.0600	714.9833			
LL	4.1179	43.4031				
(β, λ)	0.3441	16.0679				
APTLL	1.0000	2.4984	4.1179			
(α, a, b)	9.8160	2.9795	0.3441			
PBXLL	0.1780	17.3156	98.3022			
(α, β, λ)	0.0418	2.9191	136.4771			
BLL	1697.9684	0.8891	7.2891	2319.2097		
(α, β, a, b)	1682.1547	0.2489	4.0259	1509.3635		
WGLL	51.4276	0.9369	61.1452	0.0337		
(α, β, a, b)	155.7936	0.0996	1630.0540	0.1686		
McLL	0.7254	0.0557	50.2750	127.5920	104.0075	
$(\alpha, \beta, \lambda, a, b)$	0.1894	0.0704	44.3432	105.0090	120.3863	
AWLL	2.7613	9.6996	2.9893	4.9695	27.9440	1.0581
$(\alpha, \lambda, a, b, c, d)$	39.9664	13.2076	278.9846	81.3893	105.4537	14.6684

Furthermore, the visual fitting performance of the NExLL model is explored in Figs. 5 and 6. These figures show the histogram of the two data sets and the estimated PDFs, CDFs, SFs, and PP plots of the NExLL distribution. It is clear from these plots that the NExLL distribution provides the best fits to the two datasets.

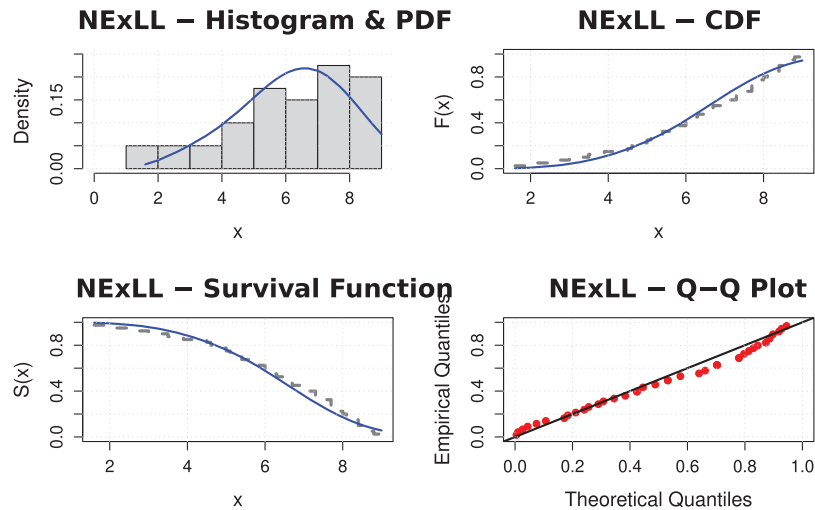
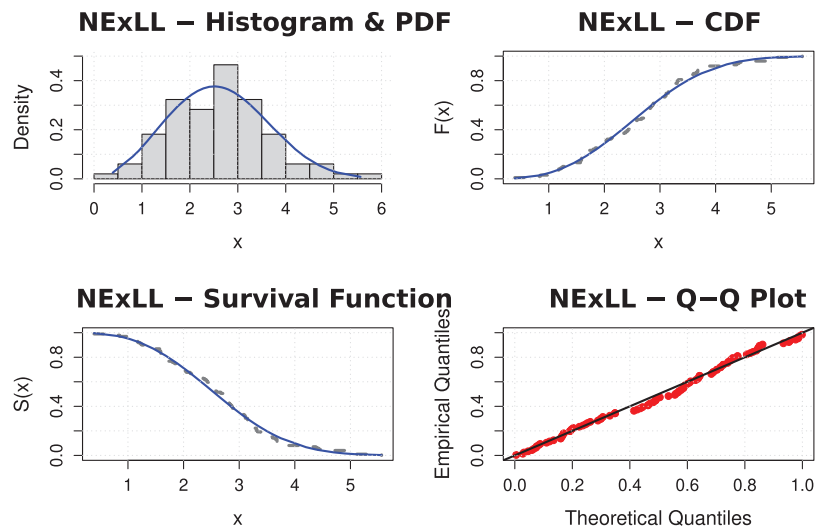


Figure 5: The fitted functions of the NExLL model for first dataset



Figures 6: The fitted functions of the NExLL model for second dataset

Fig. 7 displays the histogram and corresponding fitted density curves for the two real-life datasets. The left panel illustrates the fit of the proposed NExLL distribution to the first dataset, while the right panel shows the fit for the second dataset. In both cases, the NExLL model closely follows the empirical data, capturing the underlying distributional patterns effectively. The visual alignment between the histograms and the fitted curves highlights the model's flexibility and suitability for modeling real-world lifetime data with varying shapes and characteristics.

