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ARTICLE

Computational Modeling of Streptococcus Suis Dynamics via Stochastic Delay Differential Equations

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ABSTRACT: Streptococcus suis (S. suis) is a major disease impacting pig farming globally. It can also be transferred to humans by eating raw pork. A comprehensive study was recently carried out to determine the indices through multiple geographic regions in China. Methods: The well-posed theorems were employed to conduct a thorough analysis of the model's feasible features, including positivity, boundedness equilibria, reproduction number, and parameter sensitivity. Stochastic Euler, Runge Kutta, and Euler Maruyama are some of the numerical techniques used to replicate the behavior of the streptococcus suis infection in the pig population. However, the dynamic qualities of the suggested model cannot be restored using these techniques. Results: For the stochastic delay differential equations of the model, the non-standard finite difference approach in the sense of stochasticity is developed to avoid several problems such as negativity, unboundedness, inconsistency, and instability of the findings. Results from traditional stochastic methods either converge conditionally or diverge over time. The stochastic non-negative step size convergence nonstandard finite difference (NSFD) method unconditionally converges to the model's true states. Conclusions: This study improves our understanding of the dynamics of streptococcus suis infection using versions of stochastic with delay approaches and opens up new avenues for the study of cognitive processes and neuronal analysis. The plotted interaction behaviour and new solution comparison profiles.

KEYWORDS: Streptococcus suis disease model; stochastic delay differential equations (SDDEs); existence and uniqueness; Lyapunov function; stability results; reproduction number; computational methods

1 Introduction

In [1], the authors provide a mathematical model for streptococcus suis infection in the pig population. The technique used to evaluate the model is beneficial for overcoming the disease. In [2], the authors constructed a factual model to regulate model specifications in several antibiotic ramifications various perceptions, and instructions of infection to resist streptococcus suis. In [3], the authors studied separately from recently discovered streptococcus suis species. In [4], the author's presence of streptococcus suis

illness in seedling pigs was linked to pigs that performed averagely and had a sow effect rather than any notable disease traits. In [5], the authors' enhanced infection was only seen in the upper respiratory tract in this investigation. We used two separate models to assess the variations in streptococcus suis disease. In [6], the authors provide more convincing evidence for the beneficial effects of the drug *vs*. streptococcus suis disease by elucidating the underlying molecular process. In [7], the authors evaluate the effect of implementing an autogenous vaccination program on the emergence of illnesses linked to streptococcus suis in natural environments as a challenging undertaking. In [8], the authors illustrate that the survival of other streptococcus suis pathotypes in porcine blood is also restricted by antibody-mediated, as evidenced by the fact that the bacterial surface was usually substantially greater following development in standard piglets' plasma than following incubation in serum obtained before any colostrum adoption. In [9], the authors are shown to be the most effective solvent for a substance called separation by ultrasound-assisted extraction (UAE), and the response surface methodology (RSM) framework accurately represented the anticipated optimization of Emirati. In [10], the authors discovered during the streptococcus suis-2 disease, vimentin increased lung damage, neutrophil counts, and the production of proinflammatory cytokines and chemokines in the lungs of pigs and swine tracheal epithelial cells (STEC). In [11], the authors illustrate how the host-defense peptide cathelicidins are avoided by the Streptococcus suis pepo protease, which affects the pathophysiology of Streptococcus suis. It was discovered that Pepo cleaves the anti-Streptococcus suis cathelicidins, mouse cathelicidin mouse and human cathelicidin. In [12], the authors present the mathematical framework of climate influence on Streptococcus suis infection in pig-human populations generally. In [13], the authors developed a fractional-order mathematical framework relying on fractional derivative concepts. In [14], the authors' investigation is based on the hypotheses of further studies in this domain, especially utilizing both experimental and real-world data. The model suggested that batch-level isolation might cause a likelihood of Streptococcus suis incidence in the facility. In [15], the authors studied that Streptococcus suis is a human pathogen that is frequently responsible for meningitis in Asian nations that consume pork. In [16], the authors provide an exclusive preventative option accessible to pig breeders as a possibility to medicines for controlling the Streptococcus suis infection. In [17], the authors determine that Streptococcus suis strain extracted from an appropriate pig tonsil is aggressive and possesses multiple mechanisms that encourage niche conflict in pig tonsil. In [18], the authors create a computational model of Streptococcus suis infection in a pig community. The approach employed to analyze the model is useful in conquering the illness. In [19], the authors examine blood cortisol levels as a distress readout metric and buprenorphine therapy as a refining measure in a novel pig Streptococcus suis disease model. In [20], the authors created a scientific simulation to control model parameters in several antibiotic implications, different perspectives on infection, and guidelines for resisting Streptococcus suis. In [21], the author explores the use of Stochastic Differential Equations (SDEs) in applications of sciences and many more. In [22], the authors studied the existence and approximate controllability of the Hilfer fractional neutral stochastic hemivariational inequality with the Rosenblatt process. Stochastic or probabilistic components are included in a mathematical model of Streptococcus suis infection dissemination by numerical simulation and analysis, with an emphasis on accounting for uncertainty in disease transmission. Public health efforts for disease control and prevention are informed by this kind of modeling, which provides insights into how the illness could spread under various circumstances.

The main key point to study is the structure-preserving and dynamical analysis of the Streptococcus suis disease model. The fundamental properties of the model like positivity, boundedness, and local and global stabilities are studied rigorously. The authors used well-known methods like Euler Maruyama, stochastic Euler, and stochastic Runge Kutta for the computational analysis and made a comparison analysis with the proposed method like nonstandard finite difference in the sense of stochastic. The Nonstandard Finite

Difference (NSFD) method gives a guarantee of Structure-preserving properties of the model like positivity, boundedness, and dynamical consistency of the solution instead of other standard methods.

The paper is organized as follows: An overview of Streptococcus suis infection-like conditions and a thorough assessment of the literature is provided in Section 1. Building the delayed model and the ensuing mathematical analysis are the focus of Section 2. In Section 3, the local and global levels of the model's stability, reproduction number, and equilibria are examined. The sensitivity analysis of the model's parameters is covered in Section 4. The stochastic conceptualization phase is presented in Section 5. The numerical approach of the NSFD technique and numerical simulations and the presentation of the results are the explicit focus of Section 6. Final opinions provide a conclusive overview of the work under Section 7.

2 Model Formulation

This section presents the delay model formulation of infection spread by pigs and humans. Four classifications were used to categorize the pig population: susceptible class $S_p(t)$, infectious class $I_p(t)$, quarantine class $Q_p(t)$, and recovered class $R_p(t)$. Because Streptococcus Suis may spread from pig to people, the model includes the susceptible human class $S_h(t)$, infectious human class $I_h(t)$, and recovered class $R_h(t)$. System (1)–(7) defines the SIQR-SIR model diagram for people and pigs, as shown in Fig. 1.

$$\frac{dS_p(t)}{dt} = \Lambda_p - M\beta S_p(t-\tau) I_p(t-\tau) e^{-b\tau} - bS_p(t) \qquad t \geq 0, \tau < t \tag{1}$$

$$\frac{dI_p(t)}{dt} = M\beta S_p(t-\tau) I_p(t-\tau) e^{-b\tau} - (\delta + m + b) I_p(t) \qquad t \geq 0, \tau < t \tag{2}$$

$$\frac{dQ_p(t)}{dt} = \delta I_p(t) - (\varepsilon + m + b) Q_p(t) \qquad t \geq 0 \tag{3}$$

$$\frac{dR_p(t)}{dt} = \varepsilon Q_p(t) - bR_p(t) \qquad t \geq 0 \tag{4}$$

$$\frac{dS_h(t)}{dt} = \Lambda_h - \gamma S_h(t-\tau) I_p(t-\tau) e^{-\mu\tau} - \mu S_h(t) \qquad t \geq 0, \tau < t \tag{5}$$

$$\frac{dI_h(t)}{dt} = \gamma S_h(t-\tau) I_p(t-\tau) e^{-\mu\tau} - (\alpha + \mu + \beta_2) I_h(t) \qquad t \geq 0, \tau < t \tag{6}$$

$$\frac{dR_h(t)}{dt} = \beta_2 I_h(t) - \mu R_h(t) \qquad t \geq 0 \tag{7}$$

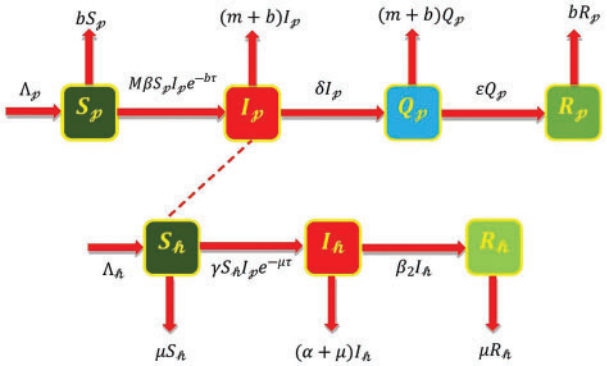

Figure 1: SIQR-SIR model diagram for people and pigs [13]

By $S_p(0) \geq 0, I_p(0) \geq 0, Q_p(0) \geq 0, R_p(0) \geq 0, S_h(0) \geq 0, I_h(0) \geq 0, R_h(0) \geq 0$ initial conditions.

The pig model attribute can be expressed as follows: β is the rate of transmission, M is the relative humidity; m is the disease-induced pig death rate; δ is the rate from infectious class to quarantine class in pigs; ε is the pig recovered rate; μ is the human natural death rate, γ is the transmission rate from infected pig to human, α is the disease death rate, and β_2 is the human recovery rate.

3 Model Analysis

This section examines the delay model feasible region, which carries biological significance as the suggested model takes into account. Consider every parameter and variable in the delay model is non-negative. Next, the model's equilibria and the fundamental reproduction number are determined. Furthermore, we investigate each equilibrium at locally and globally stable.

3.1 Feasible Region

The feasible region of the system (1)–(7) is shown

$$\mathcal{L} = \left\{ (S_p, I_p, Q_p, R_p, S_h, I_h, R_h) \in \mathbb{R}^{+7}; N \leq \frac{\Lambda_p + \Lambda_h}{\mathcal{B}} \right\}$$

Theorem 1: *The solution of the system (1)–(7) is positive in the feasible region.*

Proof: Consider the system (1)–(7), we have

$\left.\frac{dS_p}{dt}\right|_{S_p=0} = \Lambda_p > 0$, $\left.\frac{dI_p}{dt}\right|_{I_p=0} = M\beta S_p(t) I_p(t) e^{-b\tau} > 0$, $\left.\frac{dQ_p}{dt}\right|_{Q_p=0} = \delta I_p(t) > 0$, $\left.\frac{dR_p}{dt}\right|_{R_p=0} = \varepsilon Q_p(t) > 0$, $\left.\frac{dS_h}{dt}\right|_{S_h=0} = \Lambda_h > 0$, $\left.\frac{dI_h}{dt}\right|_{I_h=0} = \gamma S_h(t) I_p(t) e^{-\mu\tau} > 0$, $\left.\frac{dR_h}{dt}\right|_{R_h=0} = \beta_2 I_h(t) > 0$.

Hence, system (1)–(7) has a positive solution with the initial condition in the feasible region. □

Theorem 2: *The solution of the model (1)–(7) is bounded in the feasible region.*

Proof: The total number of people and pigs may be written as

$$N(t) = S_p(t) + I_p(t) + Q_p(t) + R_p(t) + S_h(t) + I_h(t) + R_h(t)$$

$$\frac{dN(t)}{dt} = \frac{dS_p}{dt} + \frac{dI_p}{dt} + \frac{dQ_p}{dt} + \frac{dR_p}{dt} + \frac{dS_h}{dt} + \frac{dI_h}{dt} + \frac{dR_h}{dt}$$

$$\frac{dN(t)}{dt} \leq \Lambda_p + \Lambda_h - \mathcal{B}N(t)$$

$$\frac{dN(t)}{dt} + \mathcal{B}N(t) \leq \Lambda_p + \Lambda_h$$

Which is a linear differential equation

$$N(t) \leq \frac{\Lambda_p + \Lambda_h}{\mathcal{B}} + \left(N(0) - \frac{\Lambda_p + \Lambda_h}{\mathcal{B}} \right) e^{-\mathcal{B}t}$$

Using Grown's inequality

$\lim_{t\to\infty} SupN(t) \leq \frac{\Lambda_p + \Lambda_h}{\mathcal{B}}$ as desired. □

3.2 Model Equilibria and Reproduction Number

This section includes two types of model equilibria for Streptococcus Suis Equilibrium.

Streptococcus Suis Free Equilibrium = $SSFE = \mathfrak{D}^0 = \left(S_{\mathfrak{p}}^0, I_{\mathfrak{p}}^0, Q_{\mathfrak{p}}^0, R_{\mathfrak{p}}^0, S_{\hbar}^0, I_{\hbar}^0, R_{\hbar}^0\right) = \left(\frac{\Lambda_{\mathfrak{p}}}{b}, 0, 0, 0, \frac{\Lambda_{\hbar}}{\mu}, . 0, 0\right)$

Streptococcus Suis Endemic Equilibrium = $SSEE = \mathfrak{D}^* = \left(S_{\mathfrak{p}}^*, I_{\mathfrak{p}}^*, Q_{\mathfrak{p}}^*, R_{\mathfrak{p}}^*, S_{\hbar}^*, I_{\hbar}^*, R_{\hbar}^*\right)$

$S_{\mathfrak{p}}^* = \frac{(\delta+m+b)}{M\beta e^{-b\tau}}$, $I_{\mathfrak{p}}^* = \frac{\Lambda_{\mathfrak{p}} M\beta e^{-b\tau} - b(\delta+m+b)}{(\delta+m+b)M\beta e^{-b\tau}}$, $Q_{\mathfrak{p}}^* = \frac{\delta}{(\varepsilon+m+b)} I_{\mathfrak{p}}^*$, $R_{\mathfrak{p}}^* = \frac{\varepsilon\delta}{b(\varepsilon+m+b)} I_{\mathfrak{p}}^*$, $S_{\hbar}^* = \frac{\Lambda_{\hbar}}{\gamma I_{\mathfrak{p}}^* e^{-\mu\tau} - \mu}$, $I_{\hbar}^* = \frac{\gamma\Lambda_{\hbar} I_{\mathfrak{p}}^* e^{-\mu\tau}}{\left(\gamma I_{\mathfrak{p}}^* e^{-\mu\tau} - \mu\right)(\alpha+\mu+\beta_2)}$, $R_{\hbar}^* = \frac{\beta_2\gamma\Lambda_{\hbar} I_{\mathfrak{p}}^* e^{-\mu\tau}}{\mu\left(\gamma I_{\mathfrak{p}}^* e^{-\mu\tau} - \mu\right)(\alpha+\mu+\beta_2)}$.

The reproduction number is vastly essential in epidemiology. This determines the probability that the illness exists in the community or not. If the reproduction is less than one, disease can be prevented in the community; if the reproduction number is larger than one, disease exists in the community. Use the next-generation approach to calculate the reproduction number. Thus, $\mathcal{F}$ is the transmission matrix, while $\mathcal{V}$ is the transition matrix.

$$\mathcal{F} = \begin{bmatrix} M\beta S_{\mathfrak{p}} e^{-b\tau} & 0 \\ \gamma S_{\hbar} e^{-\mu\tau} & 0 \end{bmatrix}, \mathcal{V} = \begin{bmatrix} (\delta + m + b) & 0 \\ 0 & (\alpha + \mu + \beta_2) \end{bmatrix}$$

$$\mathcal{F}\mathcal{V}^{-1} = \begin{bmatrix} M\beta S_{\mathfrak{p}} e^{-b\tau} & 0 \\ \gamma S_{\hbar} e^{-\mu\tau} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{(\delta + m + b)} & 0 \\ 0 & \frac{1}{(\alpha + \mu + \beta_2)} \end{bmatrix}$$

$$\mathcal{F}\mathcal{V}^{-1} = \begin{bmatrix} \frac{M\beta S_{\mathfrak{p}} e^{-b\tau}}{(\delta + m + b)} & 0 \\ \frac{\gamma S_{\hbar} e^{-\mu\tau}}{(\alpha + \mu + \beta_2)} & 0 \end{bmatrix}$$

The largest eigenvalue of the matrix called the spectral radius or reproduction number at Streptococcus suis free equilibrium, follows as $\mathcal{R}_0 = \frac{M\beta\Lambda_{\mathfrak{p}} e^{-b\tau}}{b(\delta+m+b)}$.

