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ARTICLE

On a New Version of Weibull Model: Statistical Properties, Parameter Estimation and Applications

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ABSTRACT

In this paper, we introduce a new four-parameter version of the traditional Weibull distribution. It is able to provide seven shapes of hazard rate, including constant, decreasing, increasing, unimodal, bathtub, unimodal then bathtub, and bathtub then unimodal shapes. Some basic characteristics of the proposed model are studied, including moments, entropies, mean deviations and order statistics, and its parameters are estimated using the maximum likelihood approach. Based on the asymptotic properties of the estimators, the approximate confidence intervals are also taken into consideration in addition to the point estimators. We examine the effectiveness of the maximum likelihood estimators of the model's parameters through simulation research. Based on the simulation findings, it can be concluded that the provided estimators are consistent and that asymptotic normality is a good method to get the interval estimates. Three actual data sets for COVID-19, engineering and blood cancer are used to empirically demonstrate the new distribution's usefulness in modeling real-world data. The analysis demonstrates the proposed distribution's ability in modeling many forms of data as opposed to some of its well-known sub-models, such as alpha power Weibull distribution.

KEYWORDS

Weibull distribution; alpha power transformation method; maximum likelihood; entropy; order statistics

1 Introduction

In life testing experiments and reliability research, the Weibull (W) distribution is a lifetime distribution that is heavily favoured. It is a widespread distribution that can be used to model the product's lifetime, including electrical insulation, ball bearings, and automotive parts. Applications in biology and medicine also regularly use it. In place of many well-known distributions including the exponential, gamma, and inverse W distributions, the W distribution is widely used. If the probability density function (PDF) of the random variable X is given by

$$g(x;\rho,\delta)=\rho\delta x^{\delta-1}e^{-\rho x^{\delta}}, \tag{1}$$

then the random variable X is said to have a W distribution with scale parameter $\rho > 0$ and shape parameter $\delta > 0$. The cumulative distribution function (CDF) of corresponds to (1) is expressed as follows:

$$G(x;\rho,\delta) = 1 - e^{-\rho x^{\delta}}, \qquad x > 0. \tag{2}$$

However, because the W distribution lacks a bathtub or unimodal hazard rate function (HRF), it cannot be used to represent the lifetime of some systems. Researchers have recently discovered a number of generalizations of the traditional Weibull distribution to address this drawback, see for example the exponentiated W distribution by Mudholkar et al. [1], extended W distribution by Marshall et al. [2], Kumaraswamy W distribution by Cordeiro et al. [3], W-W distribution by Abouelmagd et al. [4], alpha power W (APW) distribution by Nassar et al. [5] and Alpha logarithmic transformed W distribution by Nassar et al. [6], odd Lomax W distribution by Cordeiro et al. [7], the log-normal modified W distribution by Shakhatreh et al. [8], generalized extended exponential-W distribution by [9] and logarithmic transformed W by Nassar et al. [10].

By combining the logarithmic transformed method proposed by Pappas et al. [11] with the alpha power transformation procedure proposed by Mahdavi et al. [12], Alotaibi et al. [13] introduced a new family of continuous distributions which called the logarithmic transformed alpha power (LTAP) family. This family introduces new two-shape parameters to any existing baseline distribution to add more flexibility to its PDF and HRF. The LTAP family of distributions' CDF and PDF are provided, respectively, as

$$F(x;\lambda,\alpha) = \begin{cases} 1 - \dfrac{\log\left[\lambda - \frac{\lambda-1}{\alpha-1}\left(\alpha^{G(x)} - 1\right)\right]}{\log(\lambda)}, & \lambda,\alpha > 0, \lambda,\alpha \neq 1, \\ G(x), & \lambda = \alpha = 1, \end{cases} \tag{3}$$

and

$$f(x;\lambda,\alpha) = \begin{cases} \dfrac{(\lambda-1)\log(\alpha)}{\log(\lambda)} \dfrac{g(x)\alpha^{G(x)}}{\left[\lambda\alpha - 1 - (\lambda-1)\alpha^{G(x)}\right]}, & \lambda,\alpha > 0, \lambda,\alpha \neq 1, \\ g(x), & \lambda = \alpha = 1, \end{cases} \tag{4}$$

where λ and α are shape parameters. Alotaibi et al. [13] investigated the main structure properties of the LTAP family and introduced the so-called LTAP exponential (LTAPEx) distribution as a new extension of the traditional exponential distribution.

In this paper, we offer the LTAPW distribution as a novel four-parameter modification of the W distribution. Numerous lifetime distributions, including the exponential, Rayleigh, W, alpha logarithmic transformed W, and APW distributions, are included in the suggested distribution as special instances. We are encouraged to present the LTAPW distribution because (1) it possesses more than ten lifetime distributions as sub-models. (2) It reveals seven hazard rate shapes, including constant, decreasing, increasing, unimodal, bathtub, unimodal then bathtub, and bathtub then unimodal shapes which causes this distribution to be outstanding when compared with other lifetime models. (3) It is displayed in Section 2 that the PDF of the LTAPW distribution can be presented as a mixture of W distribution and this property is beneficial when deriving its main properties. (4) It can be regarded as a practical model for fitting the skewed data and can also be utilized to model COVID-19, engineering, and blood cancer data sets.

The remainder of this paper is structured as follows. We introduce the LTAPW distribution and go through some of its characteristics in Section 2. The estimation of the unknown parameters is described in Section 3. Section 4 conducts a simulation study to assess the performance of the maximum likelihood estimators (MLEs). In Section 5, three actual data sets are utilized to illustrate how useful the LTAPW distribution is. Some concluding remarks are included in Section 6.

2 LTAPW Distribution

The primary structural characteristics of the LTAPW distribution are explored in this section. By substituting $G(x)$ of the W distribution given by (2) in the CDF of the LTAP class in (3), one can write the CDF of the LTAPW distribution with scale parameter ρ and shape parameters λ, α and δ as follows:

$$F(x;\boldsymbol{\theta}) = 1 - \frac{\log\left[\lambda - \frac{\lambda-1}{\alpha-1}\left(\alpha^{1-e^{-\rho x^{\delta}}} - 1\right)\right]}{\log(\lambda)}, \qquad x > 0, \lambda, \alpha, \rho, \delta > 0, \tag{5}$$

where $\boldsymbol{\theta} = (\lambda, \alpha, \rho, \delta)^{\top}$ is the vector of the unknown parameters, and the associated PDF is given by

$$f(x;\boldsymbol{\theta}) = \frac{(\lambda-1)\log(\alpha)}{\log(\lambda)} \frac{\rho\delta x^{\delta-1} e^{-\rho x^{\delta}}}{\left[1-\lambda+(\lambda\alpha-1)\alpha^{e^{-\rho x^{\delta}}-1}\right]}. \tag{6}$$

The reliability function (RF) and HRF of the LTAPW distribution are given, respectively, by

$$R(x;\boldsymbol{\theta}) = \frac{\log\left[\lambda - \frac{\lambda-1}{\alpha-1}\left(\alpha^{1-e^{-\rho x^{\delta}}} - 1\right)\right]}{\log(\lambda)}, \tag{7}$$

and

$$h(x;\boldsymbol{\theta}) = \frac{(\lambda-1)\log(\alpha)\rho\delta x^{\delta-1} e^{-\rho x^{\delta}}}{\left[1-\lambda+(\lambda\alpha-1)\beta^{e^{-\rho x^{\delta}}-1}\right]\log\left[\lambda - \frac{\lambda-1}{\alpha-1}\left(\alpha^{1-e^{-\rho x^{\delta}}} - 1\right)\right]}. \tag{8}$$

Numerous sub-models can be obtained directly from the LTAPW distribution. Some important special models are listed in Table 1. Fig. 1 depicts various plots of the PDF of the LTAPW distribution using $\rho = 1$ consistently and by taking into account different values for the shape parameters λ, α and δ. It reveals how the newly added shape parameters provide the PDF of the LTAPW distribution with more powerful flexibility than the traditional W distribution. It can be observed from Fig. 1 that the PDF the LTAPW distribution can be right-skewed, left-skewed, or approximately symmetric. Fig. 2 displays the different shapes for the HRF of the LTAPW distribution with $\rho = 1$. Beside the constant HRF, Fig. 2 demonstrates that the HRF of the LTAPW distribution has six different shapes including decreasing, increasing, unimodal, bathtub, unimodal then bathtub, and bathtub then unimodal shapes. One of the benefits of the LTAPW distribution over the traditional W distribution is that the latter can model only data with constant, increasing and decreasing failure rates.

Table 1: Some sub-models of LTAPW distribution

λ	α	ρ	δ	Reduced model
$\to 1$	$\to 1$	–	1	Exponential
$\to 1$	$\to 1$	–	–	W
$\to 1$	$\to 1$	–	2	Rayleigh

(Continued)

Table 1 (continued)

λ	α	ρ	δ	Reduced model
$\rightarrow 1$	–	–	1	Alpha power exponential
$\rightarrow 1$	–	–	–	APW
$\rightarrow 1$	–	–	2	Alpha power Rayleigh
–	$\rightarrow 1$	–	1	Alpha logarithmic transformed exponential
–	$\rightarrow 1$	–	–	Alpha logarithmic transformed W
–	$\rightarrow 1$	–	2	Alpha logarithmic transformed Rayleigh
–	–	–	1	LTAPEx
–	–	–	2	LTAP Rayleigh

