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The Study of Thermal Stresses of a Two Phase FGM Hollow Sphere

Baoyu Ma[1], Guansuo Dui[1], Shengyou Yang[2] and Libiao Xin[1]

Abstract: This article focuses on the analytical solution for uniform heating of a FGM hollow sphere made of two phase of different materials. It is assumed that the volume fraction of one phase is a function $f_1 = (r^n - a^n)/(b^n - a^n)$ varied in the radial direction. Based on the Voigt constant strain approximation, analytical solutions of stresses, displacements and the effective coefficient of thermal expansion are obtained. The effects of the volume fraction, Poisson's ratio, Young's moduli and coefficients of thermal expansion on the solutions are studied. Two special cases, constant elastic modulus and constant coefficient of thermal expansion, are finally discussed.

Keywords: FGM hollow sphere, uniform heating, thermal stress, effective coefficient of thermal expansion.

1 Introduction

Functionally graded materials (FGMs) are composite materials formed of two or more constituent phases with a continuously variable composition. FGMs have a lot of advantages that make them attractive in potential applications, including a potential reduction of in-plane and transverse through-the-thickness stresses, an improved residual stress distribution, enhanced thermal properties, higher fracture toughness, and reduced stress intensity factors. To aid in the design of FGM, it would be useful to have a clear understanding of the manner in which the property gradients affect the induced thermal stresses.

Many researchers studied the thermal stresses in FGM structures by using numerical methods. Fukui et al. (1993) presented a numerical solution for the problem of uniform heating of a radially inhomogeneous thick-walled cylinder. Alavi et al. (1993) and Kwon et al. (1994) investigated the mechanical behavior of a gradient sphere under inhomogeneous temperature distribution. Wang and Mai (2005)

[1] Institute of Mechanics, Beijing Jiaotong University, 100044, Beijing, P.R. China.
[2] Department of Mechanical Engineering, University of Houston, Houston, TX 77204, USA.

used the finite element method to analyze the transient heat conduction of one-dimensional problems. By the MLPG method, Sladek et al. (2008a, b, c, 2009) discussed some thermoelastic problems of FGM structures, including thermal bending and transient heat conduction. Dong L et al. (2012, 2013a, b) developed a useful and reliable procedure of stochastic computation, which was combined with the highly accurate and efficient Trefftz Computational Grains, for a direct numerical simulation of heterogeneous materials with microscopic randomness. It can be used for ground-breaking studies in micromechanical modeling of composite materials.

Analytical solutions of the thermal problems in FGM structures are presented in many previous papers. Sugano (1990) gave the analytical solutions for the thermal stresses in a hollow cylinder whose Young's modulus and coefficient of thermal expansion radially varied but with a constant Poisson ratio. Lutz and Zimmerman (1996, 1999) presented the solution for the problem of uniform heating of a sphere and a cylinder whose elastic modulus and coefficient of thermal expansion linearly vary in the radial direction.

Using perturbation techniques, Obata and Noda (1994) studied the thermal stresses in a FGM hollow sphere and cylinder with uniform temperature. In terms of the Green's function approach, Kim and Noda (2002a, b) discussed the two-dimensional unsteady thermoelastic problem of a FGM infinitely long hollow cylinder. With the assumption that the nonhomogeneous material properties vary in the thickness direction, Jeon et al. (1997) solved the thermal stresses for a slab and a semi-infinite body, while Nadeau and Ferrari (1999) presented a one-dimensional thermal stress analysis of a transversely isotropic layer.

Assuming that the material properties are expressed as power functions of radius, analytical solutions of the Navier equations are presented in the following papers. Jabbari et al. (2002, 2003) derived analytical solutions for one-dimensional and two dimensional steady-state thermoelastic problems of the functionally graded circular hollow cylinder. Considering mutual effect of thermal and mechanical loads, Eslami et al. (2005), Jabbari et al. (2008), Poultangaria et al. (2008), Sadeghian and Toussi (2011) analytically solved the governing equation mechanical and thermal stresses in the FGM structures respectively.

As discussed above, the thermal and mechanical properties of the structures are assumed as special functions (such as power functions) of the radius or thickness due to their mathematical limitations. This assumption makes it possible to obtain the analytical solutions of the Navier equations. Actually, FGMs are composite materials in which the concentrations of the various phases are controlled so as to create gradients in macroscopic physical properties such as elastic moduli, thermal conductivity and thermal expansivity. In other words, the properties gradients of FGM only depend on the ways to change the composition, hence we can study the FGM structures by the volumetric ratio of constituents instead of assuming the material properties functions.

In this paper, we study the problem of the uniform heating of a FGM hollow sphere made of two phases of different materials, and both of whose volume fractions vary radially. Based on the Voigt constant strain approximation, we present the analytical solution for the stresses, displacements and the effective coefficient of thermal expansion within the hollow sphere.

2 Theoretical model

2.1 Governing equations for radially symmetric deformations

Consider a thick hollow FGM sphere of inner radius a and outer radius b. The FGM is a two-phase composite material. The two components, denoted by material A and material B, whose volume fractions vary in the r-direction, thus the material properties are functions of r. Let u be the radial displacement. It is assumed that the volume fraction of material B is the function of r as

$$f_1 = (r^n - a^n)\big/(b^n - a^n) \tag{1}$$

For small deformation, the strain-displacement relations are

$$\varepsilon_{rr} = du\big/dr, \quad \varepsilon_{\theta\theta} = u/r \tag{2}$$

With the Voigt constant strain approximation, we have

$$\varepsilon_{kl} = \varepsilon_{kl}^0 = \varepsilon_{kl}^1 \tag{3}$$

The average stress is:

$$\sigma_{kl} = \sum f_i \sigma_{kl}^i = f_0 \sigma_{kl}^0 + f_1 \sigma_{kl}^1 \tag{4}$$

Where i=0, 1 demotes material A and B, respectively; σ_{kl} and ε_{kl} (k, $l = r$, θ) are the average stress and strain tensors, respectively; σ_{kl}^i and ε_{kl}^i (i=0, 1) are the average stress and strain tensors of the i-th material, respectively.