3.3 Stability Analysis

We will demonstrate the following well-known result about local and global stability in both model equilibrium points. Consider the function as follows:

$$A = \Lambda_{\mathfrak{p}} - M\beta S_{\mathfrak{p}}(t) I_{\mathfrak{p}}(t) e^{-b\tau} - bS_{\mathfrak{p}}(t)$$
$$B = M\beta S_{\mathfrak{p}}(t) I_{\mathfrak{p}}(t) e^{-b\tau} - (\delta + m + b) I_{\mathfrak{p}}(t)$$
$$C = \delta I_{\mathfrak{p}}(t) - (\varepsilon + m + b) Q_{\mathfrak{p}}(t)$$
$$D = \varepsilon Q_{\mathfrak{p}}(t) - bR_{\mathfrak{p}}(t)$$
$$E = \Lambda_{\hbar} - \gamma S_{\hbar}(t) I_{\mathfrak{p}}(t) e^{-\mu\tau} - \mu S_{\hbar}(t)$$
$$F = \gamma S_{\hbar}(t) I_{\mathfrak{p}}(t) e^{-\mu\tau} - (\alpha + \mu + \beta_2) I_{\hbar}(t)$$
$$G = \beta_2 I_{\hbar}(t) - \mu R_{\hbar}(t)$$

The Jacobian matrix has the following elements:

$\frac{\partial A}{\partial S_{\mathfrak{p}}} = -M\beta I_{\mathfrak{p}} e^{-b\tau} - b$, $\frac{\partial A}{\partial I_{\mathfrak{p}}} = -M\beta S_{\mathfrak{p}} e^{-b\tau}$, $\frac{\partial A}{\partial Q_{\mathfrak{p}}} = 0$, $\frac{\partial A}{\partial R_{\mathfrak{p}}} = 0$, $\frac{\partial A}{\partial S_{\hbar}} = 0$, $\frac{\partial A}{\partial I_{\hbar}} = 0$, $\frac{\partial A}{\partial R_{\hbar}} = 0$, $\frac{\partial B}{\partial S_{\mathfrak{p}}} = M\beta I_{\mathfrak{p}} e^{-n\tau}$, $\frac{\partial B}{\partial I_{\mathfrak{p}}} = M\beta S_{\mathfrak{p}} e^{-b\tau} - (\delta + m + b)$, $\frac{\partial B}{\partial Q_{\mathfrak{p}}} = 0$, $\frac{\partial B}{\partial R_{\mathfrak{p}}} = 0$, $\frac{\partial B}{\partial S_{\hbar}} = 0$, $\frac{\partial B}{\partial I_{\hbar}} = 0$, $\frac{\partial B}{\partial R_{\hbar}} = 0$, $\frac{\partial C}{\partial S_{\mathfrak{p}}} = 0$, $\frac{\partial C}{\partial I_{\mathfrak{p}}} = \delta$, $\frac{\partial C}{\partial Q_{\mathfrak{p}}} = -(\varepsilon + m + b)$, $\frac{\partial C}{\partial R_{\mathfrak{p}}} = 0$, $\frac{\partial C}{\partial S_{\hbar}} = 0$, $\frac{\partial C}{\partial I_{\hbar}} = 0$, $\frac{\partial C}{\partial R_{\hbar}} = 0$, $\frac{\partial D}{\partial S_{\mathfrak{p}}} = 0$, $\frac{\partial D}{\partial I_{\mathfrak{p}}} = 0$, $\frac{\partial D}{\partial Q_{\mathfrak{p}}} = \varepsilon$, $\frac{\partial D}{\partial R_{\mathfrak{p}}} = -n$, $\frac{\partial D}{\partial S_{\hbar}} = 0$, $\frac{\partial D}{\partial I_{\hbar}} = 0$, $\frac{\partial D}{\partial R_{\hbar}} = 0$, $\frac{\partial E}{\partial S_{\mathfrak{p}}} = 0$, $\frac{\partial E}{\partial I_{\mathfrak{p}}} = -\gamma S_{\hbar} e^{-\mu\tau}$, $\frac{\partial E}{\partial Q_{\mathfrak{p}}} = 0$, $\frac{\partial E}{\partial R_{\mathfrak{p}}} = 0$, $\frac{\partial E}{\partial S_{\hbar}} = -\gamma I_{\mathfrak{p}} e^{-\mu\tau}$, $\frac{\partial E}{\partial I_{\hbar}} = 0$, $\frac{\partial E}{\partial R_{\hbar}} = 0$, $\frac{\partial F}{\partial S_{\mathfrak{p}}} = 0$, $\frac{\partial F}{\partial I_{\mathfrak{p}}} =$

$\gamma S_{\hbar} e^{-\mu\tau}$, $\frac{\partial F}{\partial Q_{\rho}} = 0$, $\frac{\partial F}{\partial R_{\rho}} = 0$, $\frac{\partial F}{\partial S_{\hbar}} = \gamma I_{\rho} e^{-\mu\tau}$, $\frac{\partial F}{\partial I_{\hbar}} = -(\alpha + \mu + \beta_2)$, $\frac{\partial F}{\partial R_{\hbar}} = 0$, $\frac{\partial G}{\partial S_{\rho}} = 0$, $\frac{\partial G}{\partial I_{\rho}} = 0$, $\frac{\partial G}{\partial Q_{\rho}} = 0$, $\frac{\partial G}{\partial R_{\rho}} = 0$, $\frac{\partial G}{\partial S_{\hbar}} = 0$, $\frac{\partial G}{\partial I_{\hbar}} = \beta_2$, $\frac{\partial G}{\partial R_{\hbar}} = -\mu$,

$J =$

$$\begin{bmatrix} -M\beta I_{\rho} e^{-b\tau} - b & -M\beta S_{\rho} e^{-b\tau} & 0 & 0 & 0 & 0 & 0 \\ M\beta I_{\rho} e^{-b\tau} & M\beta S_{\rho} e^{-b\tau} - (\delta + m + b) & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & -(\varepsilon + m + b) & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & -b & 0 & 0 & 0 \\ 0 & -\gamma S_{\hbar} e^{-\mu\tau} & 0 & 0 & -\gamma I_{\rho} e^{-\mu\tau} - \mu & 0 & 0 \\ 0 & \gamma S_{\hbar} e^{-\mu\tau} & 0 & 0 & \gamma I_{\rho} e^{-\mu\tau} & -(\alpha + \mu + \beta_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 & -\mu \end{bmatrix}$$

Theorem 3: *The Streptococcus Suis Free Equilibrium = SSFE =* $\mathfrak{D}^0 = \left(S_{\rho}^0, I_{\rho}^0, Q_{\rho}^0, R_{\rho}^0, S_{\hbar}^0, I_{\hbar}^0, R_{\hbar}^0\right) = \left(\frac{\Lambda_{\rho}}{b}, 0, 0, 0, \frac{\Lambda_{\hbar}}{\mu}, 0, 0\right)$ *is locally asymptotical stable (LAS) if* $R_0 < 1$. *Otherwise, the system is unstable at* $\mathfrak{D}^0$ *if* $R_0 > 1$.

Proof: For stability at $\mathfrak{D}^0 = \left(S_{\rho}^0, I_{\rho}^0, Q_{\rho}^0, R_{\rho}^0, S_{\hbar}^0, I_{\hbar}^0, R_{\hbar}^0\right)$, the Jacobian matrix (8) becomes

$$J\left(\mathfrak{D}^0\right) = \begin{bmatrix} -b & -M\beta S_{\rho}^0 e^{-b\tau} & 0 & 0 & 0 & 0 & 0 \\ 0 & M\beta S_{\rho}^0 e^{-b\tau} - (\delta + m + b) & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & -(\varepsilon + m + b) & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & -b & 0 & 0 & 0 \\ 0 & -\gamma S_{\hbar}^0 e^{-\mu\tau} & 0 & 0 & -\mu & 0 & 0 \\ 0 & \gamma S_{\hbar}^0 e^{-\mu\tau} & 0 & 0 & 0 & -(\alpha + \mu + \beta_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 & -\mu \end{bmatrix}$$

$\left|J\left(\mathfrak{D}^0\right) - \lambda\right| =$

$$\begin{vmatrix} -b - \lambda & -M\beta S_{\rho}^0 e^{-b\tau} & 0 & 0 & 0 & 0 & 0 \\ 0 & M\beta S_{\rho}^0 e^{-b\tau} - (\delta + m + b) - \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & -(\varepsilon + m + b) - \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & -b - \lambda & 0 & 0 & 0 \\ 0 & -\gamma S_{\hbar}^0 e^{-\mu\tau} & 0 & 0 & -\mu - \lambda & 0 & 0 \\ 0 & \gamma S_{\hbar}^0 e^{-\mu\tau} & 0 & 0 & 0 & -(\alpha + \mu + \beta_2) - \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 & -\mu - \lambda \end{vmatrix}$$

$\lambda_1 = \lambda_5 = -b$, $\lambda_2 = \lambda_4 = -\mu$, $\lambda_3 = -(\alpha + \mu + \beta_2)$, $\lambda_6 = -(\varepsilon + m + b)$, $\lambda_7 = M\beta S_{\rho}^0 e^{-b\tau} - (\delta + m + b)$

$\lambda_7 = -(\delta + m + b)(1 - \mathcal{R}_0)$

Hence the streptococcus Suis free equilibrium of the given system (1)–(7) is stable in the sense of local if $\mathcal{R}_0 < 1$. Else, if $\mathcal{R}_0 > 1$, then, $\mathfrak{D}^0$ is unstable in the sense of local. □

Theorem 4: *The Streptococcus Suis Endemic Equilibrium = SSEE =* $\mathfrak{D}^* = \left(S_{\rho}^*, I_{\rho}^*, Q_{\rho}^*, R_{\rho}^*, S_{\hbar}^*, I_{\hbar}^*, R_{\hbar}^*\right)$ *is Locally Asymptotical Stable (LAS) if* $\mathcal{R}_0 > 1$.

Proof: Letting from (8), we get

$J\left(\mathfrak{D}^*\right) =$

$$\begin{bmatrix} -M\beta I_{\mathfrak{p}}^{*}e^{-b\tau}-b & -M\beta S_{\mathfrak{p}}^{*}e^{-b\tau} & 0 & 0 & 0 & 0 & 0 \\ M\beta I_{\mathfrak{p}}^{*}e^{-b\tau} & M\beta S_{\mathfrak{p}}^{*}e^{-b\tau}-(\delta+m+b) & 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & -(\varepsilon+m+b) & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & -b & 0 & 0 & 0 \\ 0 & -\gamma S_{\hbar}^{*}e^{-\mu\tau} & 0 & 0 & -\gamma I_{\mathfrak{p}}^{*}e^{-\mu\tau}-\mu & 0 & 0 \\ 0 & \gamma S_{\hbar}^{*}e^{-\mu\tau} & 0 & 0 & \gamma I_{\mathfrak{p}}^{*}e^{-\mu\tau} & -(\alpha+\mu+\beta_2) & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_2 & -\mu \end{bmatrix}$$

For eigenvalue, consider $|J-\lambda I|=0$

$$\lambda_1=-\mu,\ \lambda_2=-(\alpha+\mu+\beta_2),\ \lambda_3=-\gamma I_{\mathfrak{p}}^{*}e^{-\mu\tau}-\mu,\ \lambda_4=-b,\ \lambda_5=-(\varepsilon+m+b)$$

$$\begin{vmatrix} -M\beta I_{\mathfrak{p}}^{*}e^{-b\tau}-b-\lambda & -M\beta S_{\mathfrak{p}}^{*}e^{-b\tau} \\ M\beta I_{\mathfrak{p}}^{*}e^{-b\tau} & M\beta S_{\mathfrak{p}}^{*}e^{-b\tau}-(\delta+m+b)-\lambda \end{vmatrix}=0$$

$$\left[-M\beta I_{\mathfrak{p}}^{*}e^{-b\tau}-b-\lambda\right]\left[M\beta S_{\mathfrak{p}}^{*}e^{-b\tau}-(\delta+m+b)-\lambda\right]+\left[M\beta S_{\mathfrak{p}}^{*}e^{-b\tau}\right]\left[M\beta I_{\mathfrak{p}}^{*}e^{-b\tau}\right]=0$$

$$\lambda^2+\left[M\beta I_{\mathfrak{p}}^{*}e^{-b\tau}+b-M\beta S_{\mathfrak{p}}^{*}e^{-b\tau}+(\delta+m+b)\right]\lambda+\left[M\beta S_{\mathfrak{p}}^{*}e^{-b\tau}\right]\left[M\beta I_{\mathfrak{p}}^{*}e^{-b\tau}\right]=0$$

$$\lambda^2+a_1\lambda+a_o=0$$

$a_1>0$, If $M\beta I_{\mathfrak{p}}^{*}e^{-b\tau}+b+(\delta+m+b)>M\beta S_{\mathfrak{p}}^{*}e^{-b\tau}$. So, $a_1, a_0>0$

So, by the Routh-Hurwitz Criterion for a 2nd-degree polynomial, the coefficient of the characteristic equation is positive with the constraint $\mathcal{R}_0>1$. Hence the endemic equilibria (EE) of the given system (1)–(7) are stable in the sense of locally. Else, if $\mathcal{R}_0<1$, then Routh Hurwitz's condition for stability is violated. Thus, EE is unstable in the sense of local. □

3.4 Global Stability Analysis

The stability of the Streptococcus Suis infection model is demonstrated by well-known outcomes in following global sense.

Theorem 5: *The Streptococcus Suis Free Equilibrium = SSFE* $=\mathfrak{D}^0=\left(S_{\mathfrak{p}}^0, I_{\mathfrak{p}}^0, Q_{\mathfrak{p}}^0, R_{\mathfrak{p}}^0, S_{\hbar}^0, I_{\hbar}^0, R_{\hbar}^0\right)=\left(\frac{\Lambda_{\mathfrak{p}}}{b},0,0,0,\frac{\Lambda_{\hbar}}{\mu},0,0\right)$ *is globally asymptotical stable (GAS) if* $\mathcal{R}_0<1$. *Otherwise, the system (1)–(7) is unstable at* $\mathfrak{D}^0$ *if* $\mathcal{R}_0>1$.

Proof: Define the Volterra Lyapunov function $A:\mathcal{L}\to\mathbb{R}$ defined as

$$\mathcal{L}=\left[S_{\mathfrak{p}}-S_{\mathfrak{p}}^0-S_{\mathfrak{p}}^0 log\frac{S_{\mathfrak{p}}}{S_{\mathfrak{p}}^0}\right]+I_{\mathfrak{p}}+Q_{\mathfrak{p}}+R_{\mathfrak{p}}+S_{\hbar}+I_{\hbar}+R_{\hbar}$$

$$\frac{d\mathcal{L}}{dt}=\left[1-\frac{S_{\mathfrak{p}}^0}{S_{\mathfrak{p}}}\right]\frac{dS_{\mathfrak{p}}}{dt}+\frac{dI_{\mathfrak{p}}}{dt}+\frac{dQ_{\mathfrak{p}}}{dt}+\frac{dR_{\mathfrak{p}}}{dt}+\left[1-\frac{S_{\hbar}^0}{S_{\hbar}}\right]\frac{dS_{\hbar}}{dt}+\frac{dI_{\hbar}}{dt}+\frac{dR_{\hbar}}{dt}$$

$$\frac{d\mathcal{L}}{dt}=\left[\frac{S_{\mathfrak{p}}-S_{\mathfrak{p}}^0}{S_{\mathfrak{p}}}\right]\left[\Lambda_{\mathfrak{p}}-M\beta S_{\mathfrak{p}}(t)I_{\mathfrak{p}}(t)e^{-b\tau}-bS_{\mathfrak{p}}(t)\right]+\left[M\beta S_{\mathfrak{p}}(t)I_{\mathfrak{p}}(t)e^{-b\tau}-(\delta+m+b)I_{\mathfrak{p}}(t)\right]$$
$$+\left[\delta I_{\mathfrak{p}}(t)-(\varepsilon+m+b)Q_{\mathfrak{p}}(t)\right]+\left[\varepsilon Q_{\mathfrak{p}}(t)-bR_{\mathfrak{p}}(t)\right]$$
$$+\left[1-\frac{S_{\hbar}^0}{S_{\hbar}}\right]\left[\Lambda_{\hbar}-\gamma S_{\hbar}(t)I_{\mathfrak{p}}(t)e^{-\mu\tau}-\mu S_{\hbar}(t)\right]+\left[\gamma S_{\hbar}(t)I_{\mathfrak{p}}(t)e^{-\mu\tau}-(\alpha+\mu+\beta_2)I_{\hbar}(t)\right]$$
$$+\left[\beta_2 I_{\hbar}(t)-\mu R_{\hbar}(t)\right]$$

$$\frac{d\mathcal{L}}{dt} \leq -\Lambda_p \frac{\left(S_p - S_p^0\right)^2}{S_p S_p^0} - (m+b) I_p \left[1 - \frac{M\beta S_p e^{-b\tau}}{(m+b)}\right] - (m+b) Q_p - bR_p - \Lambda_h \frac{\left(S_h - S_h^0\right)^2}{S_h S_h^0}$$
$$- (\alpha+\mu) I_h(t) \left[1 - \frac{\gamma S_h(t) e^{-\mu\tau}}{(\alpha+\mu)}\right] - \mu R_h(t)$$

This implies that $\frac{d\mathcal{L}}{dt} \leq 0$ if $\mathcal{R}_0 < 1$ and $\frac{d\mathcal{L}}{dt} = 0$ if $S_p = S_p^0, S_h = S_h^0, I_p = Q_p = R_p = I_h = R_h = 0$. Therefore, $\mathfrak{D}^0$ is globally asymptotically stable. □

Theorem 6: *The Streptococcus suis Endemic Equilibrium = SSEE* $= \mathfrak{D}^* = \left(S_p^*, I_p^*, Q_p^*, R_p^*, S_h^*, I_h^*, R_h^*\right)$ *is Globally Asymptotical Stable (GAS) if* $\mathcal{R}_0 > 1$.

