Computer Modeling in Engineering & Sciences

Tech Science Press

DOI: 10.32604/cmes.2023.028773

ARTICLE

Novel Investigation of Stochastic Fractional Differential Equations Measles Model via the White Noise and Global Derivative Operator Depending on Mittag-Leffler Kernel

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Received: 06 January 2023 Accepted: 09 June 2023 Published: 11 March 2024

ABSTRACT

Because of the features involved with their varied kernels, differential operators relying on convolution formulations have been acknowledged as effective mathematical resources for modeling real-world issues. In this paper, we constructed a stochastic fractional framework of measles spreading mechanisms with dual medication immunization considering the exponential decay and Mittag-Leffler kernels. In this approach, the overall population was separated into five cohorts. Furthermore, the descriptive behavior of the system was investigated, including prerequisites for the positivity of solutions, invariant domain of the solution, presence and stability of equilibrium points, and sensitivity analysis. We included a stochastic element in every cohort and employed linear growth and Lipschitz criteria to show the existence and uniqueness of solutions. Several numerical simulations for various fractional orders and randomization intensities are illustrated.

KEYWORDS

Measles epidemic model; Atangana-Baleanu Caputo-Fabrizio differential operators; existence and uniqueness; qualitative analysis; Newton interpolating polynomial

1 Introduction

Measles is among the highly contagious airborne infections in humans, and it can result in significant sickness, life-long problems, and even fatality [1]. Paramyxovirus causes measles, an abrupt and deadly infectious infection. This infection can be transferred mostly through atmospheric spraying to mucosa in the pulmonary system, and it can survive in the phlegm of a contaminated person's nasal passages. When an infectious individual coughs or has respiratory secretions, it can be communicated via exposure to a contaminated nasopharynx. Only individuals are intermediate victims of the measles infection. It is split into four rounds of disease, including implantation, prodrome, erythema, and recuperation [2], see Fig. 1.

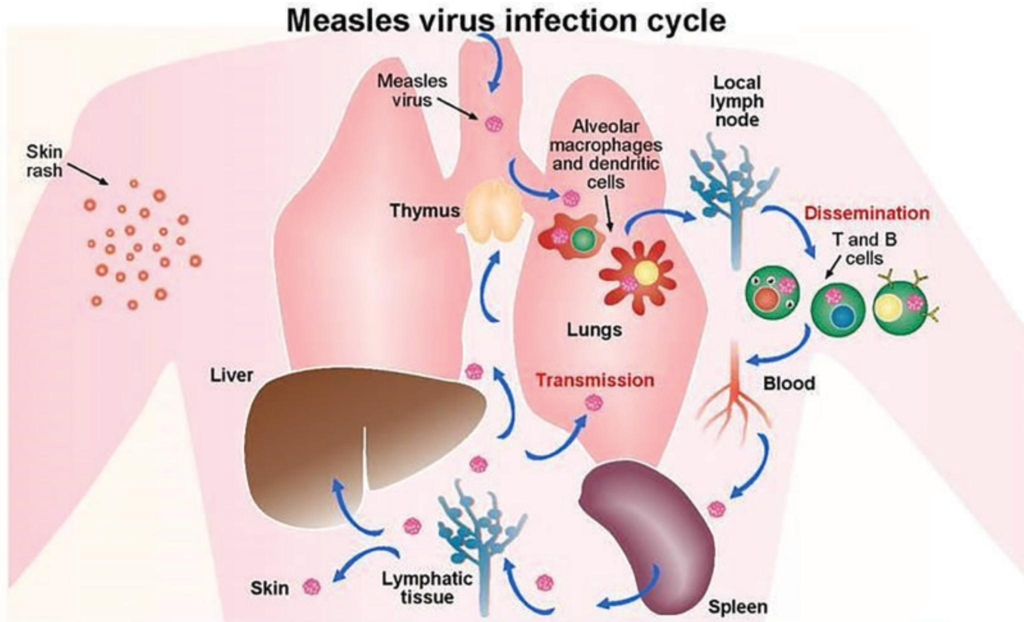


Figure 1: Measles virus cycle

Measles complications seem to be particularly likely in children under the age of five and people over the age of twenty. Tuberculosis, eardrum, and nostril problems, ulcerations, chronic diarrhea, jaundice, conjunctivitis, starvation, and cognitive impairment are just a few of them [3].

Measles is highly contagious, with an infection rate of more than 90 percent among those who are susceptible. It is, indeed, a global health issue. Several underdeveloped nations, notably Asian countries, are affected. According to the WHO, measles affects over 20 million people each year, with developing countries accounting for more than 95 percent of fatalities [4], particularly in Sub-Saharan Africa, where the disease accounts for 15 percent of all fatalities. The simplest method to avoid acquiring measles is to be immunized. It is risk-free, productive, and affordable. Youngsters who have not been inoculated and expectant mothers are more vulnerable to measles and its ramifications, which can include mortality. Immunization from measles acquired by inoculation has been proven to last about two decades and is widely considered to be a reality among healthy humans. At 9–11 months after birth, vaccine effectiveness is estimated to be 85 percent, increasing to 97 percent with a one-year intravenous infusion [4].

The goal of medication is to relieve anxiety unless the ailment is cleared by the adaptive defensive mechanism. The diagnosis of acute is not killed by any therapeutic intervention. Meditation and basic fever-reduction treatments are usually all that is required for a swift comeback. Measles patients require hospitalization, hydration, temperature and anxiety relief, skin rash, as well as medicine [2], see Fig. 2. The technique of depicting important considerations employing quantitative concepts and formulas is known as numerical simulation. On the basis of foresight, mathematical formulation can be divided into stochastic and deterministic systems. Since the beginning of the nineteenth century, numerical simulations have been used to investigate contagious infections. Various deterministic and probabilistic epidemiological approaches have been used to comprehend contagious illnesses, including measles [5], diarrhea [6], the SIRC model [7], chikungunya spread [8], and henceforth.

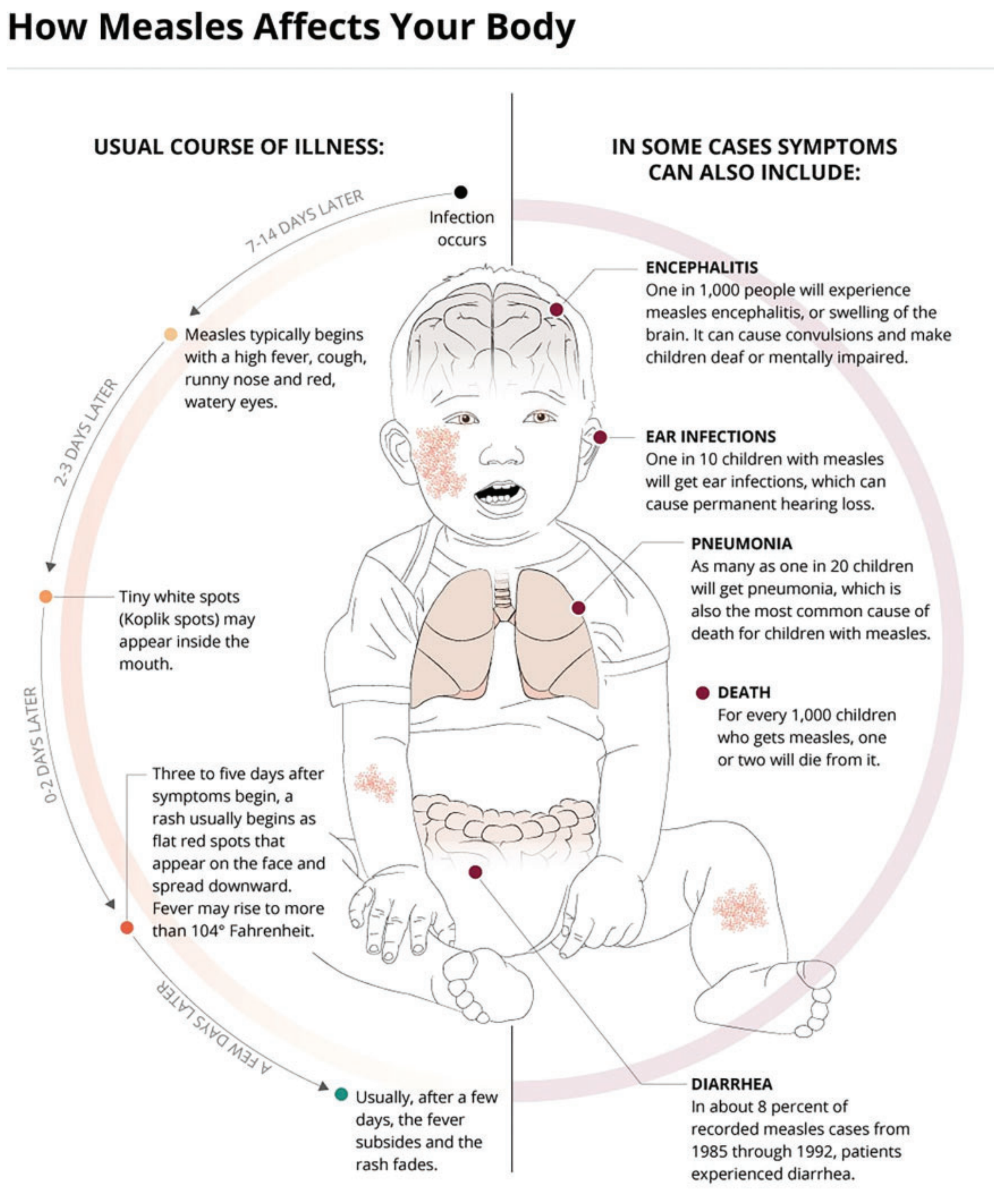


Figure 2: Measles impact on immune system

In the numerical techniques of microbiological contamination, the deterministic methodology has had significant disadvantages. They are easily interpretative, however, supply fewer insights and are difficult to estimate, so they are not stochastic and generally require modeling from very similar modeling outcomes around one manifestation. The stochastic description of a dynamic reflects the system's unpredictability in evolution. Variation, which is key to the development of the multiverse, and carelessness, which is a hallmark of people, both contribute to unpredictability. As a consequence, the random variable should specifically contain both levels of variation in order to depict ambiguity in a straightforward fashion [9]. Stochastic approaches are important in many interdisciplinary fields, notably measles prevalence and distribution, as they convey a higher sense of authenticity than deterministic approaches [10].

Owing to the robust computational formulas including index law, decay, and inversion from growth rate to index law, which can encapsulate perhaps concealed intricacies of existence, the concepts of fractional formulations have lately been coupled to generate innovative differentiation operators in several serious challenges. These novel formulations have the distinctive bonus of being able to describe phenomena that satisfy the index law, exponential decay kernel, and generalized Mittag-Leffler (M-L) kernels at the same time [11–13]. Because of their exceptional capabilities, these novel operators are well suited to modeling a wide range of complicated real-world concerns. Researchers have devised a framework known as Brownian motion, or stochastic elements, to express unpredictability [14]. This concept has been successfully implemented in a variety of disciplines in recent years, including neuroscience, automation, and epidemiology. While both have shown efficiency in simulating dynamic behavior on their own, it is important to note that differential operators, having mentioned kernels, as well as Brownian movements or stochastic ideas, never account for index law, fading memory, or overlapping influences [15–17]. However, we must keep in mind that several situations in existence are capable of exhibiting both mechanisms. Consequently, neither fractional differential operators nor stochastic techniques adequately explain them. The propagation of any contagious disease could be fully comprehended using straightforward mathematical formulae since multiple aspects influence its transmission among individuals [18–21]. Several researchers proposed various investigations for controlling the epidemics and their eradication. For example, Qureshi et al. [5] discussed the monotonic reduction of measles spread via Atangana-Baleanu-Caputo derivative operator, Qureshi et al. [6] represented the nonlinear dynamics of diarrhoea transmission via fractal-fractional operator, Rihan et al. [7] contemplated the SIRC model via Caputo fractional derivative operator, and Rashid et al. [22] investigated the oncolytic effectiveness model with M1 virus via Atangana-Baleanu fractional derivative operator.

Adopting the above propensity, we consider the transmission of the measles infection via stochastic fractional derivative operators in the Antagana-Baleanu and Caputo-Fabrizio sense. In the past decade, only a handful of important studies on the prevalence and distribution of measles have been conducted. A SIR mathematical formulation of measles incorporating inoculation and two periods of transmissibility was used in the investigation. Their research discovered that if all vulnerable people were inoculated, the ailment would be eradicated. They proposed that the measles vaccine be implemented immediately, as no child should be permitted to join a classroom lacking proof of at least two doses of measles immunization. Stochastic modeling of measles emergence and spread involving inoculation intervention was investigated in [23,24]. In comparison to a deterministic approach, stochastic interpretation proved to be more productive in investigating the evolution of measles. Furthermore, Edward et al. [25] developed a quantitative framework for controlling and eliminating measles disease propagation. Measles eradication necessitates keeping the efficient reproductive count below 1 and establishing moderate concentrations of vulnerability. In this analysis, we aim to generate stochastic and deterministic differential equation (DE) systems of measles outbreaks while taking the overall community and immunization regime into account via the fractional derivative operator techniques. For small sensitive community densities, simulated findings indicate that the probabilistic model's responses will have considerable stochastic features. For greater vulnerable population levels, the deterministic framework's result is a restriction of the stochastic counterpart's alternatives. The presence, originality, consistency, and simulation studies of a mathematical framework for measles transmission are also investigated. We verified the model's consistency, as well as the existence and uniqueness of the model's findings via the linear growth and Lipschitiz conditions. The system is numerically solved using the Newton interpolating technique. All of the aforementioned investigations have created deterministic and stochastic mathematical formulas for measles propagation and prevention. To the best of the researchers' expertise, no investigation has been performed on a stochastic framework of measles prevalence and distribution with dual dosage vaccination by segregating first and second treatment immunized groups. However, we must be certain of the interval specified.

2 Preliminaries

In this part, we will review several essential concepts for fractional calculus involving singular and non-singular kernels.

Definition 2.1. ([26]) For $\phi > 0$, then the Caputo fractional derivative of $\mathbf{f}_1 : (0,\infty) \mapsto \mathbb{R}$ is continuous and differentiable, presented as

$$ {}^{c}\mathbf{D}_{\xi}^{\phi}\mathbf{f}_1(\xi) = \frac{1}{\Gamma(1-\phi)}\int_0^{\xi}(\xi-\mathbf{x})^{-\phi}\frac{d}{d\mathbf{x}}\mathbf{f}_1(\mathbf{x})d\mathbf{x}, \quad 0 < \phi \le 1. \tag{1}$$

Definition 2.2. ([26]) For $\phi > 0$, then the Riemann-Liouville fractional integral of $\mathbf{f}_1 : (0,\infty) \mapsto \mathbb{R}$ is presented as

$$ \mathbf{I}_{\xi}^{\phi}\mathbf{f}_1(\xi) = \frac{1}{\Gamma(\phi)}\int_0^{\xi}(\xi-\mathbf{x})^{\phi-1}\mathbf{f}_1(\mathbf{x})d\mathbf{x}, \quad 0 < \phi \le 1. \tag{2}$$

Definition 2.3. ([27]) Suppose there be a function $\mathbf{f}_1 \in H^1(u,v)$, $u < v$, $\phi \in (0,1)$, then the Caputo-Fabrizio fractional derivative is defined as

$$ {}^{CF}\mathbf{D}_{\xi}^{\phi}\mathbf{f}_1(\xi) = \frac{\mathbb{M}(\phi)}{(1-\phi)}\int_u^{\xi}\exp\Big(-\phi\frac{\xi-\mathbf{x}}{1-\phi}\Big)\mathbf{f}_1'(\mathbf{x})d\mathbf{x}, \quad 0 < \phi \le 1, \tag{3}$$

where $\mathbb{M}(\phi)$ is a normalization function such that $\mathbb{M}(0) = \mathbb{M}(1) = 1$.

If $\mathbf{f}_1 \notin H^1(u,v)$, then the derivative operator is redefined as

$$ {}^{CF}\mathbf{D}_{\xi}^{\phi}\mathbf{f}_1(\xi) = \frac{\phi\mathbb{M}(\phi)}{(1-\phi)}\int_u^{\xi}\exp\Big(-\phi\frac{\xi-\mathbf{x}}{1-\phi}\Big)\big(\mathbf{f}_1(\xi)-\mathbf{f}_1(\mathbf{x})\big)d\mathbf{x}, \quad 0 < \phi \le 1, \tag{4}$$

Theorem 2.4. ([27]) For $\phi \in (0,1)$, then the following ordinary DE

$$ {}_0^{CF}\mathbf{D}_{\xi}^{\phi}\mathbf{f}_1(\xi) = \Psi(\xi) \tag{5}$$

has a unique solution by implementing the inverse Laplace transform and convolution theorem described as

$$ \mathbf{f}_1(\xi) = \frac{2(1-\phi)}{(2-\phi)\mathbb{M}(\phi)}\Psi(\xi) + \frac{2\phi}{(2-\phi)\mathbb{M}(\phi)}\int_0^{\xi}\Psi(s_1)ds_1, \quad \xi \ge 0. \tag{6}$$

Definition 2.5. ([28]) Suppose there be a function $\mathbf{f}_1 \in H^1(u, v)$, $u < v$, $\phi \in (0, 1)$, then the Atangana-Baleanu fractional derivative in the Caputo context is defined as

$$ {}^{ABC}\mathbf{D}_{\xi}^{\phi}\mathbf{f}_1(\xi) = \frac{\mathbb{B}(\phi)}{(1-\phi)} \int_u^{\xi} E_{\phi}\Big(-\phi\frac{(\xi-\mathbf{x})^{\phi}}{1-\phi}\Big)\mathbf{f}_1'(\mathbf{x})d\mathbf{x}, \quad 0 < \phi \le 1, \tag{7} $$

where E_{ϕ} is the M-L kernel and $\mathbb{B}(\phi) = \frac{\phi}{\Gamma(\phi)} + 1 - \phi$ denotes the normalized function.

Definition 2.6. ([28]) Suppose there be a function $\mathbf{f}_1 \in H^1(u, v)$, $u < v$, $\phi \in (0, 1)$, is not differentiable, then the Atangana-Baleanu fractional derivative in the Riemann context is defined as

$$ {}^{ABR}\mathbf{D}_{\xi}^{\phi}\mathbf{f}_1(\xi) = \frac{\mathbb{B}(\phi)}{(1-\phi)} \frac{d}{d\xi} \int_u^{\xi} \mathbf{f}_1(\mathbf{x}) E_{\phi}\Big(-\phi\frac{(\xi-\mathbf{x})^{\phi}}{1-\phi}\Big)d\mathbf{x}. \tag{8} $$

Definition 2.7. ([28]) For $\phi \in (0, 1)$, then the Atanagana-Baleanu fractional integral is stated as:

$$ {}_{u}^{AB}\mathbf{I}_{\xi}^{\phi}\mathbf{f}_1(\xi) = \frac{1-\phi}{\mathbb{B}(\phi)}\mathbf{f}_1(\xi) + \frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)} \int_u^{\xi} \mathbf{f}_1(\mathbf{x})(\xi-\mathbf{x})^{\phi-1}d\mathbf{x}. \tag{9} $$

Theorem 2.8. For $\phi \in (0, 1)$, then the following ordinary DE

$$ {}_{0}^{ABC}\mathbf{D}_{\xi}^{\phi}\mathbf{f}_1(\xi) = \Psi(\xi) \tag{10} $$

has a unique solution by implementing the inverse Laplace transform and convolution theorem described as

$$ \mathbf{f}_1(\xi) = \frac{1-\phi}{\mathbb{B}(\phi)}\Psi(\xi) + \frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)} \int_0^{\xi} \Psi(\mathbf{x})(\xi-\mathbf{x})^{\phi-1}d\mathbf{x}. \tag{11} $$

It is worth noting that Atangana established the aforementioned concepts with the global notion quite early. In his study [29], he offered a description of the global derivative. Let us have a glance at a few different variations of it.

Definition 2.9. ([29]) For $\phi \in (0, 1]$, then there be a continuous mapping $\mathbf{f}_1(\xi)$ and an increasing positive mapping $\mathscr{G}(\xi)$ such that there be a singular/non-singular kernel $\mathscr{K}(\xi)$, then the fractional global derivative (GD) in Caputo context is stated as follows:

$$ {}_{0}^{c}\mathbf{D}_{\mathscr{G}}^{\phi}\mathbf{f}_1(\xi) = \mathbf{D}_{\mathscr{G}}\mathbf{f}_1(\xi) * \mathscr{K}(\xi), \tag{12} $$

where $*$ denotes the convolution operator.

Next, we present the concept of fractional global derivative (GD) in Caputo form, Riemann-Liouville form, Caputo-Fabrizio form, and Atanagana-Baleanu form, respectively, which is mainly by Atangana [29].