The thermoelastic stress-strain relations for an isotropic material are

$$\sigma_{rr}^i = (\lambda_i + 2\mu_i)\varepsilon_{rr} + 2\lambda_i\varepsilon_{\theta\theta} - 3K_i\alpha_i(T - T_0) \tag{5a}$$

$$\sigma_{\theta\theta}^i = \lambda_i\varepsilon_{rr} + 2(\mu_i + \lambda_i)\varepsilon_{\theta\theta} - 3K_i\alpha_i(T - T_0) \tag{5b}$$

where λ and μ are the Lamé parameters, respectively, α is the coefficient of thermal expansion, K is the bulk modulus, T is the temperature and T_0 is the reference temperature at which the sphere is stress-free.

The equilibrium equation is

$$r\frac{d\sigma_{rr}}{dr}+2(\sigma_{rr}-\sigma_{\theta\theta})=0 \tag{6}$$

Combining Eqs. (2)-(5), the average stress-displacement relations are:

$$\sigma_{rr}=\sum_{i=0}^{1} f_i\left[(\lambda_i+2\mu_i)\frac{du}{dr}+2\lambda_i\frac{u}{r}-\kappa_i\Delta T\right] \tag{7a}$$

$$\sigma_{\theta\theta}=\sum_{i=0}^{1} f_i\left[\lambda_i\frac{du}{dr}+2(\mu_i+\lambda_i)\frac{u}{r}-\kappa_i\Delta T\right] \tag{7b}$$

where $\kappa_i=3K_i\alpha_i=(3\lambda_i+2\mu_i)\alpha_i, \Delta T=(T-T_0)$.
Substituting Eqs.(7) and (1) into the stress equilibrium equation (6), we have the following governing equation:

$$\begin{aligned}&r\frac{d^2u}{dr^2}(Sr^n-1)+\frac{du}{dr}[S(n+2)r^n-2]+\frac{u}{r}2\left\{r^nS\left[n(\lambda_0-\lambda_1)D^{-1}-1\right]+1\right\}\\&-r^nSD^{-1}(\kappa_0-\kappa_1)\left(n\Delta T(r)+r\frac{dT}{dr}\right)+r\frac{dT}{dr}SD^{-1}\left[\kappa_0-\kappa_1\left(a/b\right)^n\right]=0\end{aligned} \tag{8}$$

where

$$S=\frac{(2\mu_0+\lambda_0)-(2\mu_1+\lambda_1)}{(2\mu_0+\lambda_0)-\left(a/b\right)^n(2\mu_1+\lambda_1)} \quad D=[(2\mu_0+\lambda_0)-(2\mu_1+\lambda_1)]$$

Using the transformations $u=ry(z)$ and $z=S\bar{r}^n$, Eq. (8), a hypergeometric differential equation, reduces to

$$z(z-1)y''(z)+[(1+\alpha+\beta)z-\delta]y'(z)+\alpha\beta y(z)+P_T/n^2z=0 \tag{9}$$

where

$$\delta{=}1+\frac{3}{n},\quad \alpha=\frac{1}{2}\left(\delta-\sqrt{\delta^2-4\frac{D+2(\lambda_0-\lambda_1)}{nD}}\right),\quad \beta=\delta-\alpha$$

$$P_T=-nz\Delta T(r)(\kappa_0-\kappa_1)D^{-1}+r\frac{dT}{dr}\left\{SD^{-1}\left[\kappa_0-\kappa_1\left(a/b\right)^n\right]-zD^{-1}(\kappa_0-\kappa_1)\right\}$$

2.2 Thermal stresses of a FGM spherical shell

Now consider the thermal stresses in a hollow sphere that is uniformly heated. In this case, $\frac{dT}{dr}$ in P_T of Eq. (9) vanishes and Eq. (9) becomes:

$$z(z-1)y''(z)+[(1+\alpha+\beta)z-\delta]y'(z)+\alpha\beta y(z)=n^{-1}D^{-1}\Delta T(r)(\kappa_0-\kappa_1) \quad (10)$$

The solution of Eq.(10) is

$$y(z)=C_1F(\alpha,\beta,\delta,z)+C_2z^{1-\delta}F(\alpha+1-\delta,\beta+1-\delta,2-\delta,z)+\frac{\Delta T(r)(\kappa_0-\kappa_1)}{D+2(\lambda_0-\lambda_1)} \quad (11)$$

where C_1 and C_2 are integral constants and F is the hypergeometric function defined for

$$F(\alpha,\beta,\delta;z)=\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\delta)_m}\frac{z^m}{m!} \quad (12)$$

Here $(q)_m$ is the Pochhammer symbol, which is defined by

$$(q)_m=\begin{cases}1 & m=0\\ q(q+1)\cdots(q+m-1) & m>0\end{cases} \quad (13)$$

Recall the transformation u=$ry(z)$, the radial displacement is finally given by

$$\begin{aligned}u(r)=&C_2rz^{1-\delta}F(\alpha+1-\delta,\beta+1-\delta,2-\delta,z)\\&+C_1rF(\alpha,\beta,\delta,z)+r\Delta T(r)(\kappa_0-\kappa_1)[D+2(\lambda_0-\lambda_1)]^{-1}\end{aligned} \quad (14)$$

Substituting Eq. (14) into Eqs. (2) and (7), the strains and stresses can be obtained. The integral constants C_1 and C_2 can be determined by the boundary conditions.