Proof: Define the Volterra Lyapunov function $Z: \mathcal{L} \to \mathbb{R}$ defined as

$$Z = k_1\left(S_p - S_p^* - S_p^* \ln\left(\frac{S_p}{S_p^*}\right)\right) + k_2\left(I_p - I_p^* - I_p^* \ln\left(\frac{I_p}{I_p^*}\right)\right)$$
$$+ k_3\left(Q_p - Q_p^* - Q_p^* \ln\left(\frac{Q_p}{Q_p^*}\right)\right) + k_4\left(R_p - R_p^* - R_p^* \ln\left(\frac{R_p}{R_p^*}\right)\right)$$
$$+ k_5\left(S_h - S_h^* - S_h^* \ln\left(\frac{S_h}{S_h^*}\right)\right) + k_6\left(I_h - I_h^* - I_h^* \ln\left(\frac{I_h}{I_h^*}\right)\right) + k_5\left(R_h - R_h^* - R_h^* \ln\left(\frac{R_h}{R_h^*}\right)\right)$$

Given positive constants $k_i (i = 1, 2, 3, 4, 5, 6, 7)$, we can express the following equation:

$$\frac{dZ}{dt} = k_1\left[\frac{S_p - S_p^*}{S_p}\right]\frac{dS_p}{dt} + k_2\left[\frac{I_p - I_p^*}{I_p}\right]\frac{dI_p}{dt} + k_3\left[\frac{Q_p - Q_p^*}{Q_p}\right]\frac{dQ_p}{dt} + k_4\left[\frac{R_p - R_p^*}{R_p}\right]\frac{dR_p}{dt}$$
$$+ k_5\left[\frac{S_h - S_h^*}{S_h}\right]\frac{dS_h}{dt} + k_6\left[\frac{I_h - I_h^*}{I_h}\right]\frac{dI_h}{dt} + k_7\left[\frac{R_h - R_h^*}{R_h}\right]\frac{dR_h}{dt}$$
$$\frac{dZ}{dt} = -k_1\Lambda_p \frac{\left(S_p - S_p^*\right)^2}{S_p S_p^*} - k_2 M\beta S_p e^{-n\tau}\left(I_p - I_p^*\right)^2 - k_3\delta I_p \frac{\left(Q_p - Q_p^*\right)^2}{Q_p Q_p^*} - k_4\varepsilon Q_p \frac{\left(R_p - R_p^*\right)^2}{R_p R_p^*}$$
$$- k_5\Lambda_h \frac{\left(S_h - S_h^*\right)^2}{S_h S_h^*} - k_6\gamma S_h I_p e^{-\mu\tau}\frac{\left(I_h - I_h^*\right)^2}{I_h I_h^*} - k_7\beta_2 I_h \frac{\left(R_h - R_h^*\right)^2}{R_h R_h^*}$$

If we choose k_i, where $(i = 1, 2, 3, 4, 5, 6, 7)$

$$\frac{dZ}{dt} = -\Lambda_p \frac{\left(S_p - S_p^*\right)^2}{S_p S_p^*} - M\beta S_p e^{-b\tau}\left(I_p - I_p^*\right)^2 - \delta I_p \frac{\left(Q_p - Q_p^*\right)^2}{Q_p Q_p^*} - \varepsilon Q_p \frac{\left(R_p - R_p^*\right)^2}{R_p R_p^*} - \Lambda_h \frac{\left(S_h - S_h^*\right)^2}{S_h S_h^*}$$
$$- \gamma S_h I_p e^{-\mu\tau}\frac{\left(I_h - I_h^*\right)^2}{I_h I_h^*} - \beta_2 I_h \frac{\left(R_h - R_h^*\right)^2}{R_h R_h^*}$$

$\frac{dZ}{dt} \leq 0$ for $\mathcal{R}_0 > 1$ and $\frac{dZ}{dt} = 0$ if and only if $S_p = S_p^*, I_p = I_p^*, Q_p = Q_p^*, R_p = R_p^*, S_h = S_h^*, I_h = I_h^*, R_h = R_h^*$. Hence by Lasalle's invariance principle $\mathfrak{D}^*$ is globally asymptotical stable. □

Theorem 7. *The Streptococcus suis Free Equilibrium = SSFE* $= \mathfrak{D}^0 = \left(S_p^0, I_p^0, Q_p^0, R_p^0, S_h^0, I_h^0, R_h^0\right) = \left(\frac{\Lambda_p}{b}, 0, 0, 0, \frac{\Lambda_h}{\mu}, 0, 0\right)$ *is globally asymptotical stable (GAS) if* $\mathcal{R}_0 < 1$. *Otherwise, the system (1)–(7) is unstable at* $\mathfrak{D}^0$ if $\mathcal{R}_0 > 1$.

Proof: Define the Volterra Lyapunov function $\Phi: \mathcal{L} \to \mathbb{R}$ defined as

$$\Phi'(I_p) = \frac{1}{I_p}\frac{dI_p}{dt}$$

$$\Phi''(I_p) = \frac{1}{I_p}\frac{d^2I_p}{dt^2} - \frac{1}{I_p^2}\left(\frac{dI_p}{dt}\right)^2$$

$$\Phi''(I_p) = \frac{1}{I_p}\left(M\beta S_p e^{-b\tau} - (\delta + m + b)\right)^2 I_p - \frac{1}{I_p^2}\left(M\beta S_p I_p e^{-b\tau} - (\delta + m + b) I_p\right)^2$$

$$\Phi''(I_p) = \left(M\beta S_p e^{-b\tau} - (\delta + m + b)\right)^2 - \left(M\beta S_p e^{-b\tau} - (\delta + m + b)\right)^2$$

$$\Phi''(I_p) \le 0 \; if \; \mathcal{R}_0 < 1.$$

Thus, the system (1)–(7) is globally asymptotically stable at Streptococcus Suis Free Equilibrium. □

Theorem 8. *The Streptococcus Suis Endemic Equilibrium = SSEE* $= \mathfrak{D}^* = \left(S_p^*, I_p^*, Q_p^*, R_p^*, S_h^*, I_h^*, R_h^*\right)$ *is Globally Asymptotical Stable (GAS) if* $\mathcal{R}_0 > 1$.

Proof: Define the Volterra Lyapunov function $V: \mathcal{L} \to \mathbb{R}$ defined as

$$\frac{dV}{dt} = \left[1 - \frac{S_p^*}{S_p}\right]\frac{dS_p}{dt} + \left[1 - \frac{I_p^*}{I_p}\right]\frac{dI_p}{dt} + \left[1 - \frac{Q_p^*}{Q_p}\right]\frac{dQ_p}{dt} + \left[1 - \frac{R_p^*}{R_p}\right]\frac{dR_p}{dt} + \left[1 - \frac{S_h^*}{S_h}\right]\frac{dS_h}{dt} + \left[1 - \frac{I_h^*}{I_h}\right]\frac{dI_h}{dt} + \left[1 - \frac{R_h^*}{R_h}\right]\frac{dR_h}{dt}$$

$$\frac{d^2V}{dt^2} = \frac{S_p^*}{S_p^2}\left(\frac{dS_p}{dt}\right)^2 + \left(1 - \frac{S_p^*}{S_p}\right)\frac{d^2S_p}{dt^2} + \frac{I_p^*}{I_p^2}\left(\frac{dI_p}{dt}\right)^2 + \left(1 - \frac{I_p^*}{I_p}\right)\frac{d^2I_p}{dt^2} + \frac{Q_p^*}{Q_p^2}\left(\frac{dQ_p}{dt}\right)^2 + \left(1 - \frac{Q_p^*}{Q_p}\right)\frac{d^2Q_p}{dt^2} + \frac{R_p^*}{R_p^2}\left(\frac{dR_p}{dt}\right)^2 + \left(1 - \frac{R_p^*}{R_p}\right)\frac{d^2R_p}{dt^2} + \frac{S_h^*}{S_h^2}\left(\frac{dS_h}{dt}\right)^2 + \left(1 - \frac{S_h^*}{S_h}\right)\frac{d^2S_h}{dt^2} + \frac{I_h^*}{I_h^2}\left(\frac{dI_h}{dt}\right)^2 + \left(1 - \frac{I_h^*}{I_h}\right)\frac{d^2I_h}{dt^2} + \frac{R_h^*}{R_h^2}\left(\frac{dR_h}{dt}\right)^2 + \left(1 - \frac{R_h^*}{R_h}\right)\frac{d^2R_h}{dt^2}$$

$$\frac{d^2V}{dt^2} = \left((\Lambda_p)^2 + \left(M\beta S_p(t) I_p(t) e^{-b\tau} + bS_p(t)\right)^2\right)\frac{S_p^*}{S_p^2} - \left(2(\Lambda_p)\left(M\beta S_p(t) I_p(t) e^{-b\tau} + bS_p(t)\right)\right)\frac{S_p^*}{S_p^2} + \left((\Lambda_p)\left(M\beta I_p(t) e^{-b\tau} + b\right)\right)\frac{S_p^*}{S_p} - \left(\left(M\beta I_p(t) e^{-b\tau} + b\right)^2 S_p\right)\frac{S_p^*}{S_p} + \left(\left(M\beta I_p(t) e^{-b\tau} + b\right)^2 S_p\right) - \left((\Lambda_p)\left(M\beta I_p(t) e^{-b\tau} + b\right)\right) + \left(\left(M\beta S_p(t) I_p(t) e^{-b\tau}\right)^2 + \left((\delta + m + b) I_p(t)\right)^2\right)\frac{I_p^*}{I_p^2} - \left(2\left(M\beta S_p(t) I_p(t) e^{-b\tau}\right)(\delta + m + b) I_p(t)\right)\frac{I_p^*}{I_p^2} + \left(2\left(M\beta S_p(t) I_p(t) e^{-b\tau}\right)(\delta + m + b) I_p(t)\right)\frac{I_p^*}{I_p} - \left(\left(M\beta S_p(t) I_p(t) e^{-b\tau}\right)^2 + (\delta + m + b)^2 I_p(t)\right)\frac{I_p^*}{I_p} + \left(\left(M\beta S_p(t) I_p(t) e^{-b\tau}\right)^2 + (\delta + m + b)^2 I_p(t)\right) - \left(2\left(M\beta S_p(t) I_p(t) e^{-b\tau}\right)(\delta + m + b) I_p(t)\right) + \left((\delta I_p(t))^2 + \left((\varepsilon + m + b) Q_p(t)\right)^2\right)\frac{Q_p^*}{Q_p^2} - \left(2(\delta I_p(t))(\varepsilon + m + b) Q_p(t)\right)\frac{Q_p^*}{Q_p^2} + \left((\delta I_p(t))(\varepsilon + m + b)\right)\frac{Q_p^*}{Q_p}$$

$$
\begin{aligned}
&-\left(\left(\varepsilon+m+b\right)^{2}Q_{\mathfrak{p}}\left(t\right)\right)\frac{Q_{\mathfrak{p}}^{*}}{Q_{\mathfrak{p}}}+\left(\left(\varepsilon+m+b\right)^{2}Q_{\mathfrak{p}}\left(t\right)\right)-\left(\left(\delta I_{\mathfrak{p}}\left(t\right)\right)\left(\varepsilon+m+b\right)\right)\\
&\quad+\left(\left(\varepsilon Q_{\mathfrak{p}}\left(t\right)\right)^{2}+\left(bR_{\mathfrak{p}}\left(t\right)\right)^{2}\right)\frac{R_{\mathfrak{p}}^{*}}{R_{\mathfrak{p}}^{2}}-\left(2\left(\varepsilon Q_{\mathfrak{p}}\left(t\right)\right)\left(bR_{\mathfrak{p}}\left(t\right)\right)\right)\frac{R_{\mathfrak{p}}^{*}}{R_{\mathfrak{p}}^{2}}+\left(\left(b\varepsilon Q_{\mathfrak{p}}\left(t\right)\right)\right)\frac{R_{\mathfrak{p}}^{*}}{R_{\mathfrak{p}}}\\
&\quad-b\left(bR_{\mathfrak{p}}\left(t\right)\right)\frac{R_{\mathfrak{p}}^{*}}{R_{\mathfrak{p}}}+b\left(bR_{\mathfrak{p}}\left(t\right)\right)-\left(\left(b\varepsilon Q_{\mathfrak{p}}\left(t\right)\right)\right)\\
&+\left(\left(\Lambda_{\hbar}\right)^{2}+\left(\gamma S_{\hbar}\left(t\right)I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}+\mu S_{\hbar}\left(t\right)\right)^{2}\right)\frac{S_{\hbar}^{*}}{S_{\hbar}^{2}}-\left(2\left(\Lambda_{\hbar}\right)\left(\gamma S_{\hbar}\left(t\right)I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}+\mu S_{\hbar}\left(t\right)\right)\right)\frac{S_{\hbar}^{*}}{S_{\hbar}^{2}}\\
&\quad+\left(\left(\Lambda_{\hbar}\right)\left(\gamma I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}+\mu\right)\right)\frac{S_{\hbar}^{*}}{S_{\hbar}}-\left(\left(\gamma I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}+\mu\right)^{2}S_{\hbar}\left(t\right)\right)\frac{S_{\hbar}^{*}}{S_{\hbar}}+\left(\left(\gamma I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}+\mu\right)^{2}S_{\hbar}\left(t\right)\right)\\
&\quad-\left(\left(\Lambda_{\hbar}\right)\left(\gamma I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}+\mu\right)\right)+\left(\left(\gamma S_{\hbar}\left(t\right)I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}\right)^{2}+\left(\left(\alpha+\mu+\beta_{2}\right)I_{\hbar}\left(t\right)\right)^{2}\right)\frac{I_{\hbar}^{*}}{I_{\hbar}^{2}}-\\
&\left(2\left(\gamma S_{\hbar}\left(t\right)I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}\right)\left(\left(\alpha+\mu+\beta_{2}\right)I_{\hbar}\left(t\right)\right)\right)\frac{I_{\hbar}^{*}}{I_{\hbar}^{2}}+\left(\left(\gamma S_{\hbar}\left(t\right)I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}\right)\left(\alpha+\mu+\beta_{2}\right)\right)\frac{I_{\hbar}^{*}}{I_{\hbar}}\\
&\quad-\left(\left(\alpha+\mu+\beta_{2}\right)^{2}I_{\hbar}\left(t\right)\right)\frac{I_{\hbar}^{*}}{I_{\hbar}}+\left(\left(\alpha+\mu+\beta_{2}\right)^{2}I_{\hbar}\left(t\right)\right)-\left(\left(\gamma S_{\hbar}\left(t\right)I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}\right)\left(\alpha+\mu+\beta_{2}\right)\right)\\
&\quad+\left(\left(\beta_{2}I_{\hbar}\left(t\right)\right)^{2}+\left(\left(\mu\right)R_{\hbar}\left(t\right)\right)^{2}\right)\frac{R_{\hbar}^{*}}{R_{\hbar}^{2}}-\left(2\left(\beta_{2}I_{\hbar}\left(t\right)\right)\left(\left(\mu\right)R_{\hbar}\left(t\right)\right)\right)\frac{R_{\hbar}^{*}}{R_{\hbar}^{2}}+\left(\left(\mu\beta_{2}\right)I_{\hbar}\left(t\right)\right)\frac{R_{\hbar}^{*}}{R_{\hbar}}\\
&\quad-\left(\left(\mu\right)^{2}R_{\hbar}\left(t\right)\right)\frac{R_{\hbar}^{*}}{R_{\hbar}}+\left(\left(\mu\right)^{2}R_{\hbar}\left(t\right)\right)-\left(\left(\mu\beta_{2}\right)I_{\hbar}\left(t\right)\right)
\end{aligned}
$$

For simplification, we choose

$$\frac{d^{2}V}{dt^{2}}=\psi_{1}-\psi_{2}$$

$$
\begin{aligned}
\psi_{1}=&\left(\left(\Lambda_{\mathfrak{p}}\right)^{2}+\left(M\beta S_{\mathfrak{p}}\left(t\right)I_{\mathfrak{p}}\left(t\right)e^{-b\tau}+bS_{\mathfrak{p}}\left(t\right)\right)^{2}\right)\frac{S_{\mathfrak{p}}^{*}}{S_{\mathfrak{p}}^{2}}+\left(\left(\Lambda_{\mathfrak{p}}\right)\left(M\beta I_{\mathfrak{p}}\left(t\right)e^{-b\tau}+b\right)\right)\frac{S_{\mathfrak{p}}^{*}}{S_{\mathfrak{p}}}\\
&+\left(\left(M\beta I_{\mathfrak{p}}\left(t\right)e^{-b\tau}+b\right)^{2}S_{\mathfrak{p}}\right)+\left(\left(M\beta S_{\mathfrak{p}}\left(t\right)I_{\mathfrak{p}}\left(t\right)e^{-b\tau}\right)^{2}+\left(\left(\delta\ +m+\ b\right)I_{\mathfrak{p}}\left(t\right)\right)^{2}\right)\frac{I_{\mathfrak{p}}^{*}}{I_{\mathfrak{p}}^{2}}\\
&+\left(2\left(M\beta S_{\mathfrak{p}}\left(t\right)I_{\mathfrak{p}}\left(t\right)e^{-b\tau}\right)\left(\delta+m+b\right)I_{\mathfrak{p}}\left(t\right)\right)\frac{I_{\mathfrak{p}}^{*}}{I_{\mathfrak{p}}}\\
&+\left(\left(M\beta S_{\mathfrak{p}}(t)I_{\mathfrak{p}}(t)e^{-b\tau}\right)^{2}+\left(\delta\ +m+\ b\right)^{2}I_{\mathfrak{p}}(t)\right)+\left(\left(\delta I_{\mathfrak{p}}(t)\right)^{2}\right.\\
&\left.+\left(\left(\varepsilon+m+b\right)Q_{\mathfrak{p}}(t)\right)^{2}\right)\frac{Q_{\mathfrak{p}}^{*}}{Q_{\mathfrak{p}}^{2}}+\left(\left(\delta I_{\mathfrak{p}}(t)\right)\left(\varepsilon+m+b\right)\right)\frac{Q_{\mathfrak{p}}^{*}}{Q_{\mathfrak{p}}}\\
&+\left(\left(\varepsilon+m+b\right)^{2}Q_{\mathfrak{p}}\left(t\right)\right)+\left(\left(\varepsilon Q_{\mathfrak{p}}\left(t\right)\right)^{2}+\left(bR_{\mathfrak{p}}\left(t\right)\right)^{2}\right)\frac{R_{\mathfrak{p}}^{*}}{R_{\mathfrak{p}}^{2}}\\
&+\left(\left(b\varepsilon Q_{\mathfrak{p}}\left(t\right)\right)\right)\frac{R_{\mathfrak{p}}^{*}}{R_{\mathfrak{p}}}+b\left(bR_{\mathfrak{p}}\left(t\right)\right)+\left(\left(\Lambda_{\hbar}\right)^{2}+\left(\gamma S_{\hbar}\left(t\right)I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}+\mu S_{\hbar}\left(t\right)\right)^{2}\right)\frac{S_{\hbar}^{*}}{S_{\hbar}^{2}}\\
&+\left(\left(\Lambda_{\hbar}\right)\left(\gamma I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}+\mu\right)\right)\frac{S_{\hbar}^{*}}{S_{\hbar}}+\left(\left(\gamma I_{\mathfrak{p}}\left(t\right)e^{-\mu\tau}+\mu\right)^{2}S_{\hbar}\left(t\right)\right)
\end{aligned}
$$