Definition 2.10. ([29]) For $0 < \phi \le 1$, then the GD in the Caputo sense is defined as

$$ {}_{0}^{c}\mathbf{D}_{\mathscr{G}}^{\phi}\mathbf{f}_1(\xi) = \frac{1}{\Gamma(1-\phi)} \int_0^{\xi} (\xi-\mathbf{x})^{-\phi}\mathbf{D}_{\mathscr{G}}\mathbf{f}_1(\mathbf{x})d\mathbf{x}. \tag{13} $$

Definition 2.11. ([29]) For $0 < \phi \leq 1$, then the GD in the Riemann-Liouville sense is defined as

$$ {}_0^{RL}\mathbf{D}_{\mathscr{G}}^{\phi}\mathbf{f}_1(\xi) = \frac{1}{\Gamma(1-\phi)}\mathbf{D}_{\mathscr{G}}\int_0^{\xi}(\xi-\mathbf{x})^{-\phi}\mathbf{f}_1(\mathbf{x})d\mathbf{x}. \tag{14} $$

Definition 2.12. ([29]) For $0 < \phi \leq 1$, then the GD in the Caputo-Fabrizio sense is defined as

$$ {}_0^{CF}\mathbf{D}_{\mathscr{G}}^{\phi}\mathbf{f}_1(\xi) = \frac{\mathbb{M}(\phi)}{(1-\phi)}\int_0^{\xi}\mathbf{D}_{\mathscr{G}}\mathbf{f}_1(\mathbf{x})exp\Big(-\phi\frac{\xi-\mathbf{x}}{1-\phi}\Big)d\mathbf{x}. \tag{15} $$

Definition 2.13. ([29]) For $0 < \phi \leq 1$, then the GD in the Atangana-Baleanu in the Caputo sense is defined as

$$ {}_0^{ABC}\mathbf{D}_{\mathscr{G}}^{\phi}\mathbf{f}_1(\xi) = \frac{\mathbb{B}(\phi)}{(1-\phi)}\int_0^{\xi}\mathbf{D}_{\mathscr{G}}\mathbf{f}_1(\mathbf{x})E_{\phi}\Big(-\phi\frac{(\xi-\mathbf{x})^{\phi}}{1-\phi}\Big)d\mathbf{x}. \tag{16} $$

Definition 2.14. ([29]) For $0 < \phi \leq 1$, then the GD in the Atangana-Baleanu in the Riemann sense is defined as

$$ {}_0^{ABR}\mathbf{D}_{\mathscr{G}}^{\phi}\mathbf{f}_1(\xi) = \frac{\mathbb{B}(\phi)}{(1-\phi)}\mathbf{D}_{\mathscr{G}}\int_0^{\xi}\mathbf{f}_1(\mathbf{x})E_{\phi}\Big(-\phi\frac{(\xi-\mathbf{x})^{\phi}}{1-\phi}\Big)d\mathbf{x}. \tag{17} $$

Integral representations of the derivatives here are employed in numerical demonstrations, hence integral operators with GD in the Riemann-Liouville form are supplied as:

Definition 2.15. ([29]) For $0 < \phi \leq 1$, then the integral version of RiemannLiouville in GD form is defined as

$$ {}_0\mathbf{I}_{\mathscr{G}}^{\phi}\mathbf{f}_1(\xi) = \frac{1}{\Gamma(\phi)}\int_0^{\xi}\mathscr{G}'(\xi)(\xi-\mathbf{x})^{\phi-1}\mathbf{f}_1(\mathbf{x})d\mathbf{x}. \tag{18} $$

Definition 2.16. ([29]) For $0 < \phi \leq 1$, then the integral version of Caputo-Fabrizio in GD form is defined as

$$ {}_0^{CF}\mathbf{I}_{\mathscr{G}}^{\phi}\mathbf{f}_1(\xi) = \frac{(1-\phi)}{\mathbb{M}(\phi)}\mathscr{G}'(\mathbf{x})\mathbf{f}_1(\mathbf{x}) + \frac{\phi}{\mathbb{M}(\phi)}\int_0^{\xi}\mathscr{G}'(\mathbf{x})\mathbf{f}_1(\mathbf{x})d\mathbf{x}. \tag{19} $$

Definition 2.17. ([29]) For $0 < \phi \leq 1$, then the integral version of Atanagana-Baleanu in GD form is defined as

$$ {}_0^{ABC}\mathbf{I}_{\mathscr{G}}^{\phi}\mathbf{f}_1(\xi) = \frac{(1-\phi)}{\mathbb{B}(\phi)}\mathscr{G}'(\mathbf{x})\mathbf{f}_1(\mathbf{x}) + \frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)}\int_0^{\xi}\mathscr{G}'(\mathbf{x})(\xi-\mathbf{x})^{\phi-1}\mathbf{f}_1(\mathbf{x})d\mathbf{x}. \tag{20} $$

Definition 2.18. ([29]) For $0 < \phi \leq 1$, then the GD in the Atangana-Baleanu in the Caputo sense is defined as

$$ {}_0^{ABC}\mathbf{D}_{\mathscr{G}}^{\phi}\mathbf{f}_1(\xi) = \frac{\mathbb{B}(\phi)}{(1-\phi)}\int_0^{\xi}\mathbf{D}_{\mathscr{G}}\mathbf{f}_1(\mathbf{x})E_{\phi}\Big(-\phi\frac{(\xi-\mathbf{x})^{\phi}}{1-\phi}\Big)d\mathbf{x}. \tag{21} $$

3 Configuration of Stochastic Measles Epidemic Model

Despite the fact that deterministic DEs have been frequently employed to simulate the transmission of several contagious ailments, daily information gathering revealed that their distribution occasionally follows non-locality and unpredictability. This shows that neither fractional DEs nor stochastic differential equations (SDEs) can duplicate such a dispersion. However, SDEs are appropriate for simulating complex issues if the dissemination maintains random noise. Some examples of studies conducted that involve SDEs are [30–33]. The following is a brief description of how to differentiate when considering unpredictability:

$$d\mathbf{x} = \mathbf{f}_1(\xi, \mathbf{x}, \mathbf{y})d\xi + \mathbf{f}_2(\xi, \mathbf{x}, \mathbf{y})d\rho_1, \tag{22}$$

where $\rho_j = \left[\rho_1, ..., \rho_{\mathbf{n}}\right], for\, j = 1, ..., \mathbf{n}$ represents the independent Wiener process. In the measles epidemic model, the whole population is divided into five cohorts as follows: susceptible $\mathbf{S}(\xi)$, infected $\mathbf{I}(\xi)$, $\mathbf{V}_1(\xi)$ initial prescription of vaccination, $\mathbf{V}_2(\xi)$ second prescription of vaccination and recovered $\mathbf{R}(\xi)$, respectively are presented as follows:

$$\begin{cases} \dot{\mathbf{S}}(\xi) = \Xi + \vartheta\mathbf{V}_1(\xi) - \omega\mathbf{S}(\xi)\mathbf{I}(\xi) - \chi_1\mathbf{S}(\xi) - \varphi\mathbf{S}(\xi), \\ \dot{\mathbf{V}}_1(\xi) = \Omega + \chi_1\mathbf{S}(\xi) - \vartheta\mathbf{V}_1(\xi) - \chi_3\mathbf{V}_1(\xi) - \varphi\mathbf{V}_1(\xi), \\ \dot{\mathbf{V}}_2(\xi) = \chi_3\mathbf{V}_1(\xi) - \chi_2\mathbf{V}_2(\xi) - \varphi\mathbf{V}_2(\xi), \\ \dot{\mathbf{I}}(\xi) = \omega\mathbf{S}(\xi)\mathbf{I}(\xi) - \delta\mathbf{I}(\xi) - \eta\mathbf{I}(\xi) - \varphi\mathbf{I}(\xi), \\ \dot{\mathbf{R}}(\xi) = \delta\mathbf{I}(\xi) + \chi_2\mathbf{V}_2(\xi) - \varphi\mathbf{R}(\xi). \end{cases} \tag{23}$$

Therefore, the aforesaid (23) can be transformed into Itôs type SDEs, by inserting of noise environment.

$$\begin{cases} d\mathbf{S}(\xi) = \left[\Xi + \vartheta\mathbf{V}_1(\xi) - \omega\mathbf{S}(\xi)\mathbf{I}(\xi) - \chi_1\mathbf{S}(\xi) - \varphi\mathbf{S}(\xi)\right]d\xi + \rho_1\mathbf{S}(\xi)dB_1(\xi), \\ d\mathbf{V}_1(\xi) = \left[\Omega + \chi_1\mathbf{S}(\xi) - \vartheta\mathbf{V}_1(\xi) - \chi_3\mathbf{V}_1(\xi) - \varphi\mathbf{V}_1(\xi)\right]d\xi + \rho_2\mathbf{V}_1(\xi)dB_2(\xi), \\ d\mathbf{V}_2(\xi) = \left[\chi_3\mathbf{V}_1(\xi) - \chi_2\mathbf{V}_2(\xi) - \varphi\mathbf{V}_2(\xi)\right]d\xi + \rho_3\mathbf{V}_2(\xi)dB_3(\xi), \\ d\mathbf{I}(\xi) = \left[\omega\mathbf{S}(\xi)\mathbf{I}(\xi) - \delta\mathbf{I}(\xi) - \eta\mathbf{I}(\xi) - \varphi\mathbf{I}(\xi)\right]d\xi + \rho_4\mathbf{I}(\xi)dB_4(\xi), \\ d\mathbf{R}(\xi) = \left[\delta\mathbf{I}(\xi) + \chi_2\mathbf{V}_2(\xi) - \varphi\mathbf{R}(\xi)\right]d\xi + \rho_5\mathbf{R}(\xi)dB_5(\xi), \end{cases} \tag{24}$$

supplemented with initial conditions (ICs) $\left(\mathbf{S}(0), \mathbf{V}_1(0), \mathbf{V}_2(0), \mathbf{I}(0), \mathbf{R}(0)\right)^{\mathbb{T}} = \left(\mathbf{S}_0, \mathbf{V}_{1_0}, \mathbf{V}_{2_0}, \mathbf{I}_0, \mathbf{R}_0\right)^{\mathbb{T}} \in \mathbb{R}^5_+$ while $(\rho_j)_{j=1,2,...,5}$ represents the densities of unpredictability and $B_j(\xi)_{j=1,...,5}$ are Brownian motions of every cohort.

Susceptible class $\mathbf{S}(\xi)$ is grown at frequency Ξ, and diminishing for first dosage of immune at speed $\vartheta\mathbf{V}_1$, and lowered at rate $\omega\mathbf{SI}$, those who acquire first prescription of vaccine to vulnerable at rate $\chi_1\mathbf{S}$. Interaction involving the vulnerable group increases the contaminated category by incidence $\omega\mathbf{SI}$, while the infectious group recovers at rate $\delta\mathbf{I}$. The healed group is boosted because the contaminated group survived at a rate of $\delta\mathbf{I}$ and added a new treatment of vaccination to recuperate at a rate of $\chi_2\mathbf{V}_2$. The Immunizations attracted of newborns at a rate Ω, get initial dose of immune to susceptible at a rate $\chi_1\mathbf{S}$, and diminished owing to dwindling for initial dosage of immune to susceptible at rate $\vartheta\mathbf{V}_1$, acquire first dosage of vaccine to intravenous infusion of vaccine at a rate $\chi_3\mathbf{V}_1$. The immunized subsequent dose group is enhanced by receiving the initial prescription of immuneÂ and decreasing by receiving the subsequent prescription $\chi_3\mathbf{V}_1$ of immune and healing at a rate $\chi_2\mathbf{V}_2$. Background extinction rate φ and infectious mortality rate η for the contaminated group only fell in all sub-classes. $\mathbf{N}(\xi)$ represents the entire population count at time ξ, where $\mathbf{N}(\xi) = \mathbf{S}(\xi) + \mathbf{V}_1(\xi) + \mathbf{V}_2(\xi) + \mathbf{I}(\xi) + \mathbf{R}(\xi)$ is consistent and combines adequately. Because the incubation phase is not important for the highly vulnerable contact, the unprotected cohort is excluded. Recruitment of new infants who have completed their first dosage of vaccine is placed in the immunocompetent group, while anyone who has not gotten their initial dosage of vaccine is placed in the vulnerable category. When an infectious agent comes into contact with another person, the virus can be transmitted. There is no breakdown of therapy. A person will either recuperate or perish after receiving the initial and subsequent doses of vaccination. The progression of the measles sickness is represented in Fig. 3.

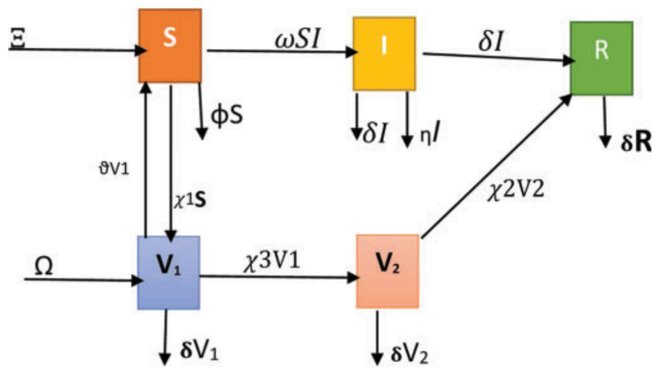


Figure 3: Flow chart for measles infection model

3.1 Qualitative Aspects of Measles Epidemic Model

Here, we will use the structural evaluation of deterministic dynamics to analyze the evolution of stochastic frameworks in this part. Additionally, the (23) solution pertains to the (24) mean. To get equilibria, we first investigate a measles epidemic model (23), and then we examine adequate requirements wherein the equilibria are locally stable.

Theorem 3.1. If there be a viable domain $\Omega = \big(\mathbf{S}(0), \mathbf{V}_1(0), \mathbf{V}_2(0), \mathbf{I}(0), \mathbf{R}(0)\big)$ of the systems (23) and (24), then $\Omega^* = \left\{\big(\mathbf{S}(0), \mathbf{V}_1(0), \mathbf{V}_2(0), \mathbf{I}(0), \mathbf{R}(0)\big) \in \mathbb{R}^5_+ : 0 \leq \mathbf{N} \leq \dfrac{\Xi + \Omega}{\varphi}\right\}$ is bounded.

Proof. Considering, the overall population in the system under discussion is provided by

$$\mathbf{N}(\xi) = \mathbf{S}(\xi) + \mathbf{V}_1(\xi) + \mathbf{V}_2(\xi) + \mathbf{I}(\xi) + \mathbf{R}(\xi).$$

After differentiation with respect to ξ and utilizing (23), yield

$$\begin{aligned}\dot{\mathbf{N}}(\xi) &= \Xi + \Omega - \varphi\mathbf{N} - \eta\mathbf{I}\\ &\leq \Xi + \Omega - \varphi\mathbf{N}.\end{aligned}$$

Simple computations yield

$$\ln(\Xi + \Omega - \varphi\mathbf{N}) \leq -(\xi + C_1), \quad \text{where } C_1 \text{ is a constant.}$$

It follows that

$$\Xi + \Omega - \varphi\mathbf{N} \geq \exp(-\varphi\xi)\exp(-\varphi C_1).$$

Assume that $\mathbf{N}_0 = \exp(-\varphi C_1)$ and $\lim_{\xi\mapsto\infty}\exp(-\varphi\xi) = 0$. Therefore $(\Xi + \Omega - \varphi\mathbf{N}) \geq 0$ gives

$$\mathbf{N} \leq \frac{\Xi + \Omega}{\varphi}, \quad \textit{for all } \xi \geq 0.$$

Hence, $\Omega^* = \left\{ \big(\mathbf{S}(0), \mathbf{V}_1(0), \mathbf{V}_2(0), \mathbf{I}(0), \mathbf{R}(0)\big) \in \mathbb{R}^5_+ : 0 \leq \mathbf{N} \leq \dfrac{\Xi + \Omega}{\varphi} \right\}$ is positive invariant for the frameworks (23) and (24), so that (24) introduce stochastic perturbations to (23).

3.2 Disease-Free Equilibrium Point (DFEP)

In order to find the EP at which the outbreak is eliminated from the community is found in this section. Allowing the right hand sides of (23) to zero and $\mathbf{I} = \mathbf{R} = 0$, yields

$$\begin{cases} \Xi + \vartheta\mathbf{V}_1^* - (\chi_1 + \varphi)\mathbf{S}_0 = 0, \\ \Omega + \chi_1\mathbf{S}_0 - (\vartheta + \chi_3 + \varphi)\mathbf{V}_1^* = 0, \\ \chi_3\mathbf{V}_1^* - (\chi_2 + \varphi)\mathbf{V}_2^* = 0. \end{cases} \tag{25}$$

After simplification, we have

$$(\mathbf{S}_0, \mathbf{V}_1^*, \mathbf{V}_2^*, 0, 0) = \Bigg(\frac{\Xi(\vartheta + \chi_3 + \varphi) + \vartheta\Omega}{(\chi_1 + \varphi)(\vartheta + \chi_3 + \varphi) - \chi_1\vartheta}, \frac{\chi_1\Xi + \Omega(\chi_1 + \varphi)}{(\chi_1 + \varphi)(\vartheta + \chi_3 + \varphi) - \chi_1\vartheta}, \frac{\Xi\chi_1\chi_3 + \Omega\chi_3(\chi_1 + \varphi)}{(\chi_1 + \varphi)(\chi_2 + \varphi)(\vartheta + \chi_3 + \varphi) - \chi_1\vartheta} \Bigg),$$

i.e., is the phase where no disease exists in the environment.

3.3 Stochastic Fundamental Reproductive Number

Here, the fundamental reproductive number for (24) can be evaluated by employing Itôs formula for twice differentiable mapping on $[0, \mathbb{T}]$ to $\mathbf{f}_1(\mathbf{I}) = \ln(\mathbf{I})$. Utilizing the fact of Taylor series expansion so that we have

$$d\mathbf{f}_1(\xi, \mathbf{I}(\xi)) = \frac{\partial\mathbf{f}_1}{d\xi}d\xi + \frac{\partial\mathbf{f}_1}{d\mathbf{I}}d\mathbf{I} + \frac{1}{2}\frac{\partial^2\mathbf{f}_1}{\partial\mathbf{I}^2}(d\mathbf{I})^2 + \frac{1}{2}\frac{\partial^2\mathbf{f}_1}{\partial\xi^2}(d\xi)^2 + \frac{\partial^2\mathbf{f}_1}{\partial\xi\partial\mathbf{I}}d\xi d\mathbf{I}.$$

Now, by means of system (24), we have

$$d\mathbf{I} = \big[\omega\mathbf{S}\mathbf{I} - (\varphi + \eta + \delta)\mathbf{I}\big]d\xi + \rho_4\mathbf{I}dB_4.$$

It follows that

$$d\mathbf{f}_1(\xi, \mathbf{I}(\xi)) = \big(\big[\omega\mathbf{S} - (\varphi + \eta + \delta)\big]d\xi + \rho_4 dB_4\big) - \frac{1}{2}\rho_4^2(dB_4^2).$$

Higher order differentials $(d\xi, dB)$ approach quickly zero; $(d\xi)^2 \mapsto 0$ and $d\xi dB(\xi) \mapsto 0$. The stochastic component $dB^2(\xi)$ is supplied as $dB^2(\xi) = d\xi$ pertaining to Brownian motion principles.

where we implement the underlying features to compute $(d\mathbf{I}(\xi))^2$, we have

$$d\mathbf{f}_1(\xi,\mathbf{I}(\xi)) = \left(\left[\omega\mathbf{S} - (\varphi+\eta+\delta) - \frac{1}{2}\rho_4^2\right]d\xi + \rho_4 dB_4\right).$$

Taking into account the next-generation matrix, we computed $\mathbb{E}_0 = \left(\mathbf{S},\mathbf{V}_1^*,\mathbf{V}_2^*,0,0\right)$, where $\mathbf{S}_0 = \dfrac{\Xi(\vartheta+\chi_3+\varphi)+\vartheta\Omega}{(\chi_1+\varphi)(\vartheta+\chi_3+\varphi)-\chi_1\vartheta}$.

Hence, the stochastic fundamental reproductive number is

$$\mathbb{R}_0^s = \frac{2\omega\mathbf{S}_0 - \rho_4^2}{2(\varphi+\eta+\chi_1)}. \tag{26}$$

3.4 Local Stability of DFEP

For the (24), we deliver an analogous stochastic eradication of transmission. In the long term, if $\mathbb{R}_0 < 1$, the population of the contaminated group will approach zero. If there is an $\mathbb{R}_0 < 1$, then the DFE of the system (24) is locally asymptotically stable.

Theorem 3.2. If there be $\mathbb{R}_0 < 1$, then $\mathbf{I}(\xi)$ will approaches to zero almost certainly tremendously stable of system (24) is locally asymptotically stable, i.e., $\limsup_{\xi\mapsto\infty}\dfrac{\ln(\xi)}{\xi} < 0$.

