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ARTICLE

Mathematical Model of the Monkeypox Virus Disease via $\mathcal{ABC}$ Fractional Order Derivative

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ABSTRACT: The Department of Economic and Social Affairs of the United Nations has released seventeen goals for sustainable development and SDG No. 3 is "Good Health and Well-being", which mainly emphasizes the strategies to be adopted for maintaining a healthy life. The Monkeypox Virus disease was first reported in 1970. Since then, various health initiatives have been taken, including by the WHO. In the present work, we attempt a fractional model of Monkeypox virus disease, which we feel is crucial for a better understanding of this disease. We use the recently introduced $\mathcal{ABC}$ fractional derivative to closely examine the Monkeypox virus disease model. The evaluation of this model determines the existence of two equilibrium states. These two stable points exist within the model and include a disease-free equilibrium and endemic equilibrium. The disease-free equilibrium has undergone proof to demonstrate its stability properties. The system remains stable locally and globally whenever the effective reproduction number remains below one. The effective reproduction number becoming greater than unity makes the endemic equilibrium more stable both globally and locally than unity. To comprehensively study the model's solutions, we employ the Picard-Lindelof approach to investigate their existence and uniqueness. We investigate the Ulam-Hyers and UlamHyers Rassias stability of the fractional order nonlinear framework for the Monkeypox virus disease model. Furthermore, the approximate solutions of the $\mathcal{ABC}$ fractional order Monkeypox virus disease model are obtained with the help of a numerical technique combining the Lagrange polynomial interpolation and fundamental theorem of fractional calculus with the $\mathcal{ABC}$ fractional derivative.

KEYWORDS: $\mathcal{ABC}$ fractional derivative; monkeypox virus disease; existence and uniqueness; fixed point theory

1 Introduction

World Health Organisation (WHO), in its news release dated 28th November 2022, renamed Zonotonic Orthopox virus disease or the Monkeypox disease using a new synonym as Mpox [1]. MPox virus was first discovered in monkeys in Denmark as early as 1958 [2]. However, after the first infected case was reported in humans in 1970, it became endemic in Central and West Africa [3]. It is noteworthy that cases of Mpox have been reported since May 2022 even in countries that have no epidemiological connections. In July 2022, the WHO declared that Mpox is a public health emergency of international concern. The global spread of the Mpox in August 2022, forced the WHO to declare the disease a threat to moderate public health [3,4]. Approximately 86000 infected cases of Mpox and 112 confirmed deaths spread over 110 countries were reported as of March 2023 across the globe, with Brazil, Spain, and the USA being the worst affected [4]. The symptoms of Mpox infection usually start with a flu-like fever, fatigue, head and muscle aches, followed by rashes starting from the face and spreading all over the body. The rash then progresses from small pustules, which aggravate to vesicles, which are blisters with fluid, with symptoms of cough and sore throat, Gastrointestinal disorders include diarrhea, nausea and vomiting and even lymphadenopathy. The incubation period of Mpox varies between 7 to 21 days. For more details, one can refer to [5,6].

Out of various transmission channels, primarily through animal-human direct contact, for example, contact with infected monkeys, squirrels, rats, etc. [7]. It is also noticed that Mpox gets transmitted through close contact with infected persons through sexual contact, especially among men having male-to-male sexual activity [8]. It is also noteworthy that the virus survives on surfaces for a longer duration and even the bed, cloth, or equipment can act as a transmission mode. Like Animal-to-human transmission, the reverse process does occur when an infected human comes into close contact with animals, especially pets, through kissing or cuddling or even sharing food [9]. Recently, a case of an infected dog through human transmission has been reported in [10]. The above studies reveal the importance of making a thorough analysis of the disease, which will help in attaining not only UN SDG No. 3 but also Goals No. 15 and 17. In the next section, we present the formulation of the mathematical model of our study.

2 Model Formulation

Many mathematicians have studied the model for analyzing the spread of Mpox, the readers can see, [11–15]. Ahmad Yu et al. [16] proposed a new deterministic mathematical model that incorporates various situations such as pre- and post-vaccination exposures, and isolation, while describing the human-human transmission of Mpox. They analyzed the dynamics of transmission in six categories of humans: Susceptible $\check{L}$, exposed $\check{Q}$, asymptomatic $\check{T}$, symptomatic $\check{H}$, isolated $\check{E}$, recovered $\check{F}$. The monkeypox virus model is shown as follows:

$$
\begin{aligned}
\frac{d\check{L}}{dj} &= \Lambda - \nu\check{L} - (\epsilon_1 + \varrho)\check{L} \\
\frac{d\check{Q}}{dj} &= \nu\check{L} - (\varrho + \vartheta)\check{Q} \\
\frac{d\check{T}}{dj} &= \vartheta(1 - \mathsf{p})\check{Q} - (\varrho + \varsigma)\check{T}, \\
\frac{d\check{H}}{dj} &= \varsigma(1 - \epsilon_2)\check{T} + \vartheta\mathsf{p}\check{Q} - (\varrho + \omega + \chi + \tau)\check{H}, \\
\frac{d\check{E}}{dj} &= \chi\check{H} - (\varrho + \omega + \wp)\check{E}, \\
\frac{d\check{F}}{dj} &= \varsigma\,\epsilon_2\,\check{T} + \epsilon_1\,\check{L} + \tau\check{H} + \wp\check{E} - \varrho\,\check{F},
\end{aligned}
$$

along with the following non-negative initial conditions.

$$\check{L}(0) = \check{L}_0, \check{Q}(0) = \check{Q}_0, \check{T}(0) = \check{T}_0, \check{H}(0) = \check{H}_0, \check{E}(0) = \check{E}_0, \check{F}(0) = \check{F}_0.$$

Peter et al. [17] proposed a deterministic mathematical model of monkeypox virus by using both classical and fractional-order differential equations. Al Qurashi et al. [18] investigated using a deterministic mathematical framework within the Atangana-Baleanu fractional derivative that depends on the generalized Mittag-Leffler kernel. Bansal et al. [19] investigated a mathematical model to analyze the transmission dynamics of Mpox in human and non-human denizens and utilized the fractional Atangana-Baleanu operator in the Caputo sense to study the characteristics and various epidemiological aspects of Mpox. Okyere, Ackora-Prah [20] proposed that Atangana-Baleanu fractional-order derivatives are defined in the Caputo sense to investigate the kinetics of Monkeypox transmission in Ghana. Liu et al. [21] examined various epidemiological aspects of the monkeypox viral infection using a fractional-order mathematical model. Inspired by our present work, we investigate the existence and unique solution of human-to-human monkeypox virus using $\mathcal{ABC}$ fractional derivative. Moreover, we investigate the stability of the solutions using Ulam-Hyers and Ulam-Hyers-Rassias stability criteria. We also perform numerical simulations and present the results graphically. The Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative is given by

$$\begin{cases} {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\check{L} = \Lambda - \nu_a \check{L} - (\epsilon_1 + \varrho)\check{L}, \\ {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\check{Q} = \nu_a \check{L} - (\varrho + \vartheta)\check{Q}, \\ {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\check{T} = \vartheta(1-\mathsf{p})\check{Q} - (\varrho + \varsigma)\check{T}, \\ {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\check{H} = \varsigma(1-\epsilon_2)\check{T} + \vartheta\mathsf{p}\check{Q} - (\varrho + \omega + \chi + \tau)\check{H}, \\ {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\check{E} = \chi\check{H} - (\varrho + \omega + \wp)\check{E}, \\ {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\check{F} = \varsigma\,\epsilon_2\,\check{T} + \epsilon_1\,\check{L} + \tau\check{H} + \wp\check{E} - \varrho\,\check{F}, \end{cases} \tag{1}$$

with ζ being the fractional order $0 < \zeta < 1$ subject to the following initial conditions

$$\check{L}(0) = \check{L}_0, \check{Q}(0) = \check{Q}_0, \check{T}(0) = \check{T}_0, \check{H}(0) = \check{H}_0, \check{E}(0) = \check{E}_0, \check{F}(0) = \check{F}_0.$$

$$\begin{cases} {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\check{L} = \Lambda - \dfrac{\varpi\check{H}}{\mathcal{N}}\check{L} - (\epsilon_1 + \varrho)\check{L}, \\ {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\check{Q} = \dfrac{\varpi\check{H}}{\mathcal{N}}\check{L} - (\varrho + \vartheta)\check{Q}, \\ {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\check{T} = \vartheta(1-\mathsf{p})\check{Q} - (\varrho + \varsigma)\check{T}, \\ {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\check{H} = \varsigma(1-\epsilon_2)\check{T} + \vartheta\mathsf{p}\check{Q} - (\varrho + \omega + \chi + \tau)\check{H}, \\ {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\check{E} = \chi\check{H} - (\varrho + \omega + \wp)\check{E}, \\ {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\check{F} = \varsigma\,\epsilon_2\,\check{T} + \epsilon_1\,\check{L} + \tau\check{H} + \wp\check{E} - \varrho\,\check{F}. \end{cases} \tag{2}$$

The following Table 1 describes various parameters and variables used in model (1).

Table 1: Description of various Parameters and variables used in the model (1)

Parameter	Description	Values	Source
Λ	Recruitment rate	73648	[16]
ϱ	Natural death rate	0.000350	[16]
ϖ	effective contact rate	0.022325	[16]
ϑ	Rate of leaving exposed compartment	0.01674	[16]

(Continued)

Table 1 (continued)

Parameter	Description	Values	Source
p	proportion of exposed individuals progresses to symptomatic individuals	0.4	[16]
ς	Rate of leaving Asymptomatic compartment	0.8020	[16]
χ	Isolation rate	0.5	[16]
$\wp$	Treatment rate	0.036246	[16]
τ	Recovery rate of non-isolated infected individuals	0.003226	[16]
ϵ_1	Pre-exposure effective vaccination rate	0.802	[16]
ϵ_2	Post-exposure effective vaccination rate	0.3	[16]
ν_a	Infection force	0.022325	Estimated
ω	Monkeypox virus induced death rate	0.008836	[16]

Model basic preliminaries

In this section, we provide mathematical preliminaries in the form of theorems that will be utilized to demonstrate the positivity and uniqueness of solutions for the Monkeypox virus disease model with the $\mathcal{ABC}$ fractional derivative (1), as defined in the literature [22]. These theorems serve as foundational tools for establishing key properties of the model and are essential for rigorous analysis and validation. We assume the space $\{\chi(j) \in \mathcal{C}([0,1] \to \mathbb{R})\}$ with $\|\chi\| = \max_{[0,1]} |\chi(j)|$. The flowchart of the considered model is shown in Fig. 1 and basic definitions are stated as follows. Table 2 shows the comparison between the Caputo and ABC fractional derivatives.

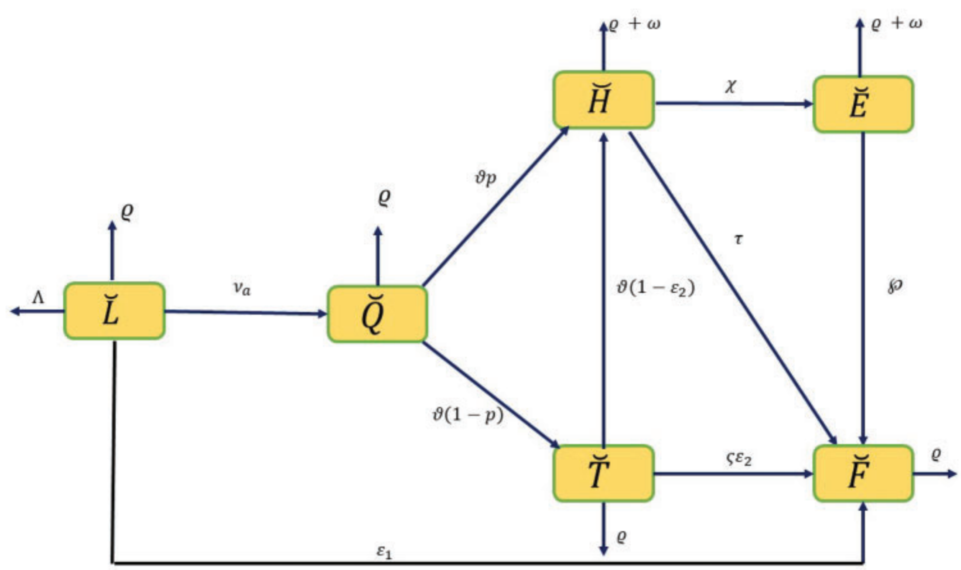

Figure 1: Flow chart of proposed model

Table 2: Comparison of Caputo and ABC fractional derivatives

Feature	Caputo derivative	ABC fractional derivative
Memory kernel	Power-law decay $(j-\varsigma)^{-\zeta}$	Exponential decay exp $\left(-\zeta\frac{j-\varsigma}{1-\varsigma}\right)$
Decay rate	Slow (long-term memory)	Faster decay (shorter memory)
Influence on present	Depends on entire past state	Adjustable memory influence
Computational efficiency	High due to singularity at $j=\varsigma$	More efficient for short-memory processes

Definition 1: [22] *On the interval* $\zeta \in [0,1]$, *by taking the function* $\chi \in \mathcal{H}^1(\mathfrak{l}_1, \mathfrak{l}_2)$, $\mathfrak{l}_2 > \mathfrak{l}_1$, *so the* $\mathcal{ABC}$ *derivative is given by*

$$\mathcal{D}_j^{\zeta}(\chi(j)) = \frac{\mathcal{M}(\zeta)}{(1-\zeta)}\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}\chi'(\varsigma)exp\left(-\zeta\frac{j-\varsigma}{1-\varsigma}\right)d(\varsigma), \tag{3}$$

with $\mathcal{M}(0) = \mathcal{M}(1) = 1$, *here* $\mathcal{M}(\zeta)$ *is a normalization function.*

Remark 1: *The Caputo derivative and ABC fractional derivative differ primarily in memory effects, decay behavior, and computational efficiency:*

Memory Behavior:

Caputo follows a power-law decay, leading to long-term memory effects. ABC follows exponential decay, making it better suited for short-memory processes.

Decay Rate:

Caputo has a slow decay, meaning past states strongly influence the present. ABC has a faster decay, reducing historical influence more quickly.

Impact on Present State:

Caputo depends on the entire past state, making it ideal for processes with persistent memory effects. ABC allows for adjustable memory influence, offering more flexibility.

Computational Efficiency:

Caputo is less efficient due to singularities in calculations. ABC is computationally efficient, especially for short-memory systems.

Definition 2: [22] *On the interval* $\zeta \in [0,1]$, *by taking the function* $\chi \in \mathcal{H}^1(\mathfrak{l}_1, \mathfrak{l}_2)$, $\mathfrak{l}_2 > \mathfrak{l}_1$, *which is not differentiable, therefore the Atangana-Baleanu fractional derivative in Riemann-Liouville sense is defined as*

$$^{\mathcal{ABR}}_{\mathfrak{z}}\mathcal{D}_j^{\zeta}(\chi(j)) = \frac{\mathcal{M}(\zeta)}{(1-\zeta)}\frac{\mathrm{d}}{\mathrm{d}j}\int_{\mathfrak{l}_1}^{\mathfrak{l}_2}\chi(\varsigma)\mathrm{E}_{\zeta}\left(-\zeta\frac{(j-\varsigma)^{\zeta}}{1-\zeta}\right)d\varsigma. \tag{4}$$

Definition 3: [22] *The* $\mathcal{ABC}$ *fractional derivative for the fractional integral of order* ζ *is given by*

$$^{\mathcal{ABC}}_{\mathfrak{z}}\check{J}_j^{\zeta}(\chi(j)) = \frac{1-\zeta}{\mathcal{M}(\zeta)}\chi(j) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_{\mathfrak{z}}^{j}\chi(\varsigma)(j-\varsigma)^{\zeta-1}d\varsigma. \tag{5}$$

If $\zeta = 0$ and $\zeta = 1$, the initial function and ordinary integral are obtained, respectively.

Definition 4: [22] *The Laplace transform of* ${}^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\chi(j)$ *is represented as follows:*

$$\mathcal{L}[{}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\chi(j)] = \frac{\mathcal{M}(\zeta)}{1-\zeta}\frac{\kappa^{\zeta}\mathcal{L}[\chi(j)](\kappa) - \kappa^{\zeta^{\zeta-1}}\chi(0)}{\kappa^{\zeta}+\frac{\zeta}{(1-\zeta)}}. \tag{6}$$

Definition 5: [22] *The Laplace transform of* ${}^{\mathcal{ABR}}\mathcal{D}_j^{\zeta}\chi(j)$ *is represented as follows:*

$$\mathcal{L}[{}_0^{\mathcal{ABR}}\mathcal{D}_j^{\zeta}\chi(j)] = \frac{\mathcal{M}(\zeta)}{1-\zeta}\frac{\kappa^{\zeta}\mathcal{L}[\chi(j)](\kappa)}{\kappa^{\zeta}+\dfrac{\zeta}{(1-\zeta)}}. \tag{7}$$

Definition 6: [23] *Let* $\chi \in \mathcal{H}^1(\mathfrak{l}_1, \mathfrak{l}_2)$ *and* $\mathfrak{l}_2 > \mathfrak{l}_1$ *then the fractional differential operator usually utilized in general fractional calculus is the Caputo fractional operator defined by the following integral equation.*

$${}_{\mathfrak{z}}^{C}\mathcal{D}_j^{\zeta}\chi(j) = \frac{1}{\wp(i-\zeta)}\int_{\mathfrak{z}}^{j}\chi^{(i)}(\varsigma)(j-\varsigma)^{(i-\zeta-1)}d\varsigma, \quad i-1<\zeta<i\in\mathbb{N}, \tag{8}$$

$\wp$ (.) denotes a gamma function.

Definition 7: [23] *The Caputo integral is defined as follows:*

$${}_{\mathfrak{z}}^{C}\breve{J}_j^{\zeta}[\chi(j)] = \frac{1}{\wp(\zeta)}\int_{\mathfrak{z}}^{j}\chi(\varsigma)(j-\varsigma)^{\zeta-1}d\varsigma. \tag{9}$$

Theorem 1: [24] *Let* $(\mathcal{P}, \mathsf{d})$ *be a complete metric space, and let* $\mathsf{H} : \mathcal{P} \to \mathcal{P}$ *be a contraction on* $\mathcal{P}$. *Then* H *has a unique fixed point, say* $\mathsf{u} \in \mathcal{P}$.

3 Qualitative Analysis of the Proposed Model

This section presents the equilibrium points, basic reproductive number and stability analysis, analytically.

3.1 Boundedness and Positivity of Solutions

Theorem 2: *The closed set,*

$$\mathcal{H}_1 = \left\{(\breve{L}(j), \breve{Q}(j), \breve{T}(j), \breve{H}(j), \breve{E}(j), \breve{F}(j)) \in \breve{F}_+^6 : \mathcal{N} \le \frac{\Lambda}{\varrho}\right\},$$

is positively invariant and attract all positive solutions of the model.

Proof: We must demonstrate that $\breve{F}_+^6$ maintains positive invariance because all system (1) solutions from $\mathcal{H}_1$ will stay within $\mathcal{H}_1$. The verification of positive initial values becoming positive at time j together with boundedness of solutions within $\frac{\Lambda}{\varrho}$ region establishes the positiveness of the solution set. From (1), we have

$${}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\breve{L} = \Lambda - \frac{\varpi\breve{H}}{\mathcal{N}}\breve{L} - (\epsilon_1+\varrho)\breve{L}.$$

Applying Laplace transform, we have

$$\frac{\mathcal{M}(\zeta)}{s^{\zeta}(1-\zeta)+\zeta}\left[s^{\zeta}\mathcal{L}(\breve{L}) - s^{\zeta-1}\breve{L}(0)\right] = \frac{\Lambda}{s} - \mathcal{L}\left(\frac{\varpi\breve{H}}{\mathcal{N}}\breve{L}\right) - (\epsilon_1+\varrho)\mathcal{L}(\breve{L}).$$

Rearranging for $\mathcal{L}(\check{L})$, we obtain

$$\mathcal{L}(\check{L}) \geq \frac{\mathcal{M}(\zeta)\check{L}(0)}{\mathcal{M}(\zeta) + (\epsilon_1 + \varrho)(1-\zeta)} \cdot \frac{s^{\zeta-1}}{s^{\zeta} + \dfrac{\zeta(\epsilon_1 + \varrho)}{\mathcal{M}(\zeta) + (\epsilon_1 + \varrho)(1-\zeta)}}.$$

The inverse transform shows non-negativity through the Mittag-Leffler function E_{ζ}. Repeating this process for all compartments $(\check{Q}, \check{T}, \check{H}, \check{E}, \check{F})$ preserves non-negativity by linearity of the $\mathcal{ABC}$ derivative. Recall the total population

$$\mathcal{N} = \check{L} + \check{Q} + \check{T} + \check{H} + \check{E} + \check{F}.$$

Therefore, we have

$${}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{N} = \Lambda - \varrho\,\mathcal{N} - \left[\omega(\check{H} + \check{E}) + \chi\check{H}\right].$$

Ignoring disease-induced mortality, we obtain

$${}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{N} \leq \Lambda - \varrho\,\mathcal{N}.$$

Applying Laplace transform, we have

$$\mathcal{L}\left\{{}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{N}\right\} \leq \frac{\Lambda}{s} - \varrho\,\mathcal{L}\{\mathcal{N}\}.$$

Using $\mathcal{ABC}$ derivative properties, we obtain

$$\frac{\mathcal{M}(\zeta)}{s^{\zeta}(1-\zeta) + \zeta}\left[s^{\zeta}\mathcal{N}(s) - s^{\zeta-1}\mathcal{N}(0)\right] \leq \frac{\Lambda}{s} - \varrho\,\mathcal{N}(s).$$

Rearranging for $\mathcal{N}(s)$, we obtain

$$\mathcal{N}(s) \leq \frac{\Lambda(s^{\zeta}(1-\zeta) + \zeta)}{s\left[\mathcal{M}(\zeta)s^{\zeta} + \varrho\,(s^{\zeta}(1-\zeta) + \zeta)\right]} + \frac{\mathcal{M}(\zeta)s^{\zeta-1}\mathcal{N}(0)}{\mathcal{M}(\zeta)s^{\zeta} + \varrho\,(s^{\zeta}(1-\zeta) + \zeta)}.$$

Inverse Laplace transform yields

$$\mathcal{N}(t) \leq \frac{\Lambda}{\varrho}\left[1 - \mathrm{E}_{\zeta}\left(-\frac{\varrho\,\zeta}{\mathcal{M}(\zeta)}t^{\zeta}\right)\right] + \mathcal{N}(0)\mathrm{E}_{\zeta}\left(-\frac{\varrho\,\zeta}{\mathcal{M}(\zeta)}t^{\zeta}\right).$$

As $t \to \infty$, we have

$$\lim_{t\to\infty}\mathcal{N}(t) \leq \frac{\Lambda}{\varrho}.$$

All positive solutions of the system (1) are attracted to the positively invariant region $\mathcal{N}$. □

3.2 Equilibria

We determine the equilibria of the system (1) by assuming the state variables are constant and by putting the right side equal to zero. Eq. (1) admits two types of equilibria as follows:

For the disease-free equilibrium (DFE), we assume $\check{Q}^* = \check{T}^* = \check{H}^* = \check{E}^* = 0$. Solving for $\check{L}^*$ and $\check{F}^*$

$$\check{L}^* = \frac{\Lambda}{\epsilon_1 + \varrho},$$
$$\check{F}^* = \frac{\Lambda\,\epsilon_1}{\varrho\,(\epsilon_1 + \varrho)}.$$