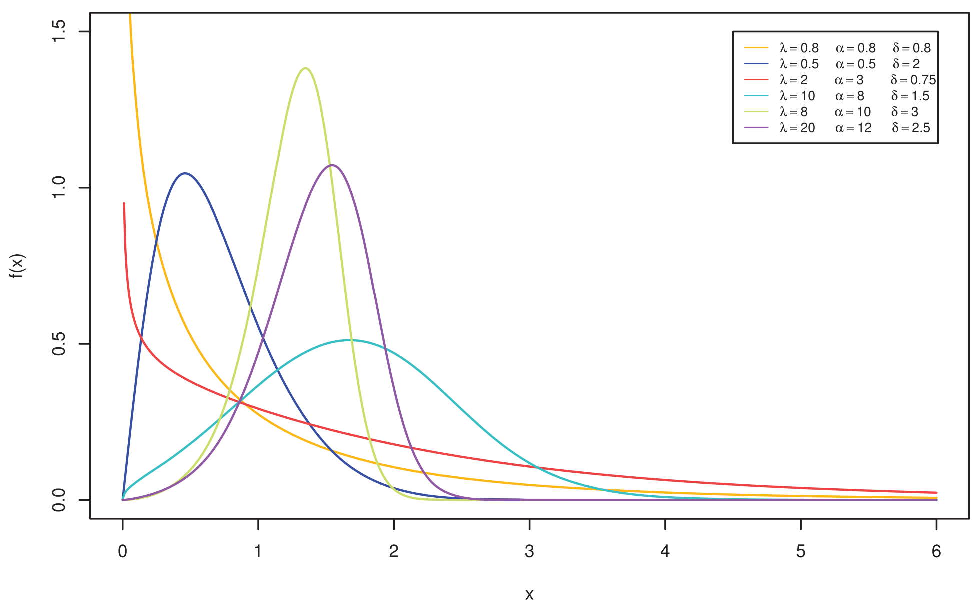

Figure 1: Different plots of the LTAPW distribution's PDF

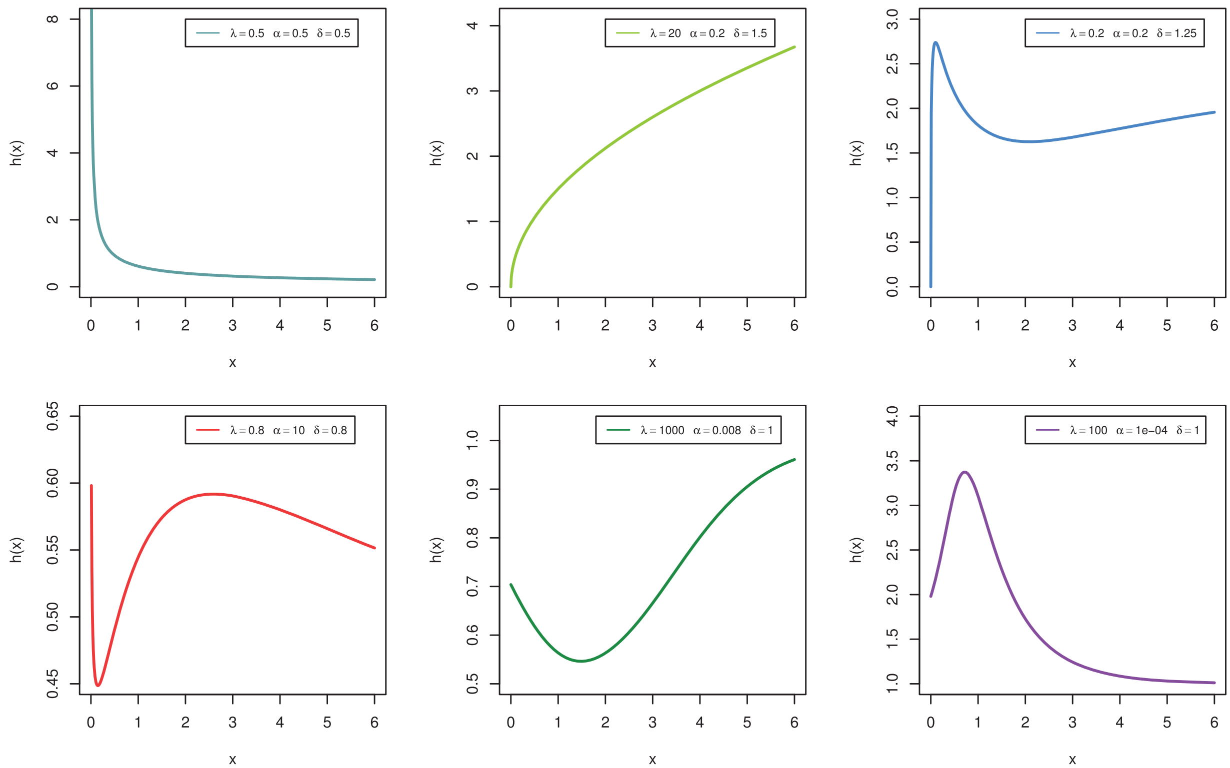

Figure 2: Different plots of the LTAPW distribution's HRF

Throughout the paper, we will use $X \sim LTAPW(\boldsymbol{\theta})$ to refer to that the random variable X follows the LTAPW distribution with PDF given by (6). The LTAPW distribution's quantile function, linear representation, moments, entropies, and other characteristics are derived in the following subsections.

2.1 Quantile Function

For estimation (for instance, quantile estimators) and simulation, quantiles are essential. For the LTAPW distribution, the q^{th} quantile $x_q, 0 < q < 1$, can be expressed as follows:

$$x_q = Q_{LTAPW}(q) = \left[\frac{-1}{\rho}\log\left\{1 - \left[\frac{\log\left(1 + \frac{\lambda(1-\lambda^{-q})(\alpha-1)}{\lambda-1}\right)}{\log\alpha}\right]\right\}\right]^{\frac{1}{\delta}}. \quad (9)$$

By substituting $q = 0.5$ in (9), the median of the LTAPW distribution can be acquired as follows:

$$M = \left[\frac{-1}{\rho}\log\left\{1 - \left[\frac{\log\left(1 + \frac{(1-\lambda^{0.5})(\alpha-1)}{\lambda-1}\right)}{\log\alpha}\right]\right\}\right]^{\frac{1}{\delta}}. \quad (10)$$

By putting $q = 0.25$ and 0.75 in (9), the first and third quartiles of the LTAPW distribution can be obtained, respectively. Allowing U to be uniform $(0, 1)$, Eq. (9) can be employed to generate a random sample of size n from the LTAPW distribution as follows:

$$x_i = \left[\frac{-1}{\rho} \log \left\{ 1 - \left[\frac{\log\left(1 + \frac{\lambda(1-\lambda^{-u_i})(\alpha-1)}{\lambda-1}\right)}{\log \alpha} \right] \right\} \right]^{\frac{1}{\delta}}, \quad i = 1, \ldots, n. \tag{11}$$

2.2 Mixture Representation

In this subsection, we offer some mixture expressions for the PDF and CDF of the LTAPW distribution. Employing the generalized binomial expansion and the power series given, respectively, by

$$(1-z)^{-r} = \sum_{k=0}^{\infty} \frac{\Gamma(k+r)}{\Gamma(r)k!} z^k, \quad -1 < z < 1, r > 0 \tag{12}$$

and

$$\alpha^z = \sum_{m=0}^{\infty} \frac{(\log \alpha)^m}{m!} z^m. \tag{13}$$

Applying (12) and (13) to the PDF in (6), then the following is a useful linear expression for the PDF of the LTAPW distribution:

$$f(x; \boldsymbol{\theta}) = \sum_{s=0}^{\infty} \sum_{i=0}^{s} \varpi_{s,i}\, g(x; (i+1)\rho, \delta). \tag{14}$$

where $g(x; (i+1)\rho, \delta)$ is the PDF of the W distribution with scale parameter $(i+1)\rho$ and shape parameter δ, and

$$\varpi_{s,i} = \sum_{k=0}^{\infty} \sum_{m=0}^{k} \sum_{j=0}^{m} \frac{(-1)^{m+j+i}(j+1)^s i\Gamma(k)\,(\log(\alpha))^{s+1}}{s(1+\lambda)^{k+1}\log(\lambda)\Gamma(i+2)\Gamma(j)\Gamma(k-m)\Gamma(m-j)\Gamma(s-i)} \left(\frac{\lambda-1}{\alpha-1}\right)^{m+1}.$$

By integrating (14), the linear expression for the CDF of the LTAPW distribution given by (5), can be obtained as

$$F(x; \boldsymbol{\theta}) = \sum_{s=0}^{\infty} \sum_{i=0}^{s} \varpi_{s,i} G(x; (i+1)\rho, \delta). \tag{15}$$

where $G(x; (i+1)\rho, \delta)$ is the CDF of the W distribution with scale parameter $(i+1)\rho$ and shape parameter δ.

2.3 Moments

Moments are crucial to statistics and the use of statistics. Moments can be utilized to examine a probability distribution's tendency, variation, and symmetry, among other essential aspects. The r^{th} moment for the LTAPW distribution can be derives from (14) as follows:

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx$$
$$= \sum_{s=0}^{\infty} \sum_{i=0}^{s} \varpi_{s,i} \int_0^{\infty} x^r g(x; (i+1)\rho, \delta) dx$$

$$= \sum_{s=0}^{\infty} \sum_{i=0}^{s} \varpi_{s,i} \frac{\Gamma\left(\frac{r}{\delta}+1\right)}{[(i+1)\rho]^{\frac{r}{\delta}}}, \qquad r = 1, 2, 3, \ldots, \tag{16}$$

where $\Gamma(.)$ is the gamma function. Particularly, the first two moments of the LTAPW distribution can be obtained directly from (16) by setting $r = 1, 2$, respectively, as

$$\mu_1^{'} = E(X) = \sum_{s=0}^{\infty} \sum_{i=0}^{s} \varpi_{s,i} \frac{\Gamma\left(\frac{1}{\delta}+1\right)}{[(i+1)\rho]^{\frac{1}{\delta}}}$$

and

$$\mu_2^{'} = E(X^2) = \sum_{s=0}^{\infty} \sum_{i=0}^{s} \varpi_{s,i} \frac{\Gamma\left(\frac{2}{\delta}+1\right)}{[(i+1)\rho]^{\frac{2}{\delta}}}.$$

Incomplete moments (IMs), moment generating functions (MGF), characteristic function (CF), and conditional moments (CMs) are four different forms of moments for the LTAPW distribution which can be expressed according to the following theorem:

Theorem. If $X \sim$ LTAPW, then

1. The r^{th} IM of X is

$$\psi_r(t) = \sum_{s=0}^{\infty} \sum_{i=0}^{s} \varpi_{s,i} \frac{\gamma(r+1, (i+1)\rho t^{\delta})}{[(i+1)\rho]^{\frac{i}{\delta}}}, \tag{17}$$

where $\gamma(.,.)$ is the lower incomplete gamma.