Proof. By means of (23), we have

$$d\ln(\mathbf{I}) = \left\{\omega\mathbf{S} - (\varphi+\eta+\delta) - \frac{1}{2}\rho_4^2\right\}d\xi + \rho_4 dB_4(\xi).$$

Performing integration from 0 to ξ, we have

$$\ln(\mathbf{I}) - \ln(\mathbf{I}_0) = \left\{\omega\mathbf{S} - (\varphi+\eta+\delta) - \frac{1}{2}\rho_4^2\right\}\xi + \int_0^{\xi}\rho_4 dB_4(\xi).$$

It follows that

$$\frac{\ln(\mathbf{I}) - \ln(\mathbf{I}_0)}{\xi} \le \left\{\omega\mathbf{S} - (\varphi+\eta+\delta) - \frac{1}{2}\rho_4^2\right\} + \frac{\mathscr{Q}(\xi)}{\xi}.$$

Assume that $\mathscr{Q}(\xi) = \int_0^{\xi}\rho_4 dB_4(s_1)$. As $\mathscr{Q}(\xi)$ represents the martingale [34] having a quadratic variation described by

$$\langle\mathscr{Q}(\xi),\mathscr{Q}(\xi)\rangle_{\xi} = \int_0^{\xi}\rho_4^2 ds_1 = \rho_4^2\xi.$$

In view of strong law, we have

$$\limsup_{\xi\mapsto\infty}\frac{\langle\mathscr{Q}(\xi),\mathscr{Q}(\xi)\rangle_{\xi}}{\xi} = \rho_4^2 < \infty,$$

arrives at

$$\lim_{\xi\mapsto\infty}\sup\frac{\mathscr{Q}(\xi)}{\xi}=0.$$

This leads to $\lim\limits_{\xi\mapsto\infty}\sup\dfrac{\ln(\xi)}{\xi}=\omega\mathbf{S}-\varphi-\eta-\delta-\dfrac{1}{2}\rho_4^2.$

If $\mathbb{R}_0<1$, then $\omega\mathbf{S}-\varphi-\eta-\delta-\dfrac{1}{2}\rho_4^2<0.$

At DFE, we have

$$\begin{aligned}\lim_{\xi\mapsto\infty}\sup\frac{\ln(\xi)}{\xi}&=\omega\mathbf{S}_0-\varphi-\eta-\delta-\frac{1}{2}\rho_4^2,\\&=(\varphi+\eta+\delta)\left(\frac{\omega_0\mathbf{S}_0}{\varphi+\eta+\delta}-\frac{\rho_4^2}{2(\varphi+\eta+\delta)}-1\right),\\&=(\varphi+\eta+\delta)(\mathbb{R}_0-1),\\&\leq(\mathbb{R}_0-1)<0.\end{aligned}$$

This concludes that $\mathbb{R}_0<1$.

3.5 Endemic Equilibrium Point (EEP)

Here, the equilibrium point at which the infection survives in the population is determined in this part. By putting all of the systems equations equal to zero, the EEP of model (23) can be determined as

$$\begin{cases}\Xi+\vartheta\mathbf{V}_1(\xi)-\omega\mathbf{S}(\xi)\mathbf{I}(\xi)-\chi_1\mathbf{S}(\xi)-\varphi\mathbf{S}(\xi)=0,\\\Omega+\chi_1\mathbf{S}(\xi)-\vartheta\mathbf{V}_1(\xi)-\chi_3\mathbf{V}_1(\xi)-\varphi\mathbf{V}_1(\xi)=0,\\\chi_3\mathbf{V}_1(\xi)-\chi_2\mathbf{V}_2(\xi)-\varphi\mathbf{V}_2(\xi)=0,\\\omega\mathbf{S}(\xi)\mathbf{I}(\xi)-\delta\mathbf{I}(\xi)-\eta\mathbf{I}(\xi)-\varphi\mathbf{I}(\xi)=0,\\\delta\mathbf{I}(\xi)+\chi_2\mathbf{V}_2(\xi)-\varphi\mathbf{R}(\xi)=0.\end{cases}\tag{27}$$

After simplifying, then system (23) has the EEP as follows:

$$\begin{aligned}\left(\mathbf{S}^*,\mathbf{V}_1^*,\mathbf{V}_2^*,\mathbf{I}^*,\mathbf{R}^*\right)=\Bigg(&\frac{\delta+\eta+\varphi}{\omega},\frac{\chi_1(\delta+\eta+\varphi)+\omega\Omega}{\omega(\chi_3+\vartheta+\varphi)},\frac{\chi_3\chi_1(\delta+\eta+\varphi)+\chi_3\omega\Omega}{\omega(\chi_2+\varphi)(\chi_3+\vartheta+\varphi)},\frac{\Xi+\Omega-\varphi\mathbf{N}}{\eta},\\&\frac{\delta(\Xi+\Omega-\varphi\mathbf{N})}{\eta}+\frac{\chi_3\chi_1(\delta+\eta+\varphi)+\chi_3\omega\Omega}{\omega(\chi_2+\varphi)(\chi_3+\vartheta+\varphi)}\Bigg).\end{aligned}$$

3.6 Sensitivity Result

This section determines the importance of each component in the spread and prevention of measles infection through sensitivity evaluation. When a criterion improves, the investigation can help measure and compare the variation in a factor. This data is critical for studying the disease's emergence and spread. With regard to a factor $\mathscr{M}$, the sensitivity criterion is determined by

$$\mathscr{M}_{\mathbf{x}_j}=\frac{\partial\mathbb{R}_0}{\partial\mathbf{x}_j}*\frac{\mathbf{x}_j}{\mathbb{R}_0},\quad j=1,2,...,9,$$

$$\mathscr{M}_{\omega} = \frac{\partial \mathbb{R}_0}{\partial \omega} * \frac{\omega}{\mathbb{R}_0} = \frac{2\omega(\Xi(\vartheta + \chi_3 + \varphi) + \vartheta\Omega}{2\omega(\Xi(\vartheta + \chi_3 + \varphi) + \vartheta\Omega) - \rho_4^2(\chi_3\chi_1 + \varphi(\varphi + \vartheta + \chi_3 + \chi_1))} > 0,$$

$$\mathscr{M}_{\Xi} = \frac{\partial \mathbb{R}_0}{\partial \Xi} * \frac{\Xi}{\mathbb{R}_0} = \frac{2\Xi\omega(\vartheta + \chi_3 + \varphi + \chi_3)}{2\omega(\Xi(\vartheta + \chi_3 + \varphi) + \vartheta\Omega) - \rho_4^2(\chi_3\chi_1 + \varphi(\varphi + \vartheta + \chi_3 + \chi_1))} > 0,$$

$$\mathscr{M}_{\Omega} = \frac{\partial \mathbb{R}_0}{\partial \Omega} * \frac{\Omega}{\mathbb{R}_0}$$

$$= \frac{\omega\vartheta\Omega}{(\varphi + \eta + \delta)(\varphi(\varphi + \vartheta + \chi_3 + \chi_1))\big(2\omega(\Xi(\vartheta + \chi_3 + \varphi) + \vartheta\Omega) - \rho_4^2(\chi_3\chi_1 + \varphi(\varphi + \vartheta + \chi_3 + \chi_1))\big)} > 0,$$

$$\mathscr{M}_{\vartheta} = \frac{\partial \mathbb{R}_0}{\partial \vartheta} * \frac{\vartheta}{\mathbb{R}_0}$$

$$= \frac{4\vartheta(\varphi + \eta + \delta)\big(\varphi(\varphi + \vartheta + \chi_3 + \chi_1)(\omega\chi_3\Xi + \omega\varphi\Xi + \omega\Omega\Xi) - \omega\rho_4^2(\varphi(\varphi + \chi_3 + \chi_1) + \chi_3\chi_1 - \chi_1)\big)}{(\varphi + \eta + \delta)(\varphi(\varphi + \vartheta + \chi_3 + \chi_1))\big(2\omega(\Xi(\vartheta + \chi_3 + \varphi) + \vartheta\Omega) - \rho_4^2(\chi_3\chi_1 + \varphi(\varphi + \vartheta + \chi_3 + \chi_1))\big)}$$

$$- \frac{4\vartheta(\varphi + \eta + \delta)(\Xi(\vartheta + \chi_3 + \varphi) + \vartheta\Omega)(\varphi(\varphi + \chi_3 + \chi_1 + \chi_3) - \chi_1)}{(\varphi + \eta + \delta)(\varphi(\varphi + \vartheta + \chi_3 + \chi_1))\big(2\omega(\Xi(\vartheta + \chi_3 + \varphi) + \vartheta\Omega) - \rho_4^2(\chi_3\chi_1 + \varphi(\varphi + \vartheta + \chi_3 + \chi_1))\big)} > 0,$$

$$\mathscr{M}_{\varphi} = \frac{\partial \mathbb{R}_0}{\partial \varphi} * \frac{\varphi}{\mathbb{R}_0}$$

$$= \frac{1}{2(\varphi + \eta + \delta)(\varphi(\varphi + \vartheta + \chi_3 + \chi_1))\big(2\omega(\Xi(\vartheta + \chi_3 + \varphi) + \vartheta\Omega) + 2\omega\vartheta\Omega - \rho_4^2(\chi_3\chi_1 + \varphi)(\chi_1 + \varphi) - \vartheta\chi_1)\big)}$$

$$\begin{cases} 4\omega\Xi\varphi + 2\rho_4^2\varphi(2\varphi + \vartheta + \chi_3 + \chi_1)(\varphi + \eta + \delta)(\varphi(\varphi + \vartheta + \chi_3 + \chi_1) + \chi_3\chi_1) \\ +6\varphi^3 + 2\varphi^2(\vartheta + \chi_3 + \chi_1 + \eta + \delta) + (2\varphi\delta + 2\varphi\delta)(\vartheta + \chi_3 + \chi_1 + \chi_1\chi_3)\big(2\omega(\Xi(\vartheta + \chi_3 + \varphi) + \vartheta\Omega) \\ -\rho_4^2(\varphi(\varphi + \vartheta + \chi_3 + \chi_1) + \chi_3\chi_1)\big) < 0, \end{cases}$$

$$\mathscr{M}_{\eta} = \frac{\partial \mathbb{R}_0}{\partial \eta} * \frac{\eta}{\mathbb{R}_0} = \frac{-\eta}{(\varphi + \eta + \delta)} < 0,$$

$$\mathscr{M}_{\delta} = \frac{\partial \mathbb{R}_0}{\partial \delta} * \frac{\phi}{\mathbb{R}_0} = \frac{-\delta}{(\varphi + \eta + \delta)} < 0,$$

$$\mathscr{M}_{\rho_4} = \frac{\partial \mathbb{R}_0}{\partial \rho_4} * \frac{\rho_4}{\mathbb{R}_0} = \frac{2\rho_4^2(\varphi(\varphi + \vartheta + \chi_3 + \chi_1 + \chi_3\chi_1))}{2\omega(\Xi(\vartheta + \chi_3 + \varphi) + \vartheta\Omega) - \rho_4^2(\varphi(\varphi + \vartheta + \chi_3 + \chi_1) + \chi_3\chi_1)} < 0. \tag{28}$$

The sensitivity values of the components of the measles illness framework are shown in the aforementioned evaluation. The following is a summary of sensitivity index comprehension.

The fundamental reproducing number's responsiveness levels to the essential factors are addressed and presented in the previous analysis. As a result, the factors ω, Ξ, Ω, ϑ and χ_1 have high sensitivity values, indicating that increasing their levels has a significant effect on the spread of infection in the population. However, because the factors φ, η, δ and ρ_4 have deleterious sensitivity values, diminishing their level will result in the infection outbreak spreading more.

4 A Fractional Stochastic Model of Measles Epidemic

Next, we investigate a generic measles stochastic model in which the standard temporal derivative is transformed into the global derivative in this part. It is specified as the GD of a differentiable mapping $\mathbf{f}_1$ with respect to an increasing non-negative continuous mapping $\mathscr{G}$ as follows:

$$\mathbf{D}_{\mathscr{G}}\mathbf{f}_1(\xi) = \lim_{t\mapsto\xi} \frac{\mathbf{f}_1(t) - \mathbf{f}_1(\xi)}{\mathscr{G}(t) - \mathscr{G}(\xi)}.$$

In fact, if $\mathscr{G}$ is differentiable then

$$\mathbf{D}_{\mathscr{G}}\mathbf{f}_1(\xi) = \frac{\mathbf{f}_1'(\xi)}{\mathscr{G}'(\xi)}.$$

Considering the system (24), we can simply evaluate the disease's elimination and prevalence. For this, we assume the system (24) with respect to the global derivative as follows:

$$\begin{cases} \mathbf{D}_{\mathscr{G}}\mathbf{S}(\xi) = \left[\Xi + \vartheta\mathbf{V}_1(\xi) - \omega\mathbf{S}(\xi)\mathbf{I}(\xi) - \chi_1\mathbf{S}(\xi) - \varphi\mathbf{S}(\xi)\right] - \sigma\mathbf{S}(\xi)\mathbf{I}(\xi), \\ \mathbf{D}_{\mathscr{G}}\mathbf{V}_1(\xi) = \left[\Omega + \chi_1\mathbf{S}(\xi) - \vartheta\mathbf{V}_1(\xi) - \chi_3\mathbf{V}_1(\xi) - \varphi\mathbf{V}_1(\xi)\right], \\ \mathbf{D}_{\mathscr{G}}\mathbf{V}_2(\xi) = \left[\chi_3\mathbf{V}_1(\xi) - \chi_2\mathbf{V}_2(\xi) - \varphi\mathbf{V}_2(\xi)\right], \\ \mathbf{D}_{\mathscr{G}}\mathbf{I}(\xi) = \left[\omega\mathbf{S}(\xi)\mathbf{I}(\xi) - \delta\mathbf{I}(\xi) - \eta\mathbf{I}(\xi) - \varphi\mathbf{I}(\xi)\right] + \sigma\mathbf{S}(\xi)\mathbf{I}(\xi), \\ \mathbf{D}_{\mathscr{G}}\mathbf{R}(\xi) = \left[\delta\mathbf{I}(\xi) + \chi_2\mathbf{V}_2(\xi) - \varphi\mathbf{R}(\xi)\right]. \end{cases}$$

Since $\mathscr{G}$ is differentiable, we have

$$\begin{cases} d\mathbf{S}(\xi) = \left[\Xi + \vartheta\mathbf{V}_1(\xi) - \omega\mathbf{S}(\xi)\mathbf{I}(\xi) - \chi_1\mathbf{S}(\xi) - \varphi\mathbf{S}(\xi)\right]\mathscr{G}'(\xi)d\xi - \rho_1\mathbf{S}(\xi)dB_1(\xi), \\ d\mathbf{V}_1(\xi) = \left[\Omega + \chi_1\mathbf{S}(\xi) - \vartheta\mathbf{V}_1(\xi) - \chi_3\mathbf{V}_1(\xi) - \varphi\mathbf{V}_1(\xi)\right]\mathscr{G}'(\xi)d\xi + \rho_2\mathbf{V}_1(\xi)\mathscr{G}'(\xi)dB_2(\xi), \\ d\mathbf{V}_2(\xi) = \left[\chi_3\mathbf{V}_1(\xi) - \chi_2\mathbf{V}_2(\xi) - \varphi\mathbf{V}_2(\xi)\right]\mathscr{G}'(\xi)d\xi + \rho_3\mathbf{V}_2(\xi)\mathscr{G}'(\xi)dB_3(\xi), \\ d\mathbf{I}(\xi) = \left[\omega\mathbf{S}(\xi)\mathbf{I}(\xi) - \delta\mathbf{I}(\xi) - \eta\mathbf{I}(\xi) - \varphi\mathbf{I}(\xi)\right]\mathscr{G}'(\xi)d\xi + \rho_4\mathbf{I}(\xi)\mathscr{G}'(\xi)dB_4(\xi), \\ d\mathbf{R}(\xi) = \left[\delta\mathbf{I}(\xi) + \chi_2\mathbf{V}_2(\xi) - \varphi\mathbf{R}(\xi)\right]\mathscr{G}'(\xi)d\xi + \rho_5\mathbf{R}(\xi)\mathscr{G}'(\xi)dB_5(\xi). \end{cases} \tag{29}$$

It is important to mention that the framework is simple to determine if the ambient noise (ρ_j), $j = 1, 2, ..., 5$.

Taking the integral on both sides, we have

$$\begin{cases} \mathbf{S}(\xi) = \mathbf{S}(0) + \int_0^\xi \left(\Xi + \vartheta\mathbf{V}_1(\tau) - \omega\mathbf{S}(\tau)\mathbf{I}(\tau) - \chi_1\mathbf{S}(\tau) - \varphi\mathbf{S}(\tau)\right)\mathscr{G}'(\tau)d\tau - \int_0^\xi \sigma\mathbf{S}(\xi)\mathbf{I}(\tau)\mathscr{G}'(\tau)d\tau, \\ \mathbf{V}_1(\xi) = \mathbf{V}_1(0) + \int_0^\xi \left(\Omega + \chi_1\mathbf{S}(\tau) - \vartheta\mathbf{V}_1(\tau) - \chi_3\mathbf{V}_1(\tau) - \varphi\mathbf{V}_1(\tau)\right)\mathscr{G}'(\tau)d\tau + \int_0^\xi \rho_2\mathbf{V}_1(\tau)\mathscr{G}'(\tau)dB_2(\tau), \\ \mathbf{V}_2(\xi) = \mathbf{V}_2(0) + \int_0^\xi \left(\chi_3\mathbf{V}_1(\tau) - \chi_2\mathbf{V}_2(\tau) - \varphi\mathbf{V}_2(\tau)\right)\mathscr{G}'(\tau)d\tau + \int_0^\xi \rho_3\mathbf{V}_2(\tau)\mathscr{G}'(\tau)dB_3(\tau), \\ \mathbf{I}(\xi) = \mathbf{I}(0) + \int_0^\xi \left(\omega\mathbf{S}(\tau)\mathbf{I}(\tau) - \delta\mathbf{I}(\tau) - \eta\mathbf{I}(\tau) - \varphi\mathbf{I}(\tau)\right)\mathscr{G}'(\tau)d\tau + \int_0^\xi \sigma\mathbf{S}(\xi)\mathbf{I}(\tau)\mathscr{G}'(\tau)d\tau, \\ \mathbf{R}(\xi) = \mathbf{R}(0) + \int_0^\xi \left(\delta\mathbf{I}(\tau) + \chi_2\mathbf{V}_2(\tau) - \varphi\mathbf{R}(\tau)\right)\mathscr{G}'(\tau)d\tau + \int_0^\xi \rho_5\mathbf{R}(\tau)\mathscr{G}'(\tau)dB_5(\tau). \end{cases} \tag{30}$$

In view of the Brownian motion, we have

$$
\begin{cases}
\mathbf{S}(\xi) = \mathbf{S}(0) + \int_0^{\xi} \big(\Xi + \vartheta \mathbf{V}_1(\tau) - \omega \mathbf{S}(\tau)\mathbf{I}(\tau) - \chi_1 \mathbf{S}(\tau) - \varphi \mathbf{S}(\tau)\big)\mathscr{G}'(\tau)d\tau - \int_0^{\xi} \sigma \mathbf{S}(\xi)\mathbf{I}(\tau)\mathscr{G}'(\tau)dB(\tau),\\
\mathbf{V}_1(\xi) = \mathbf{V}_1(0) + \int_0^{\xi} \big(\Omega + \chi_1 \mathbf{S}(\tau) - \vartheta \mathbf{V}_1(\tau) - \chi_3 \mathbf{V}_1(\tau) - \varphi \mathbf{V}_1(\tau)\big)\mathscr{G}'(\tau)d\tau + \int_0^{\xi} \rho_2 \mathbf{V}_1(\tau)\mathscr{G}'(\tau)dB_2(\tau),\\
\mathbf{V}_2(\xi) = \mathbf{V}_2(0) + \int_0^{\xi} \big(\chi_3 \mathbf{V}_1(\tau) - \chi_2 \mathbf{V}_2(\tau) - \varphi \mathbf{V}_2(\tau)\big)\mathscr{G}'(\tau)d\tau + \int_0^{\xi} \rho_3 \mathbf{V}_2(\tau)\mathscr{G}'(\tau)dB_3(\tau),\\
\mathbf{I}(\xi) = \mathbf{I}(0) + \int_0^{\xi} \big(\omega \mathbf{S}(\tau)\mathbf{I}(\tau) - \delta \mathbf{I}(\tau) - \eta \mathbf{I}(\tau) - \varphi \mathbf{I}(\tau)\big)\mathscr{G}'(\tau)d\tau + \int_0^{\xi} \sigma \mathbf{S}(\xi)\mathbf{I}(\tau)\mathscr{G}'(\tau)dB(\tau),\\
\mathbf{R}(\xi) = \mathbf{R}(0) + \int_0^{\xi} \big(\delta \mathbf{I}(\tau) + \chi_2 \mathbf{V}_2(\tau) - \varphi \mathbf{R}(\tau)\big)\mathscr{G}'(\tau)d\tau + \int_0^{\xi} \rho_5 \mathbf{R}(\tau)\mathscr{G}'(\tau)dB_5(\tau).
\end{cases}
\tag{31}
$$

With the classic GD, we obtain the complex stochastic equation below. Let us describe the requirement in which the nonlinear problem has a specific value, based on Atangana's work [29,35].