$$+\left(\left(\gamma S_{\hbar}\left(t\right)I_{\not p}\left(t\right)e^{-\mu\tau}\right)^{2}+\left(\left(\alpha+\mu+\beta_{2}\right)I_{\hbar}\left(t\right)\right)^{2}\right)\frac{I_{\hbar}^{*}}{I_{\hbar}^{2}}$$
$$+\left(\left(\gamma S_{\hbar}\left(t\right)I_{\not p}\left(t\right)e^{-\mu\tau}\right)\left(\alpha+\mu+\beta_{2}\right)\right)\frac{I_{\hbar}^{*}}{I_{\hbar}}$$
$$+\left(\left(\alpha+\mu+\beta_{2}\right)^{2}I_{\hbar}\left(t\right)\right)+\left(\left(\beta_{2}I_{\hbar}\left(t\right)\right)^{2}+\left(\left(\mu\right)R_{\hbar}\left(t\right)\right)^{2}\right)\frac{R_{\hbar}^{*}}{R_{\hbar}^{2}}$$
$$+\left(\left(\mu\beta_{2}\right)I_{\hbar}\left(t\right)\right)\frac{R_{\hbar}^{*}}{R_{\hbar}}+\left(\left(\mu\right)^{2}R_{\hbar}\left(t\right)\right).$$

$$\psi_{2}=\left(2\left(\Lambda_{\not p}\right)\left(M\beta S_{\not p}\left(t\right)I_{\not p}\left(t\right)e^{-b\tau}+bS_{\not p}\left(t\right)\right)\right)\frac{S_{\not p}^{*}}{S_{\not p}^{2}}$$
$$+\left(\left(M\beta I_{\not p}\left(t\right)e^{-b\tau}+b\right)^{2}S_{\not p}\right)\frac{S_{\not p}^{*}}{S_{\not p}}$$
$$+\left(\left(\Lambda_{\not p}\right)\left(M\beta I_{\not p}\left(t\right)e^{-b\tau}+b\right)\right)+\left(2\left(M\beta S_{\not p}\left(t\right)I_{\not p}\left(t\right)e^{-b\tau}\right)\left(\delta+m+b\right)I_{\not p}\left(t\right)\right)\frac{I_{\not p}^{*}}{I_{\not p}^{2}}$$
$$+\left(\left(M\beta S_{\not p}\left(t\right)I_{\not p}\left(t\right)e^{-b\tau}\right)^{2}+\left(\delta+m+b\right)^{2}I_{\not p}\left(t\right)\right)\frac{I_{\not p}^{*}}{I_{\not p}}$$
$$+\left(2\left(M\beta S_{\not p}\left(t\right)I_{\not p}\left(t\right)e^{-b\tau}\right)\left(\delta+m+b\right)I_{\not p}\left(t\right)\right)+\left(2\left(\delta I_{\not p}\left(t\right)\right)\left(\varepsilon+m+b\right)Q_{\not p}\left(t\right)\right)\frac{Q_{\not p}^{*}}{Q_{\not p}^{2}}$$
$$+\left(\left(\varepsilon+m+b\right)^{2}Q_{\not p}\left(t\right)\right)\frac{Q_{\not p}^{*}}{Q_{\not p}}+\left(\left(\delta I_{\not p}\left(t\right)\right)\left(\varepsilon+m+b\right)\right)$$
$$+\left(2\left(\varepsilon Q_{\not p}\left(t\right)\right)\left(bR_{\not p}\left(t\right)\right)\right)\frac{R_{\not p}^{*}}{R_{\not p}^{2}}+b\left(bR_{\not p}\left(t\right)\right)\frac{R_{\not p}^{*}}{R_{\not p}}$$
$$+\left(\left(b\varepsilon Q_{\not p}\left(t\right)\right)\right)+\left(2\left(\Lambda_{\hbar}\right)\left(\gamma S_{\hbar}\left(t\right)I_{\not p}\left(t\right)e^{-\mu\tau}+\mu S_{\hbar}\left(t\right)\right)\right)\frac{S_{\hbar}^{*}}{S_{\hbar}^{2}}$$
$$+\left(\left(\gamma I_{\not p}\left(t\right)e^{-\mu\tau}+\mu\right)^{2}S_{\hbar}\left(t\right)\right)\frac{S_{\hbar}^{*}}{S_{\hbar}}+\left(\left(\Lambda_{\hbar}\right)\left(\gamma I_{\not p}\left(t\right)e^{-\mu\tau}+\mu\right)\right)$$
$$+\left(2\left(\gamma S_{\hbar}\left(t\right)I_{\not p}\left(t\right)e^{-\mu\tau}\right)\left(\left(\alpha+\mu+\beta_{2}\right)I_{\hbar}\left(t\right)\right)\right)\frac{I_{\hbar}^{*}}{I_{\hbar}^{2}}$$
$$+\left(\left(\alpha+\mu+\beta_{2}\right)^{2}I_{\hbar}\left(t\right)\right)\frac{I_{\hbar}^{*}}{I_{\hbar}}+\left(\left(\gamma S_{\hbar}\left(t\right)I_{\not p}\left(t\right)e^{-\mu\tau}\right)\left(\alpha+\mu+\beta_{2}\right)\right)$$
$$+\left(2\left(\beta_{2}I_{\hbar}\left(t\right)\right)\left(\left(\mu\right)R_{\hbar}\left(t\right)\right)\right)\frac{R_{\hbar}^{*}}{R_{\hbar}^{2}}+\left(\left(\mu\right)^{2}R_{\hbar}\left(t\right)\right)\frac{R_{\hbar}^{*}}{R_{\hbar}}$$
$$+\left(\left(\mu\beta_{2}\right)I_{\hbar}\left(t\right)\right).$$

It can see that

$$\psi_{1}>\psi_{2},\frac{d^{2}V}{dt^{2}}>0$$

$$\psi_{1}<\psi_{2},\frac{d^{2}V}{dt^{2}}<0$$

$\psi_1 = \psi_2, \dfrac{d^2V}{dt^2} = 0.$ □

4 Sensitivity Analysis

This section examined the streptococcus suis model's sensitivity. Sensity analysis is a study of how various factors related to input uncertainty may be attributed to the inconsistency of a mathematical model's output outcomes. We calculate the sensitivity of the reproduction number concerning the model's parameter. This technique provided the most sensitive measure for the reproduction number, which helped the infection spread. The basic format for sensitivity is as follows:

$$\mathfrak{D}_{\mathfrak{p}}^{\mathfrak{R}} = \frac{\mathfrak{p}}{\mathfrak{R}} \times \frac{\partial \mathfrak{R}}{\partial \mathfrak{p}}$$

where $\mathfrak{R}$ depict the reproduction number while the $\mathfrak{p}$ present the parameter of the reproduction.

$\mathcal{U}_{\Lambda_{\mathfrak{p}}} = \frac{\Lambda_{\mathfrak{p}}}{\mathcal{R}_0} \times \frac{\partial \mathcal{R}_0}{\partial \Lambda_{\mathfrak{p}}} = 1 > 0, \mathcal{U}_M = \frac{M}{\mathcal{R}_0} \times \frac{\partial \mathcal{R}_0}{\partial M} = 1 > 0, \mathcal{U}_\beta = \frac{\beta}{\mathcal{R}_0} \times \frac{\partial \mathcal{R}_0}{\partial \beta} = 1 > 0, \mathcal{U}_\delta = \frac{\delta}{\mathcal{R}_0} \times \frac{\partial \mathcal{R}_0}{\partial \delta} = -\frac{1}{(\delta + m + b)} < 0, \mathcal{U}_b = \frac{b}{\mathcal{R}_0} \times \frac{\partial \mathcal{R}_0}{\partial b} = -\frac{(\delta + m + 2b)}{(\delta + m + b)} < 0, \mathcal{U}_m = \frac{m}{\mathcal{R}_0} \times \frac{\partial \mathcal{R}_0}{\partial m} = -\frac{1}{(\delta + m + b)} < 0.$

The values of sensitivity and signs of the model's parameters are presented in Table 1.

Table 1: Sensitivity indices

Parameter	Sensitivity indices	Signs
$\Lambda_{\mathfrak{p}}$	1	Positive
M	1	Positive
β	1	Positive
δ	−0.714285	Negative
b	−0.714285	Negative
m	−1.357	Negative

The most significant contributing aspect to the viral transmission phenomenon is human morality (b), as seen in Fig. 2, which has a negative connection with the fundamental reproduction number ($\mathcal{R}_0$). With an increase in the pig mortality rate, the fundamental reproduction number ratio loses value. It suggests that as the number of afflicted pigs rises, so does the systemic infection level. This means that more research on the pig's natural mortality rate analysis can be done, and it will become clearer why fewer pigs need to be infected. In pigs who have it, the most recent infectious virus is prevalent. "δ" represents the rate from infectious class to quarantine class in pigs, also "m" is the disease-induced pig death rate has negative effects on the reproduction number. On the other hand, there is a positive correlation between reproduction number and recruitment rate "$\Lambda_{\mathfrak{p}}$", the relative humidity rate "M" and transmission rate "β". The positive relationship shows that when the value of the parameter rises, so does the reproduction number. Consequently, it implies that decreasing the value of "$\Lambda_{\mathfrak{p}}$", "M", and "β", can reduce the possibility of losing transmitted yield.

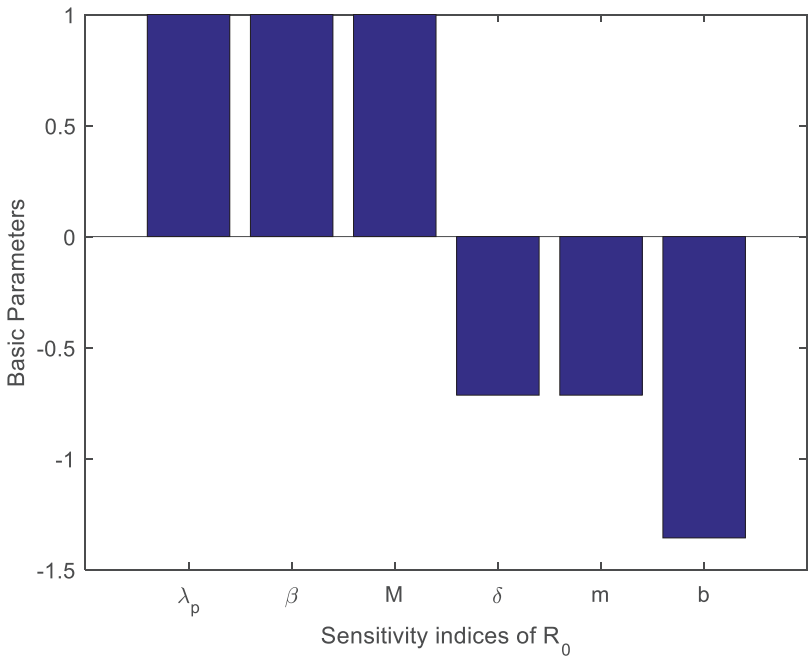

Figure 2: Analysis of the sensitive indices of reproduction number

5 Stochastic Formulation Phase 1

The Stochastic delayed differential equations (SDDEs) of the streptococcus suis model (1)–(7) may be represented by the vector $\mathcal{A} = [S_p(t), I_p(t), Q_p(t), R_p(t), S_h(t), I_h(t), R_h(t)]^T$. We wish to compute the variance $E^*\left[\Delta\mathcal{A}(\Delta\mathcal{A})^{\mathrm{T}}\right]$ and the expectation $E^*[\Delta\mathcal{A}]$. Table 2 lists the probable changes together with the associated transition probability.

$$Expectetions = E^*[\Delta\mathcal{A}] = \sum_{i=1}^{14} P_i(\Delta\mathcal{A})_i = \begin{bmatrix} \Lambda_p - M\beta S_p(t) I_p(t) e^{-b\tau} - bS_p(t) \\ M\beta S_p(t) I_p(t) e^{-b\tau} - (\delta + m + b) I_p(t) \\ \delta I_p(t) - (\varepsilon + m + b) Q_p(t) \\ \varepsilon Q_p(t) - bR_p(t) \\ \Lambda_h - \gamma S_h(t) I_p(t) e^{-\mu\tau} - \mu S_h(t) \\ \gamma S_h(t) I_p(t) e^{-\mu\tau} - (\alpha + \mu + \beta_2) I_h(t) \\ \beta_2 I_h(t) - \mu R_h(t) \end{bmatrix} \Delta t$$

$$\text{Variance} = \sum_{i=1}^{14} P_i(\Delta\mathcal{A})_i \left[(\Delta\mathcal{A})_i\right]^T.$$

$$= \begin{bmatrix} P_1 + P_2 + P_3 & -P_2 & 0 & 0 & 0 & 0 & 0 \\ -P_2 & P_2 + P_4 + P_5 & -P_4 & 0 & 0 & 0 & 0 \\ 0 & -P_4 & P_4 + P_6 + P_7 & -P_6 & 0 & 0 & 0 \\ 0 & 0 & -P_6 & P_6 + P_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_9 + P_{10} + P_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & -P_{10} & P_{10} + P_{12} + P_{13} & -P_{13} \\ 0 & 0 & 0 & 0 & 0 & -P_{13} & P_{13} + P_{14} \end{bmatrix} \Delta t$$

$$\text{Drift} = \mathrm{G}(\mathcal{A}, \mathrm{t}) = \frac{E^*[\Delta\mathcal{A}]}{\Delta t} = \begin{bmatrix} \Lambda_{\mathcal{P}} - M\beta S_{\mathcal{P}}(t) I_{\mathcal{P}}(t) e^{-b\tau} - bS_{\mathcal{P}}(t) \\ M\beta S_{\mathcal{P}}(t) I_{\mathcal{P}}(t) e^{-b\tau} - (\delta + m + b) I_{\mathcal{P}}(t) \\ \delta I_{\mathcal{P}}(t) - (\varepsilon + m + b) Q_{\mathcal{P}}(t) \\ \varepsilon Q_{\mathcal{P}}(t) - bR_{\mathcal{P}}(t) \\ \Lambda_{\mathcal{H}} - \gamma S_{\mathcal{H}}(t) I_{\mathcal{P}}(t) e^{-\mu\tau} - \mu S_{\mathcal{H}}(t) \\ \gamma S_{\mathcal{H}}(t) I_{\mathcal{P}}(t) e^{-\mu\tau} - (\alpha + \mu + \beta_2) I_{\mathcal{H}}(t) \\ \beta_2 I_{\mathcal{H}}(t) - \mu R_{\mathcal{H}}(t) \end{bmatrix} \Delta\mathrm{t} \tag{8}$$

$$\text{Diffusion} = \mathrm{H}(\mathcal{A}, \mathrm{t}) = \sqrt{\frac{E^*\left[\Delta\mathcal{A}(\Delta\mathcal{A})^{\mathrm{T}}\right]}{\Delta\mathrm{t}}} =$$

$$\sqrt{\begin{bmatrix} P_1 + P_2 + P_3 & -P_2 & 0 & 0 & 0 & 0 & 0 \\ -P_2 & P_2 + P_4 + P_5 & -P_4 & 0 & 0 & 0 & 0 \\ 0 & -P_4 & P_4 + P_6 + P_7 & -P_6 & 0 & 0 & 0 \\ 0 & 0 & -P_6 & P_6 + P_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_9 + P_{10} + P_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & -P_{10} & P_{10} + P_{12} + P_{13} & -P_{13} \\ 0 & 0 & 0 & 0 & 0 & -P_{13} & P_{13} + P_{14} \end{bmatrix}} \tag{9}$$

Table 2: Potential modifications to the model's procedure

Transition	Probabilities
$(\Delta\mathcal{A})_{\mathbf{1}} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_1 = (\Lambda_{\mathcal{P}})\,\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{2}} = \begin{bmatrix} \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_2 = \left(M\beta S_{\mathcal{P}}(t) I_{\mathcal{P}}(t) e^{-b\tau}\right)\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{3}} = \begin{bmatrix} \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_3 = (bS_{\mathcal{P}}(t))\,\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{4}} = \begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_4 = (\delta I_{\mathcal{P}}(t))\,\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{5}} = \begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_5 = ((m + b) I_{\mathcal{P}}(t))\,\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{6}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_6 = (\varepsilon Q_{\mathcal{P}}(t))\,\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{7}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_7 = ((m + b) Q_{\mathcal{P}}(t))\,\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{8}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_8 = (bR_{\mathcal{P}}(t))\,\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{9}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_9 = (\Lambda_{\mathcal{H}})\,\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{10}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_{10} = (\gamma S_{\mathcal{H}}(t) I_{\mathcal{P}}(t) e^{-\mu\tau})\,\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{11}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_{11} = (\mu S_{\mathcal{H}}(t))\,\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{12}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_{12} = ((\alpha + \mu) I_{\mathcal{H}}(t))\,\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{13}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_{13} = (\beta_2 I_{\mathcal{H}}(t))\,\Delta\mathrm{t}$
$(\Delta\mathcal{A})_{\mathbf{14}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{bmatrix}^{\mathbf{T}}$	$\mathrm{P}_{14} = (\mu R_{\mathcal{H}}(t))\,\Delta\mathrm{t}$