2.3 Convergence of the series solution

It is obvious that the hypergeometric series converges for $0\le z\le 1$, however, this is not true for $z<0$ or $z>1$. To insure the convergence of the hypergeometric series F in Eq. (11) on the interval $a\le r\le b$, the following variable transformations can be used to convert Eq.(10) into another hypergeometric differential equation of $\eta(\xi)$ with the independent variable $\xi\in[0,1]$ as:

$$z<0,\quad \xi=\frac{z}{z-1},\quad y(z)=|z-1|^{-\alpha}\eta(\xi) \quad (15a)$$

$$z>1,\quad \xi=\frac{1}{z},\quad y(z)=|z|^{-\alpha}\eta(\xi) \quad (15b)$$

3 Results and discussion

In the simulation, both the outer and the inner boundaries are stress-free. The non-dimensional expressions for the radial coordinate, the stress and the radial displacement are defined as $\underline{r} = r/b$, $\underline{\sigma}_{ij} = \sigma_{ij}/(3K_0\alpha_0\Delta T)$ and $\underline{u} = u/(b\alpha_0\Delta T)$, respectively. For simplicity, let $\underline{\alpha} = \alpha_1/\alpha_0$ and $\underline{E} = E_1/E_0$. The ratio of the inner and the outer radii is taken as *a*/*b*=0.5, throughout, which is reasonable for a thick-walled shell.

3.1 Displacements

As shown in Eqs. (14), the stresses and the radial displacement depend on the following factors: the ratio of the elastic moduli and the ratio of the thermal expansion coefficients of the two constituents $\underline{E}$ and $\underline{\alpha}$, the Poisson ratio υ_0, υ_1 and the inhomogeneity parameter *n* which determines the contents of the compositions. Hence, the effects of these variables on stresses and displacement are discussed in the following, respectively.

In Figure 1, the radial displacement $\underline{u}$ is plotted as a function of radius $\underline{r}$, with discussing dependencies on $\underline{\alpha}$ and $\underline{E}$ in Figure 1(*a*), *n* in Figure 1(*b*), and υ_1 in Figure 1(*c*).

If the sphere is homogeneous, the displacement is given by $u(r) = \alpha_0 r\Delta T$, which corresponds in Figure 1(*a*) to the case $\underline{\alpha}$ =1, which indeed appears as a straight line with a 1:l slope. When $\underline{\alpha}$>1, the local effective thermal expansion coefficient of the composite increases with radius, and the radial normal strain ε_{rr} consequently increases with *r*, causing the $u(r)$ curve to be concave upward. Conversely, when $\underline{\alpha}$<1, ε_{rr} decreases with *r*, leading to a displacement curve that is concave downward. The conclusion is the same as that of Melanie P. Lutz and Robert W. Zimmerman (1996) for a FGM sphere. This effect is more obvious when $\underline{E}$ or υ_1 increase. It is also strengthened by the decreasing *n*, which means the reduced volume fraction of material B. The radial displacement *u* is sensitive to all of these variables mentioned above; therefore the effects of them are not negligible.

3.2 Thermal stresses

It is known that an unconstrained uniformly heated homogeneous hollow sphere would incur no thermal stresses. Hence the stresses in this simulation are purely a result of the inhomogeneous properties of the hollow sphere. The radial stress $\underline{\sigma}_{rr}$ is shown in Figure 2, and the tangential (hoop) stress $\underline{\sigma}_{\theta\theta}$ is shown in Figure 3.

When $\underline{\alpha}$>1, the local effective thermal expansion coefficient increases with radius, and so the outer edges of the sphere tend to expand more than the inner edges; this causes the hollow sphere to be under tensile radial stress. Conversely, if $\underline{\alpha}$<1, the

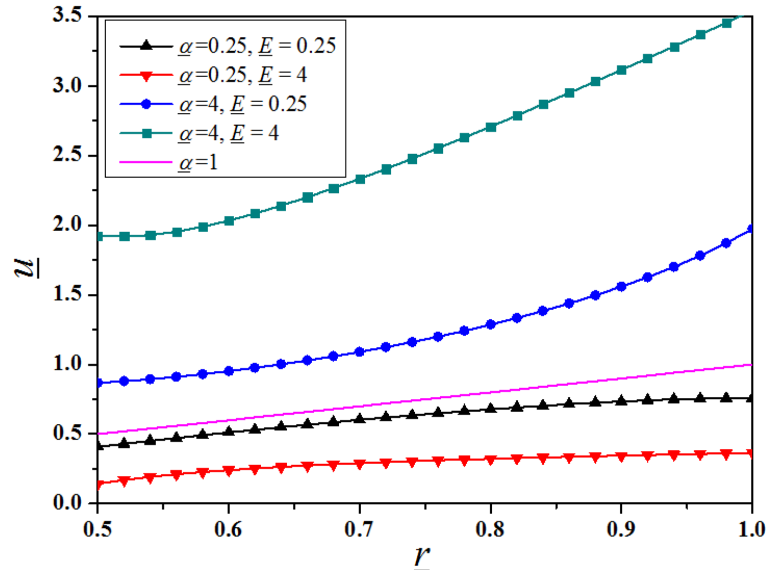

Figure 1(a): Evolution of $\underline{u}$ with different $\underline{\alpha}$ and $\underline{E}$ (υ_0=υ_1=0.25, n=4)