Theorem 4.1. Assume there be positive constants $\mathscr{A}_j$, $j = 1, 2, ..., 7$ and $\mathscr{C}_j$, $j = 1, 2, ..., 7$ such that

(a)

$$
\begin{cases}
\left\|\mathbf{f}_1(\xi,\mathbf{S}) - \mathbf{f}_1(\xi,\mathbf{S}_1)\right\|^2 \le \mathscr{A}_1 \left\|\mathbf{S} - \mathbf{S}_1\right\|^2,\\
\left\|\mathbf{f}_2(\xi,\mathbf{S}) - \mathbf{f}_2(\xi,\mathbf{S}_1)\right\|^2 \le \mathscr{A}_2 \left\|\mathbf{S} - \mathbf{S}_1\right\|^2,\\
\left\|\mathbf{d}_1(\xi,\mathbf{V}_1) - \mathbf{d}_1(\xi,\bar{\mathbf{V}}_1)\right\|^2 \le \mathscr{A}_3 \left\|\mathbf{V}_1 - \bar{\mathbf{V}}_1\right\|^2,\\
\left\|\mathbf{h}_1(\xi,\mathbf{V}_2) - \mathbf{h}_1(\xi,\bar{\mathbf{V}}_2)\right\|^2 \le \mathscr{A}_4 \left\|\mathbf{V}_2 - \bar{\mathbf{V}}_2\right\|^2,\\
\left\|\mathbf{m}_1(\xi,\mathbf{I}) - \mathbf{m}_1(\xi,\mathbf{I}_1)\right\|^2 \le \mathscr{A}_5 \left\|\mathbf{I} - \mathbf{I}_1\right\|^2,\\
\left\|\mathbf{m}_2(\xi,\mathbf{I}) - \mathbf{m}_2(\xi,\mathbf{I}_1)\right\|^2 \le \mathscr{A}_6 \left\|\mathbf{I} - \mathbf{I}_1\right\|^2,\\
\left\|\mathbf{p}_1(\xi,\mathbf{R}) - \mathbf{p}_1(\xi,\mathbf{R}_1)\right\|^2 \le \mathscr{A}_7 \left\|\mathbf{R} - \mathbf{R}_1\right\|^2.
\end{cases}
\tag{32}
$$

(b)

$$
\begin{cases}
\left|\mathbf{f}_1(\xi,\mathbf{S})\right|^2 \le \mathscr{C}_1(1 + \left|\mathbf{S}(\xi)\right|^2),\\
\left|\mathbf{f}_2(\xi,\mathbf{S})\right|^2 \le \mathscr{C}_2(1 + \left|\mathbf{S}(\xi)\right|^2),\\
\left|\mathbf{d}_1(\xi,\mathbf{V}_1)\right|^2 \le \mathscr{C}_3(1 + \left|\mathbf{V}_1(\xi)\right|^2),\\
\left|\mathbf{h}_1(\xi,\mathbf{V}_2)\right|^2 \le \mathscr{C}_4(1 + \left|\mathbf{V}_2(\xi)\right|^2),\\
\left|\mathbf{m}_1(\xi,\mathbf{I})\right|^2 \le \mathscr{C}_5(1 + \left|\mathbf{I}(\xi)\right|^2),\\
\left|\mathbf{m}_2(\xi,\mathbf{I})\right|^2 \le \mathscr{C}_6(1 + \left|\mathbf{I}(\xi)\right|^2),\\
\left|\mathbf{p}_1(\xi,\mathbf{R})\right|^2 \le \mathscr{C}_7(1 + \left|\mathbf{R}(\xi)\right|^2).
\end{cases}
$$

Proof. By means of model (29), we prove the Lipschitz condition for the proposed system as follows:

$$\begin{cases} d\mathbf{S}(\xi) = \mathbf{f}_1(\xi, \mathbf{S}(\xi))d\xi + \mathbf{f}_2(\xi, \mathbf{S}(\xi))dB\xi, \\ d\mathbf{V}_1(\xi) = \mathbf{d}_1(\xi, \mathbf{V}_1(\xi))d\xi + d_2(\xi, \mathbf{V}_1(\xi))dB\xi, \\ d\mathbf{V}_2(\xi) = \mathbf{h}_1(\xi, \mathbf{V}_2(\xi))d\xi + h_2(\xi, \mathbf{V}_2(\xi))dB\xi, \\ d\mathbf{I}(\xi) = \mathbf{m}_1(\xi, \mathbf{I}(\xi))d\xi + \mathbf{m}_2(\xi, \mathbf{I}(\xi))dB\xi, \\ d\mathbf{R}(\xi) = \mathbf{p}_1(\xi, \mathbf{R}_1(\xi))d\xi + \mathbf{p}_2(\xi, \mathbf{R}(\xi))dB\xi, \end{cases} \tag{33}$$

where,

$$\begin{aligned} &\mathbf{f}_1(\xi, \mathbf{S}(\xi)) = \Xi + \vartheta\mathbf{V}_1(\xi) - \omega\mathbf{S}(\xi)\mathbf{I}(\xi) - \chi_1\mathbf{S}(\xi) - \varphi\mathbf{S}(\xi), \\ &\mathbf{f}_2(\xi, \mathbf{S}(\xi)) = -\sigma\mathbf{S}(\xi)\mathbf{I}(\xi), \\ &\mathbf{d}_1(\xi, \mathbf{V}_1(\xi)) = \Omega + \chi_1\mathbf{S}(\xi) - \vartheta\mathbf{V}_1(\xi) - \chi_3\mathbf{V}_1(\xi) - \varphi\mathbf{V}_1(\xi), \\ &\mathbf{h}_1(\xi, \mathbf{V}_2(\xi)) = \chi_3\mathbf{V}_1(\xi) - \chi_2\mathbf{V}_2(\xi) - \varphi\mathbf{V}_2(\xi), \\ &\mathbf{m}_1(\xi, \mathbf{I}(\xi)) = \omega\mathbf{S}(\xi)\mathbf{I}(\xi) - \delta\mathbf{I}(\xi) - \eta\mathbf{I}(\xi) - \varphi\mathbf{I}(\xi), \\ &\mathbf{m}_2(\xi, \mathbf{I}(\xi)) = \sigma\mathbf{S}(\xi)\mathbf{I}(\xi), \\ &\mathbf{p}_1(\xi, \mathbf{R}(\xi)) = \delta\mathbf{I}(\xi) + \chi_2\mathbf{V}_2(\xi) - \varphi\mathbf{R}(\xi). \end{aligned} \tag{34}$$

Let us introducing the norm

$$\|\Lambda\|_\infty = \sup_{\xi\in[0,\mathbb{T}]} |\Lambda|^2, \tag{35}$$

then for all $\mathbf{S}, \mathbf{S}_1 \in \mathbb{R}^2$ and $\xi \in [0, \mathbb{T}]$, we have

$$\begin{aligned} \|\mathbf{f}_1(\xi, \mathbf{S}) - \mathbf{f}_1(\xi, \mathbf{S}_1)\|^2 &= \| - (\omega\mathbf{I}(\xi) + \varphi + \delta)(\mathbf{S} - \mathbf{S}_1)\|^2 \\ &\leq \| - (\omega \sup_{\xi\in[0,\mathbb{T}]} \mathbf{I}(\xi) + \varphi + \delta)(\mathbf{S} - \mathbf{S}_1)\|^2 \\ &\leq (2\omega^2\|\mathbf{I}(\xi)\|_\infty^2 + 2\varphi^2 + \delta^2)\|\mathbf{S} - \mathbf{S}_1)\|^2 \leq \mathscr{A}_1\|\mathbf{S} - \mathbf{S}_1\|^2, \end{aligned}$$

where $\mathscr{A}_1 = \left(2\omega^2\|\mathbf{I}(\xi)\|_\infty^2 + 2\varphi^2 + \delta^2\right)$.

$$\begin{aligned} \|\mathbf{f}_2(\xi, \mathbf{S}) - \mathbf{f}_2(\xi, \mathbf{S}_1)\|^2 &= \| - (\sigma\mathbf{I}(\xi))(\mathbf{S} - \mathbf{S}_1)\|^2 \\ &\leq \| - (\sigma \sup_{\xi\in[0,\mathbb{T}]} \mathbf{I}(\xi))(\mathbf{S} - \mathbf{S}_1)\|^2 \\ &\leq (2\sigma^2\|\mathbf{I}(\xi)\|_\infty^2)\|\mathbf{S} - \mathbf{S}_1)\|^2 \leq \mathscr{A}_2\|\mathbf{S} - \mathbf{S}_1\|^2, \end{aligned}$$

where $\mathscr{A}_2 = \left(\sigma^2\|\mathbf{I}(\xi)\|_\infty^2\right)$.

Furthermore, we show that for all $\mathbf{V}_1, \bar{\mathbf{V}}_1 \in \mathbb{R}^2$ and $\xi \in [0, \mathbb{T}]$, we have

$$\begin{aligned} \|\mathbf{d}_1(\xi, \mathbf{V}_1) - \mathbf{d}_1(\xi, \bar{\mathbf{V}}_1)\|^2 &= \| - (\vartheta + \chi_3 + \varphi)(\mathbf{V}_1 - \bar{\mathbf{V}}_1)\|^2 \\ &= (\vartheta^2 + \chi_3^2 + \varphi^2)\|(\mathbf{V}_1 - \bar{\mathbf{V}}_1)\|^2 \end{aligned}$$

$$\leq \{(\vartheta^2 + \chi_3^2 + \varphi^2) + \varepsilon\}\left\|(\mathbf{V}_1 - \bar{\mathbf{V}}_1)\right\|^2 \leq \mathscr{A}_3\left\|\mathbf{V}_1 - \bar{\mathbf{V}}_1\right\|^2,$$

where $\mathscr{A}_3 = \{(\vartheta^2 + \chi_3^2 + \varphi^2) + \varepsilon\}$.

Again, we show that for all $\mathbf{V}_2, \bar{\mathbf{V}}_2 \in \mathbb{R}^2$ and $\xi \in [0, \mathbb{T}]$, we have

$$\begin{aligned}\left\|\mathbf{h}_1(\xi, \mathbf{V}_2) - \mathbf{h}_1(\xi, \bar{\mathbf{V}}_2)\right\|^2 &= \left\| -(\chi_2 - \varphi)(\mathbf{V}_2 - \bar{\mathbf{V}}_2)\right\|^2 \\ &= (\chi_2^2 + \varphi^2)\left\|(\mathbf{V}_2 - \bar{\mathbf{V}}_2)\right\|^2 \\ &\leq \{(\chi_2^2 + \varphi^2) + \varepsilon\}\left\|(\mathbf{V}_2 - \bar{\mathbf{V}}_2)\right\|^2 \leq \mathscr{A}_4\left\|\mathbf{V}_1 - \bar{\mathbf{V}}_1\right\|^2,\end{aligned}$$

where $\mathscr{A}_4 = \{(\chi_2^2 + \varphi^2) + \varepsilon\}$.

Now, we find for all $\mathbf{I}, \mathbf{I}_1 \in \mathbb{R}^2$ and $\xi \in [0, \mathbb{T}]$, we have

$$\begin{aligned}\left\|\mathbf{m}_1(\xi, \mathbf{I}) - \mathbf{m}_1(\xi, \mathbf{I}_1)\right\|^2 &= \left\| -(\omega\mathbf{S}(\xi) + \varphi + \delta + \eta)(\mathbf{I} - \mathbf{I}_1)\right\|^2 \\ &\leq \left\| -(\omega \sup_{\xi\in[0,\mathbb{T}]} \mathbf{S}(\xi) + \varphi + \delta + \eta)(\mathbf{I} - \mathbf{I}_1)\right\|^2 \\ &\leq (2\omega^2\left\|\mathbf{S}(\xi)\right\|_\infty^2 + 2\varphi^2 + \delta^2 + 2\eta^2)\left\|\mathbf{I} - \mathbf{I}_1)\right\|^2 \leq \mathscr{A}_5\left\|\mathbf{I} - \mathbf{I}_1\right\|^2,\end{aligned}$$

where $\mathscr{A}_5 = \left(2\omega^2\left\|\mathbf{S}(\xi)\right\|_\infty^2 + 2\varphi^2 + 2\delta^2 + 2\eta^2\right)$.

$$\begin{aligned}\left\|\mathbf{m}_2(\xi, \mathbf{I}) - \mathbf{m}_2(\xi, \mathbf{I}_1)\right\|^2 &= \left\|(\sigma\mathbf{S}(\xi))(\mathbf{I} - \mathbf{I}_1)\right\|^2 \\ &\leq \left\|(\sigma \sup_{\xi\in[0,\mathbb{T}]} \mathbf{S}(\xi))(\mathbf{I} - \mathbf{I}_1)\right\|^2 \\ &\leq (2\sigma^2\left\|\mathbf{S}(\xi)\right\|_\infty^2)\left\|\mathbf{I} - \mathbf{I}_1)\right\|^2 \leq \mathscr{A}_6\left\|\mathbf{I} - \mathbf{I}_1\right\|^2,\end{aligned}$$

where $\mathscr{A}_6 = \left(\sigma^2\left\|\mathbf{S}(\xi)\right\|_\infty^2\right)$.

Now, we find for all $\mathbf{R}, \mathbf{R}_1 \in \mathbb{R}^2$ and $\xi \in [0, \mathbb{T}]$, we have

$$\begin{aligned}\left\|\mathbf{p}_1(\xi, \mathbf{R}) - \mathbf{p}_1(\xi, \mathbf{R}_1)\right\|^2 &= \left\| -\varphi(\mathbf{R} - \mathbf{R}_1)\right\|^2 \\ &= \left\| -(\varphi^2(\mathbf{R} - \mathbf{R}_1)\right\|^2 \\ &\leq (\varphi^2 + \varepsilon)\left\|\mathbf{R} - \mathbf{R}_1)\right\|^2 \leq \mathscr{A}_7\left\|\mathbf{R} - \mathbf{R}_1\right\|^2,\end{aligned}$$

where $\mathscr{A}_7 = \left\{\varphi^2 + \varepsilon\right\}$. which shows that the condition **(a)** is satisfied.

Next, in order to prove the linear growth conditions for model (29). To do this, for all $(\xi, \mathbf{S}) \in \mathbb{R}^2 \times [t_0, \mathbb{T}]$, we have

$$\begin{aligned}\left|\mathbf{f}_1(\xi, \mathbf{S})\right|^2 &= \left|\Xi + \vartheta\mathbf{V}_1(\xi) - \omega\mathbf{S}(\xi)\mathbf{I}(\xi) - \chi_1\mathbf{S}(\xi) - \varphi\mathbf{S}(\xi)\right|^2 \\ &\leq \left(5\Xi^2 + 5\vartheta^2 \sup_{\xi\in[0,\mathbb{T}]}\left|\mathbf{V}_1\right|^2 + 5(\omega^2 \sup_{\xi\in[0,\mathbb{T}]}\left|\mathbf{I}(\xi)\right|^2 + \delta^2 + \varphi^2)\left|\mathbf{S}(\xi)\right|^2\right) \\ &\leq 5\Xi^2\left(1 + \frac{\vartheta^2\left\|\mathbf{V}_1\right\|_\infty^2}{\Xi^2} + \frac{(\omega^2\left\|\mathbf{I}(\xi)\right\|_\infty^2 + \delta^2 + \varphi^2)}{\Xi^2}\right)\left|\mathbf{S}(\xi)\right|^2) \\ &\leq \mathscr{C}_1(1 + \left|\mathbf{S}(\xi)\right|^2),\end{aligned} \tag{36}$$

under supposition $\mathscr{C}_1 = \frac{\vartheta^2 \left\| \mathbf{V}_1 \right\|_\infty^2}{\Xi^2} + \frac{(\omega^2 \left\| \mathbf{I}(\xi) \right\|_\infty^2 + \delta^2 + \varphi^2)}{\Xi^2} < 1.$

Moreover,

$$\begin{aligned}
\left|\mathbf{f}_2(\xi, \mathbf{S})\right|^2 &= \left| - \sigma \mathbf{S}(\xi) \mathbf{I}(\xi)\right|^2 \\
&\leq \left(\sigma^2 \left|\mathbf{I}(\xi)\right|^2\right) \left|\mathbf{S}(\xi)\right|^2 \\
&\leq \left(\sigma^2 \sup_{\xi \in [0,\mathbb{T}]} \left|\mathbf{I}(\xi)\right|^2\right) (1 + \left|\mathbf{S}(\xi)\right|^2) \\
&\leq \left(\sigma^2 \left\|\mathbf{I}(\xi)\right\|_\infty^2\right) (1 + \left|\mathbf{S}(\xi)\right|^2) \\
&\leq \mathscr{C}_2 (1 + \left|\mathbf{S}(\xi)\right|^2),
\end{aligned} \tag{37}$$

under supposition $\mathscr{C}_2 = \sigma^2 \left\|\mathbf{I}(\xi)\right\|_\infty^2 < 1.$

Furthermore, we have

$$\begin{aligned}
\left|\mathbf{d}_1(\xi, \mathbf{V}_1)\right|^2 &= \left|\Omega + \chi_1 \mathbf{S}(\xi) - (\vartheta + \chi_3 + \varphi) \mathbf{V}_1(\xi)\right|^2 \\
&\leq \left(3\Omega^2 + 3\delta^2 \sup_{\xi \in [0,\mathbb{T}]} \left|\mathbf{S}(\xi)\right|^2 + 3(\vartheta^2 + \chi_3^2 + \varphi^2) \left|\mathbf{V}_1(\xi)\right|^2\right) \\
&\leq 3\Omega^2 \left(1 + \frac{\delta^2 \left\|\mathbf{S}\right\|_\infty^2}{\Omega^2} + \frac{(\vartheta^2 + \chi_3^2 + \varphi^2)}{\Omega^2}\right) \left|\mathbf{V}_1(\xi)\right|^2 \\
&\leq \mathscr{C}_3 (1 + \left|\mathbf{V}_1(\xi)\right|^2),
\end{aligned} \tag{38}$$

under supposition $\mathscr{C}_3 = \frac{\delta^2 \left\|\mathbf{S}\right\|_\infty^2}{\Omega^2} + \frac{(\vartheta^2 + \chi_3^2 + \varphi^2)}{\Omega^2} < 1.$

Again, we have

$$\begin{aligned}
\left|\mathbf{h}_1(\xi, \mathbf{V}_2)\right|^2 &= \left|\chi_3 - \chi_2 \mathbf{V}_1(\xi) - \varphi \mathbf{V}_2(\xi)\right|^2 \\
&\leq \left(3\chi_3^2 + 3\chi_2^2 \sup_{\xi \in [0,\mathbb{T}]} \left|\mathbf{V}_1(\xi)\right|^2 + 3\varphi^2 \left|\mathbf{V}_2(\xi)\right|^2\right) \\
&\leq \left(3\chi_3^2 + 3\chi_2^2 \left\|\mathbf{V}_1(\xi)\right\|_\infty^2 + 3\varphi^2 \left|\mathbf{V}_2(\xi)\right|^2\right) \\
&\leq \left(3\chi_3^2 + 3\chi_2^2 \left\|\mathbf{V}_1(\xi)\right\|_\infty^2\right) \left(1 + \frac{\varphi^2 \left|\mathbf{V}_2(\xi)\right|^2}{3\chi_3^2 + 3\chi_2^2 \left\|\mathbf{V}_1(\xi)\right\|_\infty^2}\right) \\
&\leq \mathscr{C}_4 \left(1 + \left|\mathbf{V}_1(\xi)\right|^2\right),
\end{aligned} \tag{39}$$

under supposition $\mathscr{C}_4 = \frac{\varphi^2}{3\chi_3^2 + 3\chi_2^2 \left\|\mathbf{V}_1(\xi)\right\|_\infty^2} < 1.$

Again, we have

$$\begin{aligned}
\left|\mathbf{m}_1(\xi, \mathbf{I})\right|^2 &= \left|\omega \mathbf{S}(\xi) - \delta \mathbf{I}(\xi) - \eta \mathbf{I}(\xi) - \varphi \mathbf{I}(\xi)\right|^2 \\
&\leq \left(4\omega^2 \sup_{\xi \in [0,\mathbb{T}]} \left|\mathbf{S}(\xi)\right|^2 + 4(\delta^2 + \eta^2 + \varphi^2) \left|\mathbf{I}(\xi)\right|^2\right)
\end{aligned}$$

$$\leq \left(4\omega^2\left\|\mathbf{S}(\xi)\right\|_\infty^2\left|\mathbf{I}(\xi)\right|^2 + 4(\delta^2+\eta^2+\varphi^2)\left|\mathbf{I}(\xi)\right|^2\right)$$
$$\leq \left(4\omega^2\left\|\mathbf{S}(\xi)\right\|_\infty^2\right)\left(1 + \frac{(\delta^2+\eta^2+\varphi^2)\left|\mathbf{I}(\xi)\right|^2}{\omega^2\left\|\mathbf{S}(\xi)\right\|_\infty^2}\right)$$
$$\leq \mathscr{C}_5\left(1+\left|\mathbf{I}(\xi)\right|^2\right), \tag{40}$$

under supposition $\mathscr{C}_5 = \dfrac{(\delta^2+\eta^2+\varphi^2)\left|\mathbf{I}(\xi)\right|^2}{\omega^2\left\|\mathbf{S}(\xi)\right\|_\infty^2} < 1.$

Now, we have

$$\left|\mathbf{m}_2(\xi,\mathbf{I})\right|^2 = \left|\sigma\mathbf{S}(\xi)\mathbf{I}(\xi)\right|^2$$
$$\leq \left(\sigma^2\left|\mathbf{I}(\xi)\right|^2\right)\left|\mathbf{S}(\xi)\right|^2$$
$$\leq \left(\sigma^2\sup_{\xi\in[0,\mathbb{T}]}\left|\mathbf{S}(\xi)\right|^2\right)(1+\left|\mathbf{I}(\xi)\right|^2)$$
$$\leq \left(\sigma^2\left\|\mathbf{S}(\xi)\right\|_\infty^2\right)(1+\left|\mathbf{I}(\xi)\right|^2)$$
$$\leq \mathscr{C}_6(1+\left|\mathbf{I}(\xi)\right|^2), \tag{41}$$

under supposition $\mathscr{C}_6 = \sigma^2\left\|\mathbf{S}(\xi)\right\|_\infty^2 < 1.$

Also, we have

$$\left|\mathbf{p}_1(\xi,\mathbf{I})\right|^2 = \left|\delta\mathbf{I}(\xi) + \chi_2\mathbf{V}_2(\xi) - \varphi\mathbf{R}(\xi)\right|^2$$
$$\leq \left(3\delta^2\sup_{\xi\in[0,\mathbb{T}]}\left|\mathbf{I}(\xi)\right|^2 + 3\chi_2^2\left|\mathbf{V}_2(\xi)\right|^2 + \varphi^2\left|\mathbf{R}(\xi)\right|^2\right)$$
$$\leq \left(3\delta^2\left\|\mathbf{I}(\xi)\right\|_\infty^2 + 3\chi_2^2\left\|\mathbf{V}_2(\xi)\right\|_\infty^2 + \varphi^2\left|\mathbf{R}(\xi)\right|^2\right)$$
$$\leq \left(3\delta^2\left\|\mathbf{I}(\xi)\right\|_\infty^2 + 3\chi_2^2\left\|\mathbf{V}_2(\xi)\right\|_\infty^2\right)\left(1 + \frac{\varphi^2\left|\mathbf{R}(\xi)\right|^2}{3\delta^2\left\|\mathbf{I}(\xi)\right\|_\infty^2 + 3\chi_2^2\left\|\mathbf{V}_2(\xi)\right\|_\infty^2}\right)$$
$$\leq \mathscr{C}_7\left(1+\left|\mathbf{R}(\xi)\right|^2\right), \tag{42}$$

under supposition $\mathscr{C}_7 = \dfrac{\varphi^2}{3\delta^2\left\|\mathbf{I}(\xi)\right\|_\infty^2 + 3\chi_2^2\left\|\mathbf{V}_2(\xi)\right\|_\infty^2} < 1.$

Clearly, we see that both assumptions are satisfied. Hence, according to the given hypothesis, the measles epidemic model (29) has a unique solution.