Therefore, $d\mathcal{A}(t) = G(\mathcal{A}, t) + H(\mathcal{A}, t)dB(t)$.

$$d\begin{bmatrix} S_{\mathcal{P}} \\ I_{\mathcal{P}} \\ Q_{\mathcal{P}} \\ R_{\mathcal{P}} \\ S_{\mathcal{H}} \\ I_{\mathcal{P}} \\ R_{\mathcal{H}} \end{bmatrix} = \begin{bmatrix} \Lambda_{\mathcal{P}} - M\beta S_{\mathcal{P}}(t) I_{\mathcal{P}}(t) e^{-b\tau} - bS_{\mathcal{P}}(t) \\ M\beta S_{\mathcal{P}}(t) I_{\mathcal{P}}(t) e^{-b\tau} - (\delta + m + b) I_{\mathcal{P}}(t) \\ \delta I_{\mathcal{P}}(t) - (\varepsilon + m + b) Q_{\mathcal{P}}(t) \\ \varepsilon Q_{\mathcal{P}}(t) - bR_{\mathcal{P}}(t) \\ \Lambda_{\mathcal{H}} - \gamma S_{\mathcal{H}}(t) I_{\mathcal{P}}(t) e^{-\mu\tau} - \mu S_{\mathcal{H}}(t) \\ \gamma S_{\mathcal{H}}(t) I_{\mathcal{P}}(t) e^{-\mu\tau} - (\alpha + \mu + \beta_2) I_{\mathcal{H}}(t) \\ \beta_2 I_{\mathcal{H}}(t) - \mu R_{\mathcal{H}}(t) \end{bmatrix} dt +$$

$$\sqrt{\begin{bmatrix} P_1 + P_2 + P_3 & -P_2 & 0 & 0 & 0 & 0 & 0 \\ -P_2 & P_2 + P_4 + P_5 & -P_4 & 0 & 0 & 0 & 0 \\ 0 & -P_4 & P_4 + P_6 + P_7 & -P_6 & 0 & 0 & 0 \\ 0 & 0 & -P_6 & P_6 + P_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_9 + P_{10} + P_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & -P_{10} & P_{10} + P_{12} + P_{13} & -P_{13} \\ 0 & 0 & 0 & 0 & 0 & -P_{13} & P_{13} + P_{14} \end{bmatrix}} dB(t) \tag{10}$$

By studying the relevant academic literature, the Euler Maruyama approach is employed to simulate the results of Eq. (10). Following is an outline of the data that is shown in Table 2.

$$\mathcal{A}_{n+1} = \mathcal{A}_n + G(\mathcal{A}_n, t)\Delta t + H(\mathcal{A}_n, t)dB(t).$$

$$\begin{bmatrix} S_{\mathcal{P}}^{n+1} \\ I_{\mathcal{P}}^{n+1} \\ Q_{\mathcal{P}}^{n+1} \\ R_{\mathcal{P}}^{n+1} \\ S_{\mathcal{H}}^{n+1} \\ I_{\mathcal{H}}^{n+1} \\ R_{\mathcal{H}}^{n+1} \end{bmatrix} = \begin{bmatrix} S_{\mathcal{P}}^{n} \\ I_{\mathcal{P}}^{n} \\ Q_{\mathcal{P}}^{n} \\ R_{\mathcal{P}}^{n} \\ S_{\mathcal{H}}^{n} \\ I_{\mathcal{H}}^{n} \\ R_{\mathcal{H}}^{n} \end{bmatrix} + \begin{bmatrix} \Lambda_{\mathcal{P}} - M\beta S_{\mathcal{P}}(t) I_{\mathcal{P}}(t) e^{-b\tau} - bS_{\mathcal{P}}(t) \\ M\beta S_{\mathcal{P}}(t) I_{\mathcal{P}}(t) e^{-b\tau} - (\delta + m + b) I_{\mathcal{P}}(t) \\ \delta I_{\mathcal{P}}(t) - (\varepsilon + m + b) Q_{\mathcal{P}}(t) \\ \varepsilon Q_{\mathcal{P}}(t) - bR_{\mathcal{P}}(t) \\ \Lambda_{\mathcal{H}} - \gamma S_{\mathcal{H}}(t) I_{\mathcal{P}}(t) e^{-\mu\tau} - \mu S_{\mathcal{H}}(t) \\ \gamma S_{\mathcal{H}}(t) I_{\mathcal{P}}(t) e^{-\mu\tau} - (\alpha + \mu + \beta_2) I_{\mathcal{H}}(t) \\ \beta_2 I_{\mathcal{H}}(t) - \mu R_{\mathcal{H}}(t) \end{bmatrix} \Delta t +$$

$$\sqrt{\begin{bmatrix} P_1 + P_2 + P_3 & -P_2 & 0 & 0 & 0 & 0 & 0 \\ -P_2 & P_2 + P_4 + P_5 & -P_4 & 0 & 0 & 0 & 0 \\ 0 & -P_4 & P_4 + P_6 + P_7 & -P_6 & 0 & 0 & 0 \\ 0 & 0 & -P_6 & P_6 + P_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_9 + P_{10} + P_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & -P_{10} & P_{10} + P_{12} + P_{13} & -P_{13} \\ 0 & 0 & 0 & 0 & 0 & -P_{13} & P_{13} + P_{14} \end{bmatrix}} \Delta t \, \Delta B_n$$

where the discretization parameter is denoted by Δt.

5.1 Stochastic Formulation Phase 2

Create an uncertainty parameter for the dynamical system (1)–(7) by including Brownian motion.

$$\frac{dS_{\mathcal{P}}(t)}{dt} = \Lambda_{\mathcal{P}} - M\beta S_{\mathcal{P}}(t) I_{\mathcal{P}}(t) e^{-b\tau} - bS_{\mathcal{P}}(t) + \sigma_1 S_{\mathcal{P}}(t) \frac{dB(t)}{dt} \qquad t \geq 0, \tau < t \tag{11}$$

$$\frac{dI_{\mathscr{p}}(t)}{dt} = M\beta S_{\mathscr{p}}(t) I_{\mathscr{p}}(t) e^{-b\tau} - (\delta + m + b) I_{\mathscr{p}}(t) + \sigma_2 I_{\mathscr{p}}(t) \frac{dB(t)}{dt} \qquad t \geq 0, \tau < t \tag{12}$$

$$\frac{dQ_{\mathscr{p}}(t)}{dt} = \delta I_{\mathscr{p}}(t) - (\varepsilon + m + b) Q_{\mathscr{p}}(t) + \sigma_3 Q_{\mathscr{p}}(t) \frac{dB(t)}{dt} \qquad t \geq 0 \tag{13}$$

$$\frac{dR_{\mathscr{p}}(t)}{dt} = \varepsilon Q_{\mathscr{p}}(t) - bR_{\mathscr{p}}(t) + \sigma_4 R_{\mathscr{p}}(t) \frac{dB(t)}{dt} \qquad t \geq 0 \tag{14}$$

$$\frac{dS_{\hbar}(t)}{dt} = \Lambda_{\hbar} - \gamma S_{\hbar}(t) I_{\mathscr{p}}(t) e^{-\mu\tau} - \mu S_{\hbar}(t) + \sigma_5 S_{\hbar}(t) \frac{dB(t)}{dt} \qquad t \geq 0, \tau < t \tag{15}$$

$$\frac{dI_{\hbar}(t)}{dt} = \gamma S_{\hbar}(t) I_{\mathscr{p}}(t) e^{-\mu\tau} - (\alpha + \mu + \beta_2) I_{\hbar}(t) + \sigma_6 I_{\hbar}(t) \frac{dB(t)}{dt} \qquad t \geq 0, \tau < t \tag{16}$$

$$\frac{dR_{\hbar}(t)}{dt} = \beta_2 I_{\hbar}(t) - \mu R_{\hbar}(t) + \sigma_7 R_{\hbar}(t) \frac{dB(t)}{dt} \qquad t \geq 0 \tag{17}$$

where $\sigma_i = 1, 2, 3, 4, 5, 6, 7$ denote the randomness of each compartment and $B(t)$ indicates the Brownian motion.

5.2 Fundamental Properties of the Stochastic Model

This part covers the analysis of the positivity and boundedness properties of the system (11)–(17).

Let us consider the vector as follows:

$$U(t) = (S_{\mathscr{p}}(t), I_{\mathscr{p}}(t), Q_{\mathscr{p}}(t), R_{\mathscr{p}}(t)) \text{ and } V(t) = (S_{\hbar}(t), I_{\hbar}(t), R_{\hbar}(t))$$

and norm

$$|U(t)| = \sqrt{S_{\mathscr{p}}^2(t) + I_{\mathscr{p}}^2(t) + Q_{\mathscr{p}}^2(t) + R_{\mathscr{p}}^2(t)} \tag{18}$$

And

$$|V(t)| = \sqrt{S_{\hbar}^2(t) + I_{\hbar}^2(t) + R_{\hbar}^2(t)} \tag{19}$$

Furthermore, let $M_1^{3,1}\left(\mathbb{R}^4 x(0,\infty) : \mathbb{R}_+\right)$ and $M_2^{3,1}\left(\mathbb{R}^3 x(0,\infty) : \mathbb{R}_+\right)$ represent the collection of all non-negative functions $V_1(U,t)$ and $V_2(U,t)$ that are defined on $\mathbb{R}^4 x(0,\infty)$ consequently. Additionally, the function is twice differentiable in U and V and once differentiable in it. We have established the differentiable operator T_1 and T_2 that is linked to seven-dimensional stochastic delay differential equations (SDDEs).

$$dU(t) = H_1(U,t)\, dt + k_1(U,t)\, \mathrm{dB(t)} \tag{20}$$

$$dV(t) = H_2(V,t)\, dt + k_2(V,t)\, \mathrm{dB(t)} \tag{21}$$

As

$$T_1 = \frac{\partial}{\partial t} + \sum_{i=1}^{4} H_{1i}(U,t) \frac{\partial}{\partial U_i} + \frac{1}{2} \sum_{i,j=1}^{4} {k_1}^T(U,t)\, k_1(U,t) \frac{\partial^2}{\partial U_i \partial U_j}$$

And

$$T_2 = \frac{\partial}{\partial t} + \sum_{i=1}^{3} H_{2i}(V,t) \frac{\partial}{\partial V_i} + \frac{1}{2} \sum_{i,j=1}^{3} {k_2}^T(V,t)\, k_2(V,t) \frac{\partial^2}{\partial V_i \partial V_j}$$

If T_1 and T_2 acts on function $U^*, V^* \in M_1^{3,1}\left(\mathbb{R}^4 x\,(0,\infty):\mathbb{R}_+\right)$ then we denote

$$T_1U^*\,(U,t) = U_t^*\,(U,t) + U_U^*\,(U,t)\,M_1\,(U,t) + \frac{1}{2}Trace\left(k_1{}^T\,(U,t)\,U_{UU}^*\,(U,t)\,k_1\,(U,t)\right)$$

$$T_2V^*\,(V,t) = V_t^*\,(V,t) + V_V^*\,(V,t)\,M_2\,(V,t) + \frac{1}{2}Trace\left(k_2{}^T\,(V,t)\,V_{VV}^*\,(V,t)\,k_2\,(V,t)\right)$$

where T is Transportation.

Theorem 9 *Shows that for the system (11)–(17) and any given initial conditions* $(S_{\not p}\,(0), I_{\not p}\,(0), Q_{\not p}\,(0), R_{\not p}\,(0)) \in \mathbb{R}_+^4$, *and* $(S_{\hbar}\,(0), I_{\hbar}\,(0), R_{\hbar}\,(0)) \in \mathbb{R}_+^3$ *there are unique solutions* $(S_{\not p}\,(t), I_{\not p}\,(t), Q_{\not p}\,(t), R_{\not p}\,(t))$ and $(S_{\hbar}\,(t), I_{\hbar}\,(t), R_{\hbar}\,(t))$ $t \geq 0$. *Furthermore, these solutions will always remain in* $\mathbb{R}_+^7$ *with a probability of one.*

Proof. Given that all model parameters satisfy the local Lipschitz limitations. Thus, according to Ito's formula, the provided model has a positive solution locally on the interval $[0, \tau_e]$, and the time of explosion is represented by τ_e. The model can be proven to have a global solution when τ_e is equal to infinity.

Let $n_0 = 0$ be a sufficiently big value such that $(S_{\not p}\,(0), I_{\not p}\,(0), Q_{\not p}\,(0), R_{\not p}\,(0))$ and $(S_{\hbar}\,(0), I_{\hbar}\,(0), R_{\hbar}\,(0))$ are all inside the interval $\left\{\frac{1}{n_0}, n_0\right\}$. Let's define a series for every non-negative integer n as follows:

$$\tau_n = inf\left\{\begin{array}{c} t \in [0,\tau_e]: S_{\not p}\,(t) \in \left(\frac{1}{n}, n\right), or\, I_{\not p}\,(t) \in \left(\frac{1}{n}, n\right), or\, Q_{\not p}\,(t) \in \left(\frac{1}{n}, n\right), or\, R_{\not p}\,(t) \in \left(\frac{1}{n}, n\right), \\ or\, S_{\hbar}\,(t) \in \left(\frac{1}{n}, n\right) or\, I_{\hbar}\,(t) \in \left(\frac{1}{n}, n\right), or\, R_{\hbar}\,(t) \in \left(\frac{1}{n}, n\right) \end{array}\right\} \quad (22)$$

where we set $inf\varphi = \infty\,(\varphi\, is\, empty\, set)$. Since τ_n is non-decreasing as $n \to \infty$,

$$\tau_\infty = \lim_{n\to\infty} \tau_n \quad (23)$$

The inequality states that τ_∞ is less than or equal to τ_e. Now, we aim to demonstrate that τ_∞ is equal to infinity, as intended.

If this condition fails to be satisfied, then there exist values T > 0 and $b_1 \in (0,1)$ that satisfy the statement.

$$U\left\{\tau_n \leq T\right\} \geq b_1 \forall n \geq n_1 \quad (24)$$

Define a C^4–function $f: \mathbb{R}_+^4 \to \mathbb{R}_+$ by

$$f\,(S_{\not p}\,(t), I_{\not p}\,(t), Q_{\not p}\,(t), R_{\not p}\,(t)) = (S_{\not p} - 1 - \ln S_{\not p}) + (I_{\not p} - 1 - \ln I_{\not p}) + (Q_{\not p} - 1 - \ln Q_{\not p}) + (R_{\not p} - 1 - \ln R_{\not p}) \quad (25)$$

Define a C^3–function $g: \mathbb{R}_+^3 \to \mathbb{R}_+$ by

$$g\,(S_{\hbar}\,(t), I_{\hbar}\,(t), R_{\hbar}\,(t)) = (S_{\hbar} - 1 - \ln S_{\hbar}) + (I_{\hbar} - 1 - \ln I_{\hbar}) + (R_{\hbar} - 1 - \ln R_{\hbar}) \quad (26)$$

By using Ito's formula (25), we calculate

$$df\,(S_{\not p}\,(t), I_{\not p}\,(t), Q_{\not p}\,(t), R_{\not p}\,(t)) = \left(1 - \frac{1}{S_{\not p}}\right)dS_{\not p} + \left(1 - \frac{1}{I_{\not p}}\right)dI_{\not p} + \left(1 - \frac{1}{Q_{\not p}}\right)dQ_{\not p} + \left(1 - \frac{1}{R_{\not p}}\right)dR_{\not p} + \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}{2}dt.$$

$$
\begin{aligned}
&df\left(S_{\mathfrak{p}}(t), I_{\mathfrak{p}}(t), Q_{\mathfrak{p}}(t), R_{\mathfrak{p}}(t)\right)\\
&\quad=\left(1-\frac{1}{S_{\mathfrak{p}}}\right)\left(\left(\Lambda_{\mathfrak{p}}-M\beta S_{\mathfrak{p}}(t) I_{\mathfrak{p}}(t) e^{-b\tau}-bS_{\mathfrak{p}}(t)\right)dt+\sigma_1 S_{\mathfrak{p}}(t) dB(t)\right)\\
&\quad+\left(1-\frac{1}{I_{\mathfrak{p}}}\right)\left(\left(M\beta S_{\mathfrak{p}}(t) I_{\mathfrak{p}}(t) e^{-b\tau}-(\delta+m+b) I_{\mathfrak{p}}(t)\right)dt+\sigma_2 I_{\mathfrak{p}}(t) dB(t)\right)\\
&\quad+\left(1-\frac{1}{Q_{\mathfrak{p}}}\right)\left(\left(\delta I_{\mathfrak{p}}(t)-(\varepsilon+m+b) Q_{\mathfrak{p}}(t)\right)dt+\sigma_3 Q_{\mathfrak{p}}(t) dB(t)\right)\\
&\quad+\left(1-\frac{1}{R_{\mathfrak{p}}}\right)\left(\left(\varepsilon Q_{\mathfrak{p}}(t)-bR_{\mathfrak{p}}(t)\right)dt+\sigma_4 R_{\mathfrak{p}}(t) dB(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
df\left(S_{\mathfrak{p}}(t), I_{\mathfrak{p}}(t), Q_{\mathfrak{p}}(t), R_{\mathfrak{p}}(t)\right)&=\left(\Lambda_{\mathfrak{p}}+4b+2m+\delta+\varepsilon+\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2}\right)dt\\
&+\sigma_1 S_{\mathfrak{p}}(t) d(B(t)+\sigma_2 I_{\mathfrak{p}}(t) d(B(t)+\sigma_3 Q_{\mathfrak{p}}(t) d(B(t)+\sigma_4 R_{\mathfrak{p}}(t) d(B(t)
\end{aligned}
\tag{27}
$$