2. The MGF of X is

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k) = \sum_{k=0}^{\infty} \mu_k^{'} \frac{t^k}{k!},$$

where $\mu_k^{'}$ is k^{th} moment for the LTAPW distribution and given by (16).

3. The CF of X is

$$\Phi_X(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E(X^k) = \sum_{k=0}^{\infty} \mu_k^{'} \frac{(it)^k}{k!}, \quad i^2 = -1.$$

4. The r^{th} CM of X is

$$\Psi_X(t) = E\left(X^r | T > t\right) = \frac{\psi_r(t)}{1 - F(t)},$$

where $\psi_r(t)$ is the IMs and given by (17).

2.4 Mean Deviations, Bonferroni and Lorenz Curves

The mean deviations from the mean and median are applications of the IMs. They are employed to evaluate how far a population has spread from its center. The mean deviations from the mean and the median can be derived, respectively, as follows:

$$\Delta(\mu) = \int_0^\infty |x - \mu| f(x) dx$$

and

$$\Delta(M) = \int_0^\infty |x - M| f(x) dx,$$

where $\mu = \mu_1^{'}$. These measure can be expressed as follow, see Cordeiro et al. [14],

$$\Delta(\mu) = 2\mu F(\mu) - 2\psi_1(\mu) \quad \text{and} \quad \Delta(M) = \mu - 2\psi_1(M)$$

where $\psi_1(\mu)$ and $\psi_1(M)$ are defined from (17) as

$$\psi_1(\mu) = \sum_{s=0}^{\infty}\sum_{i=0}^{s} \varpi_{s,i} \frac{\gamma(2,(i+1)\rho\mu^{\delta})}{[(i+1)\rho]^{\frac{i}{\delta}}} \text{ and } \psi_1(M) = \sum_{s=0}^{\infty}\sum_{i=0}^{s} \varpi_{s,i} \frac{\gamma(2,(i+1)\rho M^{\delta})}{[(i+1)\rho]^{\frac{i}{\delta}}}.$$

Moreover, Bonferroni and Lorenz curves, denoted by Λ_B and Λ_L, respectively, are employed in economics, reliability, demography, insurance, and medicine. They can be easily acquired using the IMs follows:

$$\Lambda_B(\tau) = \frac{\psi_1(q)}{\tau\mu} \quad \text{and} \quad \Lambda_L(\tau) = \frac{\psi_1(q)}{\mu},$$

where $q = Q_{LTAPW}(\tau)$ obtained from (9) and

$$\psi_1(q) = \sum_{s=0}^{\infty}\sum_{i=0}^{s} \varpi_{s,i} \frac{\gamma(2,(i+1)\rho q^{\delta})}{[(i+1)\rho]^{\frac{i}{\delta}}}.$$

2.5 Entropies of LTAPW Distribution

The amount of information that can be expected from a random variable is called entropy. In many fields, including physics, finance, statistics, chemistry, insurance and biological phenomena, estimating entropy is essential. Less information in a sample is directed to retaining higher entropy. For the LTAPW distribution, the Rényi entropy (RE), denoted by $I_\zeta(x)$ with $\zeta > 0, \zeta \neq 1$, can be derived as

$$\begin{aligned} I_\zeta(x) &= \frac{1}{1-\zeta} \log\left\{\int_0^\infty [f(x)]^\zeta dx\right\} \\ &= \frac{1}{1-\zeta} \log\left\{\left(\frac{\rho\delta(\lambda-1)\log(\alpha)}{\log(\lambda)}\right)^\zeta \int_0^\infty \frac{x^{\zeta(\delta-1)} e^{-\zeta\rho x^\delta} \alpha^{\zeta\left(1-e^{-\rho x^\delta}\right)}}{\left[(\lambda\alpha-1)-(\lambda-1)\alpha^{1-e^{-\rho x^\delta}}\right]^\zeta} dx\right\} \\ &= \frac{1}{1-\zeta} \log\left\{\left(\frac{\rho\delta(\lambda-1)\log(\alpha)}{\log(\lambda)}\right)^\zeta \int_0^\infty x^{\zeta(\delta-1)} e^{-\zeta\rho x^\delta} \alpha^{\zeta\left(1-e^{-\rho x^\delta}\right)} \right.\end{aligned}$$

$$\times \left[(\lambda\alpha - 1) - (\lambda - 1)\alpha^{1-e^{-\rho x^{\delta}}}\right]^{-\zeta} dx\Bigg\}. \tag{18}$$

Using series presented in (12) and (13), the RE in (18) can be written as follows:

$$I_{\zeta}(x) = \frac{1}{1-\zeta}\log\left\{\left(\frac{\delta\rho}{\log(\lambda)}\right)^{\zeta}\sum_{t=0}^{\infty}\sum_{m=0}^{t}\varpi_{t,m}\int_{0}^{\infty} x^{\zeta(\delta-1)}e^{-\rho(m+\xi)x^{\delta}}dx\right\}$$

$$= \frac{1}{1-\zeta}\log\left\{\left(\frac{\delta\rho}{\log(\lambda)}\right)^{\zeta}\sum_{t=0}^{\infty}\sum_{m=0}^{t}\varpi_{t,m}\frac{\Gamma\left(\frac{\zeta(\delta-1)}{\delta}+1\right)}{\delta\left[\rho(m+\zeta)\right]^{\frac{\zeta(\delta-1)}{\delta}+1}}\right\}$$

$$= \frac{\zeta}{1-\zeta}\log\left\{\frac{\delta\rho}{\log(\lambda)}\right\} + \frac{1}{1-\zeta}\log\left\{\sum_{t=0}^{\infty}\sum_{m=0}^{t}\varpi_{t,m}\frac{\Gamma(\frac{\zeta(\delta-1)}{\delta}+1)}{\delta\left[\rho(m+\zeta)\right]^{\frac{\zeta(\delta-1)}{\delta}+1}}\right\},$$

where

$$\varpi_{t,m} = \sum_{k=0}^{\infty}(-1)^{m}C_{k}\binom{t}{m}\frac{(j+\zeta)^{t}[\log(\alpha)]^{n+\zeta}}{t!}.$$

and

$$C_{k} = \sum_{i=0}^{k}\sum_{j=0}^{i}(-1)^{i+j}\binom{k}{i}\binom{i}{j}\frac{\Gamma(k+\zeta)}{\Gamma(\zeta)k!}\left(\frac{\lambda-1}{(\alpha-1)(\lambda+1)}\right)^{k+\zeta}$$

2.6 Order Statistics

Suppose that $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ indicate the order statistics (OS) acquired from a sample of size n selected from a population that follows the LTAPW distribution with CDF and PDF given by (5) and (6), respectively. Then, one can get the PDF of the i^{th} OS $X_{(i)}$, as follows:

$$f_{X_{(i)}}(x) = f(x)\sum_{k=0}^{n-i}d_{k}F^{k+i-1}(x) \tag{19}$$

where $d_k = \dfrac{(-1)^k}{B(i, n-i+1)}\begin{pmatrix} n-i \\ k \end{pmatrix}$ and $B(.,.)$ is the beta function. Using (5), (6) and after some simplifications, we can write

$$f(x)F^{k+i-1}(x) = \sum_{j=0}^{k+i-1}T_{j}\frac{x^{\delta-1}e^{-\rho x^{\delta}}}{\left[1+\frac{\lambda\alpha-1}{1-\lambda}\alpha^{e^{-\rho x^{\delta}}-1}\right]}\left[\log\left(\lambda-\frac{\lambda-1}{\alpha-1}\left(\alpha^{1-e^{-\rho x^{\delta}}}-1\right)\right)\right]^{j}, \tag{20}$$

where

$$T_{j} = \rho\delta\log(\alpha)\left(\frac{-1}{\log(\lambda)}\right)^{j+1}\binom{k+i-1}{j} \tag{21}$$

In this case, the PDF of the i^{th} OS for the LTAPW distribution can be expressed as

$$f_{X_{(i)}}(x) = \sum_{k=0}^{n-i} \sum_{j=0}^{k+i-1} T_{k,j} \frac{x^{\delta-1} e^{-\rho x^{\delta}}}{\left[1 + \frac{\lambda\alpha-1}{1-\lambda}\alpha^{e^{-\rho x^{\delta}}-1}\right]} \left[\log\left(\lambda - \frac{\lambda-1}{\alpha-1}\left(\alpha^{1-e^{-\rho x^{\delta}}} - 1\right)\right)\right]^{j}, \tag{22}$$

$T_{k,j} = d_k T_j$.