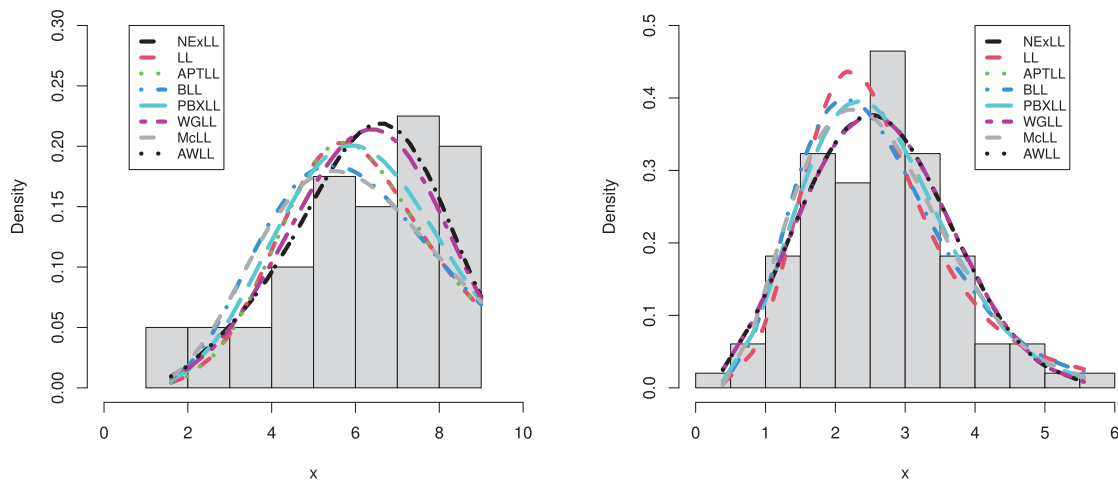


Figure 7: The histogram and fitted densities plots for the first dataset (left panel), and for the second dataset (right panel)

Additionally, the PP plots for the two datasets, presented in Figs. 8 and 9, confirm that the NExLL distribution provides a superior fit compared to competing distributions. The results demonstrate that the NExLL distribution consistently outperforms other LL extensions across all datasets, reinforcing its practical applicability for modeling real-world data.