Thus, the disease-free equilibrium is

$$\check{X}_0 = \left(\frac{\Lambda}{\epsilon_1 + \varrho}, 0, 0, 0, 0, \frac{\Lambda\,\epsilon_1}{\varrho\,(\epsilon_1 + \varrho)}\right).$$

For the endemic equilibrium, we solve for each compartment assuming $\check{Q}^*, \check{T}^*, \check{H}^*, \check{E}^* > 0$:

$$\begin{aligned}
\check{L}^* &= \frac{\Lambda}{\nu_a + \epsilon_1 + \varrho},\\
\check{Q}^* &= \frac{\nu_a \check{L}^*}{\varrho + \vartheta} = \frac{\nu_a \Lambda}{(\varrho + \vartheta)(\nu_a + \epsilon_1 + \varrho)},\\
\check{T}^* &= \frac{\vartheta(1-\mathrm{p})\check{Q}^*}{\varrho + \varsigma} = \frac{\vartheta(1-\mathrm{p})\nu_a \Lambda}{(\varrho + \varsigma)(\varrho + \vartheta)(\nu_a + \epsilon_1 + \varrho)},\\
\check{H}^* &= \frac{\varsigma(1-\epsilon_2)\check{T}^* + \vartheta \mathrm{p}\check{Q}^*}{\varrho + \omega + \chi + \tau} = \frac{\vartheta(\varsigma(1-\epsilon_2) + \mathrm{p}(\varsigma\epsilon_2 + \varrho))\nu_a \Lambda}{(\varrho + \omega + \chi + \tau)(\varrho + \varsigma)(\varrho + \vartheta)(\nu_a + \epsilon_1 + \varrho)},\\
\check{E}^* &= \frac{\chi\vartheta(\varsigma(1-\epsilon_2) + \mathrm{p}(\varsigma\epsilon_2 + \varrho))\nu_a \Lambda}{(\varrho + \omega + \wp)(\varrho + \omega + \chi + \tau)(\varrho + \varsigma)(\varrho + \vartheta)(\nu_a + \epsilon_1 + \varrho)},\\
\check{F}^* &= \frac{\varsigma\,\epsilon_2\,\check{T}^* + \epsilon_1\,\check{L}^* + \tau\check{H}^* + \wp\check{E}^*}{\varrho},
\end{aligned} \tag{10}$$

such that the force of infection is

$$\nu_a = \frac{\varpi\check{H}^*}{\mathcal{N}^*}, \tag{11}$$

where

$$\mathcal{N}^* = \check{L}^* + \check{Q}^* + \check{T}^* + \check{H}^* + \check{E}^* + \check{F}^*.$$

By using (10) and (11), we get

$$\nu_a^*(\mathcal{W}\nu_a^* + \mathcal{B}) = 0, \tag{12}$$

where the coefficients $\mathcal{W}$ and $\mathcal{B}$ are given by

$$\begin{aligned}
\mathcal{W} &= (\varrho + \varsigma)(\varrho + \omega + \chi)(\varrho + \omega + \wp) + \vartheta(1-\mathrm{p})(\varrho + \omega + \chi + \tau)(\varrho + \omega + \wp)\\
&\quad + \vartheta[\varsigma(1-\epsilon_2) + \mathrm{p}(\varsigma\epsilon_2 + \varrho)](\varrho + \omega + \chi + \wp),\\
\mathcal{B} &= (\varrho + \vartheta)(\varrho + \varsigma)(\varrho + \omega + \chi + \tau)(\varrho + \omega + \wp)\\
&\quad - \varpi\vartheta(\varrho + \omega + \wp)[\varsigma(1-\epsilon_2) + \mathrm{p}(\varsigma\epsilon_2 + \varrho)].
\end{aligned}$$

The disease-free equilibrium happens when $v_a^* = 0$ has been reached while finding a solution for $\mathcal{W}v_a^* + \mathcal{B} = 0$ leads to the endemic equilibrium. The model exhibits two equilibrium states which are the disease-free state and the endemic state.

3.3 Basic Reproduction Number

The new infection matrix F is given by

$$\mathsf{F} = \begin{bmatrix} 0 & 0 & \varpi & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The transition matrix V is

$$\mathsf{V} = \begin{bmatrix} \varrho + \vartheta & 0 & 0 & 0 \\ -\varsigma(1-\mathsf{p}) & \varrho + \varsigma & 0 & 0 \\ -\varsigma\mathsf{p} & -\vartheta(1-\epsilon_2) & \varrho + \omega + \chi + \tau & 0 \\ 0 & 0 & -\chi & \varrho + \omega + \wp \end{bmatrix}.$$

The inverse of V is

$$\mathsf{V}^{-1} = \begin{bmatrix} (\varrho+\vartheta)^{-1} & 0 & 0 & 0 \\ -\dfrac{\vartheta(-1+\mathsf{p})}{(\varrho+\vartheta)(\varrho+\varsigma)} & (\varrho+\varsigma)^{-1} & 0 & 0 \\ -\dfrac{\varsigma(1-\epsilon_2)\vartheta\mathsf{p} - \varsigma(1-\epsilon_2)\vartheta - \vartheta\mathsf{p}\,\varrho - \vartheta\mathsf{p}\varsigma}{(\varrho+\vartheta)(\varrho+\varsigma)(\varrho+\omega+\chi+\tau)} & \dfrac{\varsigma(1-\epsilon_2)}{(\varrho+\varsigma)(\varrho+\omega+\chi+\tau)} & (\varrho+\omega+\chi+\tau)^{-1} & 0 \\ -\dfrac{\chi(\varsigma(1-\epsilon_2)\vartheta\mathsf{p} - \varsigma(1-\epsilon_2)\vartheta - \vartheta\mathsf{p}\,\varrho - \vartheta\mathsf{p}\varsigma)}{(\varrho+\varsigma)(\varrho+\omega+\chi+\tau)(\varrho+\omega+\wp)} & \dfrac{\chi\varsigma(1-\epsilon_2)}{(\varrho+\varsigma)(\varrho+\omega+\chi+\tau)(\varrho+\omega+\wp)} & \dfrac{\chi}{(\varrho+\omega+\chi+\tau)(\varrho+\omega+\wp)} & (\varrho+\omega+\wp)^{-1} \end{bmatrix}.$$

Multiplying F and V^{-1}, we obtain the next-generation matrix

$$\mathsf{FV}^{-1} = \begin{bmatrix} -\dfrac{\varpi(\varsigma(1-\epsilon_2)\vartheta\mathsf{p} - \varsigma(1-\epsilon_2)\vartheta - \vartheta\mathsf{p}\,\varrho - \vartheta\mathsf{p}\varsigma)}{(\varrho+\vartheta)(\varrho+\varsigma)(\varrho+\omega+\chi+\tau)} & \dfrac{\varpi\varsigma(1-\epsilon_2)}{(\varrho+\varsigma)(\varrho+\omega+\chi+\tau)} & \dfrac{\varpi}{\varrho+\omega+\chi+\tau} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We determined the effective reproduction number by taking the dominant eigen value

$$\check{F}_{\mathsf{ef}} = \frac{\varpi\vartheta(\varsigma(1-\epsilon_2) + \mathsf{p}(\varsigma\epsilon_2 + \varrho))}{(\varrho+\vartheta)(\varrho+\varsigma)(\varrho+\omega+\chi+\tau)},$$

and the basic reproduction number after setting $\epsilon_2 = \chi = 0$,

$$\check{F}_0 = \frac{\varpi\vartheta(\varsigma + \varrho\,\mathsf{p})}{(\varrho+\vartheta)(\varrho+\varsigma)(\varrho+\omega+\chi+\tau)}.$$

3.4 Stability Analysis

In this section, we test the local stability of the system (1), considering the two defined equilibria.

Theorem 3: *(Local stability at $\check{X}_0$) The system (1) at $\check{X}_0 = \left(\check{L}^0, \check{Q}^0, \check{T}^0, \check{H}^0, \check{E}^0, \check{F}^0\right)$, is locally asymptotically stable in $\mathcal{H}_1$ if* $\mathsf{R}_{\mathsf{ef}} < 1$. *Otherwise, unstable when* $\mathsf{R}_{\mathsf{ef}} > 1$.

Proof: The Jacobian matrix at $\check{X}_0$ is computed as follows:

$$\mathsf{J}(\check{X}_0) = \begin{bmatrix} -(\epsilon_1 + \varrho) & 0 & 0 & -\varpi & 0 & 0 \\ 0 & -(\varrho + \vartheta) & 0 & 0 & 0 & 0 \\ 0 & \vartheta(1-\mathsf{p}) & -(\varrho + \varsigma) & 0 & 0 & 0 \\ 0 & \vartheta\mathsf{p} & \varsigma(1-\epsilon_2) & -(\varrho + \omega + \chi + \tau) & 0 & 0 \\ 0 & 0 & 0 & \chi & -(\varrho + \omega + \wp) & 0 \\ \epsilon_1 & 0 & \varsigma\,\epsilon_2 & 0 & \wp & -\varrho \end{bmatrix}.$$

Let, $\mathfrak{e}_1 = -(\epsilon_1 + \varrho)$, $\mathfrak{e}_2 = -(\varrho + \vartheta)$, $\mathfrak{e}_3 = \vartheta(1-\mathsf{p})$, $\mathfrak{e}_4 = -(\varrho + \varsigma)$, $\mathfrak{e}_5 = \vartheta\mathsf{p}$, $\mathfrak{e}_6 = \varsigma(1-\epsilon_2)$, $\mathfrak{e}_7 = -(\varrho + \omega + \chi + \tau)$, $\mathfrak{e}_8 = -(\varrho + \omega + \wp)$, reduce $\mathsf{J}(\check{X}_0)$ into row-echelon form

$$\mathsf{J}(\check{X}_0) = \begin{bmatrix} \mathfrak{e}_1 & 0 & 0 & -\varpi & 0 & 0 \\ 0 & \mathfrak{e}_2 & 0 & \varpi & 0 & 0 \\ 0 & 0 & -\mathfrak{e}_2\mathfrak{e}_4 & \varpi\mathfrak{e}_3 & 0 & 0 \\ 0 & 0 & 0 & \mathsf{B}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathsf{B}_1\mathfrak{e}_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\varrho\,\mathsf{B}_1^2\mathfrak{e}_1\mathfrak{e}_2\mathfrak{e}_4\mathfrak{e}_8 \end{bmatrix},$$

where $\mathsf{B}_1 = \varpi\mathfrak{e}_2\mathfrak{e}_3\mathfrak{e}_6 - \varpi\mathfrak{e}_2\mathfrak{e}_4\mathfrak{e}_5 + \mathfrak{e}_2\mathfrak{e}_2\mathfrak{e}_4\mathfrak{e}_7$. To find the eigenvalues of the Jacobian matrix $\mathsf{J}(\check{X}_0)$, we solve the characteristic equation

$$(\nu - \mathfrak{e}_1)(\nu + \mathsf{B}_1\mathfrak{e}_8)(\mathfrak{e}_2 - \nu)(\mathfrak{e}_2\mathfrak{e}_4 + \nu)(\varrho\,\mathsf{B}_1^2\mathfrak{e}_1\mathfrak{e}_2\mathfrak{e}_4\mathfrak{e}_8 + \nu)(\mathsf{B}_1 - \nu) = 0. \tag{13}$$

By computing the determinant and solving for ν, we obtain the six eigenvalues

$$\begin{aligned} \nu_1 &= \mathfrak{e}_1 = -(\epsilon_1 + \varrho) < 0, \\ \nu_2 &= \mathfrak{e}_2 = -(\varrho + \vartheta) < 0, \\ \nu_3 &= -\mathfrak{e}_2\mathfrak{e}_4 = -(\varrho + \vartheta)(\varrho + \varsigma) < 0, \\ \nu_4 &= -\mathsf{B}_1\mathfrak{e}_8 < 0 \\ \nu_5 &= -\varrho\,\mathsf{B}_1^2\mathfrak{e}_1\mathfrak{e}_2\mathfrak{e}_4\mathfrak{e}_8(\varrho + \omega + \wp) < 0, \\ \nu_6 &= \mathsf{B}_1. \end{aligned}$$

The negativity of ν_4, ν_5, ν_6 depends on B_1. Now

$$\begin{aligned} \mathsf{B}_1 &= \varpi\mathfrak{e}_2\mathfrak{e}_3\mathfrak{e}_6 - \varpi\mathfrak{e}_2\mathfrak{e}_4\mathfrak{e}_5 + \mathfrak{e}_2\mathfrak{e}_2\mathfrak{e}_4\mathfrak{e}_7 \\ &= -(\varrho + \vartheta)\varpi\vartheta\varsigma(1-\mathsf{p})(1-\epsilon_2) - (\varrho + \vartheta)\varpi\vartheta\mathsf{p}(\varrho + \varsigma) + (\varrho + \vartheta)^2(\varrho + \varsigma)(\varrho + \omega + \chi + \tau) < 0, \end{aligned}$$

which implies that

$$\check{F}_{\mathsf{ef}} = \frac{\varpi\vartheta(\varsigma(1-\epsilon_2) + \mathsf{p}(\varsigma\epsilon_2 + \varrho))}{(\varrho + \vartheta)(\varrho + \varsigma)(\varrho + \omega + \chi + \tau)} < 0.$$

Consequently, we obtain $\mathsf{B}_1 < 0$ iff $\check{F}_{\mathsf{ef}} < 1$. $\frac{\nu_a}{\varrho + \vartheta} < 1$, $\mathsf{R}_0 < 1$. Thus, $\mathsf{B}_1 < 0$, it follows that $\nu_4 < 0$ and $\nu_5 < 0$. Therefore, all the eigen values are negative. By Routh-Hurwitz criterion in [25], the disease free equilibrium is locally asymptotically stable when $\check{F}_{ef} < 1$. It is clear that, system (1) is locally asmptotically stable at $\check{X}_0$ when $R_0 < 1$, as desired. □

4 Global Stability of the Disease-Free Equilibrium

Theorem 4: *According to (1) model the disease-free equilibrium (DEF) $\check{X}_0$ achieves global asymptotic stability within the region $\mathcal{H}_1$ while the condition $\check{F}_{ef} < 1$ maintains stability but when $\check{F}_{ef} > 1$ the equilibrium becomes unstable.*

Proof: Define a Lyapunov function as in [26]

$$\mathcal{V} = \check{T}_1\check{Q} + \check{T}_2\check{T} + \check{T}_3\check{H} + \check{T}_4\check{E}, \tag{14}$$

where $\check{T}_1, \check{T}_2, \check{T}_3, \check{T}_4$ are the coefficients to be determined and must be positive.

$${}_0^{ABC}\mathcal{D}_j^{\zeta}\mathcal{V} = \check{T}_1{}_0^{ABC}\mathcal{D}_j^{\zeta}\check{Q} + \check{T}_2{}_0^{ABC}\mathcal{D}_j^{\zeta}\check{T} + \check{T}_3{}_0^{ABC}\mathcal{D}_j^{\zeta}\check{H} + \check{T}_4{}_0^{ABC}\mathcal{D}_j^{\zeta}\check{E}, \tag{15}$$

substitute for ${}_0^{ABC}\mathcal{D}_j^{\zeta}\check{Q}, {}_0^{ABC}\mathcal{D}_j^{\zeta}\check{T}, {}_0^{ABC}\mathcal{D}_j^{\zeta}\check{H}, {}_0^{ABC}\mathcal{D}_j^{\zeta}\check{E}$ in (15)

$$\begin{aligned}{}_0^{ABC}\mathcal{D}_j^{\zeta}\mathcal{V} =& \check{T}_1[\nu\check{L} - (\vartheta + \varrho)\check{Q}] + \check{T}_2[\vartheta(1-\mathsf{p})\check{Q} - (\varsigma + \varrho)\check{T}] \\ &+ \check{T}_3[\varsigma(1-\epsilon_2)\check{T} + \vartheta\mathsf{p}\check{Q} - (\varrho + \omega + \chi + \tau)\check{H}] + \check{T}_4[\chi\check{H} - (\varrho + \omega + \wp)\check{E}].\end{aligned} \tag{16}$$

Now, collect together, the coefficient of $\nu\check{L}, \check{Q}, \check{T}, \check{H}$ and $\check{E}$

$$\begin{aligned}{}_0^{ABC}\mathcal{D}_j^{\zeta}\mathcal{V} =& \check{T}_1\nu\check{L} - \check{Q}[(\vartheta + \varrho)\check{T}_1 - \vartheta(1-\mathsf{p})\check{T}_2 - \vartheta\mathsf{p}\check{T}_3] - \check{T}[(\varsigma + \varrho)\check{T}_2 - \varsigma(1-\epsilon_2)] \\ &- \check{H}[(\varrho + \omega + \chi + \tau)\check{T}_3 - \chi\check{T}_4] - \check{T}_4(\varrho + \omega + \wp)\check{E},\end{aligned} \tag{17}$$

recall that

$$\check{F}_{ef} = \frac{\varpi\vartheta[\varsigma(1-\epsilon_2) + \mathsf{p}(\varsigma\epsilon_2 + \varrho)]}{(\varsigma + \varrho)(\vartheta + \varrho)(\varrho + \omega + \chi + \tau)},$$

Our calculations determine that the values of $\check{F}_{ef}$ should use $\check{H}$ as its denominator and $\nu\check{L}$ as its numerator and the elements of $\check{Q}, \check{T}, \check{E}$ are set to zero to find the expressions for $\check{T}_1, \check{T}_2, \check{T}_3, \check{T}_4$.

$$\check{T}_1 = \vartheta[\varsigma(1-\epsilon_2) + \mathsf{p}(\varsigma\epsilon_2 + \varrho)], \tag{18}$$

$$\check{T}_2 = \frac{\varsigma(1-\epsilon_2)(\vartheta + \varrho)\check{T}_1}{\vartheta[\varsigma(1-\epsilon_2) + \mathsf{p}(\varsigma\epsilon_2 + \varrho)]}, \tag{19}$$

$$\check{T}_3 = \frac{(\vartheta + \varrho)(\varsigma + \varrho)\check{T}_1}{\vartheta[\varsigma(1-\epsilon_2) + \mathsf{p}(\varsigma\epsilon_2 + \varrho)]}, \tag{20}$$

$$\check{T}_4 = \frac{(\varrho + \omega + \chi + \tau)\check{T}_3}{\chi}. \tag{21}$$

We observed that $\check{T}_1 > 0, \check{T}_2 > 0, \check{T}_3 > 0, \check{T}_4 > 0$, substitute back into (17), $\nu = \frac{\varpi\check{H}}{\mathcal{N}}$, $\check{T}_1 = \vartheta[\varsigma(1-\epsilon_2) + \mathsf{p}(\varsigma\,\epsilon_2 + \varrho)]$, while the coefficient of $\check{Q}, \check{T}, \check{E}$ as zeros.

$${}_0^{ABC}\mathcal{D}_j^{\zeta}\mathcal{V} = \frac{\varpi\check{L}\check{H}}{\mathcal{N}}[\vartheta[\varsigma(1-\epsilon_2) + \mathsf{p}(\varsigma\epsilon_2 + \varrho)]] - \check{H}[(\varsigma + \varrho)(\vartheta + \varrho)(\varrho + \omega + \chi + \tau)]$$

$$= \breve{H}\left[\frac{\varpi\vartheta\breve{L}}{\mathcal{N}}[\varsigma(1-\epsilon_2)+\mathrm{p}(\varsigma\epsilon_2+\varrho)]-(\varsigma+\varrho)(\vartheta+\varrho)(\varrho+\omega+\chi+\tau)\right], \tag{22}$$

since $\breve{L} \leq \mathcal{N}$, we have

$$ {}_{0}^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\mathcal{V} \leq (\varsigma+\varrho)(\vartheta+\varrho)(\varrho+\omega+\chi+\tau)\breve{H}\left[\frac{\varpi\vartheta[\varsigma(1-\epsilon_2)+\mathrm{p}(\varsigma\epsilon_2+\varrho)]}{(\varsigma+\varrho)(\vartheta+\varrho)(\varrho+\omega+\chi+\tau)}-1\right], \tag{23}$$

$$ {}_{0}^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\mathcal{V} \leq \mathcal{M}\breve{H}(\breve{F}_{\mathrm{ef}}-1),$$

where $\mathcal{M}=(\varsigma+\varrho)(\vartheta+\varrho)(\varrho+\omega+\chi+\tau)$. Observed that, ${}_{0}^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\mathcal{V} \leq \mathcal{M}\breve{H}(\breve{F}_{\mathrm{ef}}-1) \leq 0$. The equality ${}_{0}^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\mathcal{V}=0$ hold only if $\breve{H}=0$ or $\breve{F}_{\mathrm{ef}}=1$, and strictly less than, ${}_{0}^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\mathcal{V}<0$ if $\breve{F}_{\mathrm{ef}}<1$. After creating a Lyapunov function $\mathcal{V}>0$ we determined that ${}_{0}^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\mathcal{V}<0$ holds true when $\breve{F}_{\mathrm{ef}}<1$. The DFE becomes globally asymptotically stable based on Lasalle's Invariant principle [27] when the effective reproduction number, $\breve{F}_{\mathrm{ef}}<1$ holds true. □

5 Endemic Equilibrium Point

Theorem 5: *The endemic equilibrium state $\breve{X}^{**}$ of the system (1) exists if the effective reproduction number, $\breve{F}_{\mathrm{ef}}>1$.*

Proof: Let, $\breve{X}^{**}$ be the endemic equilibrium point such that

$$\breve{X}^{**}=(\breve{L}^{**},\breve{Q}^{**},\breve{T}^{**},\breve{H}^{**},\breve{E}^{**},\breve{F}^{**}). \tag{24}$$

We solve for v^* in Eq. (12)

$$\mathcal{W}v^*+\mathcal{B}=0 \tag{25}$$

$$\Rightarrow v^*=\frac{-\mathcal{B}}{\mathcal{W}}, \tag{26}$$

$$v^*=\frac{\varpi\vartheta(\wp+\varrho+\omega)[\varsigma(1-\epsilon_2)+\mathrm{p}(\varsigma\epsilon_2+\varrho)-(\varsigma+\varrho)(\vartheta+\varrho)(\wp+\varrho+\omega)(\chi+\varrho+\omega+\tau)]}{\breve{T}}, \tag{27}$$

$$v^*=\frac{(\varsigma+\varrho)(\vartheta+\varrho)(\wp+\varrho+\omega)(\chi+\varrho+\omega+\tau)\left[\frac{\varpi\vartheta(\wp+\varrho+\omega)[\varsigma(1-\epsilon_2)+\mathrm{p}(\varsigma\epsilon_2+\varrho)]}{(\varsigma+\varrho)(\vartheta+\varrho)(\chi+\varrho+\omega+\tau)}-1\right]}{\mathcal{W}}, \tag{28}$$

$$v^*=\mathcal{K}(\breve{F}_{\mathrm{ef}}-1), \tag{29}$$

where

$$\mathcal{K}=\frac{(\varsigma+\varrho)(\vartheta+\varrho)(\wp+\varrho+\omega)(\chi+\varrho+\omega+\tau)}{\mathcal{W}}. \tag{30}$$