On the other hand, the CDF of the i^{th} order statistic for the LTAPW distribution corresponds to (22) is given by the following formula:

$$F_{X_{(i)}}(x) = \sum_{k=i}^{n} \binom{n}{k} [F(x)]^{k} [1 - F(x)]^{n-k}$$

$$= \sum_{k=i}^{n} \sum_{j=0}^{k} \frac{(-1)^{j}\Gamma(n+1)}{\Gamma(j+1)\Gamma(n-k+1)\Gamma(k-j+1)} \left[\frac{\log\left(\lambda - \frac{\lambda-1}{\alpha-1}\left(\alpha^{1-e^{-\rho x^{\delta}}} - 1\right)\right)}{\log(\lambda)}\right]^{n+j-k}.$$

2.7 Moments of Residual Life

The residual life (RL) function in reliability and life testing refers to the extra lifetime presumed that a component has stayed until time t. The r^{th} moment of the RL of the random variable X is defined as follows:

$$R_r(t) = E((X-t)^r | X > t) = \frac{1}{1-F(t)} \int_t^{\infty} (x-t)^r f(x)dx, \qquad r \geq 1$$

Based on (5), (14) and applying the binomial expansion of $(x-t)^n$, we have

$$R_r(t) = \frac{1}{1-F(t)} \sum_{k=0}^{r} (-t)^{r-k} \binom{r}{k} \int_t^{\infty} x^r f(x)dx$$

$$= \frac{1}{1-F(t)} \sum_{k=0}^{r} \sum_{s=0}^{\infty} \sum_{i=0}^{s} \varpi_{s,i} (-t)^{r-k} \binom{r}{k} \int_t^{\infty} x^r g(x;(i+1)\rho,\delta)dx$$

where $g(x;(i+1)\rho,\delta)$ is the PDF of the W distribution with scale parameter $(i+1)\rho$ and shape parameter δ. Then,

$$R_r(t) = \frac{\log(\lambda)}{\log\left(\lambda - \frac{\lambda-1}{\alpha-1}\left(\alpha^{1-e^{-\rho t^{\delta}}} - 1\right)\right)} \sum_{k=0}^{r} \sum_{s=0}^{\infty} \sum_{i=0}^{s} \varpi_{s,i} (-t)^{r-k} \binom{r}{k} \frac{\Gamma\left(\frac{r}{\delta}+1,(i+1)\rho t^{\delta}\right)}{[(r+1)\rho]^{\frac{r}{\delta}}},$$

where $\Gamma(a,y) = \int_y^{\infty} z^{a-1} e^{-z} dz$ is the complementary incomplete gamma function. On the other hand, one can get the r^{th} moment of the reversed RL by utilizing the following formula:

$$m_r(t) = E((t-X)^r | X \leq t) = \frac{1}{F(t)} \int_0^{t} (t-x)^r f(x)dx, \qquad r \geq 1.$$

Using (5) and (14), it follows:

$$m_r(t) = \left[1 - \frac{\log\left[\lambda - \frac{\lambda-1}{\alpha-1}\left(\alpha^{1-e^{-\rho t^{\delta}}} - 1\right)\right]}{\log(\lambda)}\right]^{-1} \sum_{k=0}^{r}\sum_{s=0}^{\infty}\sum_{i=0}^{s} \varpi_{s,i} t^k (i+1)\rho\delta$$
$$\times \int_0^t x^{r+\delta-k-1} e^{-(i+1)\rho x^{\delta}} dx$$
$$= \left[1 - \frac{\log\left[\lambda - \frac{\lambda-1}{\alpha-1}\left(\alpha^{1-e^{-\rho t^{\delta}}} - 1\right)\right]}{\log(\lambda)}\right]^{-1} \sum_{k=0}^{r}\sum_{s=0}^{\infty}\sum_{i=0}^{s} \varpi_{s,i} \frac{t^k \gamma\left(\frac{r-k+\delta}{\delta}, (i+1)\rho t^{\delta}\right)}{[(i+1)\rho]^{\frac{r-k}{\delta}}},$$

where $\gamma(a, b)$ is the lower incomplete gamma function.

3 Maximum Likelihood Estimation

The MLEs of the different unknown parameters and the associated approximate confidence intervals (ACIs) using the asymptotic traits of the MLEs are discussed in this section. Let $x_1, x_2, \ldots, x_n$ is an observed sample of size n selected from a population that follows the LTAPW distribution with PDF given by (6). Then, we can formulate the likelihood function as follows:

$$L(\boldsymbol{\theta}) = \left(\frac{\rho\delta(\lambda-1)\log(\alpha)}{\log(\lambda)}\right)^n \prod_{i=1}^{n}\left[\frac{x_i^{\delta-1} e^{-\rho x_i^{\delta}}}{(1-\lambda) + (\lambda\alpha - 1)\alpha^{e^{-\rho x_i^{\delta}} - 1}}\right]. \tag{23}$$

The log-likelihood function of (23) is expressed as follows:

$$\ell(\boldsymbol{\theta}) = n\log(\rho) + n\log(\delta) + n\log(\lambda-1) + n\log(\log(\alpha)) - n\log(\log(\lambda)) + (\delta-1)\sum_{i=0}^{n}\log(x_i)$$
$$-\rho\sum_{i=0}^{n} x_i^{\delta} - \sum_{i=0}^{n}\log\left[(\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} + (1-\lambda)\right]. \tag{24}$$

To get the MLEs of λ, α, ρ and δ denoted by $\hat{\lambda}, \hat{\alpha}, \hat{\rho}$ and $\hat{\delta}$, one can maximize the log-likelihood function in (24) with regard to λ, α, ρ and δ, or equivalently by solving the next normal equations simultaneously

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{(\lambda-1)} - \frac{n}{\lambda\log(\lambda)} - \sum_{i=0}^{n}\frac{\alpha^{e^{-\rho x_i^{\delta}}} - 1}{(\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1} = 0, \tag{25}$$

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha\log(\alpha)} - \sum_{i=0}^{n}\frac{(\alpha\lambda-1)\left(e^{-\rho x_i^{\delta}} - 1\right)\alpha^{e^{-\rho x_i^{\delta}}-2} + \lambda\alpha^{e^{-\rho x_i^{\delta}}-1}}{(\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1} = 0, \tag{26}$$

$$\frac{\partial\ell}{\partial\rho} = \frac{n}{\rho} - \sum_{i=0}^{n} x_i^{\delta} + \sum_{i=0}^{n}\frac{(\alpha\lambda-1)\log(\alpha) x_i^{\delta} e^{-\rho x_i^{\delta}}\alpha^{e^{-\rho x_i^{\delta}}-1}}{(\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1} = 0 \tag{27}$$

and

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} + \sum_{i=0}^{n} \log(x_i) - \rho \sum_{i=0}^{n} x_i^{\delta} \log(x_i) + \sum_{i=0}^{n} \frac{\rho(\alpha\lambda - 1)\log(\alpha) x_i^{\delta} \log(x_i)\, e^{-\rho x_i^{\delta}} \alpha^{e^{-\rho x_i^{\delta}}-1}}{(\alpha\lambda - 1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1} = 0. \tag{28}$$

It is clear that $\hat{\lambda}, \hat{\alpha}, \hat{\rho}$ and $\hat{\delta}$ cannot be acquired in closed forms. Therefore, some numerical methods can be implemented to solve the non-linear system of equations in (25)–(28). Then ACIs of λ, α, ρ and δ are then computed using the asymptotic properties of the MLEs. It is known that $\hat{\boldsymbol{\theta}} \sim N\left(\boldsymbol{\theta}, I^{-1}(\boldsymbol{\theta})\right)$, where $\hat{\boldsymbol{\theta}} = (\hat{\lambda}, \hat{\alpha}, \hat{\rho}, \hat{\delta})^{\top}$ and $I^{-1}(\boldsymbol{\theta})$ is the asymptotic variance-covarianc matrix. Here, we use the approximate asymptotic variance-covariance matrix of the MLEs denoted by $I_0^{-1}(\hat{\boldsymbol{\theta}})$ because the obtaining of $I_0^{-1}(\boldsymbol{\theta})$ is very complicated. In this case, we can write $I_0^{-1}(\hat{\boldsymbol{\theta}})$ as follows:

$$I_0^{-1}(\hat{\boldsymbol{\theta}}) = \begin{bmatrix} -I_{\lambda\lambda} & -I_{\lambda\alpha} & -I_{\lambda\rho} & -I_{\lambda\delta} \\ -I_{\alpha\lambda} & -I_{\alpha\alpha} & -I_{\alpha\rho} & -I_{\alpha\delta} \\ -I_{\rho\lambda} & -I_{\rho\alpha} & -I_{\rho\rho} & -I_{\rho\delta} \\ -I_{\delta\lambda} & -I_{\delta\alpha} & -I_{\delta\rho} & -I_{\delta\delta} \end{bmatrix}^{-1}_{(\lambda,\alpha,\rho,\delta)=(\hat{\lambda},\hat{\alpha},\hat{\rho},\hat{\delta})}$$
$$= \begin{bmatrix} \widehat{v}(\hat{\lambda}) & \widehat{c}(\hat{\lambda},\hat{\alpha}) & \widehat{c}(\hat{\lambda},\hat{\rho}) & \widehat{c}(\hat{\lambda},\hat{\delta}) \\ \widehat{c}(\hat{\alpha},\hat{\lambda}) & \widehat{v}(\hat{\alpha}) & \widehat{c}(\hat{\alpha},\hat{\rho}) & \widehat{c}(\hat{\alpha},\hat{\delta}) \\ \widehat{c}(\hat{\rho},\hat{\lambda}) & \widehat{c}(\hat{\rho},\hat{\alpha}) & \widehat{v}(\hat{\rho}) & \widehat{c}(\hat{\rho},\hat{\delta}) \\ \widehat{c}(\hat{\delta},\hat{\lambda}) & \widehat{c}(\hat{\delta},\hat{\alpha}) & \widehat{c}(\hat{\delta},\hat{\rho}) & \widehat{v}(\hat{\delta}) \end{bmatrix}, \tag{29}$$

where

$$I_{\lambda\lambda} = -\frac{n}{(\lambda-1)^2} + \frac{n}{\lambda^2 \log^2(\lambda)} + \frac{n}{\lambda^2 \log(\lambda)} + \sum_{i=0}^{n} \frac{\left(\alpha^{e^{-\rho x_i^{\delta}}} - 1\right)^2}{\left((\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1\right)^2},$$

$$I_{\alpha\alpha} = -\frac{n}{\alpha^2 \log^2(\alpha)} - \frac{n}{\alpha^2 \log(\alpha)} + \sum_{i=0}^{n} \frac{\left((\alpha\lambda-1)\left(e^{-\rho x_i^{\delta}} - 1\right)\alpha^{e^{-\rho x_i^{\delta}}-2} + \lambda\alpha^{e^{-\rho x_i^{\delta}}-1}\right)^2}{\left((\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1\right)^2}$$
$$- \sum_{i=0}^{n} \frac{(\alpha\lambda-1)\left(e^{-\rho x_i^{\delta}} - 2\right)\left(e^{-\rho x_i^{\delta}} - 1\right)\alpha^{e^{-\rho x_i^{\delta}}-3} + 2\lambda\left(e^{-\rho x_i^{\delta}} - 1\right)\alpha^{e^{-\rho x_i^{\delta}}-2}}{(\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1},$$