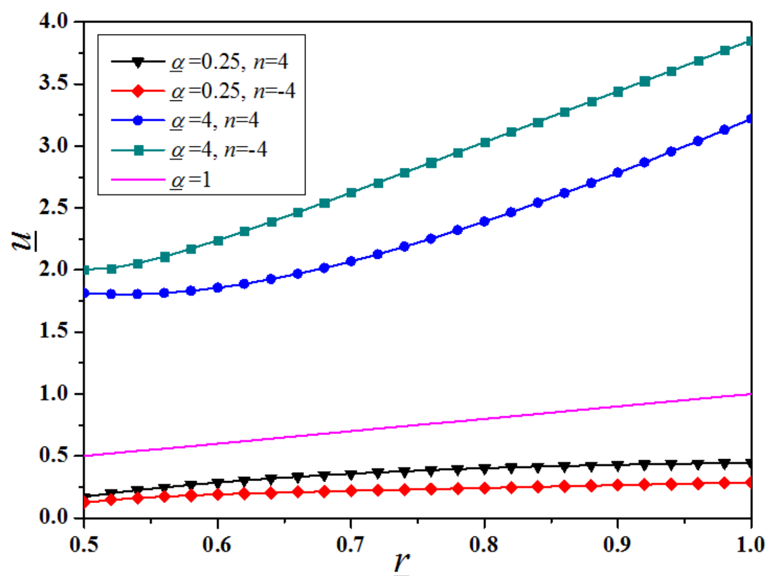

Figure 1(b): Evolution of $\underline{u}$ with different $\underline{\alpha}$ and n (υ_0=υ_1=0.25, $\underline{E}$ =4).

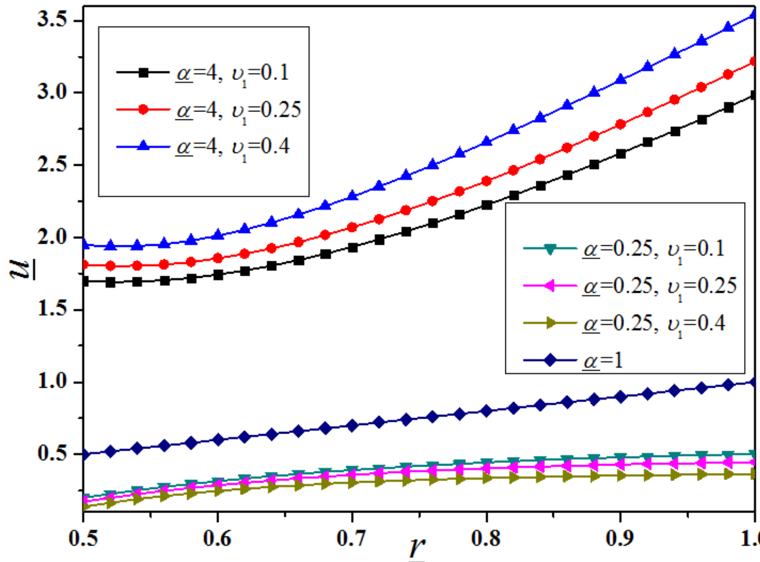

Figure 1(c): Evolution of $\underline{u}$ with different $\underline{\alpha}$ and υ_1 (υ_0=0.25, n=4, $\underline{E}$ =4)

outer edges of the sphere tend to expand less than the inner edges, the hollow sphere is consequently under compression. This dependency of the radial stress on $\underline{\alpha}$ gets even bigger when $\underline{E}$ or υ_1 rise. The radial stress σ_{rr} is the largest in magnitude at some intermediate radius, which increases with the argument n ascends.

The hoop stress is the largest in magnitude at one edge but is the opposite sign at the other edge, changing signs at some intermediate radius. These results are in qualitative agreement with the results found by Kolyano and Makhorkin (1976) for a two-shell sphere, Obata and Noda (1994) for a hollow FGM sphere and Robert W. Zimmerman and Melanie P. Lutz (1996) for a FGM sphere.

When $\underline{\alpha}$>1, the outer edges of the sphere tend to expand more than the inner edges; this causes the inner part of the hollow sphere to be under tensile hoop stress, while the outer edges are consequently under compression in the hoop, and vice versa. The hoop stresses are monotonic functions of the radius. This effect of $\underline{\alpha}$ is more obvious with the decreasing of $\underline{E}$.

The dependencies on n and υ_1 are not small. They influence not only on the magnitude but also the convexity of the hoop stress-r curves. However, it is difficult to find a simple relationship between the hoop stresses and these arguments. It is suggested that increasing n and decreasing υ_1 lead to smaller hoop stresses in magnitude at the inner edge and larger ones at the outer edge. Furthermore, when $\underline{\alpha}$>1,