4.1 Extinction Analysis

In what follows, the elimination of specific groups is discussed in this part. We did this by specifying

$$\langle\mathbf{y}(\xi)\rangle = \lim_{\xi\mapsto\infty}\frac{1}{\xi}\int_0^\xi \mathbf{y}(\tau)d\tau.$$

We shall begin using class $\mathbf{S}(\xi)$. When the integral is applied to both sides of $\mathbf{S}(\xi)$, the result is

$$\mathbf{S}(\xi)-\mathbf{S}(0)=\int_0^\xi\big(\Xi+\vartheta\mathbf{V}_1(\xi)-\omega\mathbf{S}(\xi)\mathbf{I}(\xi)-\chi_1\mathbf{S}(\xi)-\varphi\mathbf{S}(\xi)\big)d\xi+\rho_1\int_0^\xi\mathbf{S}(\xi)dB_1(\xi).$$

Multiplying both sides by $\frac{1}{\xi}$, we get

$$\frac{\mathbf{S}(\xi)-\mathbf{S}(0)}{\xi}=\frac{1}{\xi}\int_0^\xi\big(\Xi+\vartheta\mathbf{V}_1(\tau)-\omega\mathbf{S}(\tau)\mathbf{I}(\tau)-\chi_1\mathbf{S}(\tau)-\varphi\mathbf{S}(\tau)\big)d\tau-\frac{\rho_1}{\xi}\int_0^\xi\mathbf{S}(\xi)dB_1(\xi).$$

It follows that

$$\lim_{\xi\mapsto\infty}\langle\mathbf{S}(\xi)\rangle=\frac{\Xi}{\chi_1+\varphi}+\frac{1}{\chi_1+\varphi}\left\{\vartheta\langle\mathbf{V}_1(\xi)\rangle-\omega\langle\mathbf{S}(\xi)\mathbf{I}(\xi)\rangle+\frac{\mathbf{S}(\xi)-\mathbf{S}(0)}{\xi}-\frac{\rho_1}{\xi}\int_0^\xi\mathbf{S}(\xi)dB_1(\xi)\right\}.$$

Thus, we have

$$\lim_{\xi\mapsto\infty}\langle\mathbf{S}(\xi)\rangle=\frac{\Xi}{\chi_1+\varphi}.$$

Considering the class $\mathbf{I}(\xi)$, we have

$$\frac{\mathbf{I}(\xi)-\mathbf{I}(0)}{\xi}=\frac{1}{\xi}\big(\omega\langle\mathbf{S}(\xi)\mathbf{I}(\xi)\rangle-(\delta+\eta+\varphi)\langle\mathbf{I}(\xi)\rangle\big)+\frac{\rho_4}{\xi}\int_0^\xi\mathbf{I}(\tau)dB_4(\tau).$$

This leads to

$$\lim_{\xi\mapsto\infty}\langle\mathbf{I}(\xi)\rangle=\lim_{\xi\mapsto\infty}\frac{\omega}{(\delta+\eta+\varphi)}\langle\mathbf{S}(\xi)\mathbf{I}(\xi)\rangle+\lim_{\xi\mapsto\infty}\frac{\rho_4}{\xi}\int_0^\xi\mathbf{I}(\tau)dB_4(\tau)+\frac{\mathbf{I}(\xi)-\mathbf{I}(0)}{\xi}.$$

Thus, we have

$$\lim_{\xi\mapsto\infty}\langle\mathbf{I}(\xi)\rangle=0.$$

Analogously, we have

$$\lim_{\xi\mapsto\infty}\langle\mathbf{R}(\xi)\rangle=0.$$

4.2 Numerical Scheme for Stochastic Model via Global Derivative

In this part, we use the Toufik et al. [36] technique to develop a numerical scheme for the stochastic framework (29) as follows:

$$\begin{cases}{}_0\mathbf{D}^{\phi}_{\mathscr{G}}\mathbf{y}(\xi)=\mathbf{f}_1(\xi,\mathbf{y}(\xi))+\mathbf{f}_2(\xi,\mathbf{y}(\xi)),\\ \mathbf{y}(t_0)=\mathbf{y}_0.\end{cases}$$

Assuming that $\mathscr{G}$ to be a differentiable mapping, then we have

$$\mathbf{y}(\xi) = \mathbf{y}(0) + \int_0^{\xi} \mathscr{G}'(\tau)\mathbf{f}_1(\tau, \mathbf{y}(\tau))d\tau + \int_0^{\xi} \mathscr{G}'(\tau)\mathbf{f}_2(\tau, \mathbf{y}(\tau))dB\tau.$$

Since $B'(\xi)$ is differentiable, we have

$$\mathbf{y}(\xi) = \mathbf{y}(0) + \int_0^{\xi} \mathscr{G}'(\tau)\mathbf{f}_1(\tau, \mathbf{y}(\tau))d\tau + \int_0^{\xi} \mathscr{G}'(\tau)\mathbf{f}_2(\tau, \mathbf{y}(\tau))B'(\tau)d\tau.$$

At $\xi_{\mathbf{n}+1} = (\mathbf{n}+1)\Delta\xi$, we obtain

$$\mathbf{y}(\xi_{\mathbf{n}+1}) - \mathbf{y}(0) = \int_0^{\xi_{\mathbf{n}+1}} \mathscr{G}'(\tau)\mathbf{f}_1(\tau, \mathbf{y}(\tau))d\tau + \int_0^{\xi_{\mathbf{n}+1}} \mathscr{G}'(\tau)\mathbf{f}_2(\tau, \mathbf{y}(\tau))B'(\tau)d\tau.$$

Again, at $\xi_{\mathbf{n}} = \mathbf{n}\Delta\xi$, we have

$$\mathbf{y}(\xi_{\mathbf{n}+1}) - \mathbf{y}(0) = \int_0^{\xi_{\mathbf{n}}} \mathscr{G}'(\tau)\mathbf{f}_1(\tau, \mathbf{y}(\tau))d\tau + \int_0^{\xi_{\mathbf{n}}} \mathscr{G}'(\tau)\mathbf{f}_2(\tau, \mathbf{y}(\tau))B'(\tau)d\tau.$$

Letting the difference of the two successive terms as follows:

$$\mathbf{y}(\xi_{\mathbf{n}+1}) - \mathbf{y}(\xi_{\mathbf{n}}) = \int_{\xi_{\mathbf{n}}}^{\xi_{\mathbf{n}+1}} \mathscr{G}'(\tau)\mathbf{f}_1(\tau, \mathbf{y}(\tau))d\tau + \int_{\xi_{\mathbf{n}}}^{\xi_{\mathbf{n}+1}} \mathscr{G}'(\tau)\mathbf{f}_2(\tau, \mathbf{y}(\tau))B'(\tau)d\tau.$$

Taking

$$\mathscr{G}'(\tau)\mathbf{f}_1(\tau, \mathbf{y}(\tau)) = \Psi_1(\tau, \mathbf{y}(\tau)), \qquad \mathscr{G}'(\tau)\mathbf{f}_2(\tau, \mathbf{y}(\tau))B'(\tau) = \Psi_2(\tau, \mathbf{y}(\tau)).$$

Observe that

$$\mathbf{y}(\xi_{\mathbf{n}+1}) - \mathbf{y}(\xi_{\mathbf{n}}) = \int_{\xi_{\mathbf{n}}}^{\xi_{\mathbf{n}+1}} \Psi_1(\tau, \mathbf{y}(\tau))d\tau + \int_{\xi_{\mathbf{n}}}^{\xi_{\mathbf{n}+1}} \Psi_1(\tau, \mathbf{y}(\tau))d\tau.$$

Furthermore, implementing the interpolation

$$q_1(\tau) = \frac{\tau - \xi_{\mathbf{n}-1}}{\xi_{\mathbf{n}} - \xi_{\mathbf{n}-1}}\Psi_1(\xi_{\mathbf{n}}, \mathbf{x}^{\mathbf{n}}) - \frac{\tau - \xi_{\mathbf{n}-1}}{\xi_{\mathbf{n}} - \xi_{\mathbf{n}-1}}\Psi_1(\xi_{\mathbf{n}-1}, \mathbf{x}^{\mathbf{n}-1}).$$

Also, we have

$$\begin{aligned}\mathbf{y}(\xi_{\mathbf{n}+1}) - \mathbf{y}(\xi_{\mathbf{n}}) &= \left(\frac{3}{2}\Psi_1(\xi_{\mathbf{n}}, \mathbf{x}^{\mathbf{n}})\Delta\xi - \frac{1}{2}\Psi_1(\xi_{\mathbf{n}-1}, \mathbf{x}^{\mathbf{n}-1})\right) \\ &\quad + \left(\frac{3}{2}\Psi_2(\xi_{\mathbf{n}}, \mathbf{x}^{\mathbf{n}})\Delta\xi - \frac{1}{2}\Psi_2(\xi_{\mathbf{n}-1}, \mathbf{x}^{\mathbf{n}-1})\right).\end{aligned} \tag{43}$$

Inserting the values of Ψ_1 and Ψ_2 into (43), then we have

$$\mathbf{y}(\xi_{\mathbf{n}+1}) - \mathbf{y}(\xi_{\mathbf{n}}) = \left(\frac{3}{2}\mathscr{G}(\xi_{\mathbf{n}})\mathbf{f}_1(\xi_{\mathbf{n}}, \mathbf{x}^{\mathbf{n}})\Delta\xi - \frac{1}{2}\mathscr{G}(\xi_{\mathbf{n}-1})\mathbf{f}_1(\xi_{\mathbf{n}-1}, \mathbf{x}^{\mathbf{n}-1})\right) + \left(\frac{3}{2}\mathscr{G}(\xi_{\mathbf{n}})\mathbf{f}_1(\xi_{\mathbf{n}}, \mathbf{x}^{\mathbf{n}})B'(\xi_{\mathbf{n}})\Delta\xi - \frac{1}{2}\mathscr{G}(\xi_{\mathbf{n}-1})\mathbf{f}_2(\xi_{\mathbf{n}-1}, \mathbf{x}^{\mathbf{n}-1})\right).$$

and

$$\begin{aligned}\mathbf{y}(\xi_{\mathbf{n}+1}) - \mathbf{y}(\xi_{\mathbf{n}}) = {} & \frac{3}{2}\big(\mathscr{G}(\xi_{\mathbf{n}}) - \mathscr{G}(\xi_{\mathbf{n}-1})\big)\mathbf{f}_1(\xi_{\mathbf{n}}, \mathbf{x}^{\mathbf{n}}) - \frac{1}{2}\big(\mathscr{G}(\xi_{\mathbf{n}-1}) - \mathscr{G}(\xi_{\mathbf{n}-2})\big)\mathbf{f}_1(\xi_{\mathbf{n}-1}, \mathbf{x}^{\mathbf{n}-1}) \\ & + \frac{3}{2}\big(\mathscr{G}(\xi_{\mathbf{n}}) - \mathscr{G}(\xi_{\mathbf{n}-1})\big)\mathbf{f}_1(\xi_{\mathbf{n}}, \mathbf{x}^{\mathbf{n}})\big(B(\xi_{\mathbf{n}}) - B(\xi_{\mathbf{n}-1})\big) \\ & - \frac{1}{2}\big(\mathscr{G}(\xi_{\mathbf{n}-1}) - \mathscr{G}(\xi_{\mathbf{n}-2})\big)\mathbf{f}_2(\xi_{\mathbf{n}-1}, \mathbf{x}^{\mathbf{n}-1})\big(B(\xi_{\mathbf{n}-1}) - B(\xi_{\mathbf{n}-2})\big).\end{aligned}$$

5 Numerical Scheme for Fractional Stochastic Model via Global Derivative

The simulation tool for addressing the fractional order measles stochastic model involving GD is described in this section. We shall employ kernels of the exponential decay law and the M-L law to construct a highly meaningful approach. We shall employ Toufik et al. [36] numerical criteria to create the numerical results (29).

5.1 Caputo-Fabrizio Fractional Derivative Operator

First we demonstrate the numerical scheme for the Caputo-Fabrizio derivative operator:

$$\begin{cases} {}^{CF}\mathbf{D}^{\phi}_{\mathscr{G}}\mathbf{y}(\xi) = \mathbf{f}_1(\xi, \mathbf{y}(\xi)) + \mathbf{f}_2(\xi, \mathbf{y}(\xi)), \\ \mathbf{y}(t_0) = \mathbf{y}_0. \end{cases}$$

Since $\mathscr{G}$ is differentiable, we have

$${}^{CF}\mathbf{D}^{\phi}_{\mathscr{G}}\mathbf{y}(\xi) = \mathscr{G}'(\xi)\mathbf{f}_1(\xi, \mathbf{y}(\xi)) + \mathscr{G}'(\xi)\mathbf{f}_2(\xi, \mathbf{y}(\xi)).$$

By making the use of Caputo-Fabrizio integral, we apply

$$\begin{aligned}\mathbf{y}(\xi) - \mathbf{y}(0) = {} & \frac{1-\phi}{\mathbb{M}(\phi)}\mathscr{G}'(\xi)\mathbf{f}_1(\xi, \mathbf{y}(\xi)) + \frac{\phi}{\mathbb{M}(\phi)}\int_0^{\xi}\mathscr{G}'(\tau)\mathbf{f}_1(\tau, \mathbf{y}(\tau))d\tau \\ & + \frac{1-\phi}{\mathbb{M}(\phi)}\mathscr{G}'(\xi)\mathbf{f}_2(\xi, \mathbf{y}(\xi))B(\xi) + \frac{\phi}{\mathbb{M}(\phi)}\int_0^{\xi}\mathscr{G}'(\tau)\mathbf{f}_2(\tau, \mathbf{y}(\tau))dB(\tau).\end{aligned}$$

Since $B(\xi)$ is differentiable, we can write

$$\mathbf{y}(\xi) - \mathbf{y}(0) = \frac{1-\phi}{\mathbb{M}(\phi)}\mathscr{G}'(\xi)\mathbf{f}_1(\xi, \mathbf{y}(\xi)) + \frac{\phi}{\mathbb{M}(\phi)}\int_0^{\xi}\mathscr{G}'(\tau)\mathbf{f}_1(\tau, \mathbf{y}(\tau))d\tau$$

$$+\frac{1-\phi}{\mathbb{M}(\phi)}\mathscr{G}'(\xi)\mathbf{f}_2(\xi,\mathbf{y}(\xi))B(\xi)+\frac{\phi}{\mathbb{M}(\phi)}\int_0^{\xi}\mathscr{G}'(\tau)\mathbf{f}_2(\tau,\mathbf{y}(\tau))B'(\tau)d\tau.$$

At $\xi_{\mathbf{n}+1}=(\mathbf{n}+1)\Delta\xi$, we have

$$\mathbf{y}(\xi_{\mathbf{n}+1})-\mathbf{y}(0)=\frac{1-\phi}{\mathbb{M}(\phi)}\mathscr{G}'(\xi_{\mathbf{n}+1})\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{y}(\xi_{\mathbf{n}+1}))+\frac{\phi}{\mathbb{M}(\phi)}\int_0^{\xi_{\mathbf{n}+1}}\mathscr{G}'(\tau)\mathbf{f}_1(\tau,\mathbf{y}(\tau))d\tau$$
$$+\frac{1-\phi}{\mathbb{M}(\phi)}\mathscr{G}'(\xi_{\mathbf{n}+1})\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{y}(\xi_{\mathbf{n}+1}))B(\xi_{\mathbf{n}+1})+\frac{\phi}{\mathbb{M}(\phi)}\int_0^{\xi_{\mathbf{n}+1}}\mathscr{G}'(\tau)\mathbf{f}_2(\tau,\mathbf{y}(\tau))B'(\tau)d\tau.$$

Further, at $\xi_{\mathbf{n}}=\mathbf{n}\Delta\xi$, we have

$$\mathbf{y}(\xi_{\mathbf{n}})-\mathbf{y}(0)=\frac{1-\phi}{\mathbb{M}(\phi)}\mathscr{G}'(\xi_{\mathbf{n}})\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}(\xi_{\mathbf{n}}))+\frac{\phi}{\mathbb{M}(\phi)}\int_0^{\xi_{\mathbf{n}}}\mathscr{G}'(\tau)\mathbf{f}_1(\tau,\mathbf{y}(\tau))$$
$$+\frac{1-\phi}{\mathbb{M}(\phi)}\mathscr{G}'(\xi_{\mathbf{n}})\mathbf{f}_2(\xi_{\mathbf{n}},\mathbf{y}(\xi_{\mathbf{n}}))B(\xi_{\mathbf{n}})+\frac{\phi}{\mathbb{M}(\phi)}\int_0^{\xi_{\mathbf{n}}}\mathscr{G}'(\tau)\mathbf{f}_2(\tau,\mathbf{y}(\tau))B'(\tau)d\tau.$$

For the sake of simplicity, we have

$$\mathscr{G}'(\tau)\mathbf{f}_1(\tau,\mathbf{y}(\tau))=\Upsilon_1(\tau,\mathbf{y}(\tau)),\qquad \mathscr{G}'(\tau)\mathbf{f}_2(\tau,\mathbf{y}(\tau))B'(\tau)=\Upsilon_2(\tau,\mathbf{y}(\tau)).$$

Furthermore, implementing the interpolation

$$\mathscr{Q}(\tau)=\frac{\tau-\xi_{\mathbf{n}-1}}{\xi_{\mathbf{n}}-\xi_{\mathbf{n}-1}}\Upsilon_1(\xi_{\mathbf{n}},\mathbf{x}^{\mathbf{n}})-\frac{\tau-\xi_{\mathbf{n}-1}}{\xi_{\mathbf{n}}-\xi_{\mathbf{n}-1}}\Upsilon_1(\xi_{\mathbf{n}-1},\mathbf{x}^{\mathbf{n}-1}),$$
$$\mathscr{Q}(\tau)=\frac{\tau-\xi_{\mathbf{n}-1}}{\xi_{\mathbf{n}}-\xi_{\mathbf{n}-1}}\Upsilon_2(\xi_{\mathbf{n}},\mathbf{x}^{\mathbf{n}})-\frac{\tau-\xi_{\mathbf{n}-1}}{\xi_{\mathbf{n}}-\xi_{\mathbf{n}-1}}\Upsilon_2(\xi_{\mathbf{n}-1},\mathbf{x}^{\mathbf{n}-1}).$$