$$I_{\rho\rho} = -\frac{n}{\rho^2} - \sum_{i=0}^{n} \frac{(\alpha\lambda-1)\log(\alpha) x_i^{2\delta} e^{-2\rho x_i^{\delta}} \alpha^{e^{-\rho x_i^{\delta}}} \left(\log(\alpha) + e^{\rho x_i^{\delta}}\right)}{-\alpha\lambda + \alpha + (\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}}}$$
$$+ \sum_{i=0}^{n} \frac{(\alpha\lambda-1)^2 \log^2(\alpha) x_i^{2\delta} e^{-2\rho x_i^{\delta}} \alpha^{2e^{-\rho x_i^{\delta}}-2}}{\left((\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1\right)^2},$$

$$I_{\delta\delta} = -\frac{n}{\delta^2} - \rho \sum_{i=0}^{n} x_i^{\delta} \log^2(x_i) + \sum_{i=0}^{n} \frac{\rho^2(\alpha\lambda-1)^2 \log^2(\alpha) x_i^{2\delta} \log^2(x_i)\, e^{-2\rho x_i^{\delta}} \alpha^{2e^{-\rho x_i^{\delta}}-2}}{\left((\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1\right)^2}$$

$$- \sum_{i=0}^{n} \frac{\rho(\alpha\lambda-1)\log(\alpha) x_i^{\delta} \log^2(x_i)\, e^{-2\rho x_i^{\delta}} \alpha^{e^{-\rho x_i^{\delta}}} \left(\rho x_i^{\delta}\left(\log(\alpha) + e^{\rho x_i^{\delta}}\right) - e^{\rho x_i^{\delta}}\right)}{-\alpha\lambda + \alpha + (\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}}},$$

$$I_{\alpha\lambda} = \sum_{i=0}^{n} \frac{\left(\alpha^{e^{-\rho x_i^{\delta}}} - 1\right)\left((\alpha\lambda-1)\left(e^{-\rho x_i^{\delta}} - 1\right)\alpha^{e^{-\rho x_i^{\delta}}-2} + \lambda\alpha^{e^{-\rho x_i^{\delta}}-1}\right)}{\left((\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1\right)^2}$$

$$- \sum_{i=0}^{n} \frac{\left(e^{-\rho x_i^{\delta}} - 1\right)\alpha^{e^{-\rho x_i^{\delta}}-1} + \alpha^{e^{-\rho x_i^{\delta}}-1}}{(\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1},$$

$$I_{\rho\lambda} = \sum_{i=0}^{n} \left(\frac{\log(\alpha) x_i^{\delta} e^{-\rho x_i^{\delta}} \alpha^{e^{-\rho x_i^{\delta}}}}{(\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1} - \frac{(\alpha\lambda-1)\log(\alpha) x_i^{\delta} e^{-\rho x_i^{\delta}} \alpha^{e^{-\rho x_i^{\delta}}-1}\left(\alpha^{e^{-\rho x_i^{\delta}}} - 1\right)}{\left((\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1\right)^2} \right),$$

$$I_{\delta\lambda} = -\sum_{i=0}^{n} \frac{\rho(\alpha\lambda-1)\log(\alpha) x_i^{\delta} \log(x_i)\, e^{-\rho x_i^{\delta}} \alpha^{e^{-\rho x_i^{\delta}}-1}\left(\alpha^{e^{-\rho x_i^{\delta}}} - 1\right)}{\left((\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1\right)^2}$$

$$+ \sum_{i=0}^{n} \frac{\rho \log(\alpha) x_i^{\delta} \log(x_i)\, e^{-\rho x_i^{\delta}} \alpha^{e^{-\rho x_i^{\delta}}}}{(\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1},$$

$$I_{\rho\alpha} = -\sum_{i=0}^{n} \frac{x_i^{\delta} e^{-2\rho x_i^{\delta}} \left((1-\alpha\lambda)\log(\alpha) - (\alpha\lambda + \log(\alpha) - 1)e^{\rho x_i^{\delta}}\right)}{\alpha\left(\alpha\left(\lambda - (\lambda-1)\alpha^{-e^{-\rho x_i^{\delta}}}\right) - 1\right)}$$

$$- \sum_{i=0}^{n} \frac{(\alpha\lambda-1)\log(\alpha) x_i^{\delta} e^{-2\rho x_i^{\delta}} \alpha^{2e^{-\rho x_i^{\delta}}-1}\left(\alpha\lambda + e^{\rho x_i^{\delta}} - 1\right)}{\left(-\alpha\lambda + \alpha + (\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}}\right)^2},$$

$$I_{\delta\alpha} = \sum_{i=0}^{n} \frac{\rho x_i^{\delta} \log(x_i)\, e^{-2\rho x_i^{\delta}} \left((\alpha\lambda-1)\log(\alpha) + (\alpha\lambda + \log(\alpha) - 1)e^{\rho x_i^{\delta}}\right)}{\alpha\left(\alpha\left(\lambda - (\lambda-1)\alpha^{-e^{-\rho x_i^{\delta}}}\right) - 1\right)}$$

$$- \sum_{i=0}^{n} \frac{\rho(\alpha\lambda-1)\log(\alpha) x_i^{\delta} \log(x_i)\, e^{-2\rho x_i^{\delta}} \alpha^{2e^{-\rho x_i^{\delta}}-1}\left(\alpha\lambda + e^{\rho x_i^{\delta}} - 1\right)}{\left(-\alpha\lambda + \alpha + (\alpha\lambda-1)\alpha^{e^{-\rho x_i^{\delta}}}\right)^2},$$

and

$$I_{\rho\delta} = -\sum_{i=0}^{n} \frac{(\alpha\lambda - 1)\log(\alpha)x_i^{\delta}\log(x_i)\, e^{-2\rho x_i^{\delta}}\alpha^{e^{-\rho x_i^{\delta}}}\left(\rho x_i^{\delta}\left(\log(\alpha) + e^{\rho x_i^{\delta}}\right) - e^{\rho x_i^{\delta}}\right)}{-\alpha\lambda + \alpha + (\alpha\lambda - 1)\alpha^{e^{-\rho x_i^{\delta}}}}$$

$$+ \sum_{i=0}^{n} \frac{\rho(\alpha\lambda - 1)^2\log^2(\alpha)x_i^{2\delta}\log(x_i)\, e^{-2\rho x_i^{\delta}}\alpha^{2e^{-\rho x_i^{\delta}}-2}}{\left((\alpha\lambda - 1)\alpha^{e^{-\rho x_i^{\delta}}-1} - \lambda + 1\right)^2} - \sum_{i=0}^{n} x_i^{\delta}\log(x_i)\,.$$

As a result, the $(1 - \epsilon)\%$ ACIs of λ, α, ρ and δ can be computed as follows:

$$\hat{\lambda} \pm z_{\epsilon/2}\sqrt{\widehat{v}(\hat{\lambda})} \quad \hat{\alpha} \pm z_{\epsilon/2}\sqrt{\widehat{v}(\hat{\alpha})} \quad \hat{\rho} \pm z_{\epsilon/2}\sqrt{\widehat{v}(\hat{\rho})} \quad \text{and} \quad \hat{\delta} \pm z_{\epsilon/2}\sqrt{\widehat{v}(\hat{\delta})},$$

where $z_{\epsilon/2}$ is the upper $(z_{\epsilon/2}/2)^{th}$ percentile point of a standard normal distribution.

4 Simulation Study

The performance of the MLEs of the unknown parameters is examined in this section using simulation research based on 1000 samples generated from the LTAPW Distribution. The effectiveness of estimates is assessed in terms of various criteria, bias, root mean square error (RMSE) and interval length (IL), computed according to the following formulas:

$$Bias(\theta) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\theta}_j - \theta),$$

$$RMSE(\theta) = \left[\frac{1}{1000}\sum_{j=1}^{1000}(\hat{\theta}_j - \theta)^2\right]^{0.5}$$

and

$$IL(\theta) = \frac{1}{1000}\sum_{j=1}^{1000}(\theta_j^U - \theta_j^L),$$

where θ is the unknown parameter, $\hat{\theta}$ is the MLE of θ, θ_j^U and θ_j^L are the lower and upper confidence bounds of sample j. The simulation study is conducted by considering different sample sizes, i.e., $n = 25(25)150$. The true values of the parameters are selected to be $(\lambda, \alpha, \rho, \delta) = (2, 5, 1.5, 1.5)$, $(3, 2, 1, 2)$ and $(1.5, 0.5, 0.5, 2)$. The simulation outcomes are displayed in Figs. 3–5. These figures show that as sample size grows across all settings, the biases of the various parameters decrease to zero, indicating that the MLEs act as asymptotically unbiased estimators. In addition, the RMSEs of the various parameters in all the cases decrease as the sample size grows, demonstrating the consistency of the MLEs. Regarding the ACIs of the different parameters, it is noted that the ILs of the different parameters decrease and tend to zero as the sample size increases, demonstrating that the asymptotic normality is a good choice to construct the interval estimations for the unknown parameters. Finally, based on the simulation outcomes it is recommended to use the maximum likelihood estimation procedure to acquire the point and interval estimates of the unknown parameters of the LTAPW distribution.