Substituting v^* into (10), we have

$$\breve{L}^{**}=\frac{\Lambda}{\mathcal{K}(\breve{F}_{\mathrm{ef}}-1)+\varrho+\epsilon_1},$$

$$\begin{aligned}
\check{Q}^{**} &= \frac{\mathcal{K}(\check{F}_{ef}-1)\Lambda}{(\mathcal{K}(\check{F}_{ef}-1)+\varrho+\epsilon_1)(\vartheta+\varrho)},\\
\check{T}^{**} &= -\frac{(\check{F}_{ef}-1)\vartheta(\mathsf{p}-1)\mathcal{K}\Lambda}{(\varsigma+\varrho)(\varrho+\mathcal{K}(\check{F}_{ef}-1)+\epsilon_1)(\varrho+\vartheta)},\\
\check{H}^{**} &= \frac{(\check{F}_{ef}-1)\vartheta\Lambda\mathcal{K}[\varsigma(1-\epsilon_2)+\mathsf{p}(\varsigma\epsilon_2+\varrho)]}{(\varrho+\mathcal{K}(\check{F}_{ef}-1)+\epsilon_1)(\varrho+\vartheta)(\varsigma+\varrho)(\omega+\varrho+\chi+\tau)},\\
\check{E}^{**} &= \frac{(\check{F}_{ef}-1)\vartheta\Lambda\mathcal{K}[\varsigma(1-\epsilon_2)+\mathsf{p}(\varsigma\epsilon_2+\varrho)]}{(\varsigma+\varrho)(\vartheta+\varrho)(\wp+\varrho+\omega)(\mathcal{K}(\check{F}_{ef}-1)+\varrho+\epsilon_1)(\chi+\varrho+\omega+\tau)}.
\end{aligned} \tag{31}$$

□

6 Local Stability of the Endemic Equilibrium

Theorem 6: *The endemic equilibrium point $\check{X}^{**}$ of the model (1) stays locally asymptotically stable within domain $\mathcal{H}_1$ when $\check{F}_{ef} > 1$ but it becomes unstable when $\check{F}_{ef} < 1$.*

Proof: The Jacobian matrix associated with the system (1) at endemic equilibrium points is

$$\mathsf{J}(\check{X}^{**}) = \begin{bmatrix} -\frac{\varpi\check{H}^{**}}{\mathcal{N}^{**}}-\epsilon_1-\varrho & 0 & 0 & -\frac{\varpi\check{L}^{**}}{\mathcal{N}^{**}} & 0 & 0\\ -\frac{\varpi\check{H}^{**}}{\mathcal{N}^{**}} & -\varrho-\vartheta & \frac{\varpi\check{L}^{**}}{\mathcal{N}^{**}} & 0 & 0 & 0\\ 0 & \vartheta(1-\mathsf{p}) & -\varsigma-\varrho & 0 & 0 & 0\\ 0 & \vartheta\mathsf{p} & \varsigma(1-\epsilon_2) & -\varrho-\chi-\omega-\tau & 0 & 0\\ 0 & 0 & 0 & \chi & -\varrho-\omega-\wp & 0\\ \epsilon_1 & 0 & \varsigma\,\epsilon_2 & \tau & \wp & -\varrho \end{bmatrix}.$$

Let, $\mathfrak{a}_{11} = -\left(\frac{\varpi\check{H}^{**}}{\mathcal{N}^{**}}+\epsilon_1+\varrho\right)$, $\mathfrak{a}_{14} = -\frac{\varpi\check{L}^{**}}{\mathcal{N}^{**}}$, $\mathfrak{a}_{21} = \frac{\varpi\check{H}^{**}}{\mathcal{N}^{**}}$, $\mathfrak{a}_{22} = -(\varrho+\vartheta)$, $\mathfrak{a}_{24} = \frac{\varpi\check{L}^{**}}{\mathcal{N}^{**}}$, $\mathfrak{a}_{32} = \vartheta(1-\mathsf{p})$, $\mathfrak{a}_{33} = -(\varrho+\varsigma)$, $\mathfrak{a}_{42} = \vartheta\mathsf{p}$, $\mathfrak{a}_{43} = \varsigma(1-\epsilon_2)$, $\mathfrak{a}_{44} = -(\varrho+\omega+\chi+\tau)$, $\mathfrak{a}_{54} = \chi$, $\mathfrak{a}_{55} = -(\varrho+\omega+\wp)$, $\mathfrak{a}_{61} = \epsilon_1$, $\mathfrak{a}_{63} = \varsigma\,\epsilon_2$, $\mathfrak{a}_{64} = \tau$, $\mathfrak{a}_{65} = \wp$, $\mathfrak{a}_{66} = -\varrho$.

By reducing the matrix into row-echelon form, we have

$$\begin{bmatrix} \mathfrak{a}_{11} & 0 & 0 & \mathfrak{a}_{14} & 0 & 0\\ 0 & -\mathfrak{a}_{11}\mathfrak{a}_{22} & 0 & -\mathfrak{a}_{11}\mathfrak{a}_{24}+\mathfrak{a}_{14}\mathfrak{a}_{21} & 0 & 0\\ 0 & 0 & \mathfrak{a}_{11}\mathfrak{a}_{22}\mathfrak{a}_{33}\mathfrak{a}_{32} & (-\mathfrak{a}_{11}\mathfrak{a}_{24}+\mathfrak{a}_{14}\mathfrak{a}_{21}) & 0 & 0\\ 0 & 0 & 0 & \mathcal{B}_2 & 0 & 0\\ 0 & 0 & 0 & 0 & -\mathcal{B}_2\mathfrak{a}_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & \mathfrak{a}_{11}\mathfrak{a}_{22}\mathfrak{a}_{33}\mathfrak{a}_{55}\mathfrak{a}_{66} \end{bmatrix}$$

where

$$\mathcal{B}_2 = \mathfrak{a}_{43}\mathfrak{a}_{32}(\mathfrak{a}_{21}\mathfrak{a}_{14}-\mathfrak{a}_{11}\mathfrak{a}_{24}) - \mathfrak{a}_{33}\mathfrak{a}_{42}(\mathfrak{a}_{21}\mathfrak{a}_{14}-\mathfrak{a}_{11}\mathfrak{a}_{24}) - \mathfrak{a}_{11}\mathfrak{a}_{22}\mathfrak{a}_{33}\mathfrak{a}_{44}$$

The characteristic equation is given by

$$(\mathcal{B}_2-\nu)(\mathcal{B}_2\mathfrak{a}_{55}+\nu)(\mathfrak{a}_{11}-\nu)(\mathfrak{a}_{11}\mathfrak{a}_{22}+\nu)(\mathfrak{a}_{11}\mathfrak{a}_{22}\mathfrak{a}_{33}-\nu)(\mathfrak{a}_{11}\mathfrak{a}_{22}\mathfrak{a}_{33}\mathfrak{a}_{55}\mathfrak{a}_{66}-\nu) = 0. \tag{32}$$

And the eigen values are

$$\nu_1 = \mathfrak{a}_{11},$$

$$
\begin{aligned}
\nu_2 &= -\mathfrak{a}_{11}\mathfrak{a}_{22}, \\
\nu_3 &= \mathfrak{a}_{11}\mathfrak{a}_{22}\mathfrak{a}_{33}, \\
\nu_4 &= \mathcal{B}_2, \\
\nu_5 &= -\mathfrak{a}_{55}\mathcal{B}_2, \\
\nu_6 &= \mathfrak{a}_{11}\mathfrak{a}_{22}\mathfrak{a}_{33}\mathfrak{a}_{55}\mathfrak{a}_{66}.
\end{aligned}
$$

Now,

$$
\mathfrak{a}_{11} = -\left(\frac{\varpi \check{H}^{**}}{\mathcal{N}^{**}} + \epsilon_1 + \varrho\right), \tag{33}
$$

from (31), $\check{H}^{**} < 0$ if and only if $\check{F}_{\text{ef}} < 1$ and $\mathfrak{a}_{11} > 0$ if and only if $\check{H}^{**} < 0$, contrarily, $\mathfrak{a}_{11} < 0$ if and only if $\check{H}^{**} > 0$ implies that $\check{F}_{\text{ef}} > 1$. We therefore have, $\nu_1 < 0, \nu_2 < 0, \nu_3 < 0, \nu_6 < 0$ if and only if $\check{F}_{\text{ef}} > 1$. For the negativity of ν_4 and ν_5,

$$
\begin{aligned}
\mathcal{B}_2 &= \mathfrak{a}_{43}\mathfrak{a}_{32}(\mathfrak{a}_{21}\mathfrak{a}_{14} - \mathfrak{a}_{11}\mathfrak{a}_{24}) - \mathfrak{a}_{33}\mathfrak{a}_{42}(\mathfrak{a}_{21}\mathfrak{a}_{14} - \mathfrak{a}_{11}\mathfrak{a}_{24}) - \mathfrak{a}_{11}\mathfrak{a}_{22}\mathfrak{a}_{33}\mathfrak{a}_{44} \\
&= -\mathfrak{a}_{11}\mathfrak{a}_{22}\mathfrak{a}_{33}\mathfrak{a}_{44}.
\end{aligned}
$$

$\mathcal{B}_2 < 0$ if and only if $\mathfrak{a}_{11} < 0$ implies $\check{F}_{\text{ef}} > 1$. Therefore $\nu_4 < 0, \nu_5 < 0$ if and only if $\check{F}_{\text{ef}} > 1$. A local asymptotic stability analysis of system (1) exists when $\check{F}_{\text{ef}} > 1$ occurs at its endemic equilibrium. □

7 Global Stability of the Endemic Equilibrium

Theorem 7: *If $\check{F}_{\text{ef}} > 1$, the endemic equilibrium, $\check{X}^{**}$ is globally asymptotically stable ($\mathcal{GAS}$).*

Proof: The following conditions serve as assumptions for demonstrating the $\mathcal{GAS}$ of the endemic equilibrium:

1. The analysis supposes a situation where no identification is possible for asymptomatic people so $\check{T} = 0$.
2. Suppose that $\check{H} \le \check{H}^{**}$ and $\check{E}^{**} \le \check{E}$.
3. $\omega = 0$.

Under these conditions let us first define $\mathcal{N} = \frac{\Lambda}{\varrho}$ as the value of $\omega = 0$ then we can transform force infections using $\tilde{\omega} = \frac{\varpi\varrho}{\Lambda}$ to obtain $\nu_a = \tilde{\omega}\check{H}$. The steady-state values from system (1) are described in the following manner.

$$
\begin{aligned}
&\Lambda = \tilde{\omega}\check{L}^*\check{H}^* - (\epsilon_1 + \varrho)\check{L}^*, \\
&\varrho + \vartheta = \frac{\tilde{\omega}\check{L}^*\check{H}^*}{\check{Q}^*}, \\
&\varrho + \chi + \tau = \frac{\vartheta\check{Q}^*}{\check{H}^*}, \\
&\varrho + \wp = \frac{\chi\check{H}^*}{\check{E}^*}
\end{aligned}
$$

The authors present a positive definite Voltera function for the state variables in system (1) excluding $\check{T}$ and $\check{F}$ from the initial work in [28].

$$
\mathcal{V}_{\check{L}} = \check{L}^{**}\mathcal{V}\left(\frac{\check{L}}{\check{L}^{**}}\right)
$$

$$
\begin{aligned}
&= {}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\left[\left(\breve{L}^{**}\mathcal{V}\frac{\breve{L}}{\breve{L}^{**}}\right)\right]\\
&= \left(1-\frac{\breve{L}^{**}}{\breve{L}}\right)(\Lambda-\tilde{\omega}\breve{L}\breve{H}-(\epsilon_1+\varrho)\breve{L})\\
&= \left(1-\frac{\breve{L}^{**}}{\breve{L}}\right)[\tilde{\omega}\breve{L}^{**}\breve{H}^{**}+(\epsilon_1+\varrho)\breve{L}^{**}-\tilde{\omega}\breve{L}\breve{H}-(\epsilon_1+\varrho)\breve{L}]\\
&= (\epsilon_1+\varrho)\breve{L}^{**}\left(2-\frac{\breve{L}^{**}}{\breve{L}}-\frac{\breve{L}}{\breve{L}^{**}}\right)+\tilde{\omega}\breve{L}^{**}\breve{H}^{**}\left(1-\frac{\breve{L}\breve{H}}{\breve{L}^{**}\breve{H}^{**}}-\frac{\breve{L}^{**}}{\breve{L}}+\frac{\breve{H}}{\breve{H}^{**}}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}_{\breve{Q}} &= \breve{Q}^{**}\mathcal{V}\left(\frac{\breve{Q}}{\breve{Q}^{**}}\right)\\
&= {}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\left[\left(\breve{Q}^{**}\mathcal{V}\frac{\breve{Q}}{\breve{Q}^{**}}\right)\right]\\
&= \left(1-\frac{\breve{Q}^{**}}{\breve{Q}}\right)[\tilde{\omega}\breve{L}\breve{H}-(\varsigma+\varrho)\breve{Q}]\\
&= \left(1-\frac{\breve{Q}^{**}}{\breve{Q}}\right)\left[\tilde{\omega}\breve{L}\breve{H}-\frac{\tilde{\omega}\breve{L}^{**}\breve{H}^{**}\breve{Q}}{\breve{Q}^{**}}\right]\\
&= \tilde{\omega}\breve{L}^{**}\breve{H}^{**}\left(1+\frac{\breve{L}\breve{H}}{\breve{L}^{**}\breve{H}^{**}}-\frac{\breve{Q}}{\breve{Q}^{*}}-\frac{\breve{L}\breve{H}\breve{Q}^{**}}{\breve{L}^{**}\breve{H}^{**}\breve{Q}}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}_{\breve{H}} &= \breve{H}^{**}\mathcal{V}\left(\frac{\breve{H}}{\breve{H}^{**}}\right)\\
&= {}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\left[\left(\breve{H}^{**}\mathcal{V}\frac{\breve{H}}{\breve{H}^{**}}\right)\right]\\
&= \left(1-\frac{\breve{H}^{**}}{\breve{H}}\right)[\vartheta\breve{Q}-(\varrho+\omega+\chi+\tau)\breve{H}]\\
&= \left(1-\frac{\breve{H}^{**}}{\breve{H}}\right)\left[\vartheta\breve{Q}-\frac{\vartheta\breve{Q}^{**}}{\breve{H}^{**}}\right]\\
&= \vartheta\breve{Q}^{**}\left(1+\frac{\breve{Q}}{\breve{Q}^{**}-\frac{\breve{H}}{\breve{H}^{**}}}-\frac{\breve{Q}\breve{H}^{**}}{\breve{Q}^{**}\breve{H}}\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{V}_{\breve{E}} &= \breve{E}^{**}\mathcal{V}\left(\frac{\breve{E}}{\breve{E}^{**}}\right)\\
&= {}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\left[\left(\breve{E}^{**}\mathcal{V}\frac{\breve{E}}{\breve{E}^{**}}\right)\right]\\
&= \left(1-\frac{\breve{E}^{**}}{\breve{E}}\right)[\chi\breve{H}-(\varrho+\omega+\wp)\breve{Q}]\\
&= \left(1-\frac{\breve{E}^{**}}{\breve{E}}\right)\left[\chi\breve{H}-\frac{\chi\breve{H}^{**}\breve{E}}{\breve{E}^{**}}\right]\\
&= \chi\breve{H}^{**}\left(1+\frac{\breve{H}}{\breve{H}^{**}}-\frac{\breve{E}}{\breve{E}^{**}}-\frac{\breve{E}^{**}\breve{H}}{\breve{E}\breve{H}^{**}}\right).
\end{aligned}
$$

$$\mathcal{V} = \mathcal{V}_{\mathcal{S}} + \mathcal{V}_{\mathcal{E}} + \frac{\tilde{\beta}\mathcal{S}^{**}\mathcal{I}^{**}}{\alpha\mathcal{E}^{**}}\mathcal{V}_{\mathcal{I}} + \frac{\tilde{\beta}\mathcal{S}^{**}}{\phi}\mathcal{V}_{\mathcal{J}},$$

$${}_{0}^{\mathcal{ABC}}\mathcal{D}_{\mathrm{t}}^{\rho}\mathcal{V} = {}_{0}^{\mathcal{ABC}}\mathcal{D}_{\mathrm{t}}^{\rho}\mathcal{V}_{\mathcal{S}} + {}_{0}^{\mathcal{ABC}}\mathcal{D}_{\mathrm{t}}^{\rho}\mathcal{V}_{\mathcal{E}} + \frac{\tilde{\beta}\mathcal{S}^{**}\mathcal{I}^{**}}{\alpha\mathcal{E}^{**}}{}_{0}^{\mathcal{ABC}}\mathcal{D}_{\mathrm{t}}^{\rho}\mathcal{V}_{\mathcal{I}} + \frac{\tilde{\beta}\mathcal{S}^{**}}{\phi}{}_{0}^{\mathcal{ABC}}\mathcal{D}_{\mathrm{t}}^{\rho}\mathcal{V}_{\mathcal{J}},$$

Define a Lyapunov function,

$$\mathcal{V} = \mathcal{V}_{\check{L}} + \mathcal{V}_{\check{Q}} + \frac{\tilde{\omega}\check{L}^{**}\check{H}^{**}}{\vartheta\check{Q}^{**}}\mathcal{V}_{\check{H}} + \frac{\tilde{\omega}\check{L}^{**}}{\chi}\mathcal{V}_{\check{E}},$$

$${}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{V} = {}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{V}_{\check{L}} + {}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{V}_{\check{Q}} + \frac{\tilde{\omega}\check{L}^{**}\check{H}^{**}}{\vartheta\check{Q}^{**}}{}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{V}_{\check{H}} + \frac{\tilde{\omega}\check{L}^{**}}{\chi}{}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{V}_{\check{E}},$$

$$\mathcal{V} = \mathcal{V}_{\check{L}} + \mathcal{V}_{\check{Q}} + \frac{\tilde{\omega}\check{L}^{**}\check{H}^{**}}{\vartheta\check{Q}^{**}}\mathcal{V}_{\check{H}} + \frac{\tilde{\omega}\check{L}^{**}}{\chi}\mathcal{V}_{\check{E}}$$

$${}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{V} = {}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{V}_{\check{L}} + {}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{V}_{\check{Q}} + \frac{\tilde{\omega}\check{L}^{**}\check{H}^{**}}{\vartheta\check{Q}^{**}}{}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{V}_{\check{H}} + \frac{\tilde{\omega}\check{L}^{**}}{\chi}{}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{V}_{\check{E}}$$

$$\begin{aligned}{}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{V} = {} & (\epsilon_1 + \varrho)\check{L}^{**}\left(2 - \frac{\check{L}^{**}}{\check{L}} - \frac{\check{L}}{\check{L}^{**}}\right) \\ & + \tilde{\omega}\check{L}^{**}\check{H}^{**}\left(3 - \frac{\check{L}\check{H}\check{Q}^{**}}{\check{L}^{**}\check{H}^{**}\check{Q}} - \frac{\check{L}^{**}}{\check{L}} - \frac{\check{H}^{**}\check{Q}}{\check{H}\check{Q}^{**}}\right) \\ & + \tilde{\omega}\check{L}^{**}\check{H}^{**}\left(1 + \frac{\check{H}}{\check{H}^{**}} - \frac{\check{E}}{\check{E}^{**}} - \frac{\check{E}^{**}\check{H}}{\check{E}\check{H}^{**}}\right),\end{aligned}$$

which implies that

$$\left(2 - \frac{\check{L}^{**}}{\check{L}} - \frac{\check{L}}{\check{L}^{**}}\right) \le 0.$$

$$\tilde{\omega}\check{L}^{**}\check{H}^{**}\left(3 - \frac{\check{L}\check{H}\check{Q}^{**}}{\check{L}^{**}\check{H}^{**}\check{Q}} - \frac{\check{L}^{**}}{\check{L}} - \frac{\check{H}^{**}\check{Q}}{\check{H}\check{Q}^{**}}\right) \le 0,$$

$$\tilde{\omega}\check{L}^{**}\check{H}^{**}\left[\left(1 - \frac{\check{E}}{\check{E}^{**}}\right) + \frac{\check{H}}{\check{H}^{**}}\left(1 - \frac{\check{E}^{**}}{\check{E}}\right)\right] \le 0.$$

Therefore ${}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\mathcal{V} \le 0$ with equality only when $\check{L}^{**} = \check{Q}^{**} = \check{H}^{**} = \check{E}^{**}$. Under conditions 1, 2, 3 the endemic equilibrium becomes globally asymptotically stable if $\check{F}_{\mathsf{ef}} > 1$ but becomes unstable if $\check{F}_{\mathsf{ef}} \le 1$ according to Lasalle [27]. □

8 Bifurcation Analysis of the Monkeypox Model

A dynamical system experiences bifurcation even though its behavior undergoes substantial "qualitative" or topological changes. This occurs when there is a steady and modest variation in the system's parameter values. In epidemiology, there are two types of branches: local and global. We analyze the system (1) bifurcations in three directions: forward, backward and Hopf. The Hopf bifurcation is analyzed in detail using the results [29], providing insights into the dynamics of the system during its transitions between steady states. Now, we assess the feasibility of both forward and backward bifurcations using the center manifold theory as presented in [30]. The model's vector form (1) is then shown, in which the variable names are appropriately transferred as

$\check{L} = x_1, \check{Q} = x_2, \check{T} = x_3, \check{H} = x_4, \check{E} = x_5, \check{F} = x_6.$

It follows that

$${}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}(\mathcal{Z}(j)) = \mathcal{U}(\mathcal{Z}),$$

where $\mathcal{Z}$ and $\mathcal{U}$ can be expressed as

$$\mathcal{Z} = (x_1, x_2, x_3, x_4, x_5, x_6)^T$$
$$\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_5, \mathcal{U}_6)^T.$$

Then, the system (1) becomes

$$\begin{cases} \mathcal{U}_1 = {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta} x_1 & = \Lambda - \left(\dfrac{\varpi x_4}{\mathcal{N}}\right) x_1 - (\epsilon_1 + \varrho) x_1, \\ \mathcal{U}_2 = {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta} x_2 & = \left(\dfrac{\varpi x_4}{\mathcal{N}}\right) x_1 - (\varrho + \vartheta) x_2, \\ \mathcal{U}_3 = {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta} x_3 & = \vartheta(1-\mathrm{p}) x_2 - (\varrho + \varsigma) x_3, \\ \mathcal{U}_4 = {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta} x_4 & = \varsigma(1-\epsilon_2) x_3 + \vartheta \mathrm{p} x_2 - (\varrho + \omega + \chi + \tau) x_4, \\ \mathcal{U}_5 = {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta} x_5 & = \chi x_4 - (\varrho + \omega + \wp) x_5, \\ \mathcal{U}_6 = {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta} x_6 & = \varsigma \epsilon_2 x_3 + \epsilon_1 x_1 + \tau x_4 + \wp x_5 - \varrho x_6. \end{cases} \tag{34}$$

Recall that

$$\check{F}_{\mathrm{ef}} = \frac{\varpi \vartheta(\varsigma(1-\epsilon_2) + \mathrm{p}(\varsigma \epsilon_2 + \varrho))}{(\varrho + \vartheta)(\varrho + \varsigma)(\varrho + \omega + \chi + \tau)}.$$

Let the bifurcation parameter ϖ be chosen so that $\check{F}_{\mathrm{ef}} = 1$. Then it follows that

$$\varpi = \varpi^* = \frac{(\varrho + \vartheta)(\varrho + \varsigma)(\varrho + \omega + \chi + \tau)}{\vartheta(\varsigma(1-\epsilon_2) + \mathrm{p}(\varsigma \epsilon_2 + \varrho))}.$$

The jacobian matrix of the system (1) at DFE is given by

$$\mathrm{J}(\varpi^*, DFE) = \begin{pmatrix} -(\epsilon_1 + \varrho) & 0 & 0 & -\dfrac{\varpi \Lambda}{\mathcal{N}(\epsilon_1 + \varrho)} & 0 & 0 \\ 0 & -(\varrho + \vartheta) & 0 & \dfrac{\varpi \Lambda}{\mathcal{N}(\epsilon_1 + \varrho)} & 0 & 0 \\ 0 & \vartheta(1-\mathrm{p}) & -(\varrho + \varsigma) & 0 & 0 & 0 \\ 0 & \vartheta \mathrm{p} & \varsigma(1-\epsilon_2) & -(\varrho + \omega + \chi + \tau) & 0 & 0 \\ 0 & 0 & 0 & \chi & -(\varrho + \omega + \wp) & 0 \\ \epsilon_1 & 0 & \varsigma \epsilon_2 & \tau & \wp & -\varrho \end{pmatrix}.$$