For simplify, we assume $M_1=\left(\Lambda_{\mathfrak{p}}+4b+2m+\delta+\varepsilon+\frac{\sigma_1^2+\sigma_2^2+\sigma_3^2+\sigma_4^2}{2}\right)$. Then Eq. (27) could be written as

$$
df\left(S_{\mathfrak{p}}(t), I_{\mathfrak{p}}(t), Q_{\mathfrak{p}}(t), R_{\mathfrak{p}}(t)\right)\leq M_1 dt+\left(\sigma_1 S_{\mathfrak{p}}(t)+\sigma_2 I_{\mathfrak{p}}(t)+\sigma_3 Q_{\mathfrak{p}}(t)+\sigma_4 R_{\mathfrak{p}}(t)\right) d(B(t)). \tag{28}
$$

where M_1 is a positive constant, after that integrating from 0 to $\tau_n\wedge\tau$, we get

$$
\begin{aligned}
\int_0^{\tau_n\wedge\tau} df\left(S_{\mathfrak{p}}(t), I_{\mathfrak{p}}(t), Q_{\mathfrak{p}}(t), R_{\mathfrak{p}}(t)\right)&\leq\int_0^{\tau_n\wedge\tau} M_1 dt\\
&+\int_0^{\tau_n\wedge\tau}\left(\sigma_1 S_{\mathfrak{p}}(t)+\sigma_2 I_{\mathfrak{p}}(t)+\sigma_3 Q_{\mathfrak{p}}(t)\,\sigma_4 R_{\mathfrak{p}}(t)\right) d(B(t))
\end{aligned}
\tag{29}
$$

where $\tau_n\wedge\tau=\min\left(\tau_n, T\right)$, the taking the expectations lead to

$$
EU^*\left(S_{\mathfrak{p}}(\tau_n\wedge\tau), I_{\mathfrak{p}}(\tau_n\wedge\tau), Q_{\mathfrak{p}}(\tau_n\wedge\tau), R_{\mathfrak{p}}(\tau_n\wedge\tau)\right)\leq U^*\left(S_{\mathfrak{p}}(0), I_{\mathfrak{p}}(0), Q_{\mathfrak{p}}(0), R_{\mathfrak{p}}(0)\right)+M_1 T. \tag{30}
$$

Set $\Omega_n=\{\tau_n\leq T\}$ for $n>n_1$ and from (18), we have $X\left(\Omega_n\geq b\right)$.

For each element a_1 in the set Ω_n, there exist certain indices i such that $U_i(\tau_n, a_1)$ is equal to either n or $\frac{1}{n}$, where i takes on the values 1, 2, 3, 4.

Hence, $U^*\left(\left(S_{\mathfrak{p}}(\tau_n, a_1), I_{\mathfrak{p}}(\tau_n, a_1), Q_{\mathfrak{p}}(\tau_n, a_1), R_{\mathfrak{p}}(\tau_n, a_1)\right)\right)$ is less than $\min\left\{n-1-\ln n, \frac{1}{n}-1-\ln\frac{1}{n}\right\}$.

Next, we obtain

$$
\begin{aligned}
U^*\left(S_{\mathfrak{p}}(0), I_{\mathfrak{p}}(0), Q_{\mathfrak{p}}(0), R_{\mathfrak{p}}(0)\right)+M_1 T&\geq E\left(I_{\Omega_m(a_1)} U^*\left(S_{\mathfrak{p}}(\tau_n), I_{\mathfrak{p}}(\tau_n), Q_{\mathfrak{p}}(\tau_n), R_{\mathfrak{p}}(\tau_n)\right)\right)\\
&\geq min\left\{\mathrm{n}-1-\ln\mathrm{n}, \frac{1}{\mathrm{n}}-1-\ln\frac{1}{\mathrm{n}}\right\}
\end{aligned}
\tag{31}
$$

The indicator function is denoted as $I_{\Omega_n(a_1)}$ within the set Ω_n. As n approaches infinity, we get there to the contradiction that infinity is equal to the value of $U^*\left(S_{\mathfrak{p}}(0), I_{\mathfrak{p}}(0), Q_{\mathfrak{p}}(0), R_{\mathfrak{p}}(0)\right)+M_1 T$, which is finite.

As desired.

Again, by using Ito's formula (26), we calculate

$$dg\left(S_{\hbar}(t), I_{\hbar}(t), R_{\hbar}(t)\right) = \left(1-\frac{1}{S_{\hbar}}\right)dS_{\hbar} + \left(1-\frac{1}{I_{\hbar}}\right)dI_{\hbar} + \left(1-\frac{1}{R_{\hbar}}\right)dR_{\hbar} + \frac{\sigma_5^2+\sigma_6^2+\sigma_7^2}{2}dt$$

$$\begin{aligned} dg\left(S_{\hbar}(t), I_{\hbar}(t), R_{\hbar}(t)\right) &= \left(1-\frac{1}{S_{\hbar}}\right)\left(\left(\Lambda_{\hbar} - \gamma S_{\hbar}(t) I_{\not p}(t) e^{-\mu\tau} - \mu S_{\hbar}(t)\right)dt + \sigma_5 S_{\hbar}(t) dB(t)\right) \\ &+ \left(1-\frac{1}{I_{\hbar}}\right)\left(\left(\gamma S_{\hbar}(t) I_{\not p}(t) e^{-\mu\tau} - (\alpha+\mu+\beta_2) I_{\hbar}(t)\right)dt + \sigma_6 I_{\hbar}(t) dB(t)\right) \\ &+ \left(1-\frac{1}{R_{\hbar}}\right)\left(\left(\beta_2 I_{\hbar}(t) - \mu R_{\hbar}(t)\right)dt + \sigma_7 R_{\hbar}(t) dB(t)\right) \end{aligned}$$

$$\begin{aligned} dg\left(S_{\hbar}(t), I_{\hbar}(t), R_{\hbar}(t)\right) &= \left(\Lambda_{\hbar} + 3\mu + \alpha + \beta_2 + \frac{\sigma_5^2+\sigma_6^2}{2}\right)dt + \sigma_5 S_{\hbar}(t) dB(t) + \sigma_6 I_{\hbar}(t) dB(t) \\ &+ \sigma_7 R_{\hbar}(t) dB(t) \end{aligned} \tag{32}$$

For simplify, we assume $M_2 = \left(\Lambda_{\hbar} + 3\mu + \alpha + \beta_2 + \frac{\sigma_5^2+\sigma_6^2+\sigma_7^2}{2}\right)$, Then Eq. (32) could be written as

$$dg\left(S_{\hbar}(t), I_{\hbar}(t), R_{\hbar}(t)\right) \le M_2 dt + \left(\sigma_5 S_{\hbar}(t) + \sigma_6 I_{\hbar}(t) + \sigma_7 R_{\hbar}(t)\right) d(B(t)) \tag{33}$$

where M_2 is a positive constant, after that integrating from 0 to $\tau_n \wedge \tau$, we get

$$\int_0^{\tau_n\wedge\tau} dg\left(S_{\hbar}(t), I_{\hbar}(t), R_{\hbar}(t)\right) \le \int_0^{\tau_n\wedge\tau} M_2 dt + \int_0^{\tau_n\wedge\tau} \left(\sigma_5 S_{\hbar}(t) + \sigma_6 I_{\hbar}(t) + \sigma_7 R_{\hbar}(t)\right) d(B(t)) \tag{34}$$

where $\tau_n \wedge \tau = \min(\tau_n, T)$, the taking the expectations lead to

$$EV^*\left(S_{\hbar}(\tau_n\wedge\tau), I_{\hbar}(\tau_n\wedge\tau), R_{\hbar}(\tau_n\wedge\tau)\right) \le V^*\left(S_{\hbar}(0), I_{\hbar}(0), R_{\hbar}(0)\right) + M_2 T \tag{35}$$

Hence, $V^*\left(\left(S_{\hbar}(\tau_n, V_1), I_{\hbar}(\tau_n, V_1), R_{\hbar}(\tau_n, V_1)\right)\right)$ is less than $\min\left\{n-1-\ln n, \frac{1}{n}-1-\ln\frac{1}{n}\right\}$.

Next, we obtain

$$\begin{aligned} &V^*\left(S_{\hbar}(0), I_{\hbar}(0), R_{\hbar}(0)\right) + M_2 T \ge E\left(I_{\hbar\,\Omega_m(\nu_1)} V^*\left(S_{\hbar}(\tau_n), I_{\hbar}(\tau_n), R_{\hbar}(\tau_n)\right)\right) \\ &\ge min\left\{\mathrm{n}-1-\ln \mathrm{n}, \frac{1}{\mathrm{n}}-1-\ln\frac{1}{\mathrm{n}}\right\} \end{aligned} \tag{36}$$

The indicator function is denoted as $I_{\hbar\,\Omega_n(\nu_1)}$ within the set Ω_n. As n approaches infinity, we get there to the contradiction that infinity is equal to the value of $V^*\left(S_{\hbar}(0), I_{\hbar}(0), R_{\hbar}(0)\right) + M_2 T$, which is finite.

As desired. □

Theorem 10. *If the spectral radius ν and the variance* $\sigma_2^2 < \frac{M\beta\Lambda_{\not p} e^{-b\tau}}{b(\delta+m+b)}$, *then the number of infected pig population in the system (11)–(17) will exponentially approach zero.*

Proof: Let's examine the initial data $\left(S_{\not p}(0), I_{\not p}(0), Q_{\not p}(0), R_{\not p}(0), S_{\hbar}(0), I_{\hbar}(0), R_{\hbar}(0)\right) \in \mathbb{R}_+^7$ and the system (11)–(17) has a solution $\left(S_{\not p}(t), I_{\not p}(t), Q_{\not p}(t), R_{\not p}(t), S_{\hbar}(t), I_{\hbar}(t), R_{\hbar}(t)\right)$ if it satisfies the stochastic delayed differential equation, where σ represents randomness and c represents drift.

$$dI_{\not p}(t) = \left(M\beta S_{\not p}(t) I_{\not p}(t) e^{-b\tau} - (\delta + m + b) I_{\not p}(t)\right)dt + c\sigma_2 I_{\not p}(t) dB(t)$$

Applying Ito's lemma to the function $f(I_p) = \ln(I_p)$, we obtain

$$d\ln\left(I_p\left(t\right)\right) = g'\left(I_p\left(t\right)\right)dP + \frac{1}{2}g''\left(I_p\right){I_p}^2\sigma_2^2 dt.$$

$$d\ln(I_p\left(t\right)) = \frac{1}{I_p\left(t\right)}dI_p + \frac{1}{2}\left(-\frac{1}{{I_p}^2}\right){I_p}^2\sigma_2^2 dt.$$

$$d\ln\left(I_p\left(t\right)\right) = \frac{1}{I_p\left(t\right)}dI_p - \frac{1}{2}\sigma_2^2 dt.$$

$$d\ln\left(I_p\left(t\right)\right) = \frac{1}{I_p\left(t\right)}\left[\left(M\beta S_p\left(t\right)I_p\left(t\right)e^{-b\tau} - \left(\delta + m + b\right)I_p\left(t\right)\right)dt + c\sigma_2 I_p\left(t\right)dB\left(t\right)\right] - \frac{1}{2}\sigma_2^2 dt.$$

$$d\ln\left(I_p\left(t\right)\right) = \left(M\beta S_p\left(t\right)e^{-b\tau} - \left(\delta + m + b\right)\right)dt + c\sigma_2 dB\left(t\right) - \frac{1}{2}\sigma_2^2 dt.$$

$$\ln(I_p\left(t\right)) = \ln I_p\left(0\right) + \left(M\beta S_p\left(t\right)e^{-b\tau} - \left(\delta + m + b\right) - \frac{1}{2}\sigma_2^2\right)dt + \int_0^t c\sigma_2 dB\left(t\right),$$

Notice that, $N\left(t\right) = \int_0^t c\sigma_2 dB\left(t\right)$ with $N\left(0\right) = 0$.

If $\sigma_2^2 > \frac{M\beta\Lambda_p e^{-b\tau}}{b(\delta+m+b)}$,

$$\ln\left(I_p\left(t\right)\right) > \left(\frac{M\beta\Lambda_p e^{-b\tau}}{b\left(\delta + m + b\right)} - \left(\delta + m + b\right) - \frac{1}{2}\frac{M\beta\Lambda_p e^{-b\tau}}{b\left(\delta + m + b\right)}\right)t + N\left(t\right) + \ln I_p\left(0\right),$$

$$\frac{\ln I_p\left(t\right)}{t} > \left(\frac{M\beta\Lambda_p e^{-b\tau}}{2b\left(\delta + m + b\right)} - \left(\delta + m + b\right)\right) + \frac{N\left(t\right)}{t} + \frac{\ln I_p\left(0\right)}{t},$$

$\lim_{t\to\infty}\frac{\ln I_p(t)}{t} > \left(\frac{M\beta\Lambda_p e^{-b\tau}}{2b(\delta+m+b)} - \left(\delta + m + b\right)\right) > 0$, with $\lim_{t\to\infty}\frac{N(t)}{t} = 0$,

If $\sigma_2^2 < \frac{M\beta\Lambda_p e^{-b\tau}}{b(\delta+m+b)}$, then

$$\ln\left(I_p\left(t\right)\right) < \left(\frac{M\beta\Lambda_p e^{-b\tau}}{b\left(\delta + m + b\right)} - \left(\delta + m + b\right) - \frac{1}{2}\sigma_2^2\right)t + N\left(t\right) + \ln I_p\left(0\right),$$

$$\frac{\ln I_p\left(t\right)}{t} < \left(\delta + m + b\right)\left(\frac{M\beta\Lambda_p e^{-b\tau}}{b\left(\delta + m + b\right)^2} - 1 - \frac{1}{2}\sigma_2^2\right) + \frac{N\left(t\right)}{t} + \frac{\ln I_p\left(0\right)}{t},$$

$\lim_{t\to\infty}\sup\frac{\ln I_p(t)}{t} < \left(\delta + m + b\right)\left(\mathcal{R}_0{}^S - 1\right)$, when $\mathcal{R}_0{}^S < 1$, we get $\lim_{t\to\infty}\sup\frac{\ln I_p(t)}{t} \le 0$,

$\lim_{t\to\infty} I_p\left(t\right) = 0$, as desired.

$$\mathcal{R}_o^S = \mathcal{R}_o^d - \frac{\sigma_2^2}{2\left(\delta + m + b\right)} < 1. \square$$

6 Stochastic Nonstandard Finite Difference Scheme

The NSFD method is used in this analysis to address the stochastic delay differential equations regulating Streptococcus suis dynamics. The discrete approximations are very carefully selected to keep stability and be able to precisely model the turbulent behavior of the system. This ensures that the theoretical analysis is kept consistent and that the numerical solutions stay physically significant. For (11)–(17), the stochastic non-standard finite difference scheme has the following equation:

$$S_{\mathcal{P}}^{n+1} = \frac{S_{\mathcal{P}}^{n} + h\left(\Lambda_{\mathcal{P}} + \sigma_1 S_{\mathcal{P}}^{n}\Delta B_n\right)}{1 + h\left(M\beta I_{\mathcal{P}}^{n} e^{-b\tau} + b\right)} \tag{37}$$

$$I_{\mathcal{P}}^{n+1} = \frac{I_{\mathcal{P}}^{n} + h\left(M\beta S_{\mathcal{P}}^{n} I_{\mathcal{P}}^{n} e^{-b\tau} + \sigma_2 I_{\mathcal{P}}^{n}\Delta B_n\right)}{1 + h\left(\delta + m + b\right)} \tag{38}$$

$$Q_{\mathcal{P}}^{n+1} = \frac{Q_{\mathcal{P}}^{n} + h\left(\delta I_{\mathcal{P}}^{n} + \sigma_3 Q_{\mathcal{P}}^{n}\Delta B_n\right)}{1 + h\left(\varepsilon + m + b\right)} \tag{39}$$

$$R_{\mathcal{P}}^{n+1} = \frac{R_{\mathcal{P}}^{n} + h\left(\varepsilon Q_{\mathcal{P}}^{n} + \sigma_4 R_{\mathcal{P}}^{n}\Delta B_n\right)}{1 + hb} \tag{40}$$

$$S_{\mathcal{H}}^{n+1} = \frac{S_{\mathcal{H}}^{n} + h\left(\Lambda_{\mathcal{H}} + \sigma_5 S_{\mathcal{H}}^{n}\Delta B_n\right)}{1 + h\left(\gamma I_{\mathcal{P}}^{n} e^{-\mu\tau} + \mu\right)} \tag{41}$$

$$I_{\mathcal{H}}^{n+1} = \frac{I_{\mathcal{H}}^{n} + h\left(\gamma S_{\mathcal{H}}^{n} I_{\mathcal{P}}^{n} e^{-\mu\tau} + \sigma_6 I_{\mathcal{H}}^{n}\Delta B_n\right)}{1 + h\left(\alpha + \mu + \beta_2\right)} \tag{42}$$

$$R_{\mathcal{H}}^{n+1} = \frac{R_{\mathcal{H}}^{n} + h\left(\beta_2 I_{\mathcal{H}}^{n} + \sigma_7 R_{\mathcal{H}}^{n}\Delta B_n\right)}{1 + h\mu} \tag{43}$$

where h represents a discretization parameter and n is a non-negative integer.