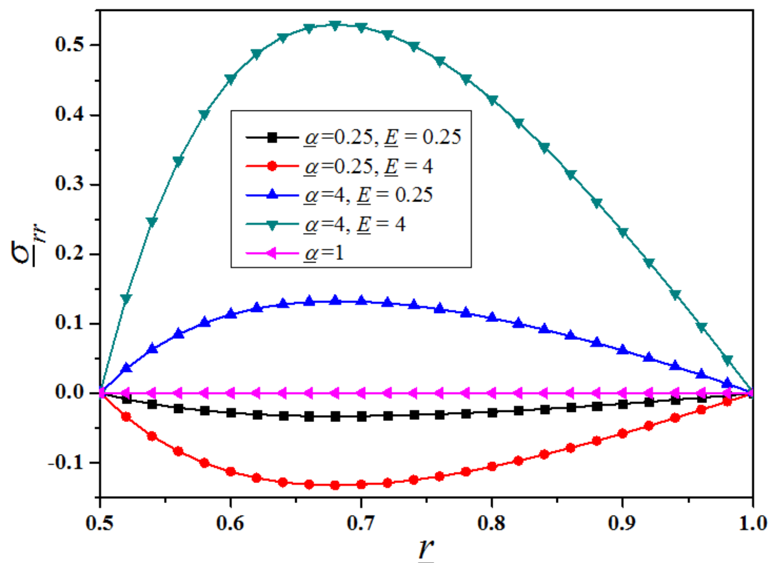

Figure 2(a): Evolution of $\underline{\sigma}_{rr}$ with different $\underline{\alpha}$ and $\underline{E}$ (υ_0=υ_1=0.25, n=4)

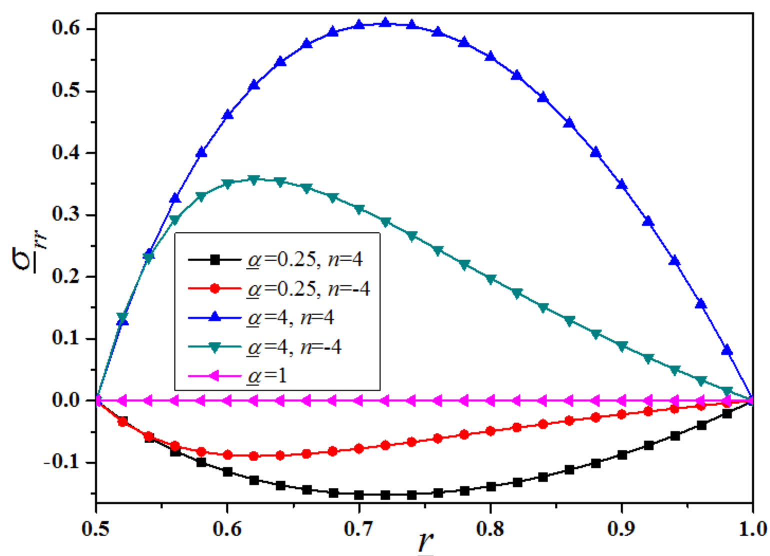

Figure 2(b): Evolution of $\underline{\sigma}_{rr}$ with different $\underline{\alpha}$ and n (υ_0=υ_1=0.25, $\underline{E}$ =4)

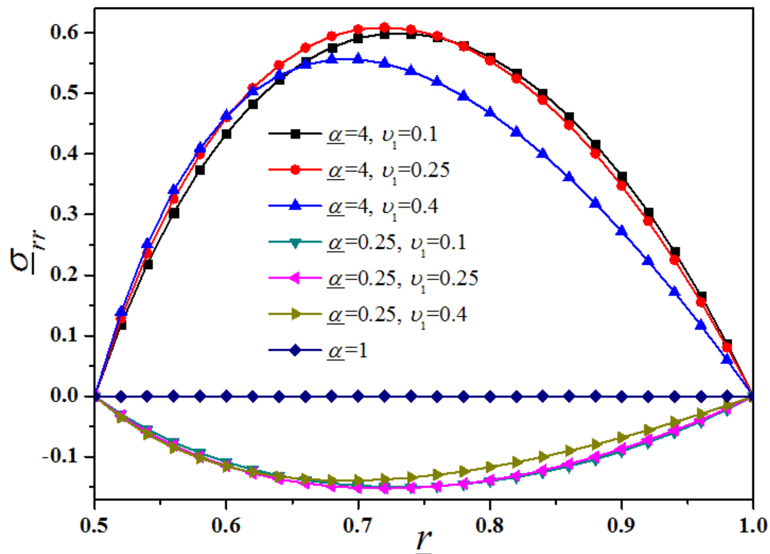

Figure 2(c): Evolution of $\underline{\sigma}_{rr}$ with different $\underline{\alpha}$ and υ_1 (υ_0=0.25, n=4, $\underline{E}$ =4)

a more strong tendency to be concave downward of the curves is shown with the decrease of n but increase υ_1. However, it is reverse when $\underline{\alpha}$<1.

The deviatoric part of the stress tensor, which controls the onset of plastic deformation, attains it largest magnitude at the inner or outer edge of the hollow sphere, depending on the particular circumstances.

4 Effective coefficient of thermal expansion

It is known that if the temperature of an unconstrained homogeneous body V increases ΔT, the total volume variation is $3\alpha\Delta TV$. Hence, the effective coefficient of thermal expansion α_{eff} is

$$\alpha_{eff} = \Delta V \big/ (3V\Delta T) \tag{16}$$

The volume variation of the hollow sphere related to the displacement field is given by

$$\Delta V = 4\pi \left[b^2 u(b) - a^2 u(a)\right] \tag{17}$$

where $u(r)$ is the radial displacement. Combining Eq. (16) and (17) leads to

$$\alpha_{eff} = \left[u(b) - (a/b)^2 u(a)\right] \Big/ \left\{\Delta T b \left[1 - (a/b)^3\right]\right\} \tag{18}$$