The right eigenvector, denoted by $\bar{a} = (w_1, w_2, w_3, w_4, w_5, w_6)^T$, can be determined by solving the equation $J(\varpi^*, DFE)\bar{a} = 0$. The simple zero eigenvalue is associated with this eigenvector. As such, we obtained the following:

$$J(\varpi^*, DFE)\bar{a} = \begin{pmatrix} -(\epsilon_1 + \varrho)w_1 + \left(-\dfrac{\varpi\Lambda}{\mathcal{N}(\epsilon_1 + \varrho)}\right)w_4 \\ -(\varrho + \vartheta)w_2 + \left(\dfrac{\varpi\Lambda}{\mathcal{N}(\epsilon_1 + \varrho)}\right)w_4 \\ \vartheta(1 - p)w_2 - (\varrho + \varsigma)w_3 \\ \vartheta p w_2 + \varsigma(1 - \epsilon_2)w_3 - (\varrho + \omega + \chi + \tau)w_4 \\ \chi w_4 - (\varrho + \omega + \wp)w_5 \\ \epsilon_1 w_1 + \varsigma\epsilon_2 w_3 + \tau w_4 + \wp w_5 - \varrho w_6 \end{pmatrix} = 0,$$

which implies that

$$w_1 = -\frac{\varpi\Lambda}{\mathcal{N}(\epsilon_1 + \varrho)^2}w_4, \quad w_2 = \frac{\varpi\Lambda}{\mathcal{N}(\epsilon_1 + \varrho)(\varrho + \vartheta)}w_4,$$
$$w_3 = \frac{1}{\varsigma(1 - \epsilon_2)}\left[(\varrho + \omega + \chi + \tau) - \frac{\vartheta p\varpi\Lambda}{\mathcal{N}(\epsilon_1 + \varrho)(\varrho + \vartheta)}\right]w_4,$$
$$w_5 = \frac{\chi w_4}{\varrho + \omega + \wp},$$
$$w_6 = \frac{1}{\varrho}\left[-\frac{\epsilon_1\varpi\Lambda}{\mathcal{N}(\epsilon_1 + \varrho)^2}w_4 + \varsigma\epsilon_2 w_3 + \tau w_4 + \frac{\wp\chi}{\varrho + \omega + \wp}w_4\right].$$

Similarly, the left eigenvector, denoted by $\bar{q} = (q_1, q_2, q_3, q_4, q_5, q_6)$, can be determined by solving the equation $\bar{q}J(\varpi^*, DFE) = 0$. The simple zero eigenvalue is associated with this eigenvector. As such, we obtained the following:

$$\begin{pmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{pmatrix} \times \begin{pmatrix} -(\epsilon_1 + \varrho) & 0 & 0 & -\dfrac{\varpi\Lambda}{\mathcal{N}(\epsilon_1 + \varrho)} & 0 & 0 \\ 0 & -(\varrho + \vartheta) & 0 & \dfrac{\varpi\Lambda}{\mathcal{N}(\epsilon_1 + \varrho)} & 0 & 0 \\ 0 & \vartheta(1 - p) & -(\varrho + \varsigma) & 0 & 0 & 0 \\ 0 & \vartheta p & \varsigma(1 - \epsilon_2) & -(\varrho + \omega + \chi + \tau) & 0 & 0 \\ 0 & 0 & 0 & \chi & -(\varrho + \omega + \wp) & 0 \\ \epsilon_1 & 0 & \varsigma\epsilon_2 & \tau & \wp & -\varrho \end{pmatrix} = 0,$$

which implies that

$$q_1 = \frac{\varpi\Lambda}{\mathcal{N}(\epsilon_1 + \varrho)^2}q_4,$$
$$q_2 = \frac{\varpi\Lambda}{(\varrho + \vartheta)\mathcal{N}(\epsilon_1 + \varrho)}q_4,$$
$$q_3 = \frac{\vartheta(1 - p)}{(\varrho + \varsigma)(\varrho + \vartheta)}\frac{\varpi\Lambda}{\mathcal{N}(\epsilon_1 + \varrho)^2}q_4,$$

$$q_4 = 0,$$
$$q_5 = \frac{\chi}{\varrho + \omega + \wp} q_4,$$
$$q_6 = \frac{q_4 \left(\epsilon_1 \frac{\varpi \Lambda}{\mathcal{N}(\epsilon_1 + \varrho)^2} + \varsigma \epsilon_2 \frac{\vartheta(1-p)}{(\varrho + \varsigma)(\varrho + \vartheta)} \frac{\varpi \Lambda}{\mathcal{N}(\epsilon_1 + \varrho)^2} + \tau + \wp \frac{\chi}{\varrho + \omega + \wp} \right)}{\varrho}.$$

Following approach in [26], we have

$$\bar{a}_1 = q_1 \sum_{i,g=1}^{6} w_i w_g \frac{\partial^2 \mathcal{U}_1}{\partial x_i \partial x_j} + q_2 \sum_{k,i,g=1}^{6} w_i w_g \frac{\partial^2 \mathcal{U}_2}{\partial x_i \partial x_j} + q_3 \sum_{k,i,g=1}^{6} w_i w_g \frac{\partial^2 \mathcal{U}_3}{\partial x_i \partial x_j} + q_4 \sum_{k,i,g=1}^{6} w_i w_g \frac{\partial^2 \mathcal{U}_4}{\partial x_i \partial x_j}$$
$$+ q_5 \sum_{k,i,g=1}^{6} w_i w_g \frac{\partial^2 \mathcal{U}_5}{\partial x_i \partial x_j} + q_6 \sum_{k,i,g=1}^{6} w_i w_g \frac{\partial^2 \mathcal{U}_6}{\partial x_i \partial x_j},$$
$$\bar{b}_1 = q_1 \sum_{i=1}^{6} w_i \frac{\partial^2 \mathcal{U}_1}{\partial x_i \partial \varpi} + q_2 \sum_{k,i=1}^{6} w_i \frac{\partial^2 \mathcal{U}_2}{\partial x_i \partial \varpi} + q_3 \sum_{k,i=1}^{6} w_i \frac{\partial^2 \mathcal{U}_3}{\partial x_i \partial \varpi} + q_4 \sum_{k,i=1}^{6} w_i \frac{\partial^2 \mathcal{U}_4}{\partial x_i \partial \varpi}$$
$$+ q_5 \sum_{k,i=1}^{6} w_i \frac{\partial^2 \mathcal{U}_5}{\partial x_i \partial \varpi} + q_6 \sum_{k,i=1}^{6} w_i \frac{\partial^2 \mathcal{U}_6}{\partial x_i \partial \varpi}.$$

Consequently,

$$\bar{a}_1 = q_1 \left(w_1 w_4 \left(\frac{\varpi}{\mathcal{N}} \right) + w_4 w_1 \left(\frac{\varpi}{\mathcal{N}} \right) \right) + q_2 \left(w_1 w_4 \left(\frac{\varpi}{\mathcal{N}} \right) + w_4 w_1 \left(\frac{\varpi}{\mathcal{N}} \right) \right) > 0,$$
$$\bar{b}_1 = -q_1 \left(\frac{\Lambda}{\mathcal{N}(\epsilon_1 + \varrho)} w_4 \right) + q_2 \left(\frac{\Lambda}{\mathcal{N}(\epsilon_1 + \varrho)} w_4 \right) < 0.$$

Therefore, no bifurcation exists.

9 Real Data and Numerical Simulation

Table 3 presents Nigeria's monkeypox virus weekly data (from week 1 to week 52) for the number of confirmed cases in [31]. The table is broken down by week ranges, corresponding to the period from 29 May 2022 to 21 May 2023. In Fig. 2, understanding the accuracy and weaknesses of the proposed model emerges from comparing real monkeypox data with model-generated forecasts in Nigeria. Both the fitted logistic model and ABC fractional model succeed in tracking the outbreak's complete pattern including its rapid infection periods mainly in the initial stages. The model prediction deviates from actual outbreak data within weeks 15 to 30 of the middle outbreak phase. The model demonstrates effective resolution modeling although unidentified elements in the current modeling framework could be influential to the epidemic dynamics during that period.

Table 3: Nigeria's monkeypox virus weekly data (week 1–52) from 29 May 2022 to 21 May 2023

Week	Number of confirmed cases
1–10	6, 10, 4, 5, 21, 17, 11, 20, 24
11–20	15, 48, 21, 36, 41, 31, 51, 56, 25, 49
21–30	42, 31, 19, 0, 14, 21, 22, 24, 24, 9

(Continued)

Table 3 (continued)

Week	Number of confirmed cases
31–40	3, 7, 3, 10, 11, 3, 7, 4, 5
41–50	2, 7, 2, 7, 4, 0, 1, 1, 0, 2
51–52	12, 1

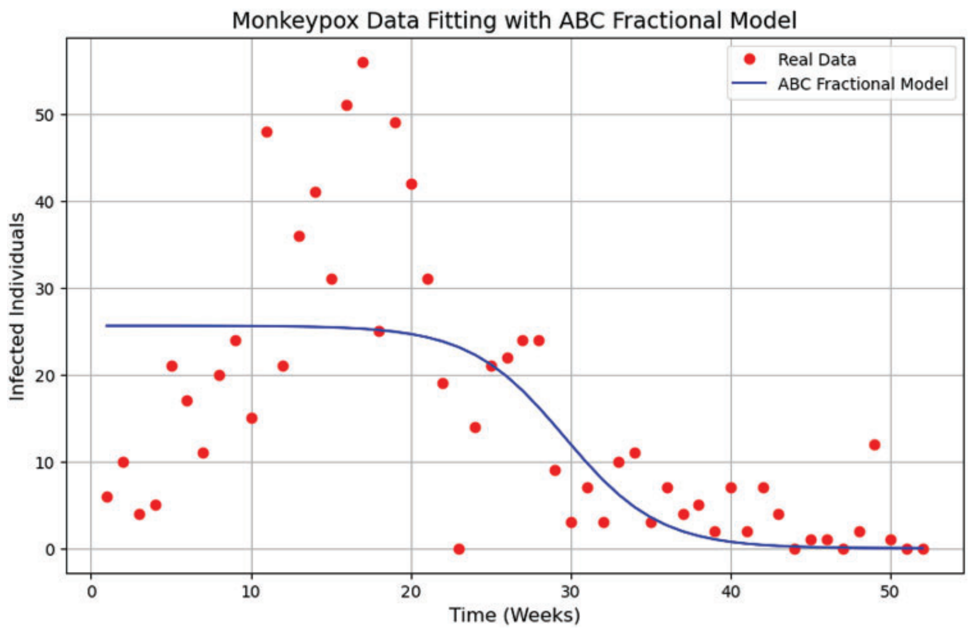

Figure 2: Real data with ABC fractional method

10 CPU Time

The implementation of fractional models (where ζ is less than one) consumes additional CPU(Python) resources because of their memory effects alongside generating superior epidemic results as shown in Fig. 3. Integer-order models that use $\zeta = 1$ perform calculations at a superior level that allows them to decrease CPU time by more than 50% in large-scale real-time simulation applications. The computational times between Caputo and ABC exhibit equivalence, thus enabling users to select based on prediction accuracy needs.

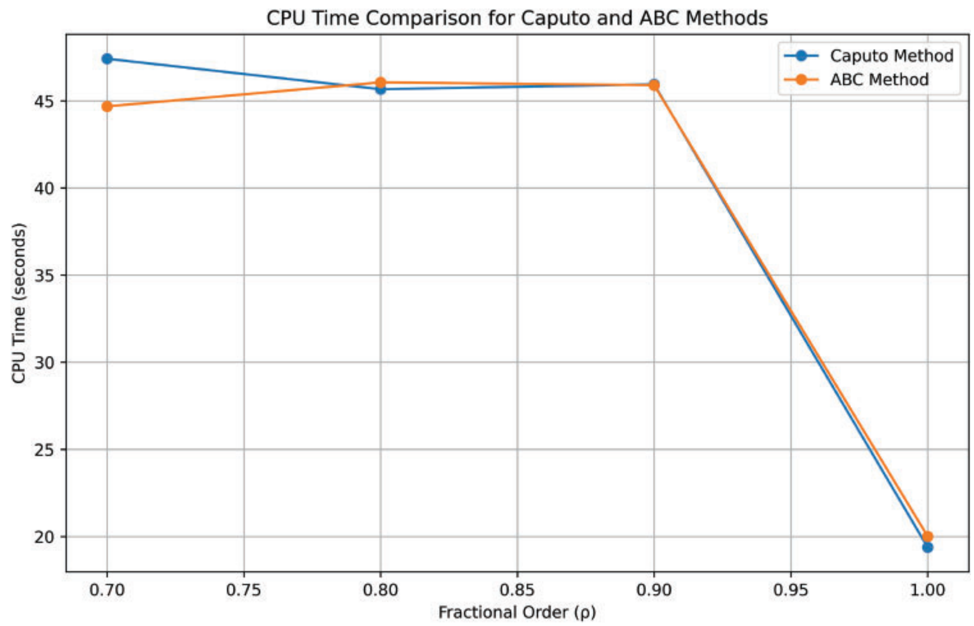

Figure 3: CPU time on fractional Caputo and ABC methods

Existence and uniqueness of solutions of the $\mathcal{ABC}$ fractional derivative model

In this section, we prove the existence and uniqueness of the solution for model (1) by applying the integral operator as defined in Atangana and Baleanu [22] which yields

$$\begin{cases} \check{L}(j) &= \check{L}(0) + {}_{0}^{\mathcal{ABC}}\check{J}_{j}^{\zeta}(\Lambda - \nu\check{L} - (\epsilon_1 + \varrho)\check{L}), \\ \check{Q}(j) &= \check{Q}(0) + {}_{0}^{\mathcal{ABC}}\check{J}_{j}^{\zeta}(\nu\check{L} - (\varrho + \vartheta)\check{Q}), \\ \check{T}(j) &= \check{T}(0) + {}_{0}^{\mathcal{ABC}}\check{J}_{j}^{\zeta}(\vartheta(1 - \mathsf{p})\check{Q} - (\varrho + \varsigma)\check{T}), \\ \check{H}(j) &= \check{H}(0) + {}_{0}^{\mathcal{ABC}}\check{J}_{j}^{\zeta}(\varsigma(1 - \epsilon_2)\check{T} + \vartheta\mathsf{p}\check{Q} - (\varrho + \omega + \chi + \tau)\check{H}), \\ \check{E}(j) &= \check{E}(0) + {}_{0}^{\mathcal{ABC}}\check{J}_{j}^{\zeta}(\chi\check{H} - (\varrho + \omega + \wp)\check{E}), \\ \check{F}(j) &= \check{F}(0) + {}_{0}^{\mathcal{ABC}}\check{J}_{j}^{\zeta}(\varsigma\epsilon_2\check{T} + \epsilon_1\check{L} + \tau\check{H} + \wp\check{E} - \varrho\check{F}). \end{cases} \tag{35}$$

Using the same notation as in [22], Eq. (35) becomes

$$\begin{cases} \breve{L}(j) &= \breve{L}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}(\Lambda - \nu\breve{L} - (\epsilon_1 + \varrho)\breve{L}), \\ &+ \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}(\Lambda - \nu\breve{L} - (\epsilon_1+\varrho)\breve{L})d\xi, \\ \breve{Q}(j) &= \breve{Q}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}(\nu\breve{L} - (\varrho+\vartheta)\breve{Q}), \\ &+ \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}(\nu\breve{L} - (\varrho+\vartheta)\breve{Q})d\xi, \\ \breve{T}(j) &= \breve{T}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}(\vartheta(1-\mathsf{p})\breve{Q} - (\varrho+\varsigma)\breve{T}), \\ &+ \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}(\vartheta(1-\mathsf{p})\breve{Q} - (\varrho+\varsigma)\breve{T})d\xi \\ \breve{H}(j) &= \breve{H}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}(\varsigma(1-\epsilon_2)\breve{T} + \vartheta\mathsf{p}\breve{Q} - (\varrho+\omega+\chi+\tau)\breve{H}), \\ &+ \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}(\varsigma(1-\epsilon_2)\breve{T} + \vartheta\mathsf{p}\breve{Q} - (\varrho+\omega+\chi+\tau)\breve{H})d\xi, \\ \breve{E}(j) &= \breve{E}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}(\chi\breve{H} - (\varrho+\omega+\wp)\breve{E}) \\ &+ \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}(\chi\breve{H} - (\varrho+\omega+\wp)\breve{E})d\xi, \\ \breve{F}(j) &= \breve{F}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}(\varsigma\epsilon_2\breve{T} + \epsilon_1\breve{L} + \tau\breve{H} + \wp\breve{E} - \varrho\breve{F}) \\ &+ \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}(\varsigma\epsilon_2\breve{T} + \epsilon_1\breve{L} + \tau\breve{H} + \wp\breve{E} - \varrho\breve{F})d\xi. \end{cases} \tag{36}$$

Without loss of generality and for simplification of notations, we shall denote

$$\begin{cases} \Psi_1(j,\breve{L}) &= \Lambda - \nu\breve{L} - (\epsilon_1+\varrho)\breve{L}, \\ \Psi_2(j,\breve{Q}) &= \nu\breve{L} - (\varrho+\vartheta)\breve{Q}, \\ \Psi_3(j,\breve{T}) &= \vartheta(1-\mathsf{p})\breve{Q} - (\varrho+\varsigma)\breve{T}, \\ \Psi_4(j,\breve{H}) &= \varsigma(1-\epsilon_2)\breve{T} + \vartheta\mathsf{p}\breve{Q} - (\varrho+\omega+\chi+\tau)\breve{H}, \\ \Psi_5(j,\breve{E}) &= \chi\breve{H} - (\varrho+\omega+\wp)\breve{E}, \\ \Psi_6(j,\breve{F}) &= \varsigma\epsilon_2\breve{T} + \epsilon_1\breve{L} + \tau\breve{H} + \wp\breve{E} - \varrho\breve{F}. \end{cases} \tag{37}$$

$(\mathcal{G}:)$ For proving our results, we consider the following assumption: For this, $\breve{L}(j), \breve{L}^\star(j), \breve{Q}(j), \breve{Q}^\star(j), \breve{T}(j), \breve{T}^\star(j), \breve{H}(j), \breve{H}^\star(j), \breve{E}(j), \breve{E}^\star(j), \breve{F}(j), \breve{F}^\star(j) \in \mathcal{L}[0,1]$ be continuous such that $\|\breve{L}(j)\| \le \mathfrak{c}_1$, $\|\breve{Q}(j)\| \le \mathfrak{c}_2$, $\|\breve{T}(j)\| \le \mathfrak{c}_3$, $\|\breve{H}(j)\| \le \mathfrak{c}_4$, $\|\breve{E}(j)\| \le \mathfrak{c}_5$ and $\|\breve{F}(j)\| \le \mathfrak{c}_6$, for some positive constants $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4, \mathfrak{c}_5, \mathfrak{c}_6 > 0$.

Theorem 8: *Each kernel* $(\Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6)$ *satisfy the Lipschitz condition and contraction under the assumption* $(\mathcal{G})$ *and*

$c_b < 1$, for $b = 1, 2, 3, 4, 5, 6$.

Proof: Now,

$$\begin{aligned}
||\Psi_1(j, \check{L}) - \Psi_1(j, \check{L}_1)|| &= ||(\check{L} - \check{L}_1)(\nu) + (\check{L} - \check{L}_1)(\epsilon_1 + \varrho))|| \\
&\leq \nu||\check{L} - \check{L}_1|| + (\epsilon_1 + \varrho)||\check{L} - \check{L}_1|| \\
&\leq (\nu + \epsilon_1 + \varrho)||\check{L} - \check{L}_1|| \\
&= c_1||\check{L} - \check{L}_1||,
\end{aligned}$$

where $c_1 = \nu + \epsilon_1 + \varrho$. Hence,

$$||\Psi_1(j, \check{L}) - \Psi_1(j, \check{L}_1)|| \leq c_1||\check{L} - \check{L}_1||.$$

Thus, Ψ_1 satisfies the Lipschitz condition. Continuing in this manner, we can prove that $(\Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6)$ satisfy the Lipschitz conditions.

$$\begin{aligned}
||\Psi_2(j, \check{Q}) - \Psi_2(j, \check{Q}_1)|| &\leq c_2||\check{Q} - \check{Q}_1|| \\
||\Psi_3(j, \check{T}) - \Psi_3(j, \check{T}_1)|| &\leq c_3||\check{T} - \check{T}_1|| \\
||\Psi_4(j, \check{H}) - \Psi_4(j, \check{H}_1)|| &\leq c_4||\check{H} - \check{H}_1|| \\
||\Psi_5(j, \check{E}) - \Psi_5(j, \check{E}_1)|| &\leq c_4||\check{E} - \check{E}_1|| \\
||\Psi_6(j, \check{F}) - \Psi_6(j, \check{F}_1)|| &\leq c_5||\check{F} - \check{F}_1||.
\end{aligned}$$

□

Now, from (36), we have

$$\begin{cases}
\check{L}(j) &= \check{L}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_1(j, \check{L}) + \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi_1(\xi, \check{L})d\xi, \\
\check{Q}(j) &= \check{Q}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_2(j, \check{Q}) + \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi_2(\xi, \check{Q})d\xi, \\
\check{T}(j) &= \check{T}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_3(j, \check{T}) + \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi_3(\xi, \check{T})d\xi, \\
\check{H}(j) &= \check{H}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_4(j, \check{H}) + \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi_4(\xi, \check{H})d\xi, \\
\check{E}(j) &= \check{E}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_5(j, \check{E}) + \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi_5(\xi, \check{E})d\xi, \\
\check{F}(j) &= \check{F}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_6(j, \check{F}) + \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi_6(\xi, \check{F})d\xi,
\end{cases}$$

with the initial conditions

$$\check{L}(0) = \check{L}_0, \check{Q}(0) = \mathcal{V}_0, \check{T}(0) = \check{Q}_0, \check{H}(0) = \check{H}_0, \check{E}(0) = \check{E}_0, \check{F}(0) = \check{F}_0.$$

Suppose, we define the iterative recursive forms below,

$$\begin{cases} \breve{L}_\nu(j) &= \breve{L}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_1(j,\breve{L}_{\nu-1}) + \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\Psi_1(\xi,\breve{L}_{\nu-1})d\xi, \\ \breve{Q}_\nu(j) &= \breve{Q}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_2(j,\breve{Q}_{\nu-1}) + \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\Psi_2(\xi,\breve{Q}_{\nu-1})d\xi, \\ \breve{T}_\nu(j) &= \breve{T}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_3(j,\breve{T}_{\nu-1}) + \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\Psi_3(\xi,\breve{T}_{\nu-1})d\xi, \\ \breve{H}_\nu(j) &= \breve{H}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_4(j,\breve{H}_{\nu-1}) + \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\Psi_4(\xi,\breve{H}_{\nu-1})d\xi \\ \breve{E}_\nu(j) &= \breve{E}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_5(j,\breve{E}_{\nu-1}) + \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\Psi_5(\xi,\breve{E}_{\nu-1})d\xi \\ \breve{F}_\nu(j) &= \breve{F}(0) + \dfrac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_6(j,\breve{F}_{\nu-1}) + \dfrac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\Psi_6(\xi,\breve{F}_{\nu-1})d\xi. \end{cases}$$

with the initial conditions

$$\breve{L}(0) = \breve{L}_0, \breve{Q}(0) = \breve{Q}_0, \breve{T}(0) = \breve{T}_0, \breve{H}(0) = \breve{H}_0, \breve{E}(0) = \breve{E}_0, \breve{F}(0) = \breve{F}_0.$$