6.1 Stability Analysis

Assuming $\Delta B_n = 0$, the system (37)–(43) consists of functions $A, B, C, D,$ and E.

$A = \frac{S_{\mathcal{P}}+h(\Lambda_{\mathcal{P}})}{1+h(M\beta I_{\mathcal{P}} e^{-b\tau}+b)}$, $B = \frac{I_{\mathcal{P}}+h\left(M\beta S_{\mathcal{P}} I_{\mathcal{P}} e^{-b\tau}\right)}{1+h(\delta+m+b)}$, $C = \frac{Q_{\mathcal{P}}+h(\delta I_{\mathcal{P}})}{1+h(\varepsilon+m+b)}$, $D = \frac{R_{\mathcal{P}}+h(\varepsilon Q_{\mathcal{P}})}{1+hb}$, $E = \frac{S_{\mathcal{H}}+h(\Lambda_{\mathcal{H}})}{1+h(\gamma I_{\mathcal{P}} e^{-\mu\tau}+\mu)}$,

$F = \frac{I_{\mathcal{H}}+h\left(\gamma S_{\mathcal{H}} I_{\mathcal{P}} e^{-\mu\tau}\right)}{1+h(\alpha+\mu+\beta_2)}$, $G = \frac{R_{\mathcal{H}}+h(\beta_2 I_{\mathcal{H}})}{1+h\mu}$.

The Jacobian matrix consists of the following elements:

$\frac{\partial A}{\partial S_{\mathcal{P}}} = \frac{1}{1+h(M\beta I_{\mathcal{P}} e^{-b\tau}+b)}$, $\frac{\partial A}{\partial I_{\mathcal{P}}} = -\frac{M\beta I_{\mathcal{P}} e^{-b\tau}(h\Lambda_{\mathcal{P}})}{(1+h(M\beta I_{\mathcal{P}} e^{-b\tau}+b))^2}$, $\frac{\partial A}{\partial Q_{\mathcal{P}}} = 0$, $\frac{\partial A}{\partial R_{\mathcal{P}}} = 0$, $\frac{\partial A}{\partial S_{\mathcal{H}}} = 0$, $\frac{\partial A}{\partial I_{\mathcal{H}}} = 0$, $\frac{\partial A}{\partial R_{\mathcal{H}}} = 0$

$\frac{\partial B}{\partial S_{\mathcal{P}}} = \frac{h\left(M\beta I_{\mathcal{P}} e^{-b\tau}\right)}{1+h(\delta+m+b)}$, $\frac{\partial B}{\partial I_{\mathcal{P}}} = \frac{1+h\left(M\beta S_{\mathcal{P}} e^{-b\tau}\right)}{1+h(\delta+m+b)}$, $\frac{\partial B}{\partial Q_{\mathcal{P}}} = 0$, $\frac{\partial B}{\partial R_{\mathcal{P}}} = 0$, $\frac{\partial B}{\partial S_{\mathcal{H}}} = 0$, $\frac{\partial B}{\partial I_{\mathcal{H}}} = 0$, $\frac{\partial B}{\partial R_{\mathcal{H}}} = 0$

$\frac{\partial C}{\partial S_{\mathcal{P}}} = 0$, $\frac{\partial C}{\partial I_{\mathcal{P}}} = \frac{h(\delta)}{1+h(\varepsilon+m+b)}$, $\frac{\partial C}{\partial Q_{\mathcal{P}}} = \frac{1}{1+h(\varepsilon+m+b)}$, $\frac{\partial C}{\partial R_{\mathcal{P}}} = 0$, $\frac{\partial C}{\partial S_{\mathcal{H}}} = 0$, $\frac{\partial C}{\partial I_{\mathcal{H}}} = 0$, $\frac{\partial C}{\partial R_{\mathcal{H}}} = 0$

$\frac{\partial D}{\partial S_{\mathcal{P}}} = 0$, $\frac{\partial D}{\partial I_{\mathcal{P}}} = 0$, $\frac{\partial D}{\partial Q_{\mathcal{P}}} = \frac{h(\varepsilon)}{1+hb}$, $\frac{\partial D}{\partial R_{\mathcal{P}}} = \frac{1}{1+hb}$, $\frac{\partial D}{\partial S_{\mathcal{H}}} = 0$, $\frac{\partial D}{\partial I_{\mathcal{H}}} = 0$, $\frac{\partial D}{\partial R_{\mathcal{H}}} = 0$

$\frac{\partial E}{\partial S_{\mathcal{P}}} = 0$, $\frac{\partial E}{\partial I_{\mathcal{P}}} = -\frac{\gamma h e^{-\mu\tau}(h\Lambda_{\mathcal{H}})}{(1+h(\gamma I_{\mathcal{P}} e^{-\mu\tau}+\mu))^2}$, $\frac{\partial E}{\partial Q_{\mathcal{P}}} = 0$, $\frac{\partial E}{\partial R_{\mathcal{P}}} = 0$, $\frac{\partial E}{\partial S_{\mathcal{H}}} = \frac{1}{1+h(\gamma I_{\mathcal{P}} e^{-\mu\tau}+\mu)}$, $\frac{\partial E}{\partial I_{\mathcal{H}}} = 0$, $\frac{\partial E}{\partial R_{\mathcal{H}}} = 0$

$\frac{\partial F}{\partial S_{\mathcal{P}}} = 0$, $\frac{\partial F}{\partial I_{\mathcal{P}}} = \frac{h\left(\gamma S_{\mathcal{H}} e^{-\mu\tau}\right)}{1+h(\alpha+\mu+\beta_2)}$, $\frac{\partial F}{\partial Q_{\mathcal{P}}} = 0$, $\frac{\partial F}{\partial R_{\mathcal{P}}} = 0$, $\frac{\partial F}{\partial S_{\mathcal{H}}} = \frac{h\left(\gamma I_{\mathcal{P}} e^{-\mu\tau}\right)}{1+h(\alpha+\mu+\beta_2)}$, $\frac{\partial F}{\partial I_{\mathcal{H}}} = \frac{1}{1+h(\alpha+\mu+\beta_2)}$, $\frac{\partial F}{\partial R_{\mathcal{H}}} = 0$

$\frac{\partial G}{\partial S_{\mathcal{P}}} = 0$, $\frac{\partial G}{\partial I_{\mathcal{P}}} = 0$, $\frac{\partial G}{\partial Q_{\mathcal{P}}} = 0$, $\frac{\partial G}{\partial R_{\mathcal{P}}} = 0$, $\frac{\partial G}{\partial S_{\hbar}} = 0$, $\frac{\partial G}{\partial I_{\hbar}} = \frac{h(\beta_2)}{1+h\mu}$, $\frac{\partial G}{\partial R_{\hbar}} = \frac{1}{1+h\mu}$

Theorem 11. *For all values of $n \geq 0$, the eigenvalues of the Jacobian matrix at the streptococcus suis-free equilibrium for the system (37)–(43) are located within the unit circle if the value of $\mathcal{R}_0 < 1$.*

Proof. The Jacobian matrix at the streptococcus suis-free equilibrium, denoted as $\left(S^0_{\mathcal{P}}, I^0_{\mathcal{P}}, Q^0_{\mathcal{P}}, R^0_{\mathcal{P}}, S^0_{\hbar}, I^0_{\hbar}, R^0_{\hbar}\right)$, can be expressed as $\left(\frac{\Lambda_{\mathcal{P}}}{b}, 0, 0, 0, \frac{\Lambda_{\hbar}}{\mu}, 0, 0\right)$.

$$J\left(\mathfrak{D}^0\right) =$$

$$\begin{bmatrix}
\frac{1}{1+hb} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1+h\left(M\beta S^0_{\mathcal{P}} e^{-b\tau}\right)}{1+h\left(\delta+m+b\right)} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{h\left(\delta\right)}{1+h\left(\varepsilon+m+b\right)} & \frac{1}{1+h\left(\varepsilon+m+b\right)} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{h\left(\varepsilon\right)}{1+hb} & \frac{1}{1+hb} & 0 & 0 & 0 \\
0 & -\frac{\gamma h e^{-\mu\tau}\left(h\Lambda_{\hbar}\right)}{\left(1+h\mu\right)^2} & 0 & 0 & \frac{1}{1+h\mu} & 0 & 0 \\
0 & \frac{h\left(\gamma S^0_{\hbar} e^{-\mu\tau}\right)}{1+h\left(\alpha+\mu+\beta_2\right)} & 0 & 0 & 0 & \frac{1}{1+h\left(\alpha+\mu+\beta_2\right)} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{h\left(\beta_2\right)}{1+h\mu} & \frac{1}{1+h\mu}
\end{bmatrix}$$

$$\left|J\left(\mathfrak{D}^0\right) - \lambda\right| =$$

$$\begin{vmatrix}
\frac{1}{1+hb}-\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1+h\left(M\beta S^0_{\mathcal{P}} e^{-b\tau}\right)}{1+h\left(\delta+m+b\right)}-\lambda & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{h\left(\delta\right)}{1+h\left(\varepsilon+m+b\right)} & \frac{1}{1+h\left(\varepsilon+m+b\right)}-\lambda & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{h\left(\varepsilon\right)}{1+hb} & \frac{1}{1+hb}-\lambda & 0 & 0 & 0 \\
0 & -\frac{\gamma h e^{-\mu\tau}\left(h\Lambda_{\hbar}\right)}{\left(1+h\mu\right)^2} & 0 & 0 & \frac{1}{1+h\mu}-\lambda & 0 & 0 \\
0 & \frac{h\left(\gamma S^0_{\hbar} e^{-\mu\tau}\right)}{1+h\left(\alpha+\mu+\beta_2\right)} & 0 & 0 & 0 & \frac{1}{1+h\left(\alpha+\mu+\beta_2\right)}-\lambda & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{h\left(\beta_2\right)}{1+h\mu} & \frac{1}{1+h\mu}-\lambda
\end{vmatrix}$$

Therefore,

$\lambda_1 = \lambda_4 = \frac{1}{1+hb} < 1$, $\lambda_2 = \frac{1+h\left(M\beta S^0_{\mathcal{P}} e^{-b\tau}\right)}{1+h\left(\delta+m+b\right)} < 1$, $\lambda_3 = \frac{1}{1+h\left(\varepsilon+m+b\right)} < 1$, $\lambda_5 = \lambda_7 = \frac{1}{1+h\mu} < 1$, $\lambda_6 = \frac{1}{1+h\left(\alpha+\mu+\beta_2\right)}$.

Using the definition of $\mathcal{R}_0$, we can show that if $\mathcal{R}_0 < 1$, then $\lambda_2 < 1$, and $\mathfrak{D}^0$ is L.A.S. on the contrary, it is obviously to verify that $\lambda_2 > 1$, if $\mathcal{R}_0 > 1$, which shows that $\mathfrak{D}^0$ is unstable. □

Theorem 12. *For all values of $n \geq 0$, the eigenvalues of the Jacobian matrix at the streptococcus suis-endemic equilibrium for the system (37)–(43) are located within the unit circle if the value of $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix at the streptococcus suis- endemic equilibrium, denoted as $\left(S_{\mathfrak{p}}^*, I_{\mathfrak{p}}^*, Q_{\mathfrak{p}}^*, R_{\mathfrak{p}}^*, S_{\mathfrak{h}}^*, I_{\mathfrak{h}}^*, R_{\mathfrak{h}}^*\right)$.

$J\left(\mathfrak{D}^*\right) =$

$$\begin{bmatrix}
\frac{1}{1+h\left(M\beta I_{\mathfrak{p}}^* e^{-b\tau}+b\right)} & -\frac{M\beta I_{\mathfrak{p}}^* e^{-b\tau}\left(h\Lambda_{\mathfrak{p}}\right)}{\left(1+h\left(M\beta I_{\mathfrak{p}}^* e^{-b\tau}+b\right)\right)^2} & 0 & 0 & 0 & 0 & 0 \\
\frac{h\left(M\beta I_{\mathfrak{p}}^* e^{-b\tau}\right)}{1+h\left(\delta+m+b\right)} & \frac{1+h\left(M\beta S_{\mathfrak{p}}^* e^{-b\tau}\right)}{1+h\left(\delta+m+b\right)} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{h\left(\delta\right)}{1+h\left(\varepsilon+m+b\right)} & \frac{1}{1+h\left(\varepsilon+m+b\right)} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{h\left(\varepsilon\right)}{1+hb} & \frac{1}{1+hb} & 0 & 0 & 0 \\
0 & -\frac{\gamma h e^{-\mu\tau}\left(h\Lambda_{\mathfrak{h}}\right)}{\left(1+h\left(\gamma I_{\mathfrak{p}}^* e^{-\mu\tau}+\mu\right)\right)^2} & 0 & 0 & \frac{1}{1+h\left(\gamma I_{\mathfrak{p}}^* e^{-\mu\tau}+\mu\right)} & 0 & 0 \\
0 & \frac{h\left(\gamma S_{\mathfrak{h}}^* e^{-\mu\tau}\right)}{1+h\left(\alpha+\mu+\beta_2\right)} & 0 & 0 & \frac{h\left(\gamma I_{\mathfrak{p}}^* e^{-\mu\tau}\right)}{1+h\left(\alpha+\mu+\beta_2\right)} & \frac{1}{1+h\left(\alpha+\mu+\beta_2\right)} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{h\left(\beta_2\right)}{1+h\mu} & \frac{1}{1+h\mu}
\end{bmatrix}$$

So, the eigenvalues of the Jacobian at $\mathfrak{D}^*$ as follows:

$\lambda_1 = \frac{1}{1+h\mu} < 1$, $\lambda_2 = \frac{1}{1+h(\alpha+\mu+\beta_2)}$, $\lambda_3 = \frac{1}{1+h\left(\gamma I_{\mathfrak{p}}^* e^{-\mu\tau}+\mu\right)}$, $\lambda_4 = \frac{1}{1+hb}$, $\lambda_5 = \frac{1}{1+h(\varepsilon+m+b)}$ provided that $R_0 > 1$.

$$\begin{vmatrix}
\frac{1}{1+h\left(M\beta I_{\mathfrak{p}}^* e^{-b\tau}+b\right)}-\lambda & -\frac{M\beta I_{\mathfrak{p}}^* e^{-b\tau}\left(h\Lambda_{\mathfrak{p}}\right)}{\left(1+h\left(M\beta I_{\mathfrak{p}}^* e^{-b\tau}+b\right)\right)^2} \\
\frac{h\left(M\beta I_{\mathfrak{p}}^* e^{-b\tau}\right)}{1+h\left(\delta+m+b\right)} & \frac{1+h\left(M\beta S_{\mathfrak{p}}^* e^{-b\tau}\right)}{1+h\left(\delta+m+b\right)}-\lambda
\end{vmatrix} = 0$$

$$A_1 = Trce\ of\ J_{\mathfrak{D}^*} = \frac{1}{1+h\left(M\beta I_{\mathfrak{p}}^* e^{-b\tau}+b\right)} + \frac{1+h\left(M\beta S_{\mathfrak{p}}^* e^{-b\tau}\right)}{1+h\left(\delta+m+b\right)}$$

$$A_2 = Determinent\ of\ J_{\mathfrak{D}^*} = \left(\left(\frac{1}{1+h\left(M\beta I_{\mathfrak{p}}^* e^{-b\tau}+b\right)}\right)\left(\frac{1+h\left(M\beta S_{\mathfrak{p}}^* e^{-b\tau}\right)}{1+h\left(\delta+m+b\right)}\right)\right)$$
$$+\left(\frac{M\beta I_{\mathfrak{p}}^* e^{-b\tau}\left(h\Lambda_{\mathfrak{p}}\right)}{\left(1+h\left(M\beta I_{\mathfrak{p}}^* e^{-b\tau}+b\right)\right)^2}\right)\left(\frac{h\left(M\beta I_{\mathfrak{p}}^* e^{-b\tau}\right)}{1+h\left(\delta+m+b\right)}\right)$$

Lemma. For the quadratic equation $\lambda^2 - A_1\lambda + A_2 = 0$, $|\lambda_i| < 1$, $i = 1, 2$. if and only if the following conditions are satisfied:

(i) $1 + A_1 + A_2 > 0$.

(ii) $1 - A_1 + A_2 > 0$.

(iii) $A_2 < 1$. □

6.2 Comparison Section

This section examines the characteristics of the graphs representing the number of infected pig population using the Euler Maruyama, stochastic Euler, and stochastic Runge Kutta schemes, in comparison to the NSFD scheme, across various step sizes and parameters values (See Table 3).