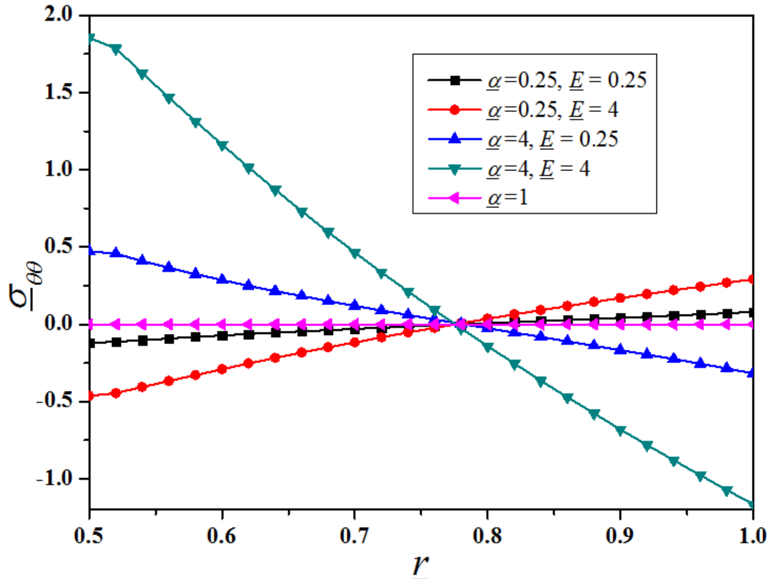

Figure 3(a): Evolution of $\underline{\sigma}_{\theta\theta}$ with different $\underline{\alpha}$ and $\underline{E}$ (υ_0=υ_1=0.25, n=4)

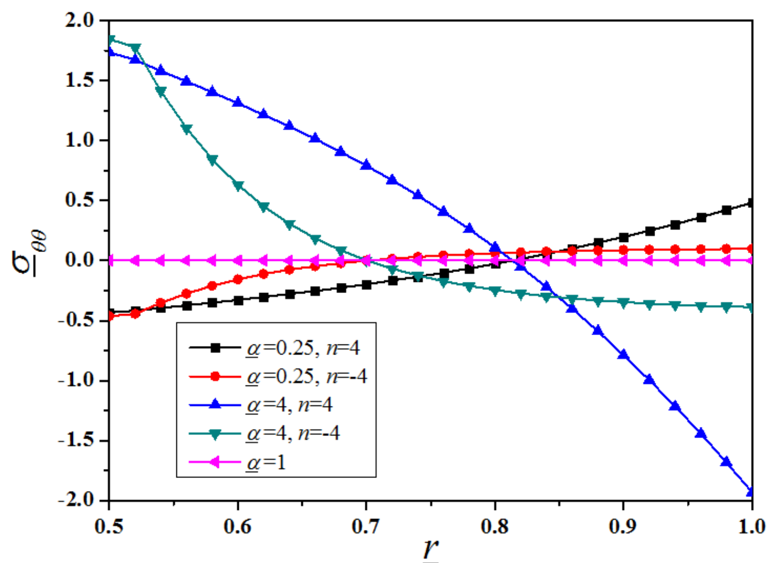

Figure 3(b): Evolution of $\underline{\sigma}_{\theta\theta}$ with different $\underline{\alpha}$ and n (υ_0=υ_1=0.25, $\underline{E}$ =4)

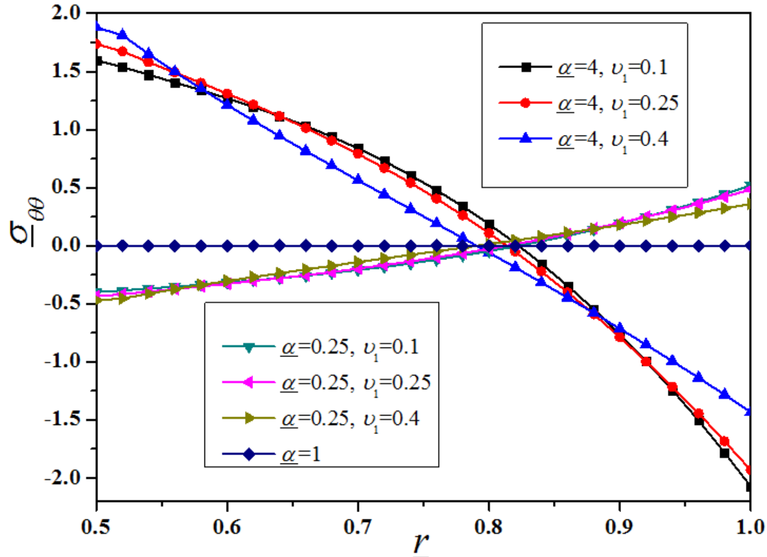

Figure 3(c): Evolution of $\underline{\sigma}_{\theta\theta}$ with different $\underline{\alpha}$ and υ_1 (υ_0=0.25, n=4, $\underline{E}$ =4)

The non-dimensional expressions for the the effective thermal expansion coefficient is defined as $\alpha_{eff} = \alpha_{eff}/\alpha_0$ and shown in Figure 4 as a function of $\underline{\alpha}$.

As shown in Eq. (18) and Figure 4, $\underline{\alpha}_{eff}$ depends on $\underline{E}$, $\underline{\alpha}$, υ_0, υ_1 and n. The $\underline{\alpha}_{eff}$-$\underline{\alpha}$ curve is substantially a straight line through the point (1, 1). The slope of the curve increases with the decrease of n, which means an increasing volume fraction of material B. The increases of the relative value $\upsilon_1$to υ_0 and $\underline{E}$ also lead to a larger slope. In addition, the magnitudes of the Poisson ratio has only a small effect on the effective thermal expansion coefficient if υ_1=υ_0.