Also, we have

$$\mathbf{y}(\xi_{\mathbf{n}+1})-\mathbf{y}(\xi_{\mathbf{n}})$$
$$=\frac{1-\phi}{\mathbb{M}(\phi)}\mathscr{G}'(\xi_{\mathbf{n}+1})\Big(\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{y}_{\mathbf{n}+1})-\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{y}_{\mathbf{n}+1})B(\xi_{\mathbf{n}+1})\Big)$$
$$+\frac{1-\phi}{\mathbb{M}(\phi)}\mathscr{G}'(\xi_{\mathbf{n}})\Big(\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}_{\mathbf{n}})-\mathbf{f}_2(\xi_{\mathbf{n}},\mathbf{y}_{\mathbf{n}})B(\xi_{\mathbf{n}})\Big)$$
$$+\frac{\phi}{\mathbb{M}(\phi)}\int_{\xi_{\mathbf{n}}}^{\xi_{\mathbf{n}+1}}\mathscr{G}'(\tau)\mathbf{f}_1(\tau,\mathbf{y}(\tau))d\tau-\frac{\phi}{\mathbb{M}(\phi)}\int_{\xi_{\mathbf{n}}}^{\xi_{\mathbf{n}+1}}\mathscr{G}'(\tau)\mathbf{f}_2(\tau,\mathbf{y}(\tau))B'(\tau)d\tau.$$

Now, the interpolation polynomials are

$$\mathbf{y}(\xi_{\mathbf{n}+1})-\mathbf{y}(\xi_{\mathbf{n}})$$

$$= \frac{1-\phi}{\mathbb{M}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n+1}})-\mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi}\Bigg(\mathbf{f}_1(\xi_{\mathbf{n+1}},\mathbf{y}_{\mathbf{n+1}})-\mathbf{f}_2(\xi_{\mathbf{n+1}},\mathbf{y}_{\mathbf{n+1}})B(\xi_{\mathbf{n+1}})\Bigg)$$
$$+ \frac{1-\phi}{\mathbb{M}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n}})-\mathscr{G}(\xi_{\mathbf{n-1}})}{\Delta\xi}\Bigg(\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}_{\mathbf{n}})-\mathbf{f}_2(\xi_{\mathbf{n}},\mathbf{y}_{\mathbf{n}})B(\xi_{\mathbf{n}})\Bigg)$$
$$+ \frac{\phi}{\mathbb{M}(\phi)}\int_{\xi_{\mathbf{n}}}^{\xi_{\mathbf{n}+1}}\Upsilon_1(\tau,\mathbf{y}(\tau))d\tau - \frac{\phi}{\mathbb{M}(\phi)}\int_{\xi_{\mathbf{n}}}^{\xi_{\mathbf{n}+1}}\Upsilon_2(\tau,\mathbf{y}(\tau))B'(\tau)d\tau.$$

It follows that

$$\mathbf{y}(\xi_{\mathbf{n+1}})-\mathbf{y}(\xi_{\mathbf{n}})$$
$$= \frac{1-\phi}{\mathbb{M}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n+1}})-\mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi}\Bigg(\mathbf{f}_1(\xi_{\mathbf{n+1}},\mathbf{y}_{\mathbf{n+1}})-\mathbf{f}_2(\xi_{\mathbf{n+1}},\mathbf{y}_{\mathbf{n+1}})B(\xi_{\mathbf{n+1}})\Bigg)$$
$$+ \frac{1-\phi}{\mathbb{M}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n}})-\mathscr{G}(\xi_{\mathbf{n-1}})}{\Delta\xi}\Bigg(\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}_{\mathbf{n}})-\mathbf{f}_2(\xi_{\mathbf{n}},\mathbf{y}_{\mathbf{n}})B(\xi_{\mathbf{n}})\Bigg)$$
$$+ \frac{\phi}{\mathbb{M}(\phi)}\Bigg(\frac{3}{2}\Upsilon_1(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}})\Delta\xi-\frac{1}{2}\Upsilon_1(\xi_{\mathbf{n-1}},\mathbf{y}^{\mathbf{n-1}})\Delta\xi\Bigg)-\frac{\phi}{\mathbb{M}(\phi)}\Bigg(\frac{3}{2}\Upsilon_2(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}})\Delta\xi-\frac{1}{2}\Upsilon_2(\xi_{\mathbf{n-1}},\mathbf{y}^{\mathbf{n-1}})\Delta\xi\Bigg),$$

where,

$$\Upsilon_1(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}}) = \mathscr{G}'(\xi_{\mathbf{n}})\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}}) = \frac{\mathscr{G}(\xi_{\mathbf{n}})-\mathscr{G}(\xi_{\mathbf{n-1}})}{\Delta\xi}\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}}),$$

$$\Upsilon_1(\xi_{\mathbf{n-1}},\mathbf{y}^{\mathbf{n-1}}) = \mathscr{G}'(\xi_{\mathbf{n}}-1)\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}}) = \frac{\mathscr{G}(\xi_{\mathbf{n}})-\mathscr{G}(\xi_{\mathbf{n-1}})}{\Delta\xi}\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}}),$$

$$\Upsilon_2(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}}) = \mathscr{G}'(\xi_{\mathbf{n}})\mathbf{f}_2(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}})B'(\xi_{\mathbf{n}}) = \frac{\mathscr{G}(\xi_{\mathbf{n}})-\mathscr{G}(\xi_{\mathbf{n-1}})}{\Delta\xi}\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}})\frac{B(\xi_{\mathbf{n}})-B(\xi_{\mathbf{n-1}})}{\Delta\xi},$$

$$\Upsilon_2(\xi_{\mathbf{n-1}},\mathbf{y}^{\mathbf{n-1}}) = \mathscr{G}'(\xi_{\mathbf{n-1}})\mathbf{f}_2(\xi_{\mathbf{n-1}},\mathbf{y}^{\mathbf{n-1}})B'(\xi_{\mathbf{n-1}})$$
$$= \frac{\mathscr{G}(\xi_{\mathbf{n-1}})-\mathscr{G}(\xi_{\mathbf{n-2}})}{\Delta\xi}\mathbf{f}_1(\xi_{\mathbf{n-1}},\mathbf{y}^{\mathbf{n-1}})\frac{B(\xi_{\mathbf{n-1}})-B(\xi_{\mathbf{n-2}})}{\Delta\xi}.$$

After rearranging all expressions, we have

$$\mathbf{y}(\xi_{\mathbf{n+1}})-\mathbf{y}(\xi_{\mathbf{n}})$$
$$= \frac{1-\phi}{\mathbb{M}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n+1}})-\mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi}\Bigg(\mathbf{f}_1(\xi_{\mathbf{n+1}},\mathbf{y}_{\mathbf{n+1}})-\mathbf{f}_2(\xi_{\mathbf{n+1}},\mathbf{y}_{\mathbf{n+1}})B(\xi_{\mathbf{n+1}})\Bigg)$$
$$+ \frac{1-\phi}{\mathbb{M}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n}})-\mathscr{G}(\xi_{\mathbf{n-1}})}{\Delta\xi}\Bigg(\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}_{\mathbf{n}})-\mathbf{f}_2(\xi_{\mathbf{n}},\mathbf{y}_{\mathbf{n}})B(\xi_{\mathbf{n}})\Bigg)$$
$$+ \frac{\phi}{\mathbb{M}(\phi)}\Bigg(\frac{3}{2}(\mathscr{G}(\xi_{\mathbf{n}})-\mathscr{G}(\xi_{\mathbf{n-1}}))\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}})-\frac{1}{2}(\mathscr{G}(\xi_{\mathbf{n}})-\mathscr{G}(\xi_{\mathbf{n-1}}))\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}})\Bigg)$$
$$- \frac{\phi}{\mathbb{M}(\phi)}\Bigg(\frac{3}{2}(\mathscr{G}(\xi_{\mathbf{n}})-\mathscr{G}(\xi_{\mathbf{n-1}}))\mathbf{f}_1(\xi_{\mathbf{n}},\mathbf{y}^{\mathbf{n}})(B(\xi_{\mathbf{n}})-B(\xi_{\mathbf{n-1}}))$$

$$- \frac{1}{2}(\mathscr{G}(\xi_{n-1}) - \mathscr{G}(\xi_{n-2}))\mathbf{f}_1(\xi_{n-1}, \mathbf{y}^{n-1})(B(\xi_{n-1}) - B(\xi_{n-2}))\Bigg),$$

Now we can employ (43) technique on measles epidemics stochastic model:

$$\begin{aligned}
&\mathbf{S}(\xi_{n+1}) - \mathbf{S}(\xi_n) \\
&= \frac{1-\phi}{\mathbb{M}(\phi)} \frac{\mathscr{G}(\xi_{n+1}) - \mathscr{G}(\xi_n)}{\Delta\xi}\Bigg(\mathbf{f}_1(\xi_{n+1}, \mathbf{S}_{n+1}) - \mathbf{f}_2(\xi_{n+1}, \mathbf{S}_{n+1})B(\xi_{n+1})\Bigg) \\
&\quad + \frac{1-\phi}{\mathbb{M}(\phi)} \frac{\mathscr{G}(\xi_n) - \mathscr{G}(\xi_{n-1})}{\Delta\xi}\Bigg(\mathbf{f}_1(\xi_n, \mathbf{S}_n) - \mathbf{f}_2(\xi_n, \mathbf{S}_n)B(\xi_n)\Bigg) \\
&\quad + \frac{\phi}{\mathbb{M}(\phi)}\Bigg(\frac{3}{2}(\mathscr{G}(\xi_n) - \mathscr{G}(\xi_{n-1}))\mathbf{f}_1(\xi_n, \mathbf{S}^n) - \frac{1}{2}(\mathscr{G}(\xi_n) - \mathscr{G}(\xi_{n-1}))\mathbf{f}_1(\xi_n, \mathbf{S}^n)\Bigg) \\
&\quad - \frac{\phi}{\mathbb{M}(\phi)}\Bigg(\frac{3}{2}(\mathscr{G}(\xi_n) - \mathscr{G}(\xi_{n-1}))\mathbf{f}_1(\xi_n, \mathbf{S}^n)(B(\xi_n) - B(\xi_{n-1})) \\
&\quad - \frac{1}{2}(\mathscr{G}(\xi_{n-1}) - \mathscr{G}(\xi_{n-2}))\mathbf{f}_1(\xi_{n-1}, \mathbf{S}^{n-1})(B(\xi_{n-1}) - B(\xi_{n-2}))\Bigg),
\end{aligned} \tag{44}$$

$$\begin{aligned}
&\mathbf{V}_1(\xi_{n+1}) - \mathbf{V}_1(\xi_n) \\
&= \frac{1-\phi}{\mathbb{M}(\phi)} \frac{\mathscr{G}(\xi_{n+1}) - \mathscr{G}(\xi_n)}{\Delta\xi}\Bigg(\mathbf{f}_1(\xi_{n+1}, \mathbf{V}_{1n+1}) - \mathbf{f}_2(\xi_{n+1}, \mathbf{V}_{1n+1})B(\xi_{n+1})\Bigg) \\
&\quad + \frac{1-\phi}{\mathbb{M}(\phi)} \frac{\mathscr{G}(\xi_n) - \mathscr{G}(\xi_{n-1})}{\Delta\xi}\Bigg(\mathbf{f}_1(\xi_n, \mathbf{V}_{1n}) - \mathbf{f}_2(\xi_n, \mathbf{V}_{1n})B(\xi_n)\Bigg) \\
&\quad + \frac{\phi}{\mathbb{M}(\phi)}\Bigg(\frac{3}{2}(\mathscr{G}(\xi_n) - \mathscr{G}(\xi_{n-1}))\mathbf{f}_1(\xi_n, \mathbf{V}_1^n) - \frac{1}{2}(\mathscr{G}(\xi_n) - \mathscr{G}(\xi_{n-1}))\mathbf{f}_1(\xi_n, \mathbf{V}_1^n)\Bigg) \\
&\quad - \frac{\phi}{\mathbb{M}(\phi)}\Bigg(\frac{3}{2}(\mathscr{G}(\xi_n) - \mathscr{G}(\xi_{n-1}))\mathbf{f}_1(\xi_n, \mathbf{V}_1^n)(B(\xi_n) - B(\xi_{n-1})) \\
&\quad - \frac{1}{2}(\mathscr{G}(\xi_{n-1}) - \mathscr{G}(\xi_{n-2}))\mathbf{f}_1(\xi_{n-1}, \mathbf{V}_1^{n-1})(B(\xi_{n-1}) - B(\xi_{n-2}))\Bigg),
\end{aligned} \tag{45}$$

$$\begin{aligned}
&\mathbf{V}_2(\xi_{n+1}) - \mathbf{V}_2(\xi_n) \\
&= \frac{1-\phi}{\mathbb{M}(\phi)} \frac{\mathscr{G}(\xi_{n+1}) - \mathscr{G}(\xi_n)}{\Delta\xi}\Bigg(\mathbf{f}_1(\xi_{n+1}, \mathbf{V}_{2n+1}) - \mathbf{f}_2(\xi_{n+1}, \mathbf{V}_{2n+1})B(\xi_{n+1})\Bigg) \\
&\quad + \frac{1-\phi}{\mathbb{M}(\phi)} \frac{\mathscr{G}(\xi_n) - \mathscr{G}(\xi_{n-1})}{\Delta\xi}\Bigg(\mathbf{f}_1(\xi_n, \mathbf{V}_{2n}) - \mathbf{f}_2(\xi_n, \mathbf{V}_{2n})B(\xi_n)\Bigg) \\
&\quad + \frac{\phi}{\mathbb{M}(\phi)}\Bigg(\frac{3}{2}(\mathscr{G}(\xi_n) - \mathscr{G}(\xi_{n-1}))\mathbf{f}_1(\xi_n, \mathbf{V}_2^n) - \frac{1}{2}(\mathscr{G}(\xi_n) - \mathscr{G}(\xi_{n-1}))\mathbf{f}_1(\xi_n, \mathbf{V}_2^n)\Bigg) \\
&\quad - \frac{\phi}{\mathbb{M}(\phi)}\Bigg(\frac{3}{2}(\mathscr{G}(\xi_n) - \mathscr{G}(\xi_{n-1}))\mathbf{f}_1(\xi_n, \mathbf{V}_2^n)(B(\xi_n) - B(\xi_{n-1}))
\end{aligned}$$

$$- \frac{1}{2}(\mathscr{G}(\xi_{\mathbf{n-1}}) - \mathscr{G}(\xi_{\mathbf{n-2}}))\mathbf{f}_1(\xi_{\mathbf{n-1}}, \mathbf{V}_2^{\mathbf{n-1}})(B(\xi_{\mathbf{n-1}}) - B(\xi_{\mathbf{n-2}}))\Bigg), \tag{46}$$

$$\begin{aligned}
&\mathbf{I}(\xi_{\mathbf{n+1}}) - \mathbf{I}(\xi_{\mathbf{n}}) \\
&= \frac{1-\phi}{\mathbb{M}(\phi)} \frac{\mathscr{G}(\xi_{\mathbf{n+1}}) - \mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi} \Bigg(\mathbf{f}_1(\xi_{\mathbf{n+1}}, \mathbf{I}_{\mathbf{n+1}}) - \mathbf{f}_2(\xi_{\mathbf{n+1}}, \mathbf{I}_{\mathbf{n+1}})B(\xi_{\mathbf{n+1}})\Bigg) \\
&\quad + \frac{1-\phi}{\mathbb{M}(\phi)} \frac{\mathscr{G}(\xi_{\mathbf{n}}) - \mathscr{G}(\xi_{\mathbf{n-1}})}{\Delta\xi} \Bigg(\mathbf{f}_1(\xi_{\mathbf{n}}, \mathbf{I}_{\mathbf{n}}) - \mathbf{f}_2(\xi_{\mathbf{n}}, \mathbf{I}_{\mathbf{n}})B(\xi_{\mathbf{n}})\Bigg) \\
&\quad + \frac{\phi}{\mathbb{M}(\phi)} \Bigg(\frac{3}{2}(\mathscr{G}(\xi_{\mathbf{n}}) - \mathscr{G}(\xi_{\mathbf{n-1}}))\mathbf{f}_1(\xi_{\mathbf{n}}, \mathbf{I}^{\mathbf{n}}) - \frac{1}{2}(\mathscr{G}(\xi_{\mathbf{n}}) - \mathscr{G}(\xi_{\mathbf{n-1}}))\mathbf{f}_1(\xi_{\mathbf{n}}, \mathbf{I}^{\mathbf{n}})\Bigg) \\
&\quad - \frac{\phi}{\mathbb{M}(\phi)} \Bigg(\frac{3}{2}(\mathscr{G}(\xi_{\mathbf{n}}) - \mathscr{G}(\xi_{\mathbf{n-1}}))\mathbf{f}_1(\xi_{\mathbf{n}}, \mathbf{I}^{\mathbf{n}})(B(\xi_{\mathbf{n}}) - B(\xi_{\mathbf{n-1}})) \\
&\quad - \frac{1}{2}(\mathscr{G}(\xi_{\mathbf{n-1}}) - \mathscr{G}(\xi_{\mathbf{n-2}}))\mathbf{f}_1(\xi_{\mathbf{n-1}}, \mathbf{I}^{\mathbf{n-1}})(B(\xi_{\mathbf{n-1}}) - B(\xi_{\mathbf{n-2}}))\Bigg),
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
&\mathbf{R}(\xi_{\mathbf{n+1}}) - \mathbf{R}(\xi_{\mathbf{n}}) \\
&= \frac{1-\phi}{\mathbb{M}(\phi)} \frac{\mathscr{G}(\xi_{\mathbf{n+1}}) - \mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi} \Bigg(\mathbf{f}_1(\xi_{\mathbf{n+1}}, \mathbf{R}_{\mathbf{n+1}}) - \mathbf{f}_2(\xi_{\mathbf{n+1}}, \mathbf{R}_{\mathbf{n+1}})B(\xi_{\mathbf{n+1}})\Bigg) \\
&\quad + \frac{1-\phi}{\mathbb{M}(\phi)} \frac{\mathscr{G}(\xi_{\mathbf{n}}) - \mathscr{G}(\xi_{\mathbf{n-1}})}{\Delta\xi} \Bigg(\mathbf{f}_1(\xi_{\mathbf{n}}, \mathbf{R}_{\mathbf{n}}) - \mathbf{f}_2(\xi_{\mathbf{n}}, \mathbf{R}_{\mathbf{n}})B(\xi_{\mathbf{n}})\Bigg) \\
&\quad + \frac{\phi}{\mathbb{M}(\phi)} \Bigg(\frac{3}{2}(\mathscr{G}(\xi_{\mathbf{n}}) - \mathscr{G}(\xi_{\mathbf{n-1}}))\mathbf{f}_1(\xi_{\mathbf{n}}, \mathbf{R}^{\mathbf{n}}) - \frac{1}{2}(\mathscr{G}(\xi_{\mathbf{n}}) - \mathscr{G}(\xi_{\mathbf{n-1}}))\mathbf{f}_1(\xi_{\mathbf{n}}, \mathbf{R}^{\mathbf{n}})\Bigg) \\
&\quad - \frac{\phi}{\mathbb{M}(\phi)} \Bigg(\frac{3}{2}(\mathscr{G}(\xi_{\mathbf{n}}) - \mathscr{G}(\xi_{\mathbf{n-1}}))\mathbf{f}_1(\xi_{\mathbf{n}}, \mathbf{R}^{\mathbf{n}})(B(\xi_{\mathbf{n}}) - B(\xi_{\mathbf{n-1}})) \\
&\quad - \frac{1}{2}(\mathscr{G}(\xi_{\mathbf{n-1}}) - \mathscr{G}(\xi_{\mathbf{n-2}}))\mathbf{f}_1(\xi_{\mathbf{n-1}}, \mathbf{R}^{\mathbf{n-1}})(B(\xi_{\mathbf{n-1}}) - B(\xi_{\mathbf{n-2}}))\Bigg).
\end{aligned} \tag{48}$$

5.2 Atangana-Baleanu Fractional Derivative Operator

Here, we illustrate the numerical scheme for the Atangana-Baleanu derivative operator:

$$\begin{cases} {}^{ABC}\mathbf{D}_{\mathscr{G}}^{\phi}\mathbf{y}(\xi) = \mathbf{f}_1(\xi, \mathbf{y}(\xi)) + \mathbf{f}_2(\xi, \mathbf{y}(\xi)), \\ \mathbf{y}(t_0) = \mathbf{y}_0. \end{cases}$$

Since $\mathscr{G}$ is differentiable, we have

$${}^{ABC}\mathbf{D}_{\mathscr{G}}^{\phi}\mathbf{y}(\xi) = \mathscr{G}'(\xi)\mathbf{f}_1(\xi, \mathbf{y}(\xi)) + \mathscr{G}'(\xi)\mathbf{f}_2(\xi, \mathbf{y}(\xi)).$$