Table 3: Values of parameter

Parameters	Values	Source [13]
$\Lambda_{\mathscr{P}}$	0.5	Fitted
$\Lambda_{\mathscr{h}}$	5	Fitted
β	2.3 (SSEE) 1.3 (SSFE)	Estimated
β_2	0.1	Estimated
μ	0.01	Estimated
M	1	Fitted
γ	0.1	Fitted
ε	0.3	Fitted
b	0.5	Estimated
m	0.3	Fitted
δ	0.6	Estimated
α	0.1	Estimated

6.3 Discussion

Fig. 3a,b provides a comparison between the infected class of the Stochastic NSFD and the Euler Maryama Method. Fig. 3a shows convergence for both approaches at $h = 0.01$. When the step size was raised to $h = 1.0$, the Euler Maryama Method diverged whereas the Stochastic NSFD Method remained convergent, as shown in Fig. 3b. Similarly, Fig. 3c,d compares the infected class of the Stochastic NSFD and the Stochastic Euler Method. At $h = 0.01$, both techniques converged in Fig. 3c. However, when the step size was increased to $h = 1.0$, the Stochastic Euler Method diverged, while the Stochastic NSFD method-maintained convergence, as shown in Fig. 3d. Similarly, Fig. 3e,f compares the infected class of the Stochastic NSFD and Stochastic RK Method. At $h = 0.01$, both methods converged, as shown in Fig. 3e. However, when the step size increased to $h = 2.0$, the Stochastic RK Method diverged, while the Stochastic NSFD method continued to converge, as shown in Fig. 3f. Fig. 4a shows how delay affects the model's susceptible class at different τ values (0.1, 0.2, 0.3, 0.4, 0.5). Fig. 4b shows the effect of delay on the infected class of the model at various values $\tau = 0.1$, 0.2, 0.3, 0.4, and 0.5, indicating a gradual decline in disease from the infected class over time. Finally, Fig. 5 shows the behavior of delay on the reproduction number of the model.

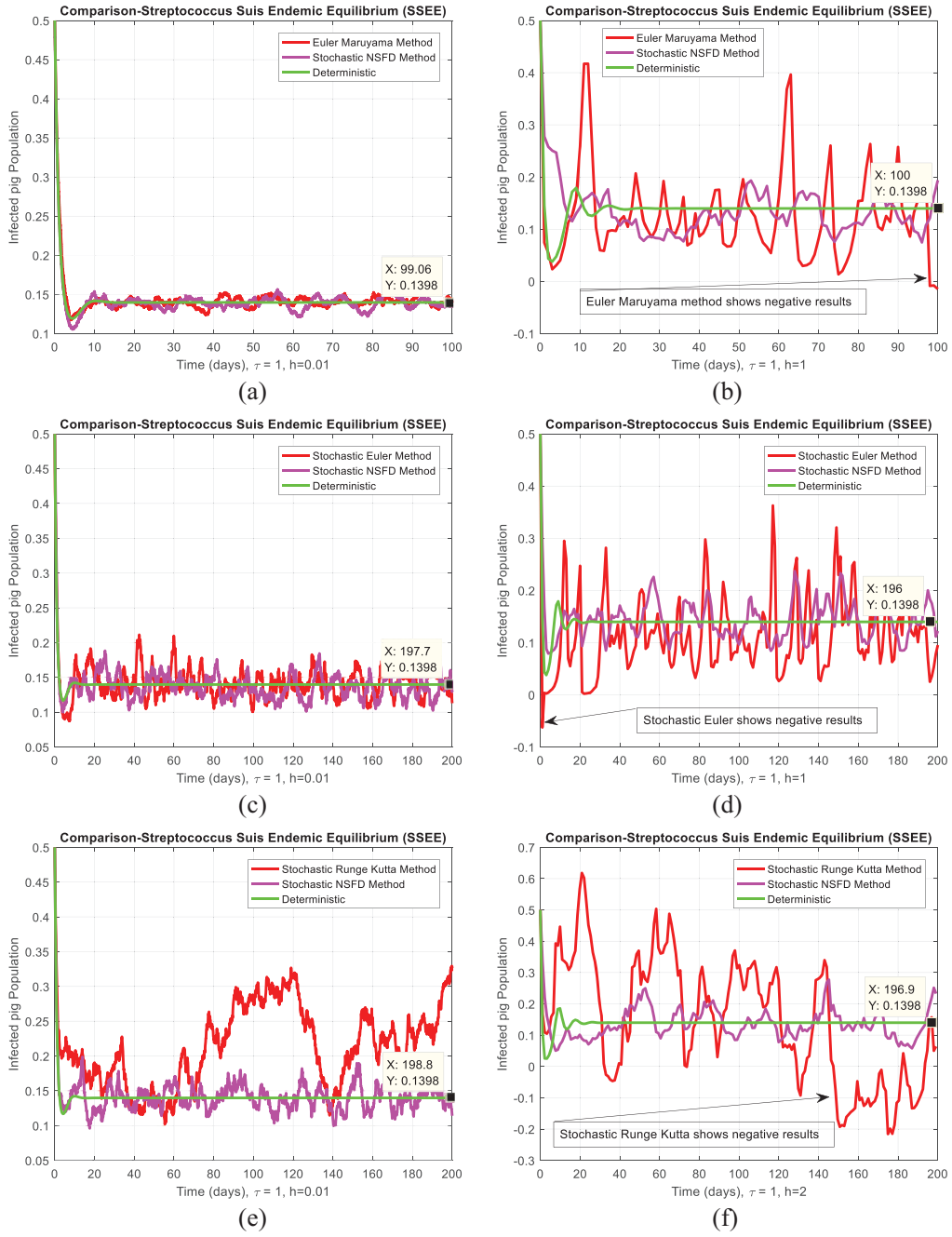

Figure 3: Comparison graph of computational methods at the Streptococcus Suis endemic equilibrium of the model (a) The comparison behavior of the infected pig population through Euler Maruyama and stochastic NSFD methods at $h = 0.01$ (convergent) (b) The comparison behavior of the infected pig population through Euler Maruyama and stochastic NSFD methods at $h = 1$ (divergent) (c) The comparison behavior of the infected pig population through stochastic Euler and stochastic NSFD methods at $h = 0.01$ (convergent) (d) The comparison behavior of the infected pig population through stochastic Euler and stochastic NSFD methods at $h = 1$ (divergent) (e) The comparison behavior of the infected pig population through stochastic Runge Kutta and stochastic NSFD methods at $h = 0.01$ (convergent) (f) The comparison behavior of the infected pig population through stochastic Runge Kutta and stochastic NSFD methods at $h = 2$ (divergent)

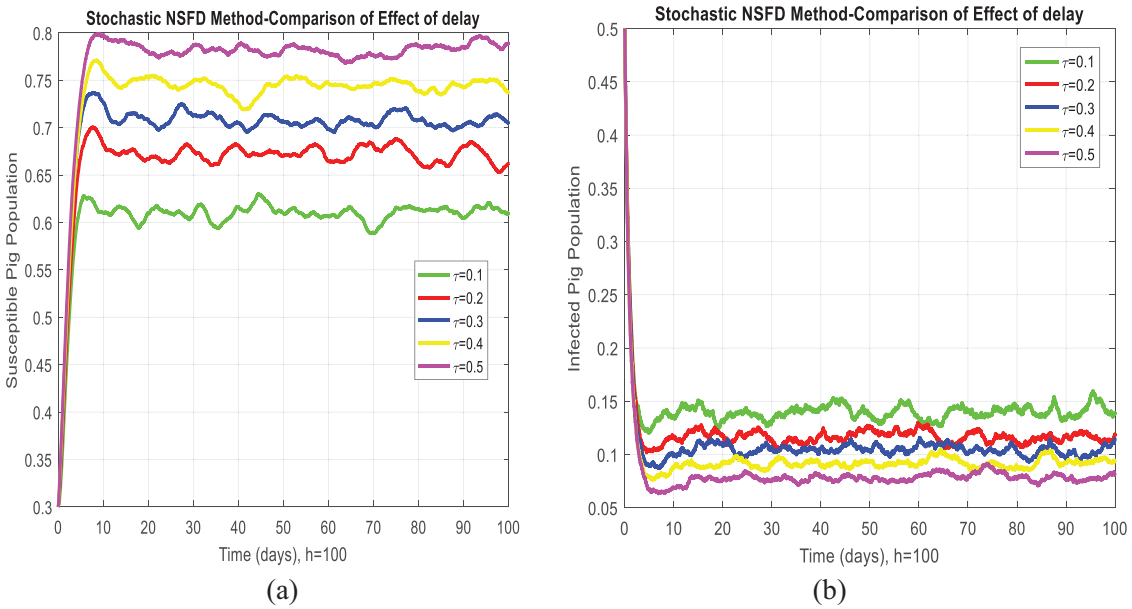

Figure 4: Time-Plot with the time delay on susceptible and infected population. (a) The effect of different values of delay on susceptible pig population (b) The effect of different values of delay on infected pig population

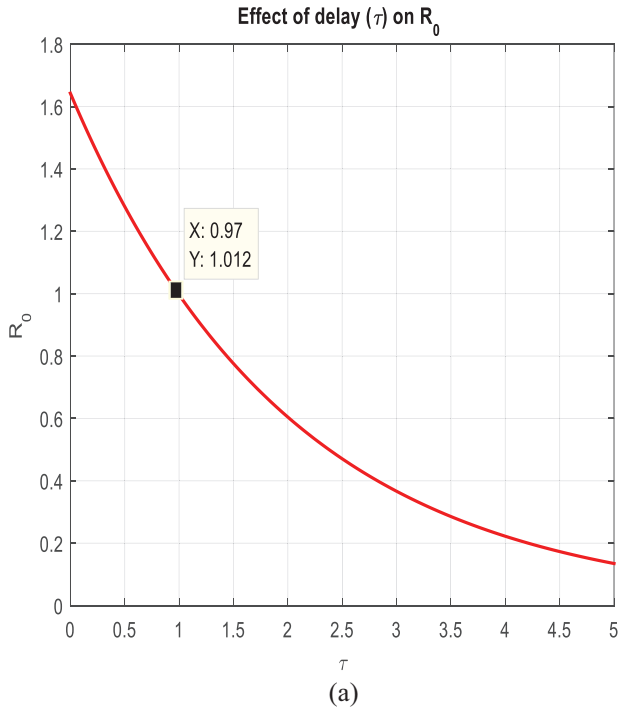

Figure 5: Time plot of the effect of time delay (τ) with reproduction number ($\mathcal{R}_0$)

7 Conclusion

This paper provides a comprehensive assessment of the mathematical analysis, including trustworthy delay techniques, of the delayed model for streptococcus suis infection. Subpopulations are classified by the model into four categories: susceptible class $S_p(t)$, infectious class $I_p(t)$, quarantine class $Q_p(t)$, and recovered class $R_p(t)$. Because Streptococcus Suis may spread from pig to people, the model includes the susceptible human class $S_h(t)$, infectious human class $I_h(t)$, and recovered class $R_h(t)$. The model's dynamic analysis examines positivity, boundedness, equilibria, and the threshold parameter. The sensitivity of the parameters is revealed by the outcomes of the model. The linearization of the model is based on existing concepts such as the Routh-Hurwitz criteria and the Jacobian. The focus of the research is on the use of Lassalle's invariance principle and Lyapunov's theory to ensure the global stability of the model. It is discovered that the Stochastic delayed NSFD method is the most accurate, successful, and efficient method. In these models, stability is necessary to avoid unpredictable behavior and incorrect results in terms of stability, optimism, and staying within normal bounds even with enormous time increments. Stochastic delayed NFSD performs exceptionally well. Other methods such as Stochastic delayed NFSD Euler Maryama, Stochastic Euler, and Stochastic RK-4 are considered valuable tools in our toolbox, however, at high time scales, they break down, leading to a loss of stability and consistency. After a great deal of testing and comparison, Stochastic delayed NFSD has emerged as the champion in stability and reliability, passing important tests like "local stability" and the Routh-Hurwitz criterion in the study of accurate predictions. This model can be taken forward by adding some dynamism in terms of space to real-world data for parameter estimation, thereby making it more applicable. It could also be taken up in terms of other diseases so that the applicability of the model further validates its relevance.

To fully capture the real-world complexities, there is a need to develop a model without the assumption of disease transmissibility and parameter estimation. Moreover, validation based on data is needed to enhance the reliability of the model.

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References

1. Alqahtani AM, Mishra MN. Mathematical analysis of Streptococcus suis infection in pig-human population by Riemann-Liouville fractional operator. Progr Fract Diff Appl. 2024;10(1):119–35. doi:10.18576/pfda/100112.
2. Mi K, Sun L, Zhang L, Tang A, Tian X, Hou Y, et al. A physiologically based pharmacokinetic/pharmacodynamic model to determine dosage regimens and withdrawal intervals of aditoprim against Streptococcus suis. Front Pharmacol. 2024;15:1378034. doi:10.3389/fphar.2024.1378034.
3. Wu CF, Chen SH, Chou CC, Wang CM, Huang SW, Kuo HC. Serotype and multilocus sequence typing of Streptococcus suis from diseased pigs in Taiwan. Sci Rep. 2023;13(1):8263. doi:10.1038/s41598-023-33778-9.

4. Fabà L, Aragon V, Litjens R, Galofré-Milà N, Segura M, Gottschalk M, et al. Metabolic insights and background from naturally affected pigs during Streptococcus suis outbreaks. Transl Anim Sci. 2023;7(1):txad126. doi:10.1093/tas/txad126.
5. Hau SJ, Nielsen DW, Brockmeier SL. Prior infection with Bordetella bronchiseptica enhanced colonization but not disease with Streptococcus suis. Vet Microbiol. 2023;284(4):109841. doi:10.1016/j.vetmic.2023.109841.
6. Lou F, Huang H, Li Y, Yang S, Shi Y. Investigation of the inhibitory effect and mechanism of epigallocatechin-3-gallate against Streptococcus suis sortase A. J Appl Microbiol. 2023;134(9):lxad191. doi:10.1093/jambio/lxad191.
7. Jeffery A, Gilbert M, Corsaut L, Gaudreau A, Obradovic MR, Cloutier S, et al. Immune response induced by a Streptococcus suis multi-serotype autogenous vaccine used in sows to protect post-weaned piglets. Vet Res. 2024;55(1):1–21. doi:10.1186/s13567-024-01313-x.
8. Zhu H, Müller U, Baums CG, Öhlmann S. Comparative analysis of the interactions of different Streptococcus suis strains with monocytes, granulocytes and the complement system in porcine blood. Vet Res. 2024;55(1):14. doi:10.1186/s13567-024-01268-z.
9. Bai J, Xie Y, Li M, Huang X, Guo Y, Sun J, et al. Ultrasound-assisted extraction of emodin from *Rheum officinale* Baill and its antibacterial mechanism against Streptococcus suis based on CcpA. Ultrason Sonochem. 2024;102:106733. doi:10.1016/j.ultsonch.2023.106733.
10. Meng Y, Lin S, Niu K, Ma Z, Lin H, Fan H. Vimentin affects inflammation and neutrophil recruitment in airway epithelium during Streptococcus suis serotype 2 infection. Vet Res. 2023;54(1):7. doi:10.1186/s13567-023-01135-3.
11. Jin M, Liang S, Wang J, Zhang H, Zhang Y, Zhang W, et al. Endopeptidase O promotes Streptococcus suis immune evasion by cleaving the host-defence peptide cathelicidins. Virulence. 2023;14(1):2283896. doi:10.1080/21505594.2023.2283896.
12. Chaiya I, Trachoo K, Nonlaopon K, Prathumwan D. The mathematical model for streptococcus suis infection in pig-human population with humidity effect. Comput Mater Contin. 2022;71(2):2981–98. doi:10.32604/cmc.2022.021856.
13. Prathumwan D, Chaiya I, Trachoo K. Study of transmission dynamics of Streptococcus suis infection mathematical model between pig and human under ABC fractional-order derivative. Symmetry. 2022;14(10):2112. doi:10.3390/sym14102112.
14. Giang E, Hetman BM, Sargeant JM, Poljak Z, Greer AL. Examining the effect of host recruitment rates on the transmission of Streptococcus suis in nursery swine populations. Pathogens. 2020;9(3):174. doi:10.3390/pathogens9030174.
15. Arenas Jús, Zomer A, Harders-Westerveen J, Bootsma HJ, De Jonge MI, Stockhofe-Zurwieden N, et al. Identification of conditionally essential genes for Streptococcus suis infection in pigs. Virulence. 2020;11(1):446–64. doi:10.1080/21505594.2020.1764173.
16. Corsaut L, Misener M, Canning P, Beauchamp G, Gottschalk M, Segura M. Field study on the immunological response and protective effect of a licensed autogenous vaccine to control Streptococcus suis infections in post-weaned piglets. Vaccines. 2020;8(3):384. doi:10.3390/vaccines8030384.
17. Liang Z, Wu H, Bian C, Chen H, Shen Y, Gao X, et al. The antimicrobial systems of Streptococcus suis promote niche competition in pig tonsils. Virulence. 2022;13(1):781–93. doi:10.1080/21505594.2022.2069390.
18. Huan H, Jiang L, Tang L, Wang Y, Guo S. Isolation and characterization of Streptococcus suis strains from swine in Jiangsu province. China J Appl Microbiol. 2020;128(6):1606–12. doi:10.1111/jam.14591.
19. Liedel C, Mayer L, Einspanier A, Völker I, Ulrich R, Rieckmann K, et al. A new S. suis serotype 3 infection model in pigs: lack of effect of buprenorphine treatment to reduce distress. BMC Vet Res. 2022;18(1):435. doi:10.1186/s12917-022-03532-w.
20. Tan M-F, Zou G, Wei Y, Liu W-Q, Li H-Q, Hu Q, et al. Protein-protein interaction network and potential drug target candidates of Streptococcus suis. J Appl Microbiol. 2021;131(2):658–70. doi:10.1111/jam.14950.
21. Oksendal B. Stochastic differential equations: an introduction with applications. Springer Science & Business Media; 2013. [cited 2025 Jan 15]. https://link.springer.com/book/10.1007/978-3-642-14394-6.
22. Gokul G, Udhayakumar R. Existence and approximate controllability for the Hilfer fractional neutral stochastic hemivariational inequality with Rosenblatt process. J Contr Decis. 2024;4(50):1–14. doi:10.1080/23307706.2024.2403492.