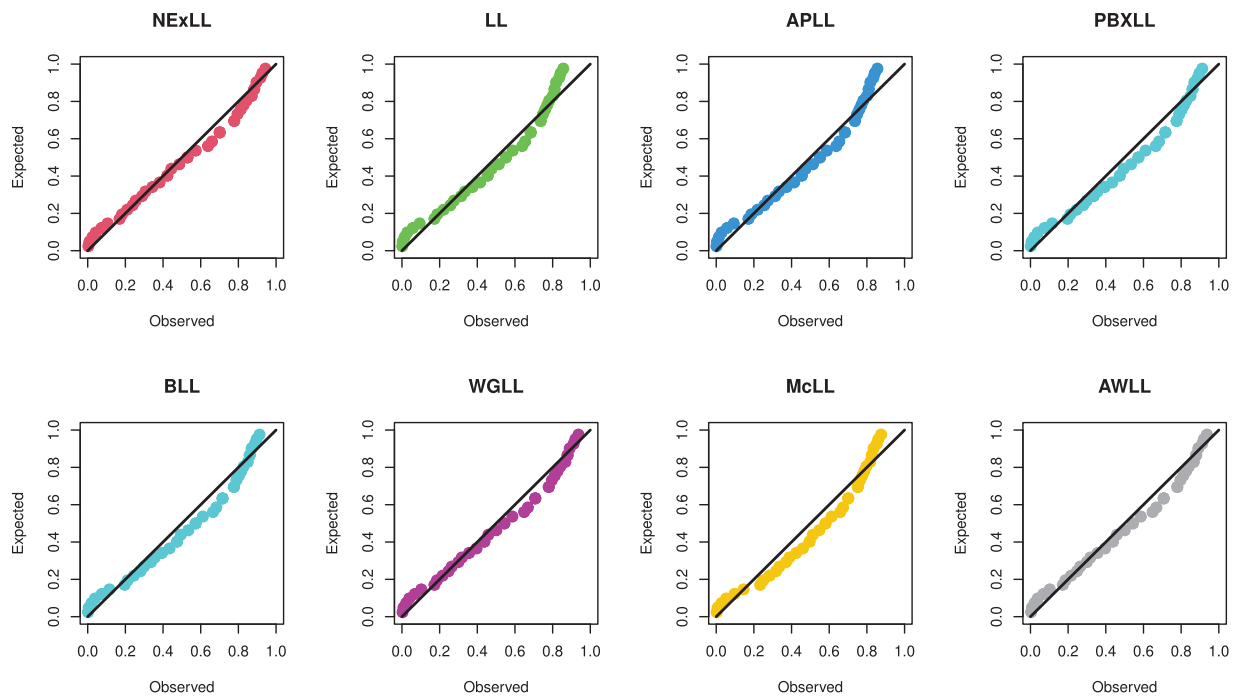


Figure 8: The PP plots of the fitted models for first dataset

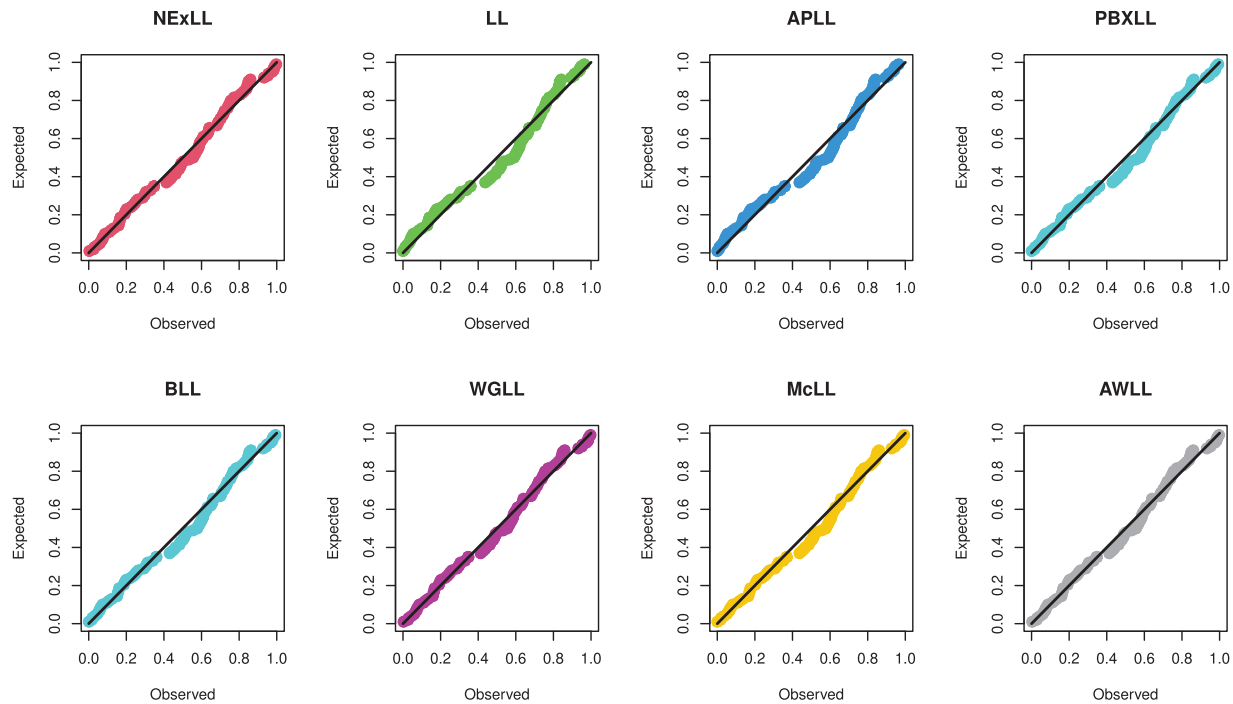


Figure 9: The PP plots of the fitted models for second dataset

7 Conclusions

In this study, we proposed a novel and flexible three-parameter distribution, called the NExLL distribution, as an extension of the classical log-logistic model. Unlike general NEx-H models, the NExLL distribution provides a fully closed-form, interpretable, and practically useful framework that can be directly applied to real-world survival and reliability data without requiring numerical approximation of its core functions. The proposed model exhibits remarkable flexibility, with a hazard rate function capable of capturing both monotonic and non-monotonic behaviors, and a density function that accommodates symmetric and asymmetric shapes—making it suitable for diverse survival and reliability applications. We thoroughly examined the mathematical properties of the NExLL distribution, and estimated its parameters using seven distinct estimation techniques. A comprehensive simulation study was conducted to evaluate the performance of these methods, revealing that the maximum product of spacings approach consistently yields the most efficient and accurate parameter estimates.

To demonstrate the practical utility of the model, we applied the NExLL distribution to two real-world lifetime datasets from the fields of engineering and reliability. The empirical results, supported by goodness-of-fit statistics and graphical analysis, confirm the NExLL model's superior fitting capability over competing Log-Logistic-based models. The findings presented in this work contribute to the ongoing advancement of parametric survival modeling by offering a versatile distribution that can be further explored within broader frameworks, including the incorporation of trigonometric-based structures and machine learning-enhanced estimation strategies. As such, this study aligns with the objectives of the current special issue by pushing the boundaries of traditional parametric survival models and opening pathways for future interdisciplinary applications. Despite its promising performance, the proposed NExLL model is limited to complete and uncensored datasets, and its estimation under censoring or covariate settings remains unexplored. Future work may extend this model to handle censored data or incorporate covariates through regression frameworks for broader applicability.

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Ethics Approval: This study does not involve any human participants, animals, or identifiable personal data. The analysis is based entirely on publicly available datasets and simulated data. Therefore, ethics approval and consent to participate were not required.

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