5 Special cases of uniform moduli or thermal expansion coefficient

It is interesting to consider two special cases where either the elastic moduli or the thermal expansion coefficient varies, but not both. In the case where $\underline{\alpha}$=1 throughout the hollow sphere, manipulating Eqs.(8) to (14), we get the solution u=$r\alpha_0\Delta T$. Combining Eq. (18), α_{eff}=α_0 is found. Further, no thermal stress is induced, regardless of the variation in the elastic moduli.

Now consider the case of uniform elastic modulus but varied coefficient of thermal expansion. Then Eq. (8) can be written as:

$$r\frac{d^2u}{dr^2}+2\frac{du}{dr}-2\frac{u}{r}+\frac{n(1+\upsilon_0)\Delta T(\alpha_0-\alpha_1)}{\left[1-(a/b)^n\right](1-\upsilon_0)}r^n=0 \tag{19}$$

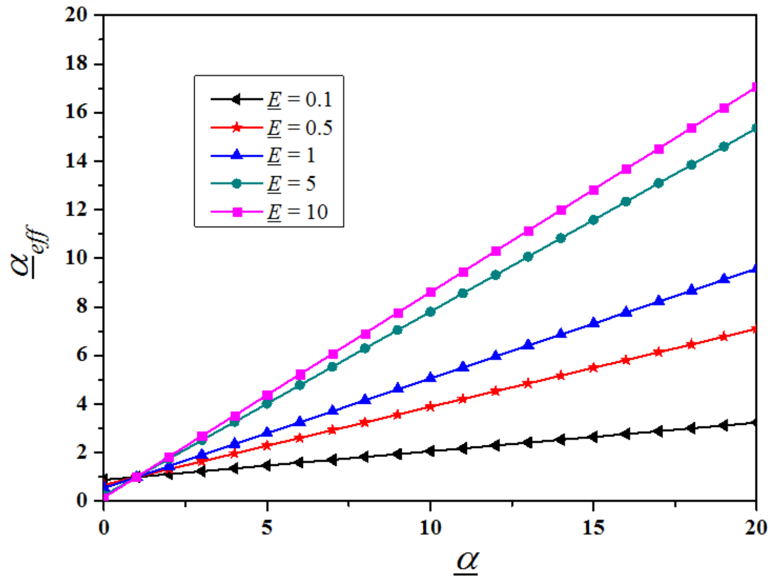

Figure 4(a): Evolution of $\underline{\alpha}_{eff}$ with different values of $\underline{E}$ (υ_0=υ_1=0.25, n=4)

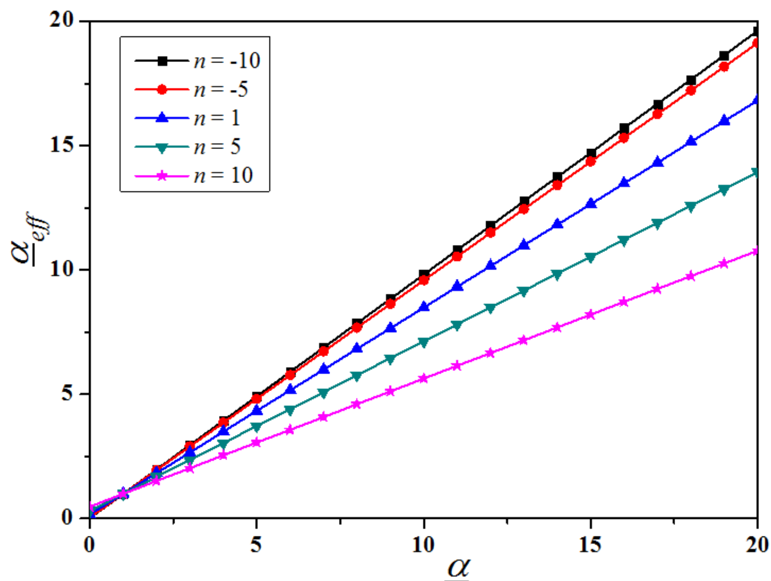

Figure 4(b): Evolution of $\underline{\alpha}_{eff}$ with different values of n (υ_0=υ_1=0.25, $\underline{E}$=4)

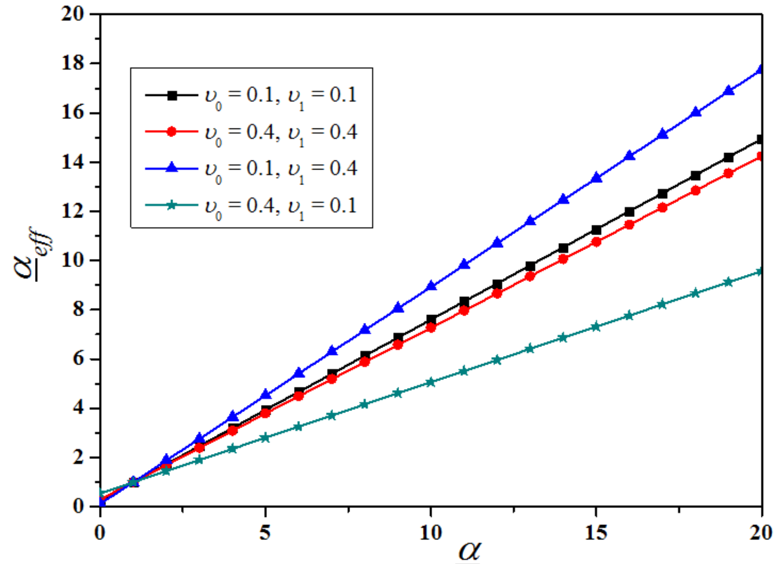

Figure 4(c): Evolution of $\underline{\alpha}_{eff}$ with different Poisson's ratio ($\underline{E}$=4, n=4)