The following system of equations is formed by using the initial conditions and the difference between the succeeding terms.

$$\begin{aligned} \omega_\nu(j) &= \breve{L}_\nu(j) - \breve{L}_{\nu-1}(j) = \frac{1-\zeta}{\mathcal{M}(\zeta)}(\Psi_1(j,\breve{L}_{\nu-1}) - \Psi_1(j,\breve{L}_{\nu-2})) \\ &\quad + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}(\Psi_1(j,\breve{L}_{\nu-1}) - \Psi_1(j,\breve{L}_{\nu-2}))d\xi, \\ \aleph_\nu(j) &= \breve{Q}_\nu(j) - \breve{Q}_{\nu-1}(j) = \frac{1-\zeta}{\mathcal{M}(\zeta)}(\Psi_2(j,\breve{Q}_{\nu-1}) - \Psi_2(j,\breve{Q}_{\nu-2})) \\ &\quad + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}(\Psi_2(j,\breve{Q}_{\nu-1}) - \Psi_2(j,\breve{Q}_{\nu-2}))d\xi, \\ \Upsilon_\nu(j) &= \breve{T}_\nu(j) - \breve{T}_{\nu-1}(j) = \frac{1-\zeta}{\mathcal{M}(\zeta)}(\Psi_3(j,\breve{T}_{\nu-1}) - \Psi_3(j,\breve{T}_{\nu-2})) \\ &\quad + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}(\Psi_3(j,\breve{T}_{\nu-1}) - \Psi_3(j,\breve{T}_{\nu-2}))d\xi, \\ \Xi_\nu(j) &= \breve{H}_\nu(j) - \breve{H}_{\nu-1}(j) = \frac{1-\zeta}{\mathcal{M}(\zeta)}(\Psi_4(j,\breve{H}_{\nu-1}) - \Psi_4(j,\breve{H}_{\nu-2})) \\ &\quad + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}(\Psi_4(j,\breve{H}_{\nu-1}) - \Psi_4(j,\breve{H}_{\nu-2}))d\xi, \\ \Theta_\nu(j) &= \breve{E}_\nu(j) - \breve{E}_{\nu-1}(j) = \frac{1-\zeta}{\mathcal{M}(\zeta)}(\Psi_5(j,\breve{E}_{\nu-1}) - \Psi_5(j,\breve{E}_{\nu-2})) \\ &\quad + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}(\Psi_5(j,\breve{E}_{\nu-1}) - \Psi_5(j,\breve{E}_{\nu-2}))d\xi, \end{aligned} \tag{38}$$

$$\mathcal{F}_\nu(j) = \check{F}_\nu(j) - \check{F}_{\nu-1}(j) = \frac{1-\zeta}{\mathcal{M}(\zeta)}(\Psi_6(j, \check{F}_{\nu-1}) - \Psi_6(j, \check{F}_{\nu-2})) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}(\Psi_6(j, \check{F}_{\nu-1}) - \Psi_6(j, \check{F}_{\nu-2}))d\xi.$$

It should be noted that

$$\begin{aligned}
\check{L}_\nu(j) &= \sum_{\mathfrak{r}=1}^{\zeta} \omega_{\mathfrak{r}}(j), \\
\check{Q}_\nu(j) &= \sum_{\mathfrak{r}=1}^{\zeta} \aleph_{\mathfrak{r}}(j), \\
\check{T}_\nu(j) &= \sum_{\mathfrak{r}=1}^{\zeta} \Upsilon_{\mathfrak{r}}(j), \\
\check{H}_\nu(j) &= \sum_{\mathfrak{r}=1}^{\zeta} \Xi_{\mathfrak{r}}(j), \\
\check{E}_\nu(j) &= \sum_{\mathfrak{r}=1}^{\zeta} \Theta_{\mathfrak{r}}(j), \\
\check{F}_\nu(j) &= \sum_{\mathfrak{r}=1}^{\zeta} \mathcal{F}_{\mathfrak{r}}(j).
\end{aligned} \tag{39}$$

By taking (38), the triangular inequality is taken into account. After applying the norm to (39).

$$||\omega_\nu(j)|| = ||\check{L}_\nu(j) - \check{L}_{\nu-1}(j)|| = \frac{1-\zeta}{\mathcal{M}(\zeta)}||\Psi_1(j, \check{L}_{\nu-1}) - \Psi_1(j, \check{L}_{\nu-2})|| + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \left\| \int_0^j (j-\xi)^{\zeta-1}(\Psi_1(j, \check{L}_{\nu-1}) - \Psi_1(j, \check{L}_{\nu-2}))d\xi \right\|. \tag{40}$$

As the Lipschitz condition is satisfied by the kernel, the following equation is obtained

$$||\check{L}_\nu(j) - \check{L}_{\nu-1}(j)|| \leq \frac{1-\zeta}{\mathcal{M}(\zeta)}||\check{L}_{\nu-1} - \check{L}_{\nu-2}|| + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}||\check{L}_{\nu-1} - \check{L}_{\nu-2}||d\xi,$$

also,

$$||\omega_\nu(j)|| \leq \frac{1-\zeta}{\mathcal{M}(\zeta)}c_1||\omega_{\nu-1}(j)|| + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_1 \int_0^j (j-\xi)^{\zeta-1}||\omega_{\nu-1}(\xi)||d\xi. \tag{41}$$

In the same manner, we get

$$\begin{aligned}
||\aleph_\nu(j)|| &\leq \frac{1-\zeta}{\mathcal{M}(\zeta)}c_2||\aleph_{\nu-1}(j)|| + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_2 \int_0^j (j-\xi)^{\zeta-1}||\aleph_{\nu-1}(\xi)||d\xi, \\
||\Upsilon_\nu(j)|| &\leq \frac{1-\zeta}{\mathcal{M}(\zeta)}c_3||\Upsilon_{\nu-1}(j)|| + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_3 \int_0^j (j-\xi)^{\zeta-1}||\Upsilon_{\nu-1}(\xi)||d\xi,
\end{aligned}$$

$$
\begin{aligned}
||\Xi_{\nu}(j)|| &\leq \frac{1-\zeta}{\mathcal{M}(\zeta)}c_4||\Xi_{\nu-1}(j)|| + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_4\int_0^j (j-\xi)^{\zeta-1}||\Xi_{\nu-1}(\xi)||d\xi,\\
||\Theta_{\nu}(j)|| &\leq \frac{1-\zeta}{\mathcal{M}(\zeta)}c_5||\Theta_{\nu-1}(j)|| + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_5\int_0^j (j-\xi)^{\zeta-1}||\Theta_{\nu-1}(\xi)||d\xi,\\
||\mathcal{F}_{\nu}(j)|| &\leq \frac{1-\zeta}{\mathcal{M}(\zeta)}c_6||\mathcal{F}_{\nu-1}(j)|| + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_6\int_0^j (j-\xi)^{\zeta-1}||\mathcal{F}_{\nu-1}(\xi)||d\xi.
\end{aligned}
$$

Theorem 9: *The Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative (1) has a unique solution if for $j_{\max}$ the following condition satisfied*

$$\left(\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{j_{\max}^{\zeta}\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\right)c_b \leq 1, \text{ for } b = 1,2,3,4,5,6.$$

Proof: It is clear that $\check{L}(j)$, $\check{Q}(j)$, $\check{T}(j)$, $\check{H}(j)$, $\check{E}(j)$ and $\check{F}(j)$ are bounded. The kernels of these functions also satisfy the Lipschitz condition. Therefore, employing the succeeding relation with the application of the (41), we get

$$
\begin{aligned}
||\omega_{\nu}(j)|| &\leq ||\check{L}(0)||\left(\frac{1-\zeta}{\mathcal{M}(\zeta)}c_1 + \frac{\mathrm{b}^{\zeta}\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_1\right)^{\zeta}\\
||\aleph_{\nu}(j)|| &\leq ||\check{Q}(0)||\left(\frac{1-\zeta}{\mathcal{M}(\zeta)}c_2 + \frac{\mathrm{b}^{\zeta}\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_2\right)^{\zeta}\\
||\Upsilon_{\nu}(j)|| &\leq ||\check{T}(0)||\left(\frac{1-\zeta}{\mathcal{M}(\zeta)}c_3 + \frac{\mathrm{b}^{\zeta}\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_3\right)^{\zeta}\\
||\Xi_{\nu}(j)|| &\leq ||\check{H}(0)||\left(\frac{1-\zeta}{\mathcal{M}(\zeta)}c_4 \frac{\mathrm{b}^{\zeta}\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_4\right)^{\zeta}\\
||\Theta_{\nu}(j)|| &\leq ||\check{E}(0)||\left(\frac{1-\zeta}{\mathcal{M}(\zeta)}c_5 \frac{\mathrm{b}^{\zeta}\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_5\right)^{\zeta}\\
||\mathcal{F}_{\nu}(j)|| &\leq ||\check{F}(0)||\left(\frac{1-\zeta}{\mathcal{M}(\zeta)}c_6 + \frac{\mathrm{b}^{\zeta}\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_6\right)^{\zeta}.
\end{aligned}
$$

Hence, (39) is a smooth function and exists. Let us suppose that the aforementioned functions represent the model's solutions.

$$
\begin{aligned}
\check{L}(j) - \check{L}(0) &= \check{L}_{\zeta}(j) - \Theta_{1(\zeta)}(j),\\
\check{Q}(j) - \check{Q}(0) &= \check{Q}_{\zeta}(j) - \Theta_{2(\zeta)}(j),\\
\check{T}(j) - \check{T}(0) &= \check{T}_{\zeta}(j) - \Theta_{3(\zeta)}(j),\\
\check{H}(j) - \check{H}(0) &= \check{H}_{\zeta}(j) - \Theta_{4(\zeta)}(j),\\
\check{E}(j) - \check{E}(0) &= \check{E}_{\zeta}(j) - \Theta_{5(\zeta)}(j),\\
\check{F}(j) - \check{F}(0) &= \check{F}_{\zeta}(j) - \Theta_{6(\zeta)}(j).
\end{aligned}
$$

The term $\|\Theta_\infty(j)\| \to 0$ at infinity. For this,

$$
\begin{aligned}
\|\Theta_{1(\zeta)}(j)\| \le & \left\| \frac{1-\zeta}{\mathcal{M}(\zeta)}(\Psi_1(j,\check{L}) - \Psi_1(j,\check{L}_{\nu-1})) \right. \\
& \left. + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}(\Psi_1(j,\check{L}) - \Psi_1(j,\check{L}_{\nu-1}))d\xi \right\| \\
& \le \frac{1-\zeta}{\mathcal{M}(\zeta)}\|\Psi_1(j,\check{L}) - \Psi_1(j,\check{L}_{\nu-1})\| \\
& + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\|\Psi_1(j,\check{L}) - \Psi_1(j,\check{L}_{\nu-1})\|d\xi \\
& \le \frac{1-\zeta}{\mathcal{M}(\zeta)}c_1\|\check{L} - \check{L}_{\zeta-1}\| + \frac{\zeta j^\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_1\|\check{L} - \check{L}_{\zeta-1}\|.
\end{aligned}
$$

By recursively repeating the process, we get

$$
\|\Theta_{1(\zeta)}(j)\| \le \left(\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \right)^{\zeta+1} c_1^\zeta \mathcal{Y}.
$$

Then, for $j_{\max}$, we get

$$
\|\Theta_{1(\zeta)}(j)\| \le \left(\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j_{\max}^\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \right)^{\zeta+1} c_1^\zeta \mathcal{Y}.
$$

By taking the limit on both sides as $\zeta \to \infty$, we get $\|\Theta_\infty(j)\| \to 0$. To prove uniqueness of the solution. For this, assume that there exists another solution $\check{L}^\star(j)$, $\check{Q}^\star(j)$, $\check{T}^\star(j)$, $\check{H}^\star(j)$, $\check{E}^\star(j)$, $\check{F}^\star(j)$ with initial values such that

$$
\begin{aligned}
\check{L}^\star(j) &= \check{L}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_1(j,\check{L}^\star) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi_1(\xi,\check{L}^\star)d\xi, \\
\check{Q}^\star(j) &= \check{Q}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_2(j,\check{Q}^\star) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi_2(\xi,\check{Q}^\star)d\xi, \\
\check{T}^\star(j) &= \check{Q}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_3(j,\check{T}^\star) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi_3(\xi,\check{T}^\star)d\xi, \\
\check{H}^\star(j) &= \check{H}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_4(j,\check{H}^\star) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi_4(\xi,\check{H}^\star)d\xi, \\
\check{E}^\star(j) &= \check{H}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_5(j,\check{E}^\star) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi_5(\xi,\check{E}^\star)d\xi, \\
\check{F}^\star(j) &= \check{F}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_6(j,\check{F}^\star) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi_6(\xi,\check{F}^\star)d\xi.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\|\check{L} - \check{L}^\star\| = & \left\| \frac{1-\zeta}{\mathcal{M}(\zeta)}(\Psi_1(j,\check{L}) - \Psi_1(j,\check{L}^\star)) \right. \\
& \left. + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}(\Psi_1(\xi,\check{L}) - \Psi_1(\xi,\check{L}^\star))d\xi \right\|
\end{aligned}
$$

$$\leq \frac{1-\zeta}{\mathcal{M}(\zeta)}\left\|\Psi_1(j,\check{L})-\Psi_1(j,\check{L}^\star)\right\|$$
$$+\frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\left\|\Psi_1(\xi,\check{L})-\Psi_1(\xi,\check{L}^\star))\right\|d\xi$$
$$\leq \frac{1-\zeta}{\mathcal{M}(\zeta)}c_1\|\check{L}-\check{L}^\star\|$$
$$+\frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j c_1(j-\xi)^{\zeta-1}\|\check{L}-\check{L}^\star\|d\xi$$
$$=\frac{1-\zeta}{\mathcal{M}(\zeta)}c_1\|\check{L}-\check{L}^\star\|$$
$$+\frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}c_1\int_0^j (j-\xi)^{\zeta-1}\|\check{L}-\check{L}^\star\|d\xi$$
$$\leq \left(\frac{1-\zeta}{\mathcal{M}(\zeta)}+\frac{\zeta j^\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\right)c_1\|\check{L}-\check{L}^\star\|$$
$$\leq \left(\frac{1-\zeta}{\mathcal{M}(\zeta)}+\frac{\zeta j^\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\right)c_1\|\check{L}-\check{L}^\star\|,$$

which implies that

$$\left(1-\left(\frac{1-\zeta}{\mathcal{M}(\zeta)}+\frac{\zeta j^\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\right)c_1\right)\|\check{L}-\check{L}^\star\|\leq 0.$$

Therefore, $\|\check{L}-\check{L}^\star\|=0$. Hence, $\check{L}=\check{L}^\star$. Similarly, we can prove

$\check{Q}=\check{Q}^\star, \check{T}=\check{T}^\star, \check{H}=\check{H}^\star, \check{E}=\check{E}^\star, \check{F}=\check{F}^\star$.

Hence, Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative (1) has a unique solution. □

11 Ulam Hyers and Ulam Hyers Rassias Stability of the Monkeypox Virus Disease Model

In this section, we obtain the Ulam Hyers and Ulam Hyers Rassias stability of the Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative (1). We state the required definition.

Definition 8: *The Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative (1) has Ulam Hyers stability if there exist constants $\Psi_b>0$, $b=1,2,3,4,5,6$ satisfying: For every $\epsilon_b>0$, $b=1,2,3,4,5,6$, if*

$$\begin{cases}\left|{}_0^{\mathcal{ABC}}\mathcal{D}_j^\zeta\check{L}(j)-\Psi_1(j,\check{L})\right|\leq\epsilon_1,\\ \left|{}_0^{\mathcal{ABC}}\mathcal{D}_j^\zeta\check{Q}(j)-\Psi_2(j,\check{Q})\right|\leq\epsilon_2,\\ \left|{}_0^{\mathcal{ABC}}\mathcal{D}_j^\zeta\check{T}(j)-\Psi_3(j,\check{T})\right|\leq\epsilon_3,\\ \left|{}_0^{\mathcal{ABC}}\mathcal{D}_j^\zeta\check{H}(j)-\Psi_4(j,\check{H})\right|\leq\epsilon_4,\\ \left|{}_0^{\mathcal{ABC}}\mathcal{D}_j^\zeta\check{E}(j)-\Psi_4(j,\check{E})\right|\leq\epsilon_5,\\ \left|{}_0^{\mathcal{ABC}}\mathcal{D}_j^\zeta\check{F}(j)-\Psi_5(j,\check{F})\right|\leq\epsilon_6,\end{cases} \tag{42}$$

and there exists a solution of the Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative (1), $\check{L}^{\star}(j), \check{Q}^{\star}(j), \check{T}^{\star}(j), \check{H}^{\star}(j), \check{E}^{\star}(j)$ and $\check{F}^{\star}(j)$ that statisfying the given model, such that

$$\|\check{L}-\check{L}^{\star}\| \leq \eta_1\epsilon_1, \|\check{Q}-\check{Q}^{\star}\| \leq \eta_2\epsilon_2, \|\check{T}-\check{T}^{\star}\| \leq \eta_3\epsilon_3,$$
$$\|\check{H}-\check{H}^{\star}\| \leq \eta_4\epsilon_4, \|\check{E}-\check{E}^{\star}\| \leq \eta_5\epsilon_5, \|\check{F}-\check{F}^{\star}\| \leq \eta_6\epsilon_6$$

Remark 2: *Consider that function $\check{L}$ is a solution of the first inequality (42) iff a continuous function $\mathfrak{h}_1$ exists (depending on $\check{L}_1$) so that*

(A1) $|\mathfrak{h}_1(j)| < \epsilon_1$, and

(A2) ${}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\check{L}(j) = \Psi_1(j,\check{L}) + \mathfrak{h}_1(j)$.

Similarly, we can define for other classes of the model (42) for some $\mathfrak{h}_b$ where $b = 2,3,4,5,6$.

Theorem 10: *Assume that the hypothesis $(\mathcal{G})$ holds true. Then the Monkeypox virus disease model with $\mathcal{ABC}$ (1) is Ulam Hyers stable if*

$$\left(\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{j^{\zeta}\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\right)c_b \leq 1, \text{ for } b = 1,2,3,4,5,6.$$

Proof: Let $\epsilon_1 > 0$ and the function $\check{L}$ be arbitary such that

$$\left|{}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\check{L}(j) - \Psi_1(j,\check{L})\right| \leq \epsilon_1 .$$

According to Remark 2, we have a function $\mathfrak{h}_1$ with $|\mathfrak{h}_1| < \epsilon_1$, which satisfies

$${}_{0}^{\mathcal{ABC}}\mathcal{D}_{j}^{\zeta}\check{L}(j) = \Psi_1(j,\check{L}) + \mathfrak{h}_1(j).$$

Accordingly, we get

$$\begin{aligned}\check{L}(j) &= \check{L}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_1(j,\check{L}) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\Psi_1(\xi,\check{L})d\xi \\ &+ \frac{1-\zeta}{\mathcal{M}(\zeta)}\mathfrak{h}_1(j) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\mathfrak{h}_1(\xi)d\xi.\end{aligned}$$

Let $\check{L}^{\star}$ be the unique solution of the Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative (1). Then,

$$\check{L}^{\star}(j) = \check{L}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_1(j,\check{L}^{\star}) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\Psi_1(\xi,\check{L}^{\star})d\xi.$$

Hence,

$$\begin{aligned}|\check{L}(j)-\check{L}^{\star}(j)| \leq& \frac{1-\zeta}{\mathcal{M}(\zeta)}|\Psi_1(j,\check{L}(j)) - \Psi_1(j,\check{L}^{\star}(j))| \\ &+ \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}|\Psi_1(\xi,\check{L}(j)) - \Psi_1(\xi,\check{L}^{\star}(j))|d\xi \\ &+ \frac{1-\zeta}{\mathcal{M}(\zeta)}|\mathfrak{h}_1(j)| + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}|\mathfrak{h}_1(\xi)|d\xi\end{aligned}$$

$$
\begin{aligned}
&\le \left[\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^{\zeta}}{\mathcal{M}(\zeta)\wp(\zeta)}\right] c_1 |\check{L}(j) - \check{L}^{\star}(j)| \\
&+ \left[\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^{\zeta}}{\mathcal{M}(\zeta)\wp(\zeta)}\right] \epsilon_1
\end{aligned}
$$

$$
||\check{L}(j) - \check{L}^{\star}(j)|| \le \frac{\left[\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^{\zeta}}{\mathcal{M}(\zeta)\wp(\zeta)}\right] \epsilon_1}{1 - \left[\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^{\zeta}}{\mathcal{M}(\zeta)\wp(\zeta)}\right] c_1}.
$$

Then,

$$
||\check{L}(j) - \check{L}^{\star}(j)|| \le \eta_1 \epsilon_1,
$$

where

$$
\eta_1 = \frac{\left[\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^{\zeta}}{\mathcal{M}(\zeta)\wp(\zeta)}\right]}{1 - \left[\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^{\zeta}}{\mathcal{M}(\zeta)\wp(\zeta)}\right] c_1}.
$$

Similarly, we have

$$
\begin{aligned}
&||\check{Q} - \check{Q}^{\star}|| \le \eta_2\, \epsilon_2, ||\check{T} - \check{T}^{\star}|| \le \eta_3\, \epsilon_3, ||\check{H} - \check{H}^{\star}|| \le \eta_4\, \epsilon_4, \\
&\qquad ||\check{E} - \check{E}^{\star}|| \le \eta_5\, \epsilon_5, ||\check{F} - \check{F}^{\star}|| \le \eta_6\, \epsilon_6\ .
\end{aligned}
$$

Thus, Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative (1) is Ulam Hyers stable. □

Definition 9: *The Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative (1) has Ulam Hyers Rassias stability if there exist constants $\Psi_b > 0$ and a function $\mathrm{y}_b(\mathrm{t}) \in \mathcal{C}([0,1] \to \mathbb{R})$, $b = 1,2,3,4,5,6$ satisfying: For every $\epsilon_b > 0 b = 1,2,3,4,5,6$, if*

$$
\begin{cases}
\left| {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta} \check{L}(j) - \Psi_1(j, \check{L}) \right| \le \epsilon_1 \mathrm{y}_1(\mathrm{t}), \\
\left| {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta} \check{Q}(j) - \Psi_2(j, \check{Q}) \right| \le \epsilon_2 \mathrm{y}_2(\mathrm{t}), \\
\left| {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta} \check{T}(j) - \Psi_3(j, \check{T}) \right| \le \epsilon_3 \mathrm{y}_3(\mathrm{t}), \\
\left| {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta} \check{H}(j) - \Psi_4(j, \check{H}) \right| \le \epsilon_4 \mathrm{y}_4(\mathrm{t}), \\
\left| {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta} \check{E}(j) - \Psi_4(j, \check{E}) \right| \le \epsilon_5 \mathrm{y}_5(\mathrm{t}), \\
\left| {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta} \check{F}(j) - \Psi_5(j, \check{F}) \right| \le \epsilon_6 \mathrm{y}_6(\mathrm{t}),
\end{cases}
\tag{43}
$$

and there exists a solution of the Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative (1), $\check{L}^\star(j), \check{Q}^\star(j), \check{T}^\star(j), \check{H}^\star(j), \check{E}^\star(j)$ and $\check{F}^\star(j)$ that statisfying the given model, such that

$$\|\check{L}-\check{L}^\star\| \le \eta_1\, \epsilon_1 \mathrm{y}_1(\mathrm{t}), \|\check{Q}-\check{Q}^\star\| \le \eta_2\, \epsilon_2 \mathrm{y}_2(\mathrm{t}), \|\check{T}-\check{T}^\star\| \le \eta_3\, \epsilon_3 \mathrm{y}_3(\mathrm{t}),$$
$$\|\check{H}-\check{H}^\star\| \le \eta_4\, \epsilon_4 \mathrm{y}_4(\mathrm{t}), \|\check{E}-\check{E}^\star\| \le \eta_5\, \epsilon_5 \mathrm{y}_5(\mathrm{t}), \|\check{F}-\check{F}^\star\| \le \eta_6\, \epsilon_6 \mathrm{y}_6(\mathrm{t}).$$

Remark 3: *Consider that function $\check{L}$ is a solution of the first inequality (43) iff a continuous function $\mathfrak{h}_1$ exists (depending on $\check{L}_1$) so that*

(*B1*) $|\mathfrak{h}_1(j)| < \epsilon_1\, \mathrm{y}_1(\mathrm{t})$, *and*
(*B2*) ${}_0^{\mathcal{ABC}}\mathcal{D}_j^\zeta \check{L}(j) = \Psi_1(j,\check{L}) + \mathfrak{h}_1(j)$.