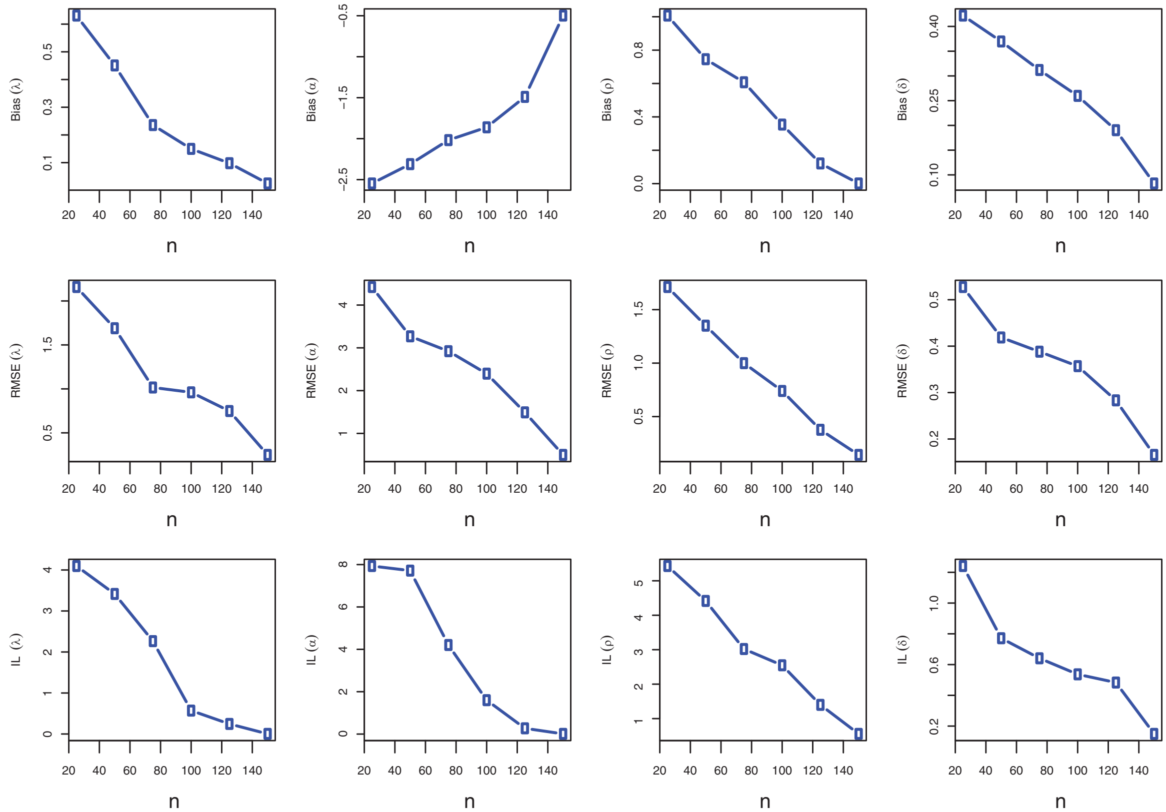

Figure 3: Bias, RMSE and ILs for $\lambda = 2, \alpha = 5, \rho = 1.5$ and $\delta = 1.5$

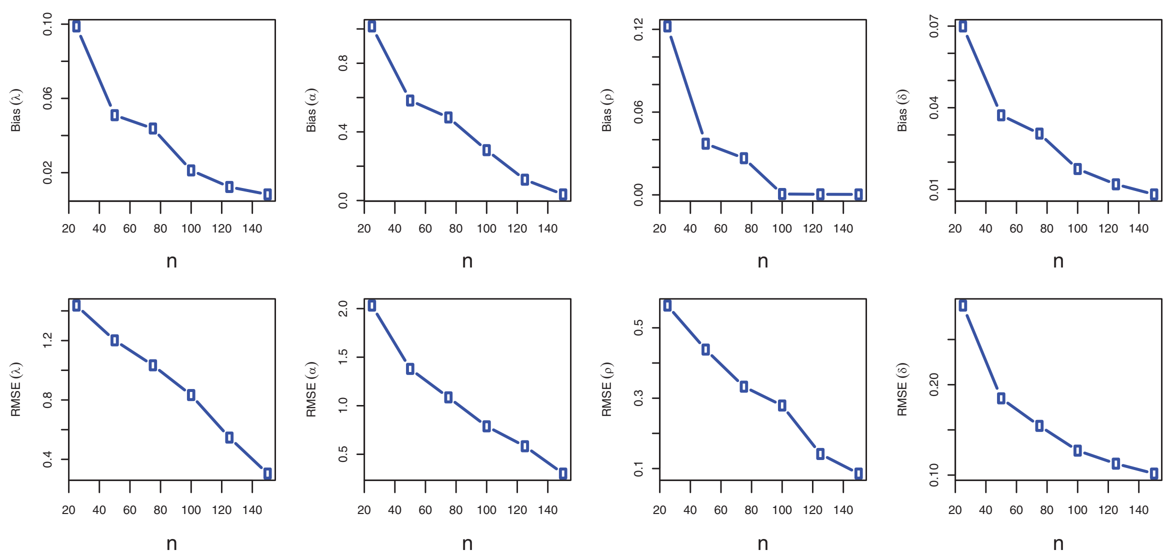

Figure 4: (Continued)

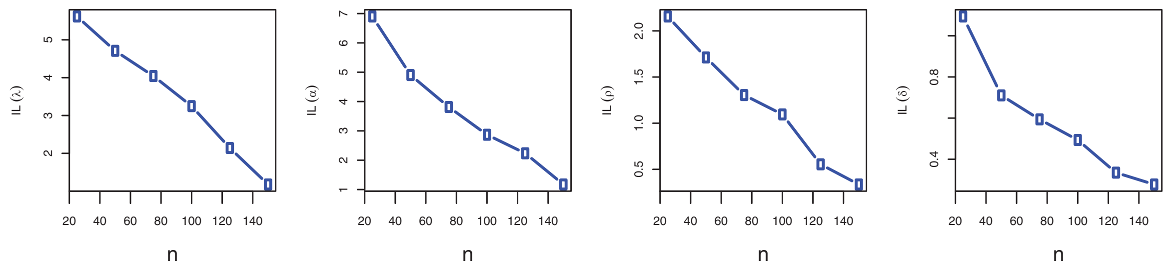

Figure 4: Bias, RMSE and ILs for $\lambda = 3, \alpha = 2, \rho = 1$ and $\delta = 2$

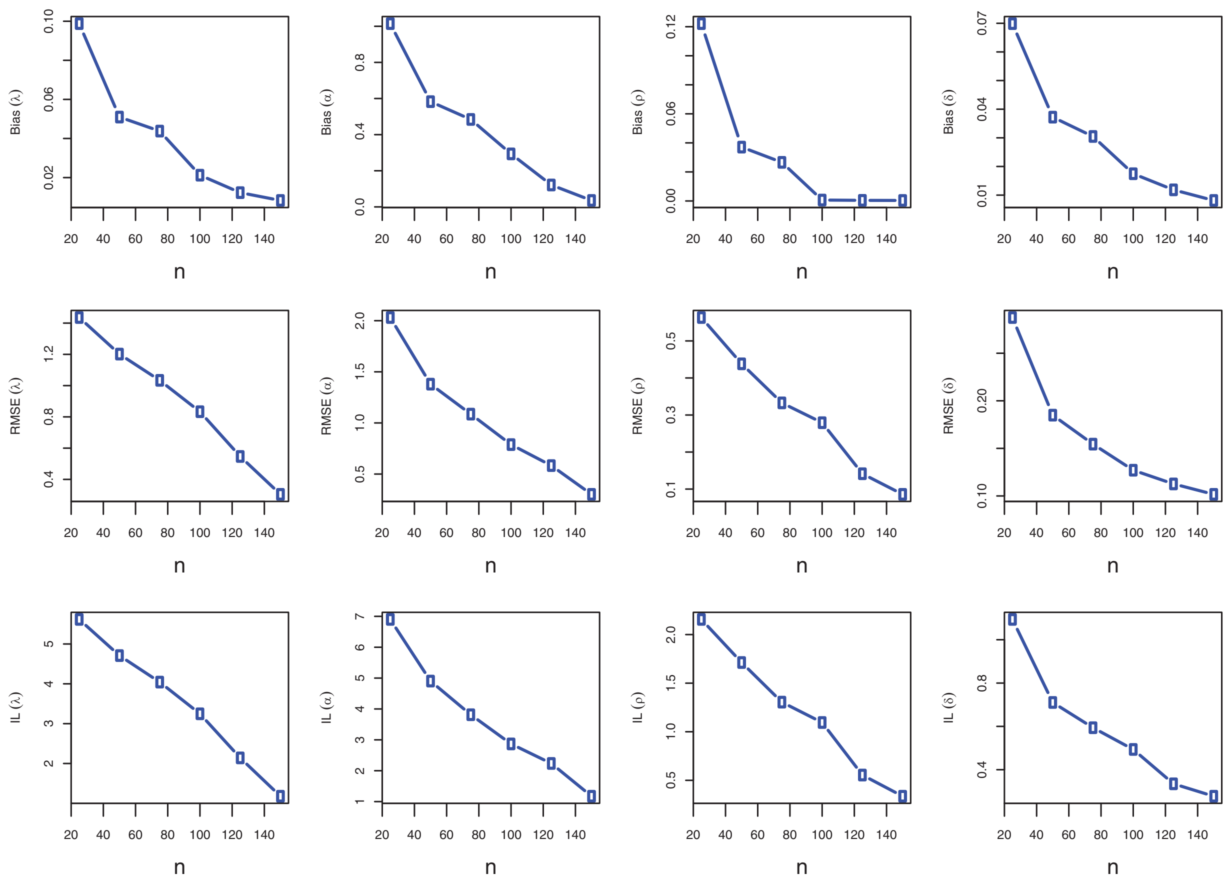

Figure 5: Bias, RMSE and IL for $\lambda = 1.5, \alpha = 0.5, \rho = 0.5$ and $\delta = 2$

5 Real Data Analysis

To show the effectiveness of our offered LTAPW distribution, three actual data sets are taken into account in this part. The first set of data (Data I) details the COVID-19 patient fatality rates in Italy for 59 days from 27 February to 27 April 2020. This data set was used by Almongy et al. [15] and analyzed recently by El-Sagheer et al. [16]. The second data set (Data II) comprises 40 observations of the airborne communication transceiver's active repair time, see for more details in Jorgensen [17] and Alotaibi et al. [18]. The lifespan (in years) of 40 patients with blood cancer (leukemia) from one of the Ministry of Health institutions in Saudi Arabia are included in the third data set (Data III) that was investigated in Al-Saiary et al. [19] and Klakattawi [20]. The actual data sets are displayed in Table 2. The box plots of the different data sets are displayed in Fig. 6. It shows that Data I and III are approximately symmetric, while Data II is positively skewed.