The general solution of the equation is

$$u = C_1 r + C_2 r^{-2} - \frac{(1+\upsilon)(\alpha_0 - \alpha_1)\Delta T}{(n+3)(1-\upsilon)\left[1-(a/b)^n\right]} r^{n+1} \tag{20}$$

Substituting Eq. (20) into Eq. (18) yields

$$\alpha_{eff} = (1-h_1)\alpha_0 + h_1\alpha_1 \tag{21}$$

$$h_1 = \frac{n(a/b)^{n+3} - (n+3)(a/b)^n + 3}{(n+3)\left[1-(a/b)^3\right]\left[1-(a/b)^n\right]} \tag{22}$$

This result is precisely equal to the volumetric average of the local thermal expansion coefficient, since

$$\langle\alpha\rangle = \frac{1}{V}\int_a^b 4\pi r^2 (f_0\alpha_0 + f_1\alpha_1)dr = (1-h_1)\alpha_0 + h_1\alpha_1 \tag{23}$$

It concurs with that found by Robert W. Zimmerman and Melanie P. Lutz for a sphere (1996) and cylinder (1999) in which the thermal expansion coefficient varied linearly with radius. It is also consistent with Levin's theorem, in the case in which the two components have identical elastic modules but different thermal expansion coefficients.

More generally, if $K_0 \neq K_1$, Levin's theorem (1967) predicts that the effective thermal expansion coefficient is given by

$$\alpha_{eff} = \alpha_0 + (\alpha_1 - \alpha_0)\,(K_1^{-1} - K_0^{-1})^{-1}(K_*^{-1} - K_0^{-1}) \tag{24}$$

Where K_* is the effective bulk module of the composite material. For convenience, take the volumetric mean as effective bulk module, thus we get

$$\alpha_{eff} = [(1-h_1)K_0\alpha_0 + h_1K_1\alpha_1] / [(1-h_1)K_0 + h_1K_1] \tag{25}$$

A comparison of the results obtained by present method and Levin's theorem (Eq. (25)) is shown in Figure 5. It shows that they are close to each other and Levin's theorem leads to a larger slope.

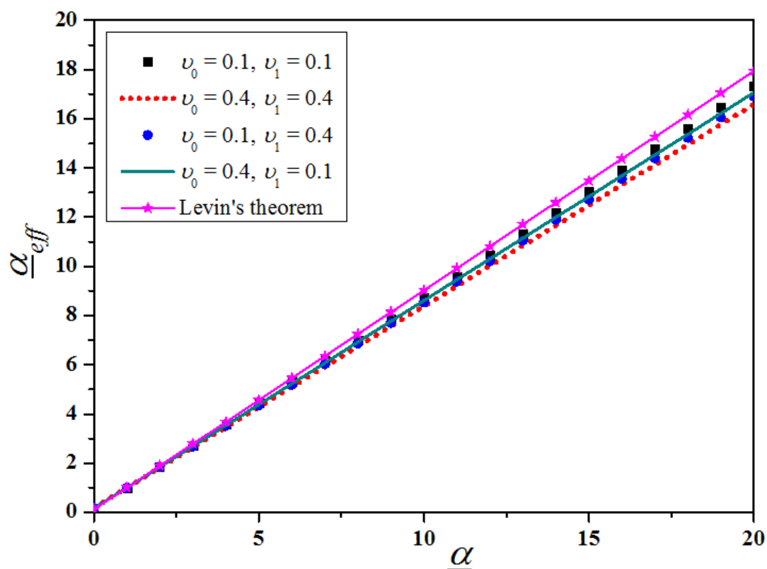

Figure 5: Comparison of $\underline{\alpha}_{eff}$ obtained with present methods and Levin's theorem (K_1/K_0=10, n=4)

6 Summary and conclusions

Based on the Voigt constant strain approximation, an exact solution has been presented for the problem of the uniform heating of a functionally gradient hollow sphere made of two phase of different materials both of whose volume fractions vary with the radius. We found analytical expressions for the stress and displacement fields as well as for the effective thermal expansion coefficient. All of them depend on the following factors: the ratio of the elastic moduli and the ratio of the thermal expansion coefficients of the two constituents $\underline{E}$ and $\underline{\alpha}$, the Poisson ratio υ_0, υ_1 and the inhomogeneity parameter n which determines the contents of the compositions.

If $\underline{\alpha}>1$, $u(r)$ curve is concave upward, the radial stress is positive (tensile), the hoop stress is positive at the inner edge and negative at the outer edge of the hollow sphere, and vice versa. The effect is strengthened by the increase values of $\underline{E}$, $\underline{\alpha}$ and υ_1, but the decrease of n and υ_0.

The effective thermal expansion coefficient is nearly equal to the weighted average by product of volume fractions and bulk modulus of the local thermal expansion coefficient. In the special case where the elastic modules are uniform, the effective thermal expansion coefficient is equal to the volume average of the local thermal expansion coefficient. It is consistent with Levin's theorem (1967) and the results presented by Robert W. Zimmerman and Melanie P. Lutz (1996, 1999).

Due to the linearity of the governing equations, this solution can be used to solve the effective bulk modulus of a gradient hollow sphere. We have shown the thermal stress of a functionally gradient hollow sphere uniform heated, that with ununiform temperature field is an interesting open question.

Acknowledgement: The work was supported by the National Natural Science Foundation of China (Nos.11132003, 11172033, 11272044, 11272136) and the support of Tianjin Natural Science Foundation (No. 15JCQNJC42600) and the National Basic Research Program of China (973 Program) (2010CB7321004).

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