By making the use of Atangana-Baleanu integral, we apply

$$\mathbf{y}(\xi) - \mathbf{y}(0) = \frac{1-\phi}{\mathbb{B}(\phi)}\mathscr{G}'(\xi)\mathbf{f}_1(\xi, \mathbf{y}(\xi)) + \frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)} \int_0^{\xi} \mathscr{G}'(\tau)\mathbf{f}_1(\tau, \mathbf{y}(\tau))(\xi - \tau)^{\phi-1} d\tau$$

$$+\frac{1-\phi}{\mathbb{B}(\phi)}\mathscr{G}'(\xi)\mathbf{f}_2(\xi,\mathbf{y}(\xi))B\xi+\frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)}\int_0^{\xi}\mathscr{G}'(\tau)\mathbf{f}_2(\tau,\mathbf{y}(\tau))(\xi-\tau)^{\phi-1}dB(\tau).$$

Since $B(\xi)$ is differentiable, we can write

$$\mathbf{y}(\xi)-\mathbf{y}(0)=\frac{1-\phi}{\mathbb{B}(\phi)}\mathscr{G}'(\xi)\mathbf{f}_1(\xi,\mathbf{y}(\xi))+\frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)}\int_0^{\xi}\mathscr{G}'(\tau)\mathbf{f}_1(\tau,\mathbf{y}(\tau))(\xi-\tau)^{\phi-1}d\tau$$

$$+\frac{1-\phi}{\mathbb{B}(\phi)}\mathscr{G}'(\xi)\mathbf{f}_2(\xi,\mathbf{y}(\xi))B\xi+\frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)}\int_0^{\xi}\mathscr{G}'(\tau)\mathbf{f}_2(\tau,\mathbf{y}(\tau))(\xi-\tau)^{\phi-1}B'(\tau)d\tau.$$

At $\xi_{\mathbf{n}+1}=(\mathbf{n}+1)\Delta\xi$, we have

$$\mathbf{y}(\xi_{\mathbf{n}+1})-\mathbf{y}(0)=\frac{1-\phi}{\mathbb{B}(\phi)}\mathscr{G}'(\xi_{\mathbf{n}+1})\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{y}(\xi_{\mathbf{n}+1}))+\frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)}\int_0^{\xi_{\mathbf{n}+1}}\mathscr{G}'(\tau)\mathbf{f}_1(\tau,\mathbf{y}(\tau))(\xi_{\mathbf{n}+1}-\tau)^{\phi-1}d\tau$$

$$+\frac{1-\phi}{\mathbb{B}(\phi)}\mathscr{G}'(\xi_{\mathbf{n}+1})\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{y}(\xi_{\mathbf{n}+1}))B(\xi_{\mathbf{n}+1})$$

$$+\frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)}\int_0^{\xi_{\mathbf{n}+1}}\mathscr{G}'(\tau)\mathbf{f}_2(\tau,\mathbf{y}(\tau))(\xi_{\mathbf{n}+1}-\tau)^{\phi-1}B'(\tau)d\tau.$$

For the sake of simplicity, we have

$$\mathscr{G}'(\tau)\mathbf{f}_1(\tau,\mathbf{y}(\tau))=\Upsilon_1(\tau,\mathbf{y}(\tau)),$$
$$\mathscr{G}'(\tau)\mathbf{f}_2(\tau,\mathbf{y}(\tau))B'(\tau)=\Upsilon_2(\tau,\mathbf{y}(\tau)).$$

Furthermore, implementing the interpolation

$$\mathscr{Q}(\tau)=\frac{\tau-\xi_{\mathbf{n}-1}}{\xi_{\mathbf{n}}-\xi_{\mathbf{n}-1}}\Upsilon_1(\xi_{\mathbf{n}},\mathbf{x}^{\mathbf{n}})-\frac{\tau-\xi_{\mathbf{n}-1}}{\xi_{\mathbf{n}}-\xi_{\mathbf{n}-1}}\Upsilon_1(\xi_{\mathbf{n}-1},\mathbf{x}^{\mathbf{n}-1}),$$
$$\mathscr{Q}(\tau)=\frac{\tau-\xi_{\mathbf{n}-1}}{\xi_{\mathbf{n}}-\xi_{\mathbf{n}-1}}\Upsilon_2(\xi_{\mathbf{n}},\mathbf{x}^{\mathbf{n}})-\frac{\tau-\xi_{\mathbf{n}-1}}{\xi_{\mathbf{n}}-\xi_{\mathbf{n}-1}}\Upsilon_2(\xi_{\mathbf{n}-1},\mathbf{x}^{\mathbf{n}-1}).$$

Also, we have

$$\mathbf{y}(\xi_{\mathbf{n}+1})-\mathbf{y}(0)$$
$$=\frac{1-\phi}{\mathbb{B}(\phi)}\mathscr{G}'(\xi_{\mathbf{n}+1})\Big(\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{y}_{\mathbf{n}+1})+\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{y}_{\mathbf{n}+1})B(\xi_{\mathbf{n}+1})\Big)$$
$$+\frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)}\int_0^{\xi_{\mathbf{n}+1}}\mathscr{G}'(\tau)\mathbf{f}_1(\tau,\mathbf{y}(\tau))(\xi_{\mathbf{n}+1}-\tau)^{\phi-1}d\tau$$

$$+\frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)}\int_{0}^{\xi_{\mathbf{n}+1}}\mathscr{G}'(\tau)\mathbf{f}_2(\tau,\mathbf{y}(\tau))(\xi_{\mathbf{n}+1}-\tau)^{\phi-1}B'(\tau)d\tau.$$

Now, the interpolation polynomials are

$$\begin{aligned}\mathbf{y}(\xi_{\mathbf{n}+1})-\mathbf{y}(0)&\\ =\frac{1-\phi}{\mathbb{B}(\phi)}&\frac{\mathscr{G}(\xi_{\mathbf{n}+1})-\mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi}\Big(\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{y}_{\mathbf{n}+1})+\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{y}_{\mathbf{n}+1})B(\xi_{\mathbf{n}+1})\Big)\\ &+\frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)}\int_{0}^{\xi_{\mathbf{n}+1}}\Upsilon_1(\tau,\mathbf{y}(\tau))(\xi_{\mathbf{n}+1}-\tau)^{\phi-1}d\tau\\ &+\frac{\phi}{\mathbb{B}(\phi)\Gamma(\phi)}\int_{0}^{\xi_{\mathbf{n}+1}}\Upsilon_2(\tau,\mathbf{y}(\tau))(\xi_{\mathbf{n}+1}-\tau)^{\phi-1}B'(\tau)d\tau.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\mathbf{y}(\xi_{\mathbf{n}+1})-\mathbf{y}(0)\\ &=\frac{1-\phi}{\mathbb{B}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n}+1})-\mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi}\Big(\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{y}_{\mathbf{n}+1})+\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{y}_{\mathbf{n}+1})B(\xi_{\mathbf{n}+1})\Big)\\ &\quad+\frac{\phi}{\mathbb{B}(\phi)\Gamma(\delta)}\sum_{\iota=2}^{\mathbf{n}}\int_{\xi_{\mathbf{n}}}^{\xi_{\mathbf{n}+1}}\Upsilon_1(\tau,\mathbf{y}(\tau))(\xi_{\mathbf{n}+1}-\tau)^{\phi-1}d\tau\\ &\quad+\frac{\phi}{\mathbb{B}(\phi)\Gamma(\delta)}\sum_{\iota=2}^{\mathbf{n}}\int_{\xi_{\mathbf{n}}}^{\xi_{\mathbf{n}+1}}\Upsilon_2(\tau,\mathbf{y}(\tau))(\xi_{\mathbf{n}+1}-\tau)^{\phi-1}d\tau.\end{aligned}$$

In view of the Lagrange interpolation polynomial technique, we have

$$\begin{aligned}\mathbf{y}^{\mathbf{n}+1}=\mathbf{y}^0&+\frac{1-\phi}{\mathbb{B}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n}+1})-\mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi}\Big(\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{y}_{\mathbf{n}+1})+\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{y}_{\mathbf{n}+1})B(\xi_{\mathbf{n}+1})\Big)\\ &+\frac{\phi(\Delta\xi)^{\phi}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\Upsilon_1(\xi_\iota,\mathbf{y}^\iota)\begin{cases}(\mathbf{n}-\iota+1)^{\phi}(\mathbf{n}-\iota+2+\phi)\\ \quad-(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+2+2\phi)\end{cases}\\ &+\frac{\phi(\Delta\xi)^{\phi}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\Upsilon_1(\xi_{\iota-1},\mathbf{y}^{\iota-1})\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ \quad-(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+1+\phi)\end{cases}\\ &+\frac{\phi(\Delta\xi)^{\phi}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\Upsilon_2(\xi_\iota,\mathbf{y}^\iota)\begin{cases}(\mathbf{n}-\iota+1)^{\phi}(\mathbf{n}-\iota+2+\phi)\\ \quad-(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+2+2\phi)\end{cases}\\ &+\frac{\phi(\Delta\xi)^{\phi}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\Upsilon_2(\xi_{\iota-1},\mathbf{y}^{\iota-1})\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ \quad-(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+1+\phi)\end{cases}\quad.\end{aligned}$$

After plugging the values of $\Upsilon_1(\xi, \mathbf{y}(\xi))$ and $\Upsilon_2(\xi, \mathbf{y}(\xi))$, then we have

$$\begin{aligned}
\mathbf{y}^{\mathbf{n}+1} = \mathbf{y}^0 &+ \frac{1-\phi}{\mathbb{B}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n}+1})-\mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi}\Big(\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{y}_{\mathbf{n}+1})+\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{y}_{\mathbf{n}+1})B(\xi_{\mathbf{n}+1})\Big)\\
&+\frac{\delta(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_1(\xi_\iota,\mathbf{y}^\iota)(\mathscr{G}(\xi_{\iota+1})-\mathscr{G}(\xi_\iota))\begin{cases}(\mathbf{n}-\iota+1)^\phi(\mathbf{n}-\iota+2+\phi)\\ -(\mathbf{n}-\iota)^\phi(\mathbf{n}-\iota+2+2\phi)\end{cases}\\
&+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_1(\xi_{\iota-1},\mathbf{y}^{\iota-1})(\mathscr{G}(\xi_\iota)-\mathscr{G}(\xi_{\iota-1}))\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ -(\mathbf{n}-\iota)^\phi(\mathbf{n}-\iota+1+\phi)\end{cases}\\
&+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_2(\xi_\iota,\mathbf{y}^\iota)(\mathscr{G}(\xi_{\iota+1})-\mathscr{G}(\xi_\iota))\big(B(\xi_{\iota+1})-B(\xi_\iota)\big)\begin{cases}(\mathbf{n}-\iota+1)^\phi(\mathbf{n}-\iota+2+\phi)\\ -(\mathbf{n}-\iota)^\phi(\mathbf{n}-\iota+2+2\phi)\end{cases}\\
&+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_2(\xi_{\iota-1},\mathbf{y}^{\iota-1})(\mathscr{G}(\xi_\iota)-\mathscr{G}(\xi_{\iota-1}))\big(B(\xi_\iota)-B(\xi_{\iota-1})\big)\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ -(\mathbf{n}-\iota)^\phi(\mathbf{n}-\iota+1+\phi)\end{cases}.
\end{aligned}$$

Now we can employ (43) technique on measles epidemics stochastic model:

$$\begin{aligned}
\mathbf{S}^{\mathbf{n}+1} = \mathbf{S}^0 &+ \frac{1-\phi}{\mathbb{B}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n}+1})-\mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi}\Big(\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{S}_{\mathbf{n}+1})+\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{S}_{\mathbf{n}+1})B(\xi_{\mathbf{n}+1})\Big)\\
&+\frac{\delta(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_1(\xi_\iota,\mathbf{S}^\iota)(\mathscr{G}(\xi_{\iota+1})-\mathscr{G}(\xi_\iota))\begin{cases}(\mathbf{n}-\iota+1)^\phi(\mathbf{n}-\iota+2+\phi)\\ -(\mathbf{n}-\iota)^\phi(\mathbf{n}-\iota+2+2\phi)\end{cases}\\
&+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_1(\xi_{\iota-1},\mathbf{S}^{\iota-1})(\mathscr{G}(\xi_\iota)-\mathscr{G}(\xi_{\iota-1}))\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ -(\mathbf{n}-\iota)^\phi(\mathbf{n}-\iota+1+\phi)\end{cases}\\
&+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_2(\xi_\iota,\mathbf{S}^\iota)(\mathscr{G}(\xi_{\iota+1})-\mathscr{G}(\xi_\iota))\big(B(\xi_{\iota+1})-B(\xi_\iota)\big)\begin{cases}(\mathbf{n}-\iota+1)^\phi(\mathbf{n}-\iota+2+\phi)\\ -(\mathbf{n}-\iota)^\phi(\mathbf{n}-\iota+2+2\phi)\end{cases}\\
&+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_2(\xi_{\iota-1},\mathbf{S}^{\iota-1})(\mathscr{G}(\xi_\iota)-\mathscr{G}(\xi_{\iota-1}))\big(B(\xi_\iota)-B(\xi_{\iota-1})\big)\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ -(\mathbf{n}-\iota)^\phi(\mathbf{n}-\iota+1+\phi)\end{cases},\\
\mathbf{V}_1^{\mathbf{n}+1} = \mathbf{V}_1^0 &+ \frac{1-\phi}{\mathbb{B}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n}+1})-\mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi}\Big(\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{V}_{1\mathbf{n}+1})+\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{V}_{1\mathbf{n}+1})B(\xi_{\mathbf{n}+1})\Big)\\
&+\frac{\delta(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_1(\xi_\iota,\mathbf{V}_1^\iota)(\mathscr{G}(\xi_{\iota+1})-\mathscr{G}(\xi_\iota))\begin{cases}(\mathbf{n}-\iota+1)^\phi(\mathbf{n}-\iota+2+\phi)\\ -(\mathbf{n}-\iota)^\phi(\mathbf{n}-\iota+2+2\phi)\end{cases}\\
&+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_1(\xi_{\iota-1},\mathbf{V}_1^{\iota-1})(\mathscr{G}(\xi_\iota)-\mathscr{G}(\xi_{\iota-1}))\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ -(\mathbf{n}-\iota)^\phi(\mathbf{n}-\iota+1+\phi)\end{cases}\\
&+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_2(\xi_\iota,\mathbf{V}_1^\iota)(\mathscr{G}(\xi_{\iota+1})-\mathscr{G}(\xi_\iota))\big(B(\xi_{\iota+1})-B(\xi_\iota)\big)\begin{cases}(\mathbf{n}-\iota+1)^\phi(\mathbf{n}-\iota+2+\phi)\\ -(\mathbf{n}-\iota)^\phi(\mathbf{n}-\iota+2+2\phi)\end{cases}
\end{aligned}$$

$$+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_2(\xi_{\iota-1},\mathbf{V}_1^{\iota-1})(\mathscr{G}(\xi_\iota)-\mathscr{G}(\xi_{\iota-1}))\big(B(\xi_\iota)-B(\xi_{\iota-1})\big)\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+1+\phi)\end{cases},$$

$$\mathbf{V}_2^{\mathbf{n}+1}=\mathbf{V}_2^0+\frac{1-\phi}{\mathbb{B}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n}+1})-\mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi}\bigg(\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{V}_{2\mathbf{n}+1})+\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{V}_{2\mathbf{n}+1})B(\xi_{\mathbf{n}+1})\bigg)$$
$$+\frac{\delta(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_1(\xi_\iota,\mathbf{V}_2^{\iota})(\mathscr{G}(\xi_{\iota+1})-\mathscr{G}(\xi_\iota))\begin{cases}(\mathbf{n}-\iota+1)^{\phi}(\mathbf{n}-\iota+2+\phi)\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+2+2\phi)\end{cases}$$
$$+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_1(\xi_{\iota-1},\mathbf{V}_2^{\iota-1})(\mathscr{G}(\xi_\iota)-\mathscr{G}(\xi_{\iota-1}))\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+1+\phi)\end{cases}$$
$$+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_2(\xi_\iota,\mathbf{V}_2^{\iota})(\mathscr{G}(\xi_{\iota+1})-\mathscr{G}(\xi_\iota))\big(B(\xi_{\iota+1})-B(\xi_\iota)\big)\begin{cases}(\mathbf{n}-\iota+1)^{\phi}(\mathbf{n}-\iota+2+\phi)\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+2+2\phi)\end{cases}$$
$$+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_2(\xi_{\iota-1},\mathbf{V}_2^{\iota-1})(\mathscr{G}(\xi_\iota)-\mathscr{G}(\xi_{\iota-1}))\big(B(\xi_\iota)-B(\xi_{\iota-1})\big)\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+1+\phi)\end{cases},$$

$$\mathbf{I}^{\mathbf{n}+1}=\mathbf{I}^0+\frac{1-\phi}{\mathbb{B}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n}+1})-\mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi}\bigg(\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{I}_{\mathbf{n}+1})+\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{I}_{\mathbf{n}+1})B(\xi_{\mathbf{n}+1})\bigg)$$
$$+\frac{\delta(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_1(\xi_\iota,\mathbf{I}^{\iota})(\mathscr{G}(\xi_{\iota+1})-\mathscr{G}(\xi_\iota))\begin{cases}(\mathbf{n}-\iota+1)^{\phi}(\mathbf{n}-\iota+2+\phi)\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+2+2\phi)\end{cases}$$
$$+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_1(\xi_{\iota-1},\mathbf{I}^{\iota-1})(\mathscr{G}(\xi_\iota)-\mathscr{G}(\xi_{\iota-1}))\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+1+\phi)\end{cases}$$
$$+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_2(\xi_\iota,\mathbf{I}^{\iota})(\mathscr{G}(\xi_{\iota+1})-\mathscr{G}(\xi_\iota))\big(B(\xi_{\iota+1})-B(\xi_\iota)\big)\begin{cases}(\mathbf{n}-\iota+1)^{\phi}(\mathbf{n}-\iota+2+\phi)\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+2+2\phi)\end{cases}$$
$$+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_2(\xi_{\iota-1},\mathbf{I}^{\iota-1})(\mathscr{G}(\xi_\iota)-\mathscr{G}(\xi_{\iota-1}))\big(B(\xi_\iota)-B(\xi_{\iota-1})\big)\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+1+\phi)\end{cases}$$

and

$$\mathbf{R}^{\mathbf{n}+1}=\mathbf{R}^0+\frac{1-\phi}{\mathbb{B}(\phi)}\frac{\mathscr{G}(\xi_{\mathbf{n}+1})-\mathscr{G}(\xi_{\mathbf{n}})}{\Delta\xi}\bigg(\mathbf{f}_1(\xi_{\mathbf{n}+1},\mathbf{R}_{\mathbf{n}+1})+\mathbf{f}_2(\xi_{\mathbf{n}+1},\mathbf{R}_{\mathbf{n}+1})B(\xi_{\mathbf{n}+1})\bigg)$$
$$+\frac{\delta(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_1(\xi_\iota,\mathbf{R}^{\iota})(\mathscr{G}(\xi_{\iota+1})-\mathscr{G}(\xi_\iota))\begin{cases}(\mathbf{n}-\iota+1)^{\phi}(\mathbf{n}-\iota+2+\phi)\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+2+2\phi)\end{cases}$$
$$+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_1(\xi_{\iota-1},\mathbf{R}^{\iota-1})(\mathscr{G}(\xi_\iota)-\mathscr{G}(\xi_{\iota-1}))\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+1+\phi)\end{cases}$$
$$+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_2(\xi_\iota,\mathbf{R}^{\iota})(\mathscr{G}(\xi_{\iota+1})-\mathscr{G}(\xi_\iota))\big(B(\xi_{\iota+1})-B(\xi_\iota)\big)\begin{cases}(\mathbf{n}-\iota+1)^{\phi}(\mathbf{n}-\iota+2+\phi)\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+2+2\phi)\end{cases}$$

$$+\frac{\phi(\Delta\xi)^{\phi-1}}{\mathbb{B}(\phi)\Gamma(\phi+2)}\sum_{\iota=0}^{\mathbf{n}}\mathbf{f}_2(\xi_{\iota-1},\mathbf{R}^{\iota-1})(\mathscr{G}(\xi_\iota)-\mathscr{G}(\xi_{\iota-1}))\big(B(\xi_\iota)-B(\xi_{\iota-1})\big)\begin{cases}(\mathbf{n}-\iota+1)^{\phi+1}\\ -(\mathbf{n}-\iota)^{\phi}(\mathbf{n}-\iota+1+\phi)\end{cases}.$$

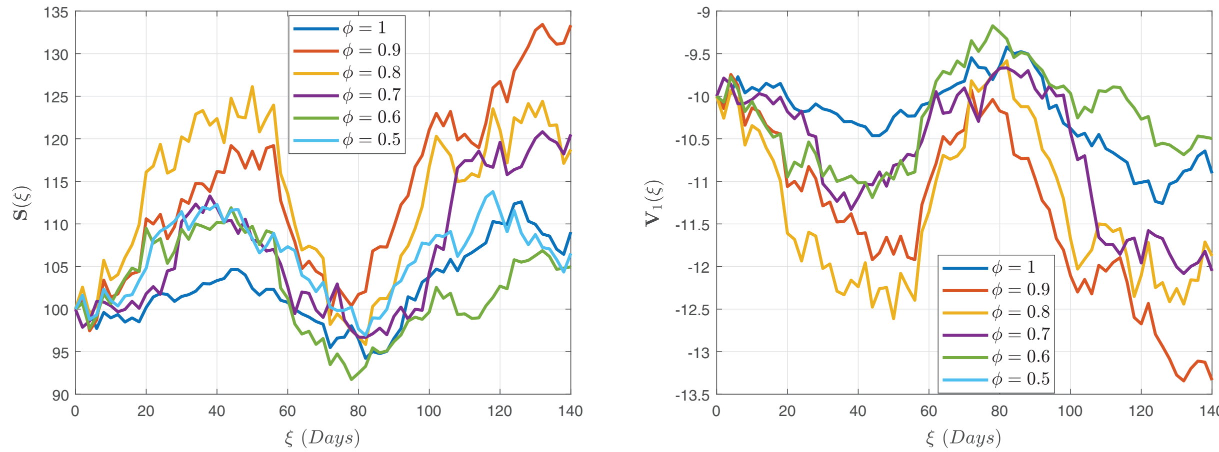
6 Results and Discussion

In what follows, we describe and investigate the system's input variables, as well as simulation studies. We applied input variables from Table 1 further to simulate the constructed framework. The fractional stochastic mathematical formulation is expressed in the context of global derivative operators in terms of Caputo differential operators of visualizations developed with MATLAB via multiple fractional orders. This investigation is being carried out to determine the mechanisms of measles spreading in the community. Taking into account the technique of Toufik et al. [36], is utilized to determine the numerical configurations.