Table 2: The actual data sets

Data	Observations							
I	4.571	7.201	3.606	8.479	11.410	8.961	10.919	10.908
	6.503	18.474	11.010	17.337	16.561	13.226	15.137	8.697
	15.787	13.333	11.822	14.242	11.273	14.330	16.046	11.950
	10.282	11.775	10.138	9.037	12.396	10.644	8.646,	8.905
	8.906	7.407	7.445	7.214	6.194	4.640	5.452	5.073
	4.416	4.859	4.408	4.639	3.148	4.040	4.253	4.011
	3.564	3.827	3.134	2.780	2.881	3.341	2.686	2.814
	2.508	2.450	1.518					
II	0.50	0.60	0.60	0.70	0.70	0.70	0.80	0.80,
	1.00	1.00	1.00	1.00	1.10	1.30	1.50	1.50
	1.50	1.50	2.00	2.00	2.20	2.50	2.70	3.00
	3.00	3.30	4.00	4.00	4.50	4.70	5.00	5.40
	5.40	7.00	7.50	8.80	9.00	10.2	22.0	24.5
III	0.315	0.496	0.616	1.145	1.208	1.263	1.414	2.025
	2.036	2.162	2.211	2.370	2.532	2.693	2.805	2.910
	2.912	3.192	3.263	3.348	3.348	3.427	3.499	3.534
	3.767	3.751	3.858	3.986	4.049	4.244	4.323	4.381
	4.392	4.397	4.647	4.753	4.929	4.973	5.074	5.381

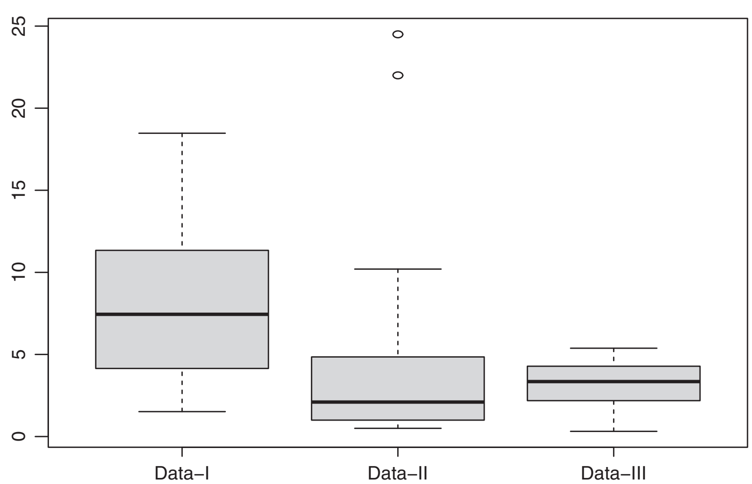

Figure 6: Box plots of various data sets

We compare the results of the proposed model with some of its sub-models namely, LTAPEx distribution proposed by Alotaibi et al. [13], APW distribution by Nassar et al. [5] and the traditional W distribution. The MLEs of the parameters of the LTAPW distribution and the other models for the given data sets along with their standard errors as well as the 95% ACIs are displayed in Tables 3–5. Moreover, the Kolmogorov-Smirnov (KS) distances with the associated p-values are presented in Tables 3–5. The results in these tables show that the three data sets are reasonably fit by the new LTAPW distribution. The fitted PDFs, RFs and probability-probability (PP) plots of the LTAPW distribution for Data I, II, and III are shown in Figs. 7–9, respectively. These figures show how the given data sets can be closely fitted using the LTAPW distribution.

Table 3: The MLEs, standard errors (in parentheses), ACIs (second row), KS and *p*-value for Data I

Model	λ	α	ρ	δ	KS	*p*-value
LTAPW	0.000092 (0.0003)	9.74 (13.09)	0.000003 (0.00001)	4.723 (1.451)	0.068	0.948
	(0, 0.0007)	(0, 35.399)	(0, 0.00003)	(1.886, 7.573)		
LTAPEx	258834.8 (163.55)	1154.8 (49.436)	0.919 (0.042)	–	0.169	0.058
	(258514.2, 259155.3)	(1057.916, 1251.7)	(0.836, 1.002)	–		
APW	–	0.6042 (0.601)	0.009 (0.007)	2.019 (0.210)	0.120	0.330
	–	(0, 1.782)	(0, 0.023)	(1.607, 2.431)		
W	–	–	0.0137 (0.006)	1.927 (0.186)	0.122	0.311
	–	–	(0.001, 026)	(1.562, 2.291)		

Table 4: The MLEs, standard errors (in parentheses), ACIs (second row), KS and *p*-value for Data II

Model	λ	α	ρ	δ	KS	*p*-value
LTAPW	0.007 (0.014)	152595.9 (16860.7)	1.017 (0.325)	0.530 (0.100)	0.098	0.832
	(0, 0.036)	(119549.4, 185642.4)	(0.381, 1.654)	(0.335, 0.725)		
LTAPEx	2.983 (5.284)	0.018 (0.044)	0.116 (0.085)	–	0.135	0.457
	(0, 13.341)	(0, 0.104)	(0.282, 0.086)	–		
APW	–	0.0281 (0.052)	0.069 (0.039)	1.187 (0.138)	0.124	0.569
	–	(0, 0.131)	(0, 0.147)	(0.917, 1.456)		
W	–	–	0.269 (0.069)	0.960 (0.109)	0.129	0.518
	–	–	(0.132, 0.406)	(0.747, 1.174)		

Table 5: The MLEs, standard errors (in parentheses), ACIs (second row), KS and *p*-value for Data III

Model	λ	α	ρ	δ	KS	*p*-value
LTAPW	412215.3 (236.18)	0.251 (0.609)	0.661 (0.511)	1.808 (0.422)	0.062	0.998
	(411752.4, 412678.2)	(0, 1.446)	(0, 1.663)	(0.982, 2.636)		
LTAPEx	583661.6 (114.57)	238.75 (122.55)	2.937 (0.144)	–	0.152	0.309
	(583437.06, 583886.2)	(0, 478.95)	(2.655, 3.219)	–		
APW	–	5.367 (6.771)	0.106 (0.083)	2.099 (0.436)	0.091	0.888
	–	(0, 8.640)	(0, 0.268)	(1.246, 2.953)		
W	–	–	0.043 (0.021)	2.499 (0.335)	0.118	0.629
	–	–	(0.001, 0.085)	(1.841, 3.157)		

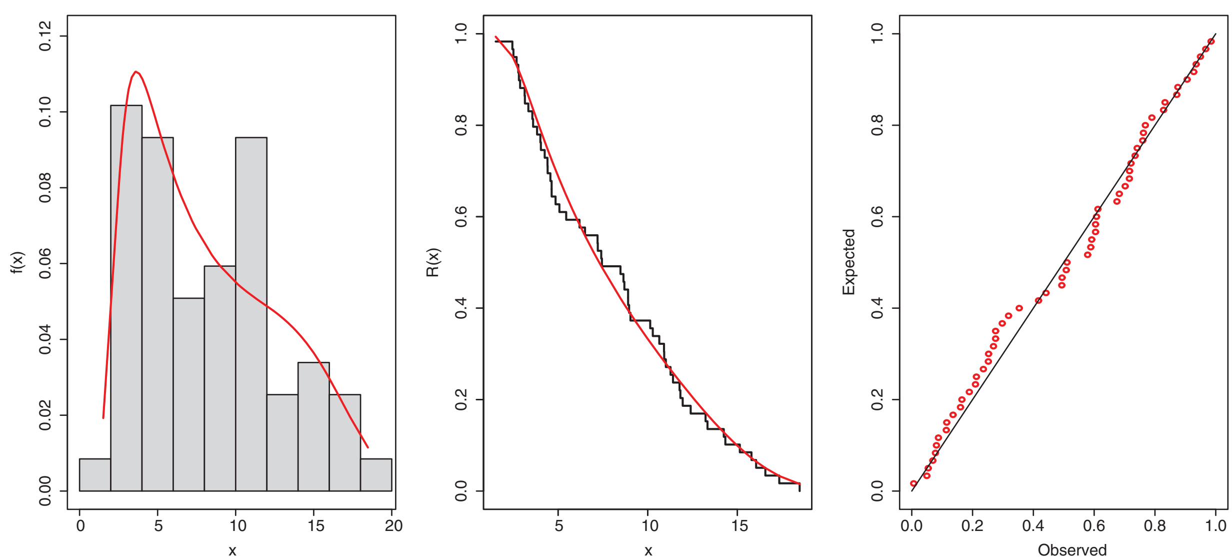

Figure 7: Histogram and fitted PDF, estimated and empirical RF and PP-plot of LTAPW distribution for Data I

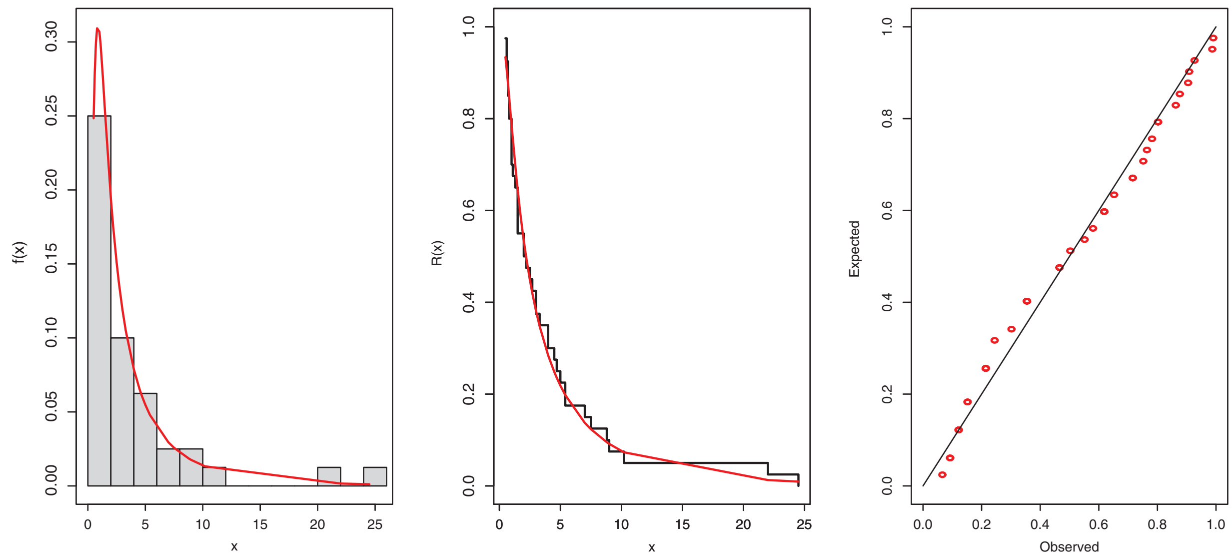

Figure 8: Histogram and fitted PDF, estimated and empirical RF and PP-plot of LTAPW distribution for Data II