Similarly, we can define for other classes of the model (43) for some $\mathfrak{h}_b$ where $b = 2,3,4,5,6$.

Theorem 11: *Assume that the hypothesis $(\mathcal{G})$ holds true. Then the Monkeypox virus disease model with $\mathcal{ABC}$ (1) is Ulam Hyers Rassias stable if*

$$\left(\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{j^\zeta \zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\right)c_b \le 1, \text{ for } b = 1,2,3,4,5,6.$$

Proof: Let $\epsilon_1 > 0$ and the function $\check{L}$ be arbitary such that

$$\left|{}_0^{\mathcal{ABC}}\mathcal{D}_j^\zeta \check{L}(j) - \Psi_1(j,\check{L})\right| \le \epsilon_1 \mathrm{y}_1(\mathrm{t}).$$

According to Remark 2, we have a function $\mathfrak{h}_1$ with $|\mathfrak{h}_1| < \epsilon_1\, \mathrm{y}_1(\mathrm{t})$, which satisfies

$${}_0^{\mathcal{ABC}}\mathcal{D}_j^\zeta \check{L}(j) = \Psi_1(j,\check{L}) + \mathfrak{h}_1(j).$$

Accordingly, we get

$$\begin{aligned}\check{L}(j) = \check{L}(0) &+ \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_1(j,\check{L}) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\Psi_1(\xi,\check{L})d\xi \\ &+ \frac{1-\zeta}{\mathcal{M}(\zeta)}\mathfrak{h}_1(j) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\mathfrak{h}_1(\xi)d\xi\end{aligned}$$

Let $\check{L}^\star$ be the unique solution of the Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative (1). Then,

$$\check{L}^\star(j) = \check{L}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_1(j,\check{L}^\star) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\Psi_1(\xi,\check{L}^\star)d\xi.$$

Hence,

$$\begin{aligned}|\check{L}(j)-\check{L}^\star(j)| \le &\frac{1-\zeta}{\mathcal{M}(\zeta)}|\Psi_1(j,\check{L}(j)) - \Psi_1(j,\check{L}^\star(j))| \\ &+ \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}|\Psi_1(\xi,\check{L}(j)) - \Psi_1(\xi,\check{L}^\star(j))|d\xi \\ &+ \frac{1-\zeta}{\mathcal{M}(\zeta)}|\mathfrak{h}_1(j)| + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}|\mathfrak{h}_1(\xi)|d\xi\end{aligned}$$

$$\leq \left[\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^{\zeta}}{\mathcal{M}(\zeta)\wp(\zeta)}\right] c_1 |\breve{L}(j) - \breve{L}^{\star}(j)|$$
$$+ \left[\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^{\zeta}}{\mathcal{M}(\zeta)\wp(\zeta)}\right] \epsilon_1 \mathrm{y}_1(\mathrm{t})$$
$$\|\breve{L}(j) - \breve{L}^{\star}(j)\| \leq \frac{\left[\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^{\zeta}}{\mathcal{M}(\zeta)\wp(\zeta)}\right] \epsilon_1 \mathrm{y}_1(\mathrm{t})}{1 - \left[\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^{\zeta}}{\mathcal{M}(\zeta)\wp(\zeta)}\right] c_1}.$$

Then,

$$\|\breve{L}(j) - \breve{L}^{\star}(j)\| \leq \eta_1\, \epsilon_1 \mathrm{y}_1(\mathrm{t}),$$

where

$$\eta_1 = \frac{\left[\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^{\zeta}}{\mathcal{M}(\zeta)\wp(\zeta)}\right]}{1 - \left[\frac{1-\zeta}{\mathcal{M}(\zeta)} + \frac{\zeta j^{\zeta}}{\mathcal{M}(\zeta)\wp(\zeta)}\right] c_1}.$$

Similarly, we have

$$\|\breve{Q} - \breve{Q}^{\star}\| \leq \eta_2\, \epsilon_2 \mathrm{y}_2(\mathrm{t}), \|\breve{T} - \breve{T}^{\star}\| \leq \eta_3\, \epsilon_3 \mathrm{y}_3(\mathrm{t}), \|\breve{H} - \breve{H}^{\star}\| \leq \eta_4\, \epsilon_4 \mathrm{y}_4(\mathrm{t}),$$
$$\|\breve{E} - \breve{E}^{\star}\| \leq \eta_5\, \epsilon_5 \mathrm{y}_5(\mathrm{t}), \|\breve{F} - \breve{F}^{\star}\| \leq \eta_6\, \epsilon_6 \mathrm{y}_1(\mathrm{t}).$$

Thus, Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative (1) is Ulam Hyers Rassias stable. □

12 Numerical Scheme on $\mathcal{ABC}$ Operator

Here, a numerical scheme for the Monkeypox virus disease model with $\mathcal{ABC}$ fractional derivative (1) is developed. For this, we use the approach related to Langrange interpolation polynomials. Consider a general Cauchy problem with fractal fractional differential operator as

$$\begin{cases} {}_0^{\mathcal{ABC}}\mathcal{D}_j^{\zeta}\chi(j) = \Psi(j, \chi(j)) \\ \chi(0) = \chi_0. \end{cases} \tag{44}$$

Utilizing the fractal fractional integral operator, we obtain

$$\chi(j) = \chi(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi(j, \chi(j)) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1}\Psi(\xi, \chi(\xi))d\xi.$$

Putting j by j_{i+1}, which gives

$$\chi_{i+1} = \chi(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi(j_i, \chi(j_i)) + \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)} \int_0^{j_{i+1}} (j_{i+1}-\xi)^{\zeta-1}\Psi(\xi, \chi(\xi))d\xi. \tag{45}$$

Over the closed interval $[j_{\mathfrak{m}}, j_{\mathfrak{m}+1}]$, the function $\Psi(\xi, \chi(\xi))$ can be approximated by the interpolation polynomial

$$\begin{aligned}\theta_{\mathfrak{m}}(\xi) &= \frac{\Psi(j_{\mathfrak{m}}, \chi(j_{\mathfrak{m}}))}{\mathfrak{h}}(\xi - j_{\mathfrak{m}-1}) - \frac{\Psi(j_{\mathfrak{m}-1}, \chi(\xi_{\mathfrak{m}-1}))}{\mathfrak{h}}(\xi - j_{\mathfrak{m}}),\\ &\cong \frac{\Psi(j_{\mathfrak{m}}, \chi_{\mathfrak{m}})}{\mathfrak{h}}(\xi - j_{\mathfrak{m}-1}) - \frac{\Psi(j_{\mathfrak{m}-1}, \chi_{\mathfrak{m}-1})}{\mathfrak{h}}(\xi - j_{\mathfrak{m}}),\end{aligned}$$

where $\mathfrak{h} = j_{\mathfrak{m}} - j_{\mathfrak{m}-1}$. By assuming (45) again for $\Psi(\xi, \chi(\xi))$ and Langrange interpolation where $\mathfrak{h}$ is step length, one gets

$$\begin{aligned}\chi_{i+1} &= \chi(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi(j_i, \chi(j_i))\\ &+ \frac{\zeta}{\mathcal{M}(\zeta)\wp(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi(j_c, \chi_c)}{\mathfrak{h}}\int_{j_c}^{j_{c+1}}(\xi - j_{c-1})(j_{i+1} - \xi)^{\zeta-1}d\xi\right.\\ &\left.- \frac{\Psi(j_{c-1}, \chi_{c-1})}{\mathfrak{h}}\int_{j_c}^{j_{c+1}}(\xi - j_c)(j_{i+1} - \xi)^{\zeta-1}d\xi\right).\end{aligned} \tag{46}$$

Now,

$$\begin{aligned}&\int_{j_c}^{j_{c+1}}(\xi - j_{c-1})(j_{i+1} - \xi)^{\zeta-1}d\xi\\ &= \mathfrak{h}^{\zeta+1}\frac{(i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)}{\zeta(\zeta+1)},\end{aligned}$$

and

$$\int_{j_c}^{j_{c+1}}(\xi - j_c)(j_{i+1} - \xi)^{\zeta-1}d\xi = \mathfrak{h}^{\zeta+1}\frac{(i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)}{\zeta(\zeta+1)}$$

Consequently,

$$\begin{aligned}\chi_{i+1} &= \chi(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi(j_i, \chi_i)\\ &+ \frac{\zeta}{\mathcal{M}(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi(j_c, \chi_c)}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta}(i-j+\zeta+2)\right.\\ &\left.- (i-c)^{\zeta}(i-c+2+2\zeta))\right)\\ &- \frac{\zeta}{\mathcal{M}(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi(j_{c-1}, \chi_{c-1})}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta+1}\right.\\ &\left.- (i-c)^{\zeta}(i+1-c+\zeta))\right).\end{aligned} \tag{47}$$

Hence, the numerical scheme for (36) is obtained as the following:

$$\begin{aligned}
\check{L}_{i+1} &= \check{L}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_1(j_i, \check{L}_i) \\
&+ \frac{\zeta}{\mathcal{M}(\zeta)}\sum_{c=0}^{i}\Bigg(\frac{\Psi_1(j_c, \check{L}_c)}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta}(i-j+\zeta+2) \\
&- (i-c)^{\zeta}(i-c+2+2\zeta))\Bigg) \\
&- \frac{\zeta}{\mathcal{M}(\zeta)}\sum_{c=0}^{i}\Bigg(\frac{\Psi_1(j_{c-1}, \check{L}_{c-1})}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta+1} \\
&- (i-c)^{\zeta}(i+1-c+\zeta))\Bigg), \\
\check{Q}_{i+1} &= \check{Q}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_2(j_i, \mathcal{V}_i) \\
&+ \frac{\zeta}{\mathcal{M}(\zeta)}\sum_{c=0}^{i}\Bigg(\frac{\Psi_2(j_c, \check{Q}_c)}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta}(i-j+\zeta+2) \\
&- (i-c)^{\zeta}(i-c+2+2\zeta))\Bigg) \\
&- \frac{\zeta}{\mathcal{M}(\zeta)}\sum_{c=0}^{i}\Bigg(\frac{\Psi_2(j_{c-1}, \check{Q}_{c-1})}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta+1} \\
&- (i-c)^{\zeta}(i+1-c+\zeta))\Bigg), \\
\check{T}_{i+1} &= \check{T}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_3(j_i, \check{T}_i) \\
&+ \frac{\zeta}{\mathcal{M}(\zeta)}\sum_{c=0}^{i}\Bigg(\frac{\Psi_3(j_c, \check{T}_c)}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta}(i-j+\zeta+2) \\
&- (i-c)^{\zeta}(i-c+2+2\zeta))\Bigg) \\
&- \frac{\zeta}{\mathcal{M}(\zeta)}\sum_{c=0}^{i}\Bigg(\frac{\Psi_3(j_{c-1}, \check{T}_{c-1})}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta+1} \\
&- (i-c)^{\zeta}(i+1-c+\zeta))\Bigg), \\
\check{H}_{i+1} &= \check{H}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)}\Psi_4(j_i, \check{H}_i) \\
&+ \frac{\zeta}{\mathcal{M}(\zeta)}\sum_{c=0}^{i}\Bigg(\frac{\Psi_4(j_c, \check{H}_c)}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta}(i-j+\zeta+2) \\
&- (i-c)^{\zeta}(i-c+2+2\zeta))\Bigg)
\end{aligned}$$

$$- \frac{\zeta}{\mathcal{M}(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_4(j_{c-1}, \check{H}_{c-1})}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)) \right),$$

$$\check{E}_{i+1} = \check{E}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)} \Psi_5(j_i, \check{H}_i) + \frac{\zeta}{\mathcal{M}(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_5(j_c, \check{E}_c)}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)) \right) - \frac{\zeta}{\mathcal{M}(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_5(j_{c-1}, \check{E}_{c-1})}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)) \right),$$

$$\check{F}_{i+1} = \check{F}(0) + \frac{1-\zeta}{\mathcal{M}(\zeta)} \Psi_6(j_i, \check{F}_i) + \frac{\zeta}{\mathcal{M}(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_6(j_c, \check{F}_c)}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)) \right) - \frac{\zeta}{\mathcal{M}(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_6(j_{c-1}, \check{F}_{c-1})}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)) \right).$$

Numerical results and discussion

We deliberated the model of (1) observing Mpox disease. Now we present the numerical outcomes after employment of the parametric values defined in Table 1 that are compatible with Mpox virus diseases for various ζ. Taking initial values as

$$\check{L}(0) = 73148, \check{Q}(0) = 300, \check{T}(0) = 200, \check{H}(0) = 100, \check{E}(0) = 50, \check{F}(0) = 0.$$

The numerical simulations for the model (1), incorporating fractional derivatives, provide detailed analyses of the progression of Monkeypox dynamics. Each compartment-Susceptible ($\check{L}$), Exposed ($\check{Q}$), Asymptomatic ($\check{T}$), Infected ($\check{H}$), Isolated ($\check{E}$), and Recovered ($\check{F}$) was examined under varying fractional orders (ζ) to explore the influence of memory effects and fractional dynamics on disease spread and control.

From Fig. 4, we can see that the simulations reveal a rapid decline in the susceptible population for higher fractional orders (ζ = 1.0), indicating accelerated exposure rates and disease progression in deterministic scenarios. Conversely, lower fractional orders (ζ = 0.80) introduce memory effects, slowing the exposure process and extending the susceptible phase. These findings suggest fractional models better capture real-world disease dynamics, where immediate transitions are less common.

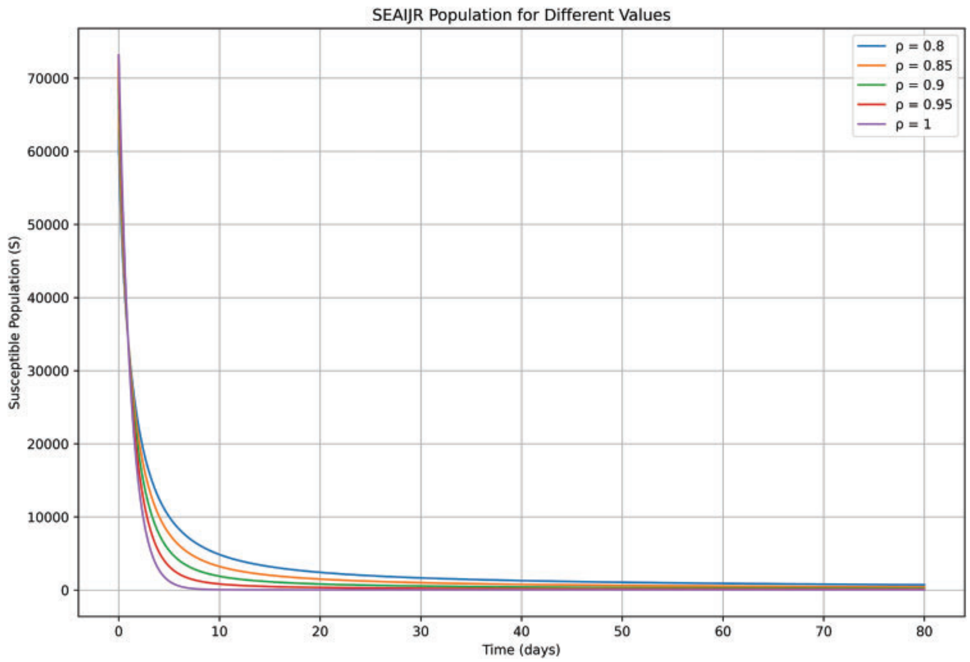

Figure 4: Susceptible class

Fig. 5 reveals that the exposed population exhibits an initial rise followed by stabilization. For higher ζ values, the peak occurs earlier, reflecting faster transitions from susceptible to exposed. Lower ζ values delay the peak, showcasing the role of fractional derivatives in extending the incubation phase. These results highlight the importance of timely interventions, such as vaccination or quarantine, to reduce the duration and intensity of exposure.

The asymptomatic compartment demonstrates significant variability under different ζ values and the same is reflected in Fig. 6. Higher fractional orders result in quicker depletion of asymptomatic individuals as they transition to symptomatic or recovery states. In contrast, lower ζ values prolong the presence of asymptomatic carriers, underscoring the need for robust asymptomatic testing and monitoring strategies.

As can be seen in Fig. 7, symptomatic cases peak later under fractional orders closer to $\zeta = 0.80$, reflecting slower disease progression due to memory effects. For ζ values approaching 1.0, symptomatic cases rise sharply and peak earlier, simulating deterministic epidemic curves. These dynamics underline the importance of rapid diagnosis and treatment to mitigate symptomatic burdens, particularly in scenarios with reduced memory effects.

Fig. 8 shows that Isolation levels steadily increase across all fractional orders, peaking earlier for higher ζ values. This trend demonstrates the effectiveness of isolation measures in curbing disease transmission in rapid-progression scenarios. Lower fractional orders highlight delayed isolation growth, emphasizing the importance of timely policy interventions.

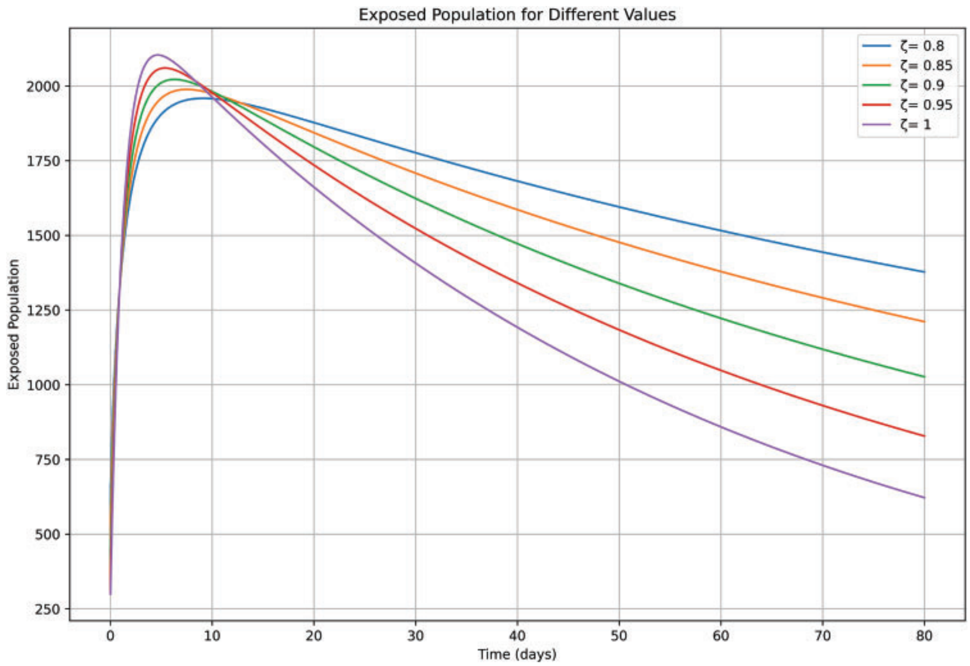

Figure 5: Exposed class

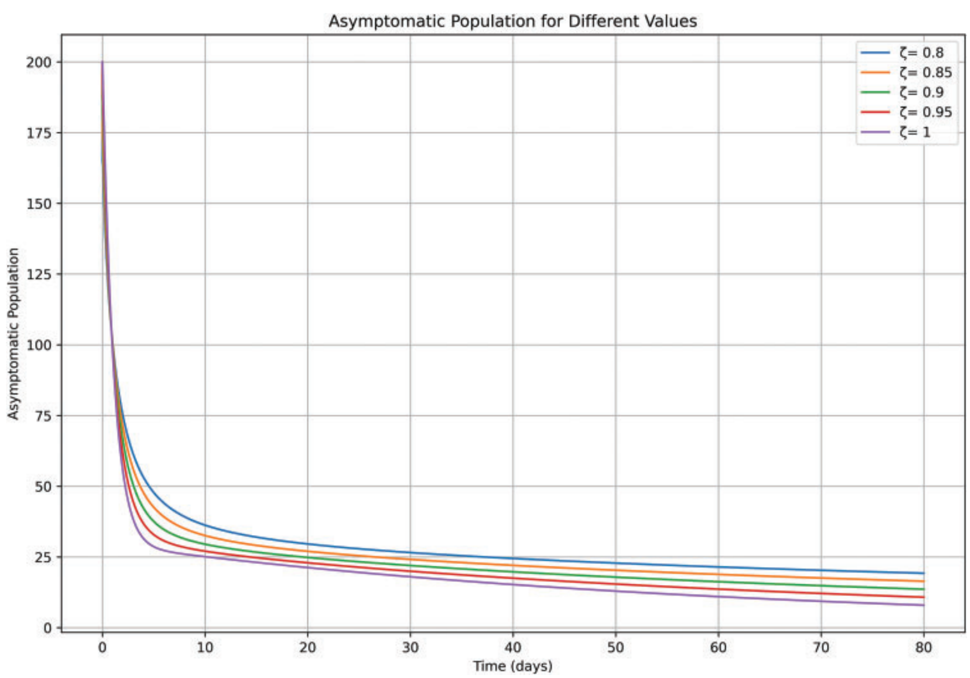

Figure 6: Asymptomatic class

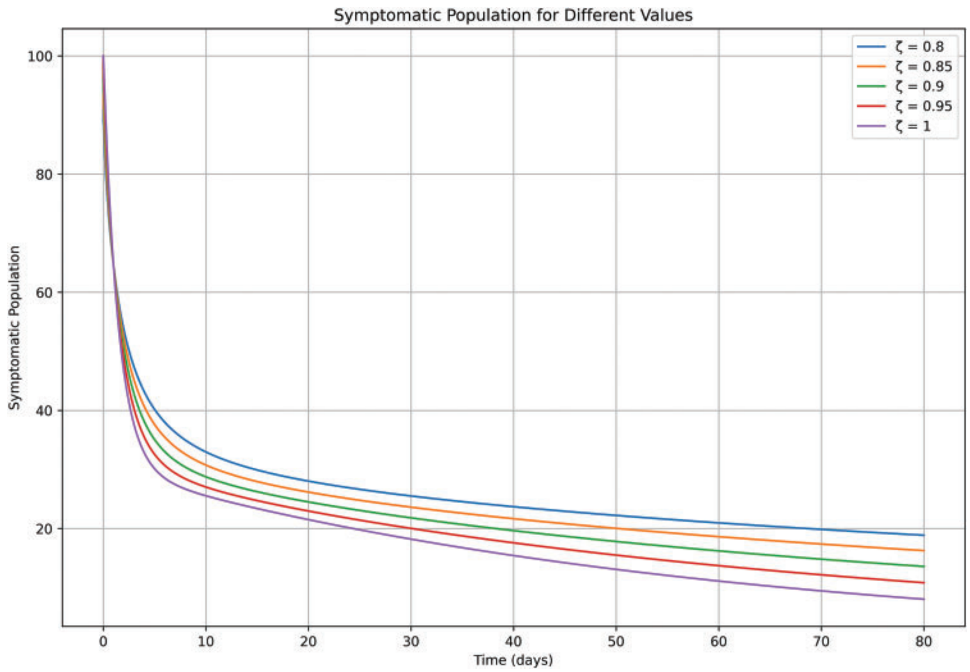

Figure 7: Symptomatic class

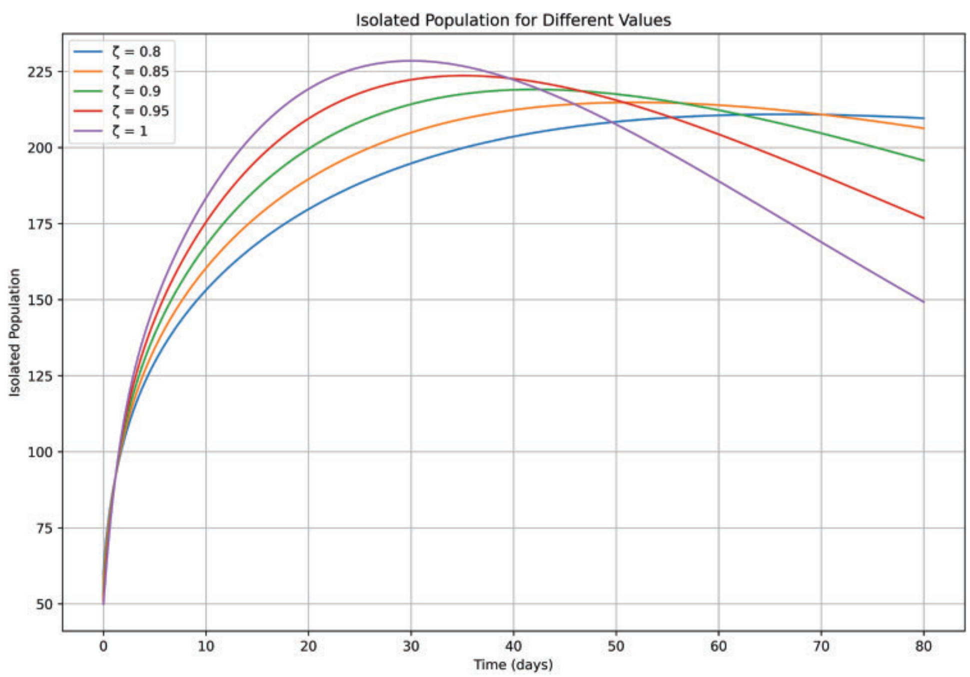

Figure 8: Isolated class

In Fig. 9 recovery trends exhibit faster growth for higher ζ values, reflecting quicker transitions from symptomatic and isolated states to recovery. In contrast, lower ζ values show delayed recovery, consistent with extended disease progression. These results highlight the role of fractional models in capturing varying recovery timelines and the potential impact of post-exposure treatments or vaccination campaigns.