Table 1: Explanation of attributed values supposed in the system

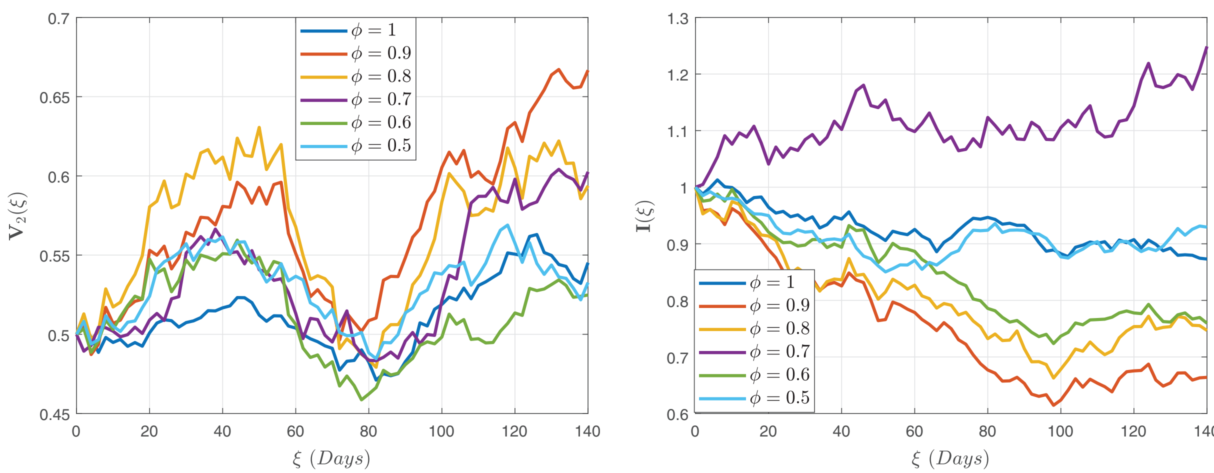
Symbols	Value	References
δ	0.14286	[3]
η	0.125	[3]
χ_3	0.7	[25]
χ_2	0.8	[25]
χ_1	0.6	[25]
φ	0.00875	[3]
ϑ	0.167	[37]
Ω	0.03755	[3]
Ξ	0.02755	[3]
ω	0.09091	[3]

Figs. 4–6 demonstrate the stochastic behavior for different fractional orders in terms of global derivative in the Caputo-Fabrizio sense, demonstrating that the number of vulnerable, contaminated individuals diminishes from the beginning stage until a specific period ξ, at which point it begins to decline to zero. In this case, the proposed ICs are $(\mathbf{S},\mathbf{V}_1,\mathbf{V}_2,\mathbf{I},\mathbf{R}) = (1000, 500, 50, 15, 3)$. Figs. 4–6 show the unpredictability densities at $\rho_1 = 0.001$, $\rho_2 = 0.003$, $\rho_3 = 0.004$, $\rho_4 = 0.006$, and $\rho_5 = 0.006$, respectively. This indicates that vaccination of both the first and second doses (dual treatment) aims to manage the measles infection and, throughout its duration, to eliminate the infection from the community while maintaining the healed community growth. Finally, we can perceive that the fractional stochastic model equations' solutions are straightforward in computation, possess randomness in behavior, and are more productive in nature.



(a) (b)

Figure 4: Stochastic behavior of the susceptible class $\mathbf{S}(\xi)$ and vaccinated after dose $\mathbf{V}_1(\xi)$ for various fractional-orders by the use of global derivative in the Caputo-Fabrizio sense



(a) (b)

Figure 5: Stochastic behaviour of the dual dose vaccination group $\mathbf{V}_2(\xi)$ and infectious class $\mathbf{V}_1$ for various fractional-orders by the use of global derivative in the Caputo-Fabrizio sense

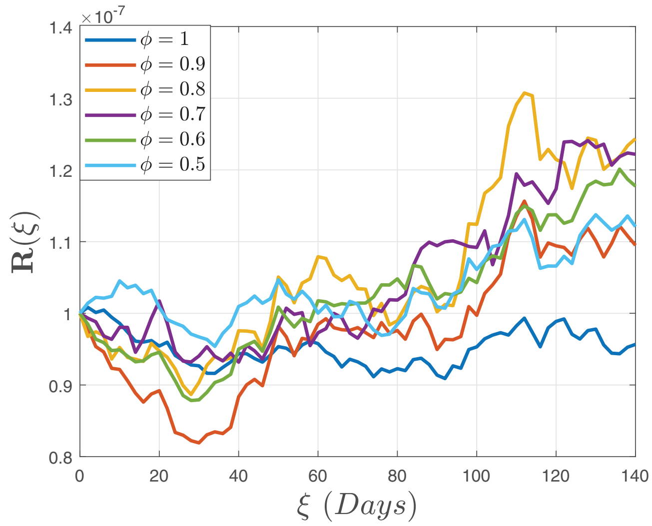
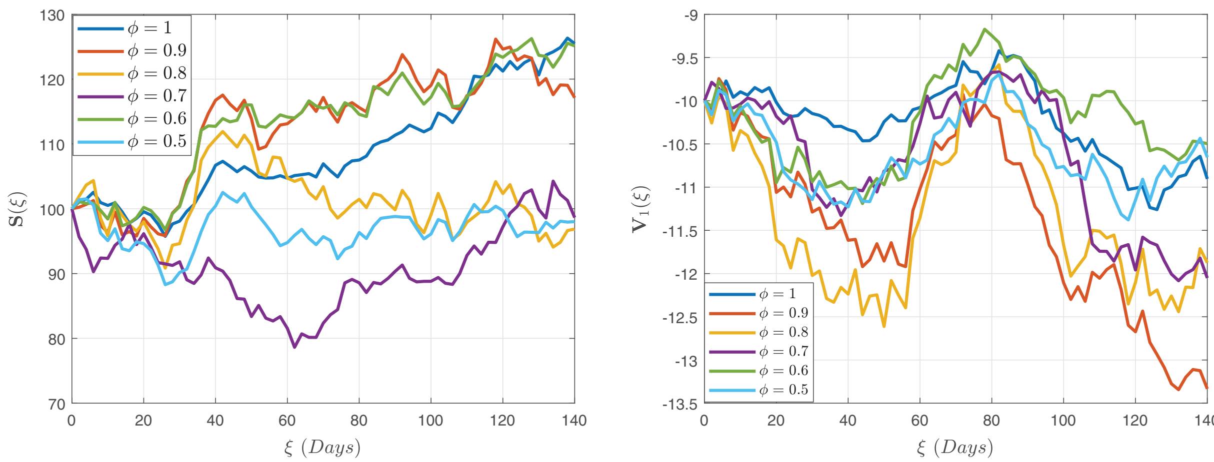


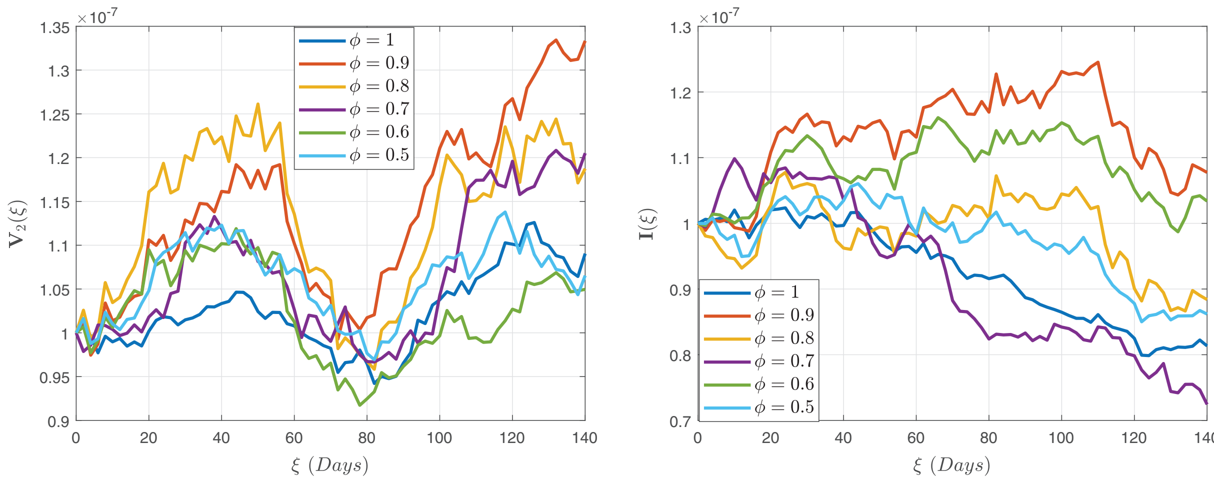
Figure 6: Stochastic behavior of the recovered class $\mathbf{R}(\xi)$ for various fractional-orders by the use of global derivative in the Caputo-Fabrizio sense

In Figs. 7–9, we attempted to demonstrate how the interaction rate ω influences the proportion of contaminated people via the global derivative in the context of Atangana-Baleanu-Caputo version. The mathematical findings were generated by modifying the interaction rate ω value with differential fractional orders while maintaining the remaining factors fixed. When the interaction rate ω is boosted in the (24) from 0.09091 to 0.3, the number of targeted people increases significantly and consistently. Furthermore, when $\omega = 0.6$ is administered, the number of highly contagious individuals apparently rises to 25 and then gradually declines, although it remains greater than in the prior two incidences. In this case, the utilized ICs are $(\mathbf{S}, \mathbf{V}_1, \mathbf{V}_2, \mathbf{I}, \mathbf{R}) = (1000, 500, 100, 10, 2)$. Figs. 7–9 depict the unpredictability densities at $\rho_1 = 0.001$, $\rho_2 = 0.003$, $\rho_3 = 0.004$, $\rho_4 = 0.006$, and $\rho_5 = 0.006$, respectively. Because of the unpredictability tendency, the findings from the fractional stochastic model are likewise accumulating while preserving their perturbing feature. Furthermore, the ultimate result suggests that incorporating the value of interaction rate ω affects the rate of contaminated people. As a result, we may deduce that as the interaction rate ω increases while the other components stay unchanged, the measles infection progresses in the population. By adjusting the value of acquiring prescribed medication at a rate χ_2 while maintaining the rest of the components fixed, Figs. 10–12 depict the number of people in the community who received a new dose of vaccination *vs*. time-frame. When the probability of acquiring a subsequent dosage of vaccination improved from $\chi_2 = 0.8$ to 1.8, the proportion of immunized second populations grew substantially and frequently. Furthermore, when $\chi_2 = 2.8$ is increased, the amount of inoculated second populations increases exponentially. In this case, the utilized ICs are $(\mathbf{S}, \mathbf{V}_1, \mathbf{V}_2, \mathbf{I}, \mathbf{R}) = (1000, 500, 100, 10, 2)$. Figs. 10–12 depict the unpredictability densities are at $\rho_1 = 0.0018$, $\rho_2 = 0.0016$, $\rho_3 = 0.0011$, $\rho_4 = 0.009$, and $\rho_5 = 0.006$, respectively. In the illustration, the inoculated (second dose) community declines gradually for the classical stochastic model [37], whereas it drops sporadically for the fractional stochastic model. As a result of acquiring an additional dosage of vaccination, the intended group significantly contributes to society's measles elimination.



(a) (b)

Figure 7: Stochastic behavior of the dual dose vaccination group $\mathbf{V}_2(\xi)$ and infectious class $\mathbf{V}_1$ for various fractional-orders by the use of global derivative in the Atangana-Baleanu-Caputo sense



(a) (b)

Figure 8: Stochastic behavior of the dual dose vaccination group $\mathbf{V}_2(\xi)$ and infectious class $\mathbf{V}_1$ for various fractional-orders by the use of global derivative in the Atangana-Baleanu-Caputo sense

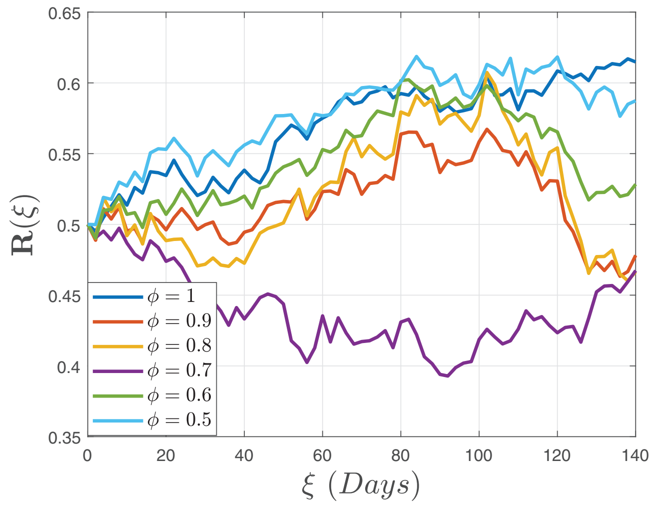
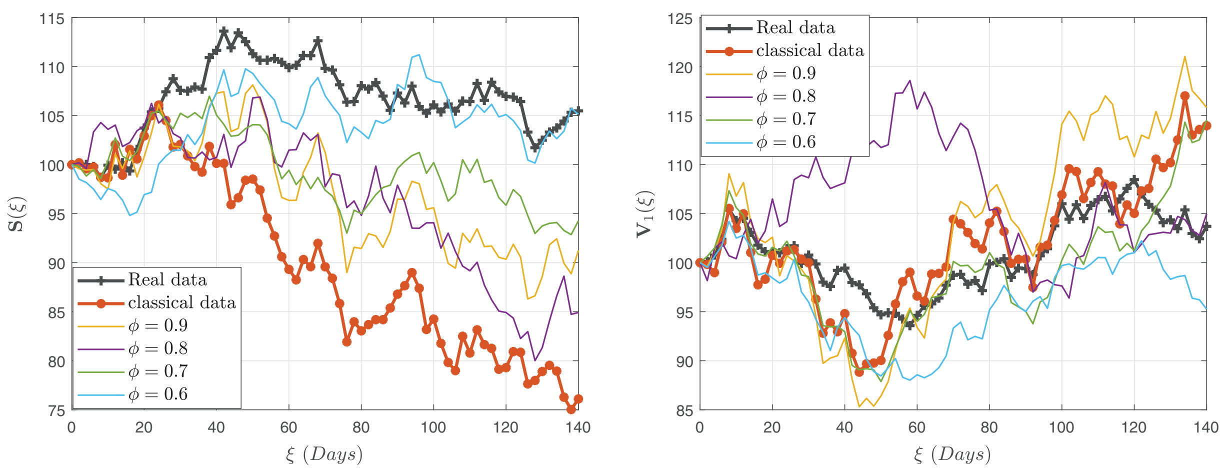
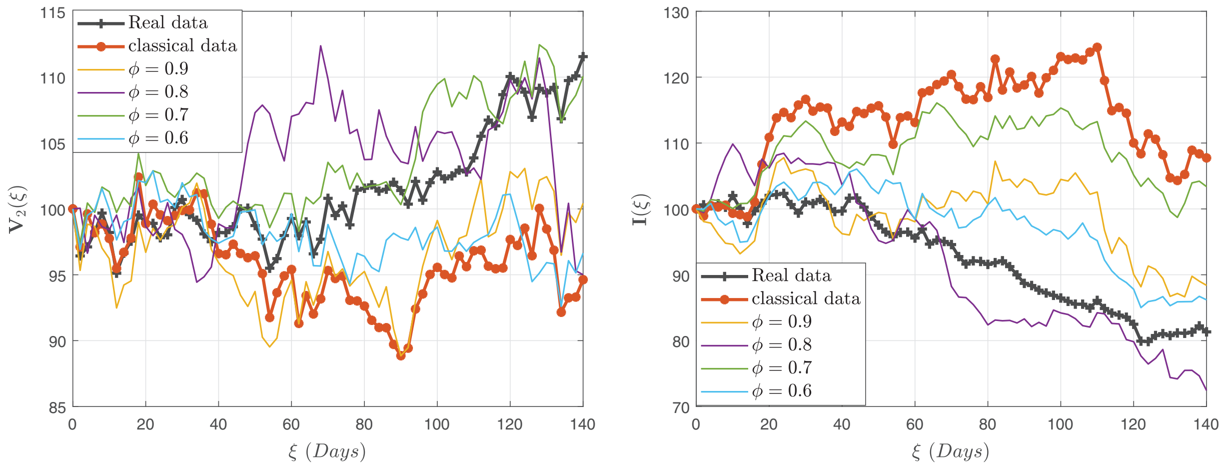


Figure 9: Stochastic behavior of the recovered class $\mathbf{R}(\xi)$ for various fractional-orders by the use of global derivative in the Atangana-Baleanu-Caputo sense



(a) (b)

Figure 10: Stochastic behavior of the dual dose vaccination group $\mathbf{V}_2(\xi)$ and infectious class $\mathbf{V}_1$ for various fractional-orders by the use of global derivative in the Atangana-Baleanu-Caputo sense



(a) (b)

Figure 11: Stochastic behavior of the dual dose vaccination group $\mathbf{V}_2(\xi)$ and infectious class $\mathbf{V}_1$ for various fractional-orders by the use of global derivative in the Atangana-Baleanu-Caputo sense

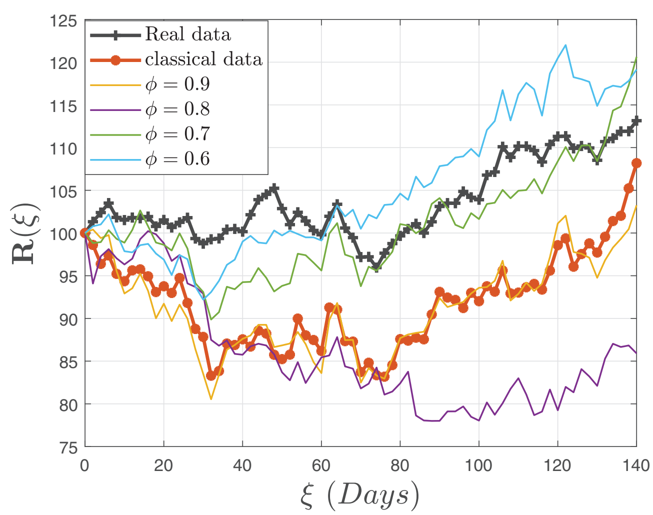


Figure 12: Stochastic behavior of the recovered class $\mathbf{R}(\xi)$ for various fractional-orders by the use of global derivative in the Atangana-Baleanu-Caputo sense

7 Conclusion

This achievement was realized through meticulous verification of settings, which facilitated the observation of linear growth patterns and Lipschitz quadratic characteristics. With a numerical technique predicated on the Newton polynomial, each version was addressed differently. The influence of fractional-order and stochastic elements was demonstrated through modeling. Owing to the modeling studies via fractional operators, serious concerns like infection are not computationally intensive, so integrating fractional stochastic influences into the system makes modeling measles outbreaks considerably more authentic than the deterministic case. Our findings underscore the superior efficiency of a randomized model approach over a deterministic model in capturing the nuances of measles transmission dynamics, especially when considering dual-dose immunization strategies. As a result, we recommend using a stochastic approach to evaluate communicable disease trends; reducing interaction between vulnerable and infectious agent populations; increasing double-dose immunization penetration; and combining understanding and acceptance with therapy to eradicate measles in communities.

Acknowledgement: The researchers would like to thank the reviewers and editors for helping to improve this article.

Funding Statement: The authors received no specific funding for this study.

Author Contributions: The authors confirm their contribution to the paper as follows: study conception and design: S. Rashid and F. Jarad; data collection: S. Rashid; analysis and interpretation of results: S. Rashid and F. Jarad; draft manuscript preparation: S. Rashid and F. Jarad; software: F. Jarad; validation: S. Rashid. All authors reviewed the results and approved the final version of the manuscript.

Availability of Data and Materials: The datasets used and/or analyzed during the current study available from the corresponding author on reasonable request.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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