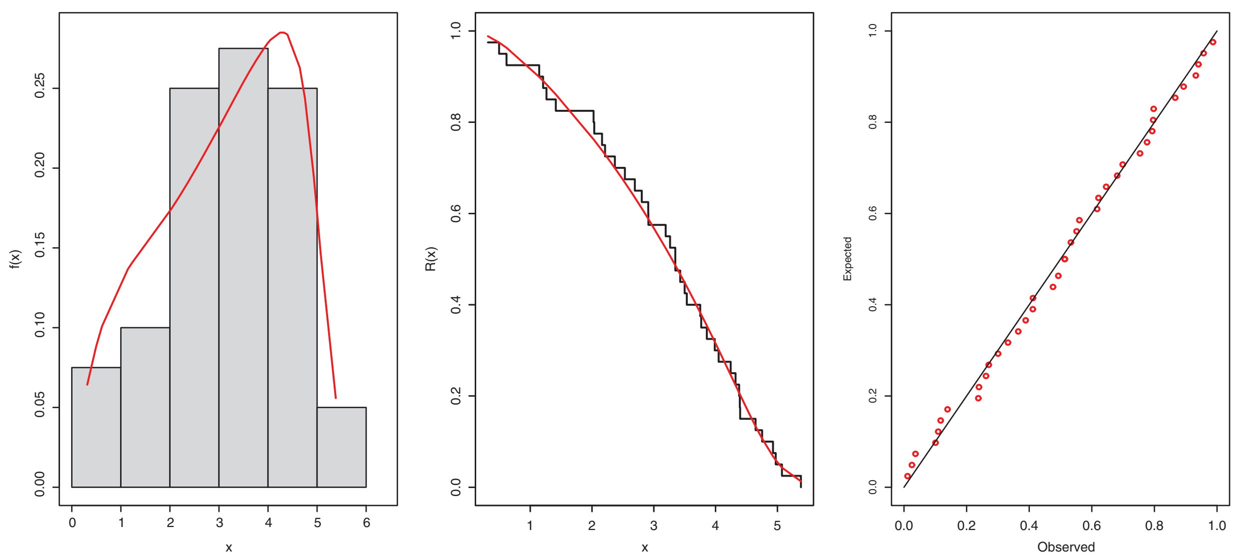

Figure 9: Histogram and fitted PDF, estimated and empirical RF and PP-plot of LTAPW distribution for Data III

Since the LTAPEx, APW and W distributions can be derived as special cases of the LTAPW distribution, then we propose to consider the likelihood ratio test to examine the suitability of the considered sub-models against the LTAPW distribution. To perform the required tests, we need to obtain $\ell(\hat{\lambda}, \hat{\alpha}, \hat{\rho}, \hat{\delta})$, $\ell(\hat{\lambda}, \hat{\alpha}, \hat{\rho})$, $\ell(\hat{\alpha}, \hat{\rho}, \hat{\delta})$ and $\ell(\hat{\rho}, \hat{\delta})$ which describe the negative log-likelihood functions computed at the MLEs of LTAPW, LTAPEx, APW and W distributions, respectively. In each test, the test statistic $\Lambda = 2[\ell(\hat{\boldsymbol{\theta}}) - \ell(\hat{\boldsymbol{\theta}}_0)]$, where $\hat{\boldsymbol{\theta}}_0$ is the MLEs of the sub-model, are computed and displayed in Table 6 for the different data sets. The results in Table 6 indicate that the LTAPW distribution is more appropriate than the other models for the given data sets.

Table 6: Likelihood ratio test outcomes for different data sets

Data	$\ell(\hat{\lambda}, \hat{\alpha}, \hat{\rho}, \hat{\delta})$	Sub-model			Λ	p-value
		$\ell(\hat{\lambda}, \hat{\alpha}, \hat{\rho})$	$\ell(\hat{\alpha}, \hat{\rho}, \hat{\delta})$	$\ell(\hat{\rho}, \hat{\delta})$		
I	162.849	166.893	–	–	8.089	0.004451
		–	167.617	–	9.536	0.002014
		–	–	167.701	9.704	0.007813
II	90.4033	94.2069	–	–	7.607	0.005814
		–	93.4720	–	5.081	0.024191
		–	–	95.5113	9.159	0.010257
III	65.0465	67.2266	–	–	4.360	0.036790
		–	68.6820	–	7.242	0.007121
		–	–	69.5579	8.994	0.011144

6 Conclusion

In this paper, we present a new lifetime distribution called the logarithmic transformed alpha power Weibull distribution. The proposed model has one scale and three shape parameters and can model left-skewed, right-skewed, and approximately symmetric data. It contains many special sub-models including Weibull, exponential, and alpha power Weibull distributions. Besides the constant, decreasing, and increasing failure rate shapes as classical forms, it is able also to model the data sets with unimodal, bathtub, unimodal then bathtub, and bathtub then unimodal shapes. The main properties of the offered distribution are derived in detail, including mixture representation, quantile function, moments, entropies, order statistics and moments of residual life. The maximum likelihood procedure is employed to acquire the point and interval estimators of the parameters. Simulation research is conducted to evaluate the functionality of the unknown parameters. The efficiency of the estimates is evaluated based on bias, root mean square error and interval length. The simulation outcomes showed that the proposed estimates perform well in terms of minimum values of the considered criteria. Three applications are considered to demonstrate the usefulness of the suggested distribution in modeling real-world data. Also, the likelihood ratio test is considered to show the superiority of the new model over some of its sub-models, namely Weibull, alpha power Weibull and logarithmic transformed alpha power Weibull distributions. Due to the flexibility of the new distribution in modeling various types of data with different failure rates, we anticipate that the new distribution will find extensive use across a variety of fields. In future work, it is of interest to investigate the estimation problems of the proposed model based on censoring schemes. Also, one may compare the classical estimation methods of the unknown parameters, including least squares and maximum product of spacing methods, with the Bayesian estimation approach.

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References

1. Mudholkar, G. S., Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability, 42(2),* 299–302.
2. Marshall, A. W., Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika, 84(3),* 641–652.
3. Cordeiro, G. M., Ortega, E. M., Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin Institute, 347(8),* 1399–1429.
4. Abouelmagd, T. H. M., Al-mualim, S., Elgarhy, M., Afify, A. Z., Ahmad, M. (2017). Properties of the four-parameter weibull distribution and its applications. *Pakistan Journal of Statistics, 33,* 449–466.
5. Nassar, M., Alzaatreh, A., Mead, M., Abo-Kasem, O. (2017). Alpha power Weibull distribution: Properties and applications. *Communications in Statistics-Theory and Methods, 46(20),* 10236–10252.
6. Nassar, M., Afify, A. Z., Dey, S., Kumar, D. (2018). A new extension of Weibull distribution: Properties and different methods of estimation. *Journal of Computational and Applied Mathematics, 336,* 439–457.
7. Cordeiro, G. M., Afify, A. Z., Ortega, E. M. M., Suzuki, A. K., Mead, M. E. (2019). The odd Lomax generator of distributions: Properties, estimation and applications. *Journal of Computational and Applied Mathematics, 347,* 222–237.
8. Shakhatreh, M. K., Lemonte, A. J., Moreno-Arenas, G. (2019). The log-normal modified Weibull distribution and its reliability implications. *Reliability Engineering & System Safety, 188,* 6–22.
9. Shakhatreh, M. K., Lemonte, A. J., Cordeiro, G. M. (2020). On the generalized extended exponential-Weibull distribution: Properties and different methods of estimation. *International Journal of Computer Mathematics, 97(5),* 1029–1057.
10. Nassar, M., Afify, A. Z., Shakhatreh, M. K., Dey, S. (2020). On a new extension of Weibull distribution: Properties, estimation, and applications to one and two causes of failures. *Quality and Reliability Engineering International, 36(6),* 2019–2043.
11. Pappas, V., Adamidis, K., Loukas, S. (2012). A family of lifetime distributions. *International Journal of Quality, Statistics, and Reliability, 2012(1),* 1–6.
12. Mahdavi, A., Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. *Communications in Statistics-Theory and Methods, 46(13),* 6543–6557.
13. Alotaibi, R., Nassar, M. (2022). A new exponential distribution to model concrete compressive strength data. *Crystals, 12(3),* 431.
14. Cordeiro, G. M., Lemonte, A. J. (2013). On the Marshall-Olkin extended Weibull distribution. *Statistical Papers, 54(2),* 333–353.
15. Almongy, H. M., Almetwally, E. M., Aljohani, H. M., Alghamdi, A. S., Hafez, E. H. (2021). A new extended Rayleigh distribution with applications of COVID-19 data. *Results in Physics, 23(60),* 104012.
16. El-Sagheer, R. M., Almuqrin, M. A., El-Morshedy, M., Eliwa, M. S., Eissa, F. H. et al. (2022). Bayesian inferential approaches and bootstrap for the reliability and hazard rate functions under progressive first-failure censoring for coronavirus data from asymmetric model. *Symmetry, 14(5),* 956.
17. Jorgensen, B. (1982). *Statistical properties of the generalized inverse gaussian distribution*. New York, USA: Springer.
18. Alotaibi, R., Okasha, H., Nassar, M., Elshahhat, A. (2022). A novel modified alpha power transformed Weibull distribution and its engineering applications. *Computer Modeling in Engineering & Sciences, 135(3),* 2065–2089. https://doi.org/10.32604/cmes.2023.023408

19. Al-Saiary, Z. A., Bakoban, R. A. (2020). The Topp-Leone generalized inverted exponential distribution with real data applications. *Entropy, 22(10),* 1144.
20. Klakattawi, H. S. (2022). Survival analysis of cancer patients using a new extended Weibull distribution. *PLoS One, 17(2),* e0264229.