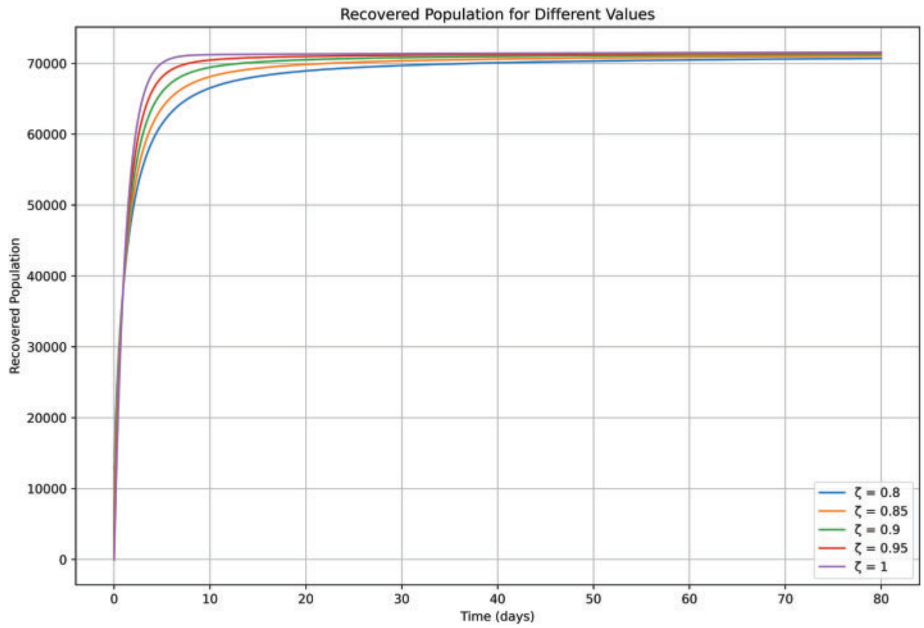

Figure 9: Recovered class

In Fig. 10, susceptible population numbers shift throughout time based on transmission speed levels. Healthcare systems become overwhelmed as the transmission rate ζ increases because the depletion of susceptible individuals happens more rapidly, which leads to rapid disease saturation or herd immunity. A slower transmission rate results in sustainability of the susceptible population during a longer period before interventions take effect. Public health measures should control ζ to determine the epidemic's speed and limit its effect on healthcare capacity and population health outcomes. The transmission rate ζ directly influences exposed population dynamics according to Fig. 11. Disease spread happens rapidly when ζ is high while the epidemic follows a fast resolution pattern. Lower values of ζ indicate a slower pace of exposed population growth that gives intervention opportunities yet extends epidemic length. Higher transmission rates ζ present in Fig. 12 reduce asymptomatic individual numbers because people swiftly move to symptomatic or isolated stages and deplete this population segment. The epidemic lasts longer because reduced transmission rates slow down the decrease of asymptomatic individuals, allowing additional time for public health responses. The control of the disease becomes more challenging when we lack comprehension of asymptomatic population behavior since these individuals maintain virus transmission despite displaying no symptoms, particularly when transmission rates are low. Fig. 13 symptomatic individuals increase rapidly and intensely in situations with high ζ while the disease progression leads to a quick decrease in cases. The epidemic duration becomes longer when ζ values decrease, while the increase of symptomatic cases occurs more slowly. Successful public health planning requires an understanding of how symptomatic populations behave. A quick epidemic demands emergency responses to protect healthcare capabilities, yet slower epidemics demand long-term, continued interventions after they start. The transmission rate ζ demonstrates substantial influence on both the isolation population development and its duration according to Fig. 14. An epidemic becomes faster and more intense when the value for ζ is higher since isolation needs to be implemented swiftly. A decreased value of ζ allows isolation measures to persist over more time so intervention teams have more opportunities to take action yet the epidemic becomes prolonged. Fig. 15 shows that raising the ζ value results in faster peak achievement of recovered patients. The disease progression along with recovery becomes both speedier when ζ values increase. Conversely, lower A slower disease transmission occurs with ζ values resulting in a systematic increase of the recovered population. Public health interventions need to adapt their disease management techniques according to transmission speed because changing the recovery process affects healthcare capacity and epidemic control measures. The response requires immediate action when transmission rates are high because the disease resolves quickly yet preparedness measures must begin right away. On the other hand, low transmission rates require long-term management to achieve complete recovery.

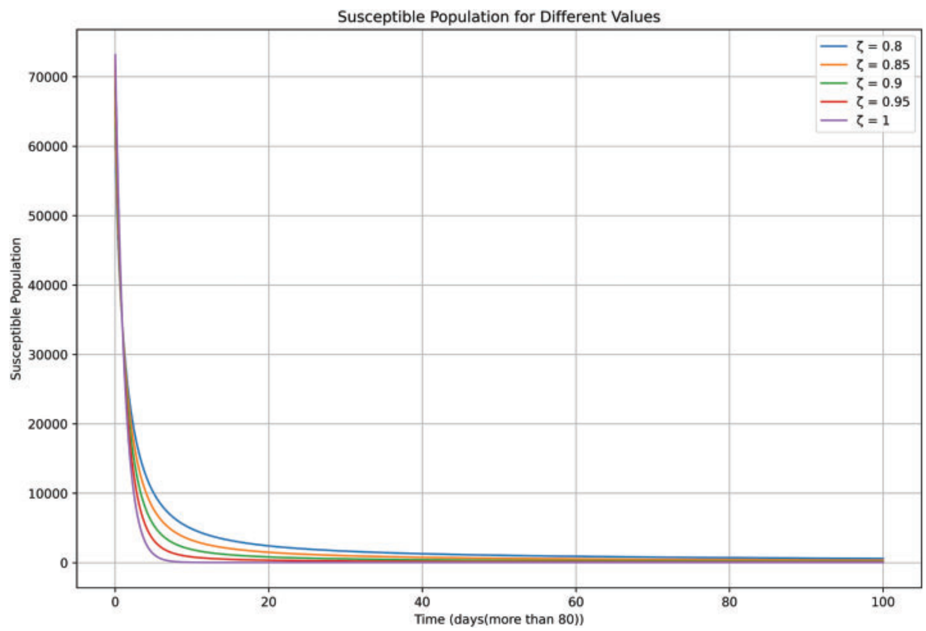

Figure 10: Susceptible class ($j > 80$)

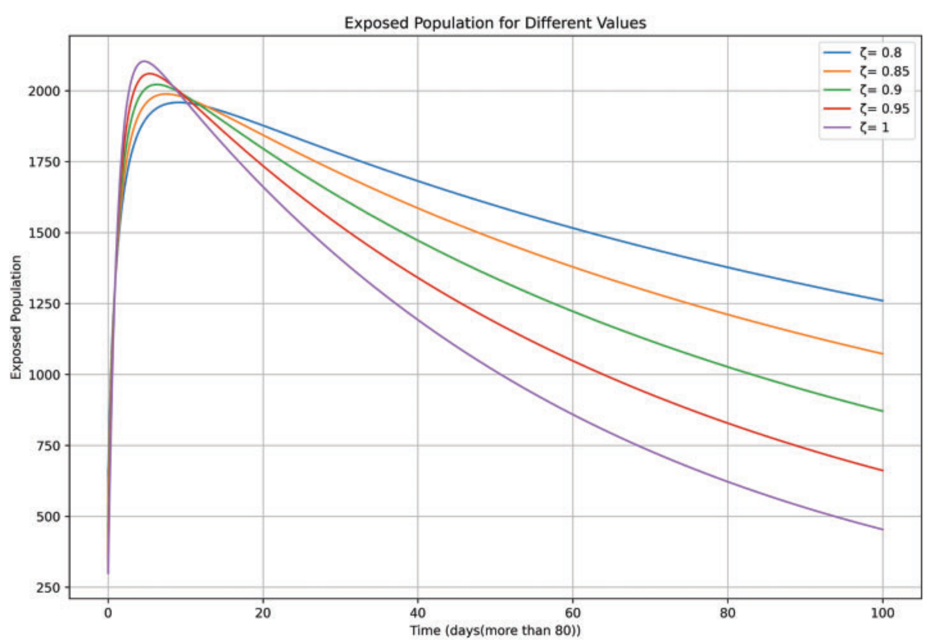

Figure 11: Exposed class ($j > 80$)

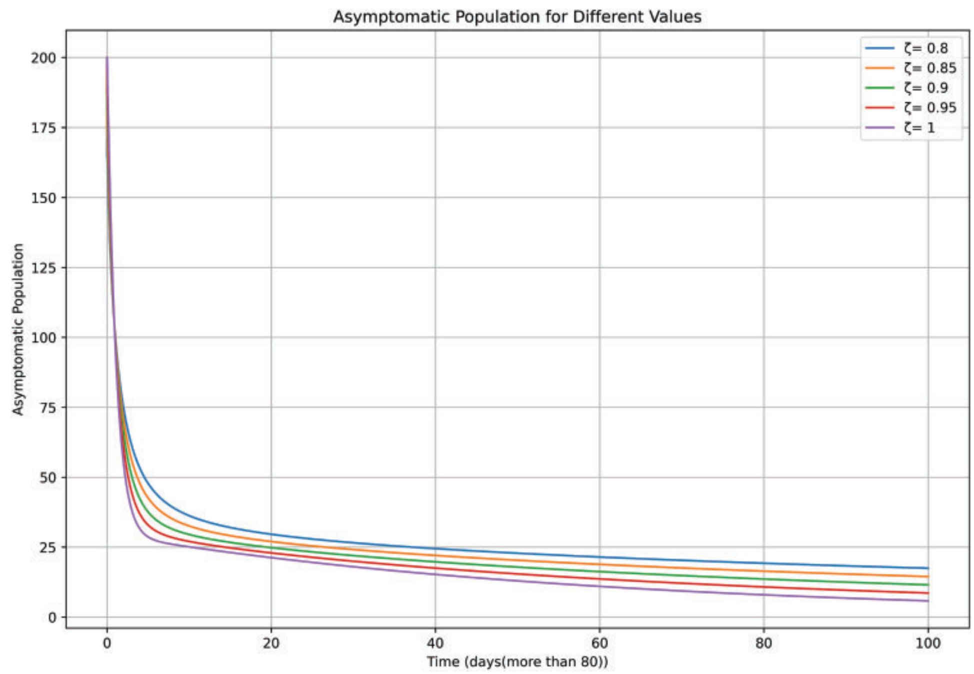

Figure 12: Asymptomatic class $(j > 80)$

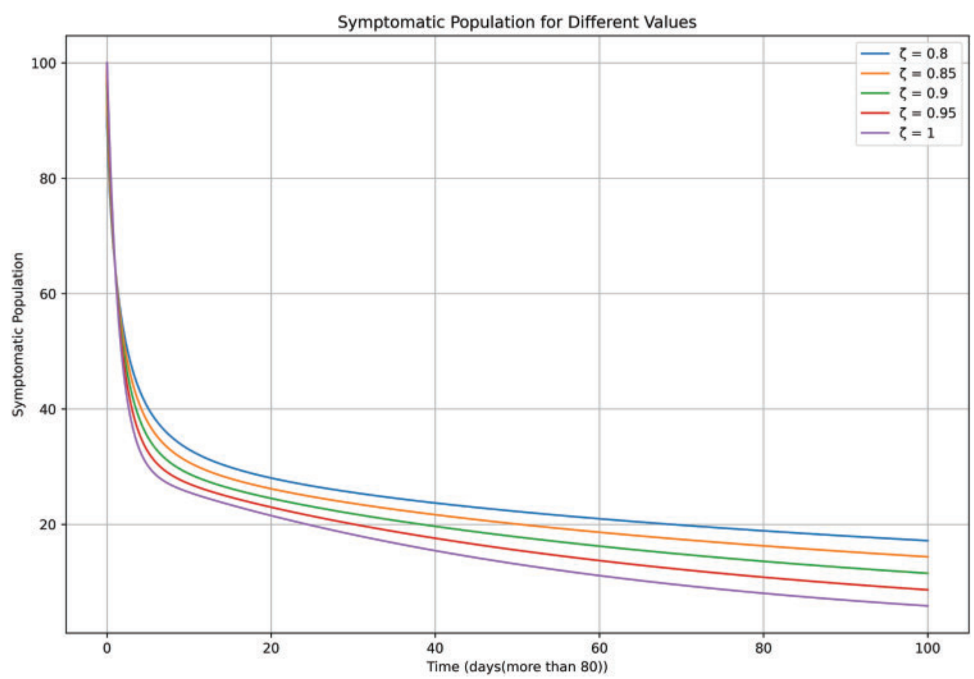

Figure 13: Symptomatic class $(j > 80)$

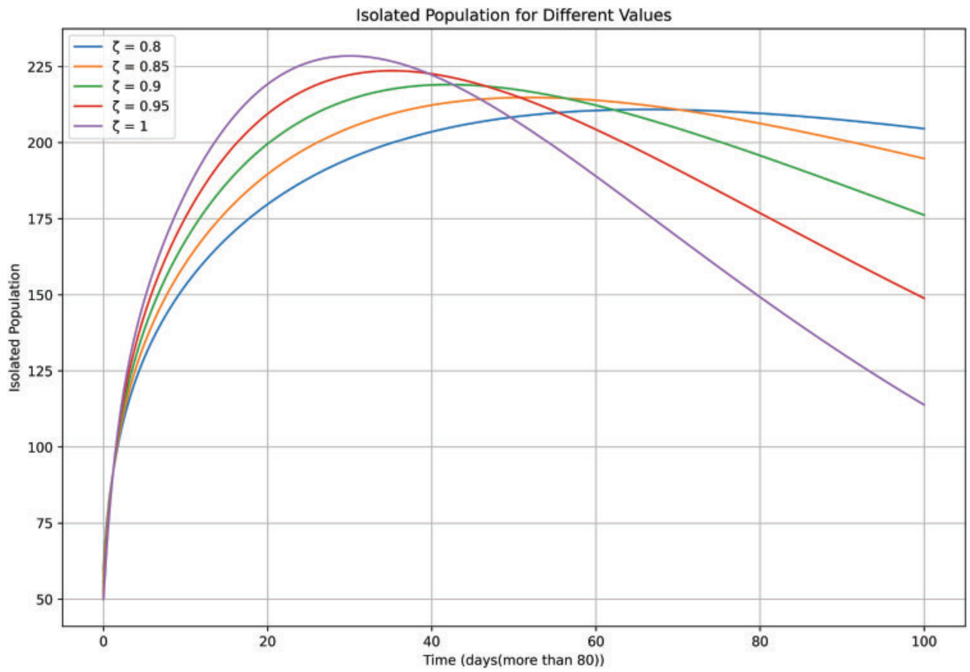

Figure 14: Isolation class ($j > 80$)

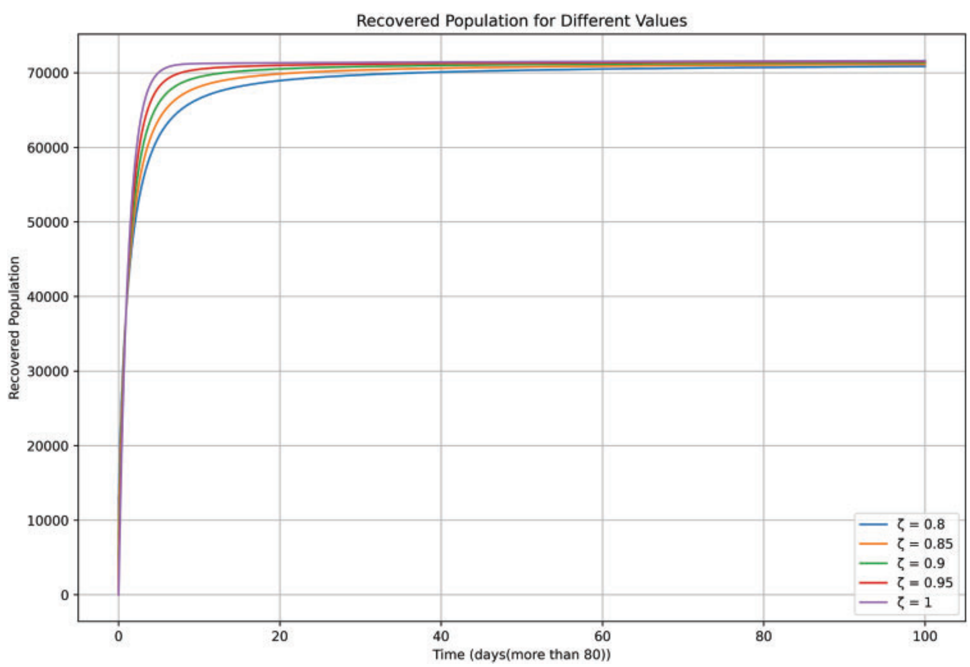

Figure 15: Recovered class ($j > 80$)

13 Numerical Scheme on Caputo Operator

We thought about a numerical method which is based upon the Caputo operator that has been discovered to describe the linear and nonlinear dynamical systems [32] and [33]. Now, we assume a fractional order initial value problem using the fractional Caputo operator

$$\begin{cases} {}_0^{C}\mathcal{D}_j^{\zeta}\chi(j) = \Psi(j, \chi(j)) \\ \chi(0) = \chi_0. \end{cases} \tag{48}$$

Utilizing the fractal fractional integral operator, we obtain

$$\chi(j) = \chi(0) + \frac{1}{\wp(\zeta)}\int_0^j (j-\xi)^{\zeta-1}\Psi(\xi, \chi(\xi))d\xi.$$

Putting j by j_{i+1}, which gives

$$\chi_{i+1} = \chi(0) + \frac{1}{\wp(\zeta)}\int_0^{j_{i+1}} (j_{i+1}-\xi)^{\zeta-1}\Psi(\xi, \chi(\xi))d\xi + \frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\int_{j_c}^{j_{c+1}} (j_{i+1}-\xi)^{\zeta-1}\Psi(\xi, \chi(\xi))d\xi. \tag{49}$$

Over the closed interval $[j_{\mathfrak{m}}, j_{\mathfrak{m}+1}]$, the function $\Psi(\xi, \chi(\xi))$ can be approximated by the interpolation polynomial

$$\begin{aligned} \theta_{\mathfrak{m}}(\xi) &= \frac{\Psi(j_{\mathfrak{m}}, \chi(j_{\mathfrak{m}}))}{\mathfrak{h}}(\xi - j_{\mathfrak{m}-1}) - \frac{\Psi(j_{\mathfrak{m}-1}, \chi(\xi_{\mathfrak{m}-1}))}{\mathfrak{h}}(\xi - j_{\mathfrak{m}}), \\ &\cong \frac{\Psi(j_{\mathfrak{m}}, \chi_{\mathfrak{m}})}{\mathfrak{h}}(\xi - j_{\mathfrak{m}-1}) - \frac{\Psi(j_{\mathfrak{m}-1}, \chi_{\mathfrak{m}-1})}{\mathfrak{h}}(\xi - j_{\mathfrak{m}}), \end{aligned}$$

where $\mathfrak{h} = j_{\mathfrak{m}} - j_{\mathfrak{m}-1}$. By assuming (53) again for $\Psi(\xi, \chi(\xi))$ and Langrange interpolation where $\mathfrak{h}$ is step length, one gets

$$\begin{aligned} \chi_{i+1} = \chi(0) + \frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\Bigg(&\frac{\Psi(j_c, \chi_c)}{\mathfrak{h}}\int_{j_c}^{j_{c+1}}(\xi - j_{c-1})(j_{i+1}-\xi)^{\zeta-1}d\xi \\ &- \frac{\Psi(j_{c-1}, \chi_{c-1})}{\mathfrak{h}}\int_{j_c}^{j_{c+1}}(\xi - j_c)(j_{i+1}-\xi)^{\zeta-1}d\xi\Bigg). \end{aligned} \tag{50}$$

Now,

$$\begin{aligned} &\int_{j_c}^{j_{c+1}}(\xi - j_{c-1})(j_{i+1}-\xi)^{\zeta-1}d\xi \\ &= \mathfrak{h}^{\zeta+1}\frac{(i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)}{\zeta(\zeta+1)}, \end{aligned}$$

and

$$\int_{j_c}^{j_{c+1}}(\xi - j_c)(j_{i+1}-\xi)^{\zeta-1}d\xi = \mathfrak{h}^{\zeta+1}\frac{(i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)}{\zeta(\zeta+1)}$$

Consequently,

$$\chi_{i+1} = \chi(0) + \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi(j_c, \chi_c)}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)) \right)$$
$$- \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi(j_{c-1}, \chi_{c-1})}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)) \right). \tag{51}$$

Hence, the numerical scheme for (36) is obtained as the following:

$$\check{L}_{i+1} = \check{L}(0) + \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_c, \check{L}_c)}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)) \right)$$
$$- \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_{c-1}, \check{L}_{c-1})}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)) \right),$$
$$\check{Q}_{i+1} = \check{Q}(0) + \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_c, \check{Q}_c)}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)) \right)$$
$$- \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_{c-1}, \check{Q}_{c-1})}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)) \right),$$
$$\check{T}_{i+1} = \check{T}(0) + \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_c, \check{T}_c)}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)) \right)$$
$$- \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_{c-1}, \check{T}_{c-1})}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)) \right),$$
$$\check{H}_{i+1} = \check{H}(0) + \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_c, \check{H}_c)}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)) \right)$$
$$- \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_{c-1}, \check{H}_{c-1})}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)) \right),$$
$$\check{E}_{i+1} = \check{E}(0) + \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_c, \check{E}_c)}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)) \right)$$
$$- \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_{c-1}, \check{E}_{c-1})}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)) \right),$$
$$\check{F}_{i+1} = \check{F}(0) + \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_c, \check{F}_c)}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)) \right)$$
$$- \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_{c-1}, \check{F}_{c-1})}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)) \right).$$

14 Numerical Scheme on Caputo Operator

We thought about a numerical method which is based upon the Caputo operator that has been discovered to describe the linear and nonlinear dynamical systems [32] and [33]. Now, we assume a fractional order initial value problem using fractional Caputo operator

$$\begin{cases} {}^{C}_{0}\mathcal{D}^{\zeta}_{j}\chi(j) = \Psi(j, \chi(j)) \\ \chi(0) = \chi_0. \end{cases} \tag{52}$$

Utilizing the fractal fractional integral operator, we obtain

$$\chi(j) = \chi(0) + \frac{1}{\wp(\zeta)} \int_0^j (j-\xi)^{\zeta-1} \Psi(\xi, \chi(\xi)) d\xi.$$

Putting j by j_{i+1}, which gives

$$\chi_{i+1} = \chi(0) + \frac{1}{\wp(\zeta)} \int_0^{j_{i+1}} (j_{i+1}-\xi)^{\zeta-1} \Psi(\xi, \chi(\xi)) d\xi + \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \int_{j_c}^{j_{c+1}} (j_{i+1}-\xi)^{\zeta-1} \Psi(\xi, \chi(\xi)) d\xi. \tag{53}$$

Over the closed interval $[j_{\mathfrak{m}}, j_{\mathfrak{m}+1}]$, the function $\Psi(\xi, \chi(\xi))$ can be approximated by the interpolation polynomial

$$\begin{aligned}\theta_{\mathfrak{m}}(\xi) &= \frac{\Psi(j_{\mathfrak{m}}, \chi(j_{\mathfrak{m}}))}{\mathfrak{h}}(\xi - j_{\mathfrak{m}-1}) - \frac{\Psi(j_{\mathfrak{m}-1}, \chi(\xi_{\mathfrak{m}-1}))}{\mathfrak{h}}(\xi - j_{\mathfrak{m}}),\\ &\cong \frac{\Psi(j_{\mathfrak{m}}, \chi_{\mathfrak{m}})}{\mathfrak{h}}(\xi - j_{\mathfrak{m}-1}) - \frac{\Psi(j_{\mathfrak{m}-1}, \chi_{\mathfrak{m}-1})}{\mathfrak{h}}(\xi - j_{\mathfrak{m}}),\end{aligned}$$

where $\mathfrak{h} = j_{\mathfrak{m}} - j_{\mathfrak{m}-1}$. By assuming (53) again for $\Psi(\xi, \chi(\xi))$ and Langrange interpolation where $\mathfrak{h}$ is step length, one gets

$$\begin{aligned}\chi_{i+1} = \chi(0) + \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \Bigg(&\frac{\Psi(j_c, \chi_c)}{\mathfrak{h}} \int_{j_c}^{j_{c+1}} (\xi - j_{c-1})(j_{i+1}-\xi)^{\zeta-1} d\xi \\ &- \frac{\Psi(j_{c-1}, \chi_{c-1})}{\mathfrak{h}} \int_{j_c}^{j_{c+1}} (\xi - j_c)(j_{i+1}-\xi)^{\zeta-1} d\xi \Bigg).\end{aligned} \tag{54}$$

Now,

$$\begin{aligned}&\int_{j_c}^{j_{c+1}} (\xi - j_{c-1})(j_{i+1}-\xi)^{\zeta-1} d\xi \\ &= \mathfrak{h}^{\zeta+1} \frac{(i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)}{\zeta(\zeta+1)},\end{aligned}$$

and

$$\int_{j_c}^{j_{c+1}} (\xi - j_c)(j_{i+1}-\xi)^{\zeta-1} d\xi = \mathfrak{h}^{\zeta+1} \frac{(i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)}{\zeta(\zeta+1)}$$

Consequently,

$$\begin{aligned}\chi_{i+1} = \chi(0) &+ \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi(j_c, \chi_c)}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)) \right) \\ &- \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi(j_{c-1}, \chi_{c-1})}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta+1} - (i-c)^{\zeta}(i+1-c+\zeta)) \right).\end{aligned} \tag{55}$$

Hence, the numerical scheme for (36) is obtained as the following:

$$\check{L}_{i+1} = \check{L}(0) + \frac{1}{\wp(\zeta)} \sum_{c=0}^{i} \left(\frac{\Psi_1(j_c, \check{L}_c)}{\wp(\zeta+2)} \mathfrak{h}^{\zeta} ((i-j+1)^{\zeta}(i-j+\zeta+2) - (i-c)^{\zeta}(i-c+2+2\zeta)) \right)$$

$$
\begin{aligned}
&-\frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi_1(j_{c-1},\check{L}_{c-1})}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta+1}-(i-c)^{\zeta}(i+1-c+\zeta))\right),\\
\check{Q}_{i+1} &= \check{Q}(0)+\frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi_1(j_c,\check{Q}_c)}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta}(i-j+\zeta+2)-(i-c)^{\zeta}(i-c+2+2\zeta))\right)\\
&-\frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi_1(j_{c-1},\check{Q}_{c-1})}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta+1}-(i-c)^{\zeta}(i+1-c+\zeta))\right),\\
\check{T}_{i+1} &= \check{T}(0)+\frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi_1(j_c,\check{T}_c)}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta}(i-j+\zeta+2)-(i-c)^{\zeta}(i-c+2+2\zeta))\right)\\
&-\frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi_1(j_{c-1},\check{T}_{c-1})}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta+1}-(i-c)^{\zeta}(i+1-c+\zeta))\right),\\
\check{H}_{i+1} &= \check{H}(0)+\frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi_1(j_c,\check{H}_c)}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta}(i-j+\zeta+2)-(i-c)^{\zeta}(i-c+2+2\zeta))\right)\\
&-\frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi_1(j_{c-1},\check{H}_{c-1})}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta+1}-(i-c)^{\zeta}(i+1-c+\zeta))\right),\\
\check{E}_{i+1} &= \check{E}(0)+\frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi_1(j_c,\check{E}_c)}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta}(i-j+\zeta+2)-(i-c)^{\zeta}(i-c+2+2\zeta))\right)\\
&-\frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi_1(j_{c-1},\check{E}_{c-1})}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta+1}-(i-c)^{\zeta}(i+1-c+\zeta))\right),\\
\check{F}_{i+1} &= \check{F}(0)+\frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi_1(j_c,\check{F}_c)}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta}(i-j+\zeta+2)-(i-c)^{\zeta}(i-c+2+2\zeta))\right)\\
&-\frac{1}{\wp(\zeta)}\sum_{c=0}^{i}\left(\frac{\Psi_1(j_{c-1},\check{F}_{c-1})}{\wp(\zeta+2)}\mathfrak{h}^{\zeta}((i-j+1)^{\zeta+1}-(i-c)^{\zeta}(i+1-c+\zeta))\right).
\end{aligned}
$$

Comparing numerical results between $\mathcal{ABC}$ and Caputo and discussion

In Fig. 16, ABC results in a faster decrease in $\check{L}(j)$, suggesting a more aggressive infection process compared to Caputo. Lower ζ values delay infection spread, making fractional-order models useful for studying real-world pandemics. The Caputo model is more stable and slower in transmission, while the ABC model shows a more rapid outbreak scenario.

In Fig. 17, ABC method provides a more realistic epidemic curve by showing a peak in exposed individuals, while Caputo remains static. Higher ζ values lead to faster exposure and peak earlier, emphasizing the need to fine-tune ζ for accurate modeling. The Caputo model may underestimate exposure, making it less reliable for analyzing the transition from susceptible to infected individuals. The ABC model is preferable for capturing early epidemic dynamics, especially for diseases with a strong incubation phase.

In Fig. 18, ABC model is more realistic, as it shows a decline in asymptomatic cases over time, reflecting real-world disease progression. The Caputo model is overly simplistic, keeping the asymptomatic population constant and failing to capture transitions. Higher ζ values lead to faster transitions in the ABC model, while lower ζ values indicate longer asymptomatic periods. The ABC model is preferable for understanding asymptomatic transmission dynamics, making it a better choice for modeling diseases with significant asymptomatic spread.

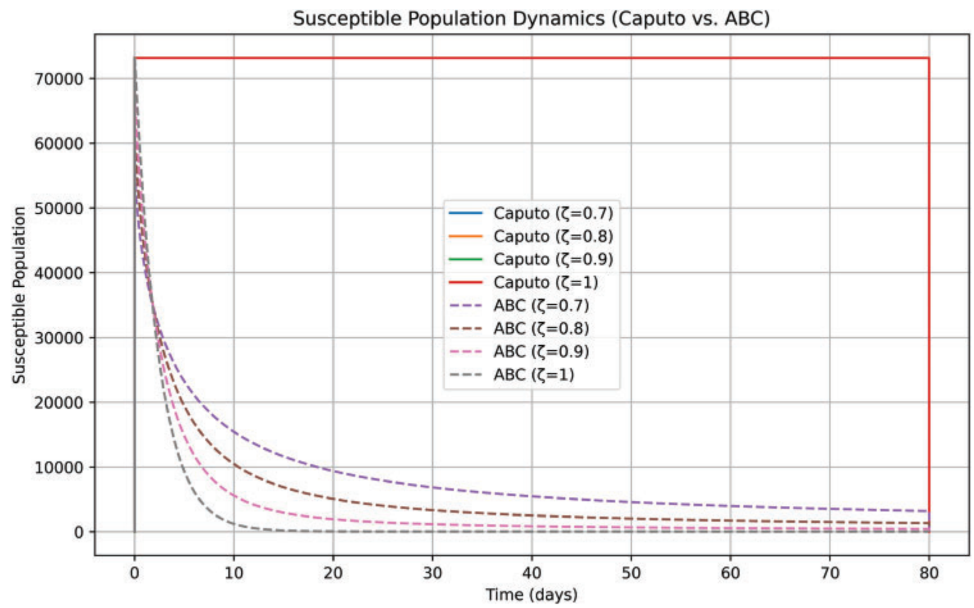

Figure 16: Comparison of ABC and Caputo on susceptible

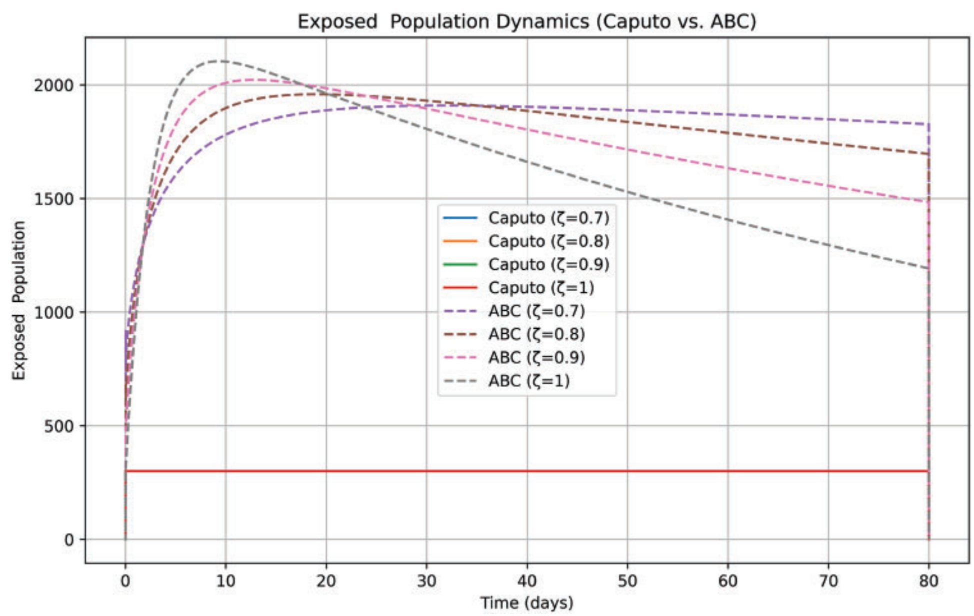

Figure 17: Comparison of ABC and Caputo on exposed

In Fig. 19, ABC model is more realistic for modeling symptomatic cases, showing a gradual decline over time. The Caputo model fails to capture symptom resolution, making it less suitable for infectious diseases where symptoms eventually disappear. Higher ζ values lead to faster symptom resolution, while lower ζ values indicate prolonged symptomatic periods. The ABC model is preferable for tracking disease progression and recovery, making it a better choice for epidemic modeling. In Fig. 20, ABC model is better suited for modeling dynamic isolation processes. The Caputo model fails to capture the evolving nature of isolation, making it less effective for real-world outbreak simulations. Higher ζ values lead to faster isolation rates, representing stronger containment measures. The ABC model aligns better with real-world quarantine strategies, making it the preferred choice for epidemiological studies. In Fig. 21, ABC model is more suitable for epidemic modeling, as it realistically captures increasing recovery rates. The Caputo model fails to show recovery, making it unrealistic for studying disease progression. Higher ζ values lead to faster recovery, reflecting better medical care and public health interventions. The ABC model aligns better with real-world epidemiological patterns, making it the preferred model for studying outbreaks.

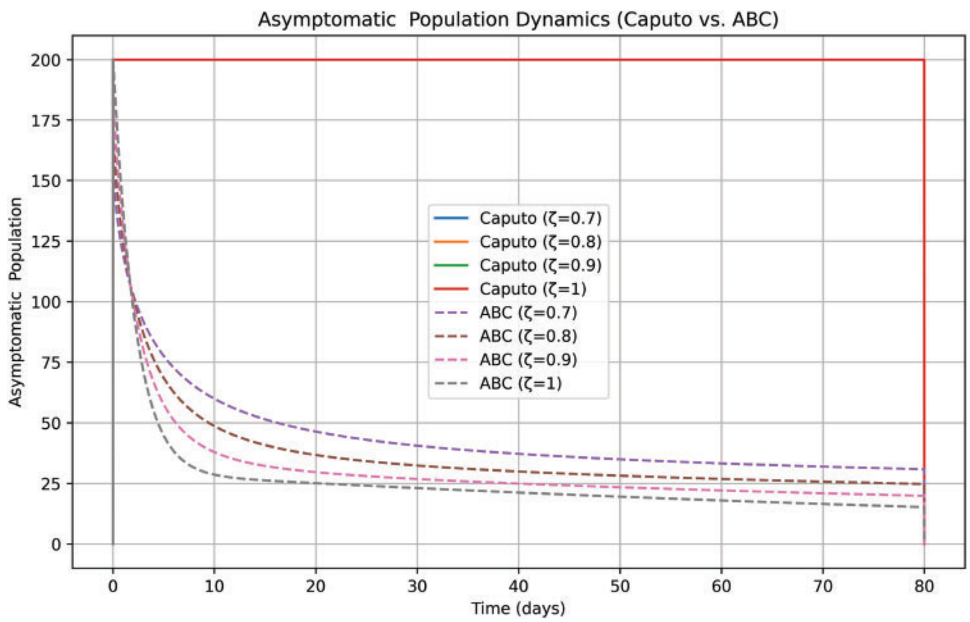

Figure 18: Comparison of ABC and Caputo on asymptomatic

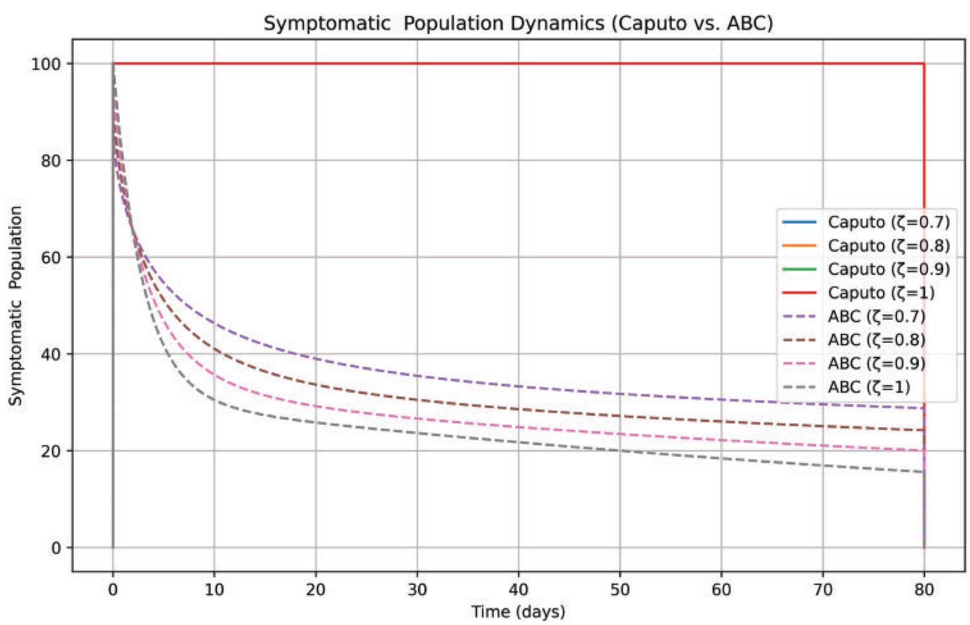

Figure 19: Comparison of ABC and Caputo on symptomatic

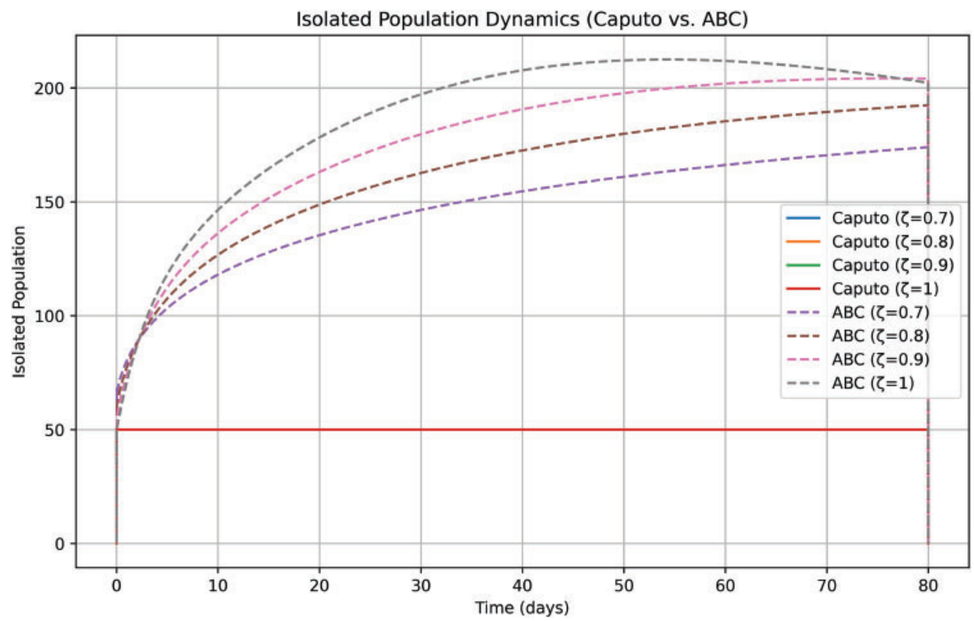

Figure 20: Comparison of ABC and Caputo on isolation

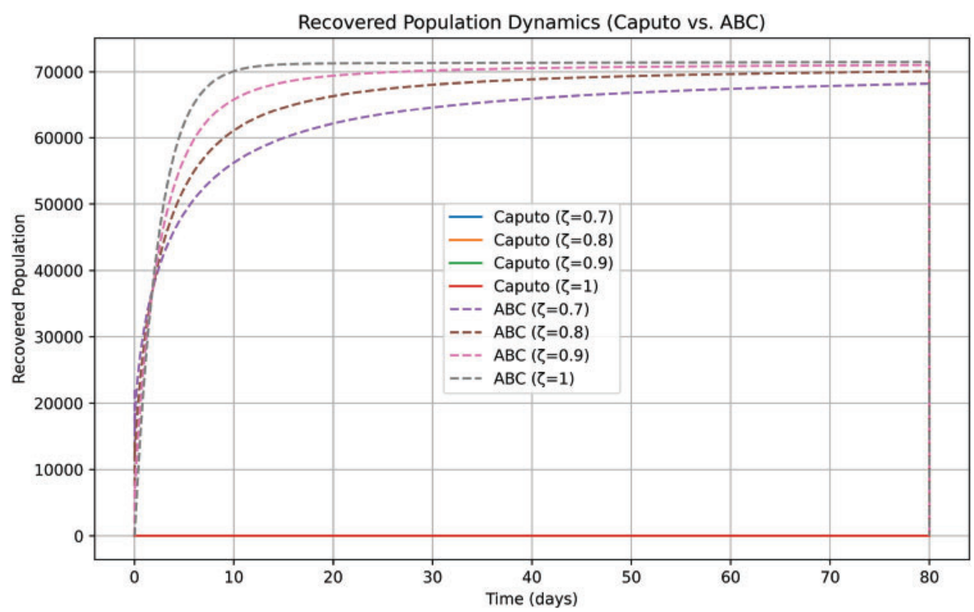

Figure 21: Comparison of ABC and Caputo on recovery

15 Conclusion

This paper presents a model for Monkeypox virus disease within the framework of the $\mathcal{ABC}$ fractional derivative to analyze the Monkeypox disease-free equilibrium, and the equilibrium exists both locally and across all scales. The disease-free equilibrium remains stable when the basic reproduction number is below unity. When $\check{F}_0$ values are below unity, the disease-free equilibrium attains local and global stability. Yet the endemic equilibrium reaches stability at both levels when $\check{F}_0$ exceeds unity and is asymptotically stable when the basic reproduction number is greater than unity. Initially, we enhance the existence theory for the model, establishing both the existence and uniqueness of solutions through a fixed-point approach. Following these results, we investigate the stability of the solutions using Ulam-Hyers and Ulam-Hyers-Rassias stability criteria. Additionally, we conduct numerical experiments to validate the model, developing a numerical scheme that is subsequently utilized to generate graphical results. Our numerical simulations produce realistic graphs, which are thoroughly explained in the numerical section of the paper across various orders of the $\mathcal{ABC}$ fractional derivatives. Thus, the model helps in formulating strategies for the attainment of Goals 3, 15, and 17 of the UN SDG. Two types of fractional derivatives based on Caputo and $\mathcal{ABC}$ were analyzed to explain the novelty of the model. We encourage readers to explore the model further using alternative numerical techniques and different fractional operators to gain deeper insights into the Monkeypox virus disease dynamics.

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Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest to report regarding the present study.

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