 CMES, vol.109-110, no.6, pp.505-517, 2015

A Note on Solving the Generalized Dirichlet to Neumann Map on Irregular Polygons using Generic Factored Approximate Sparse Inverses

E-N.G. Grylonakis[1], C.K. Filelis-Papadopoulos[1] and G.A. Gravvanis[1]

Abstract: A new transform method for solving boundary value problems in two dimensions was proposed by A.S. Fokas, namely the *unified transform.* This approach seeks a solution to the unknown boundary values by solving a *global relation*, using the known boundary data. This relation can be used to characterize the Dirichlet to Neumann map. For the numerical solution of the global relation, a collocation-type method was recently introduced. Hence, the considered method is used for solving the 2D Laplace equation in several irregular convex polygons. The linear system, resulting from the collocation-type method, was solved by the Explicit Preconditioned Generalized Minimum Residual restarted method in conjunction with the Modified Generic Factored Approximate Sparse Inverse matrix. Numerical results indicating the applicability of the proposed preconditioning scheme are provided, along with discussions on the implementation details of the method.

Keywords: Laplace Equation, Dirichlet-Neumann map, Global Relation, Collocation, Generic Factored Approximate Sparse Inverses, Preconditioned Generalized Minimum Residual restarted method.

1 Introduction

Solving boundary value problems (BVPs) is an important topic in the area of applied mathematics. Exact solutions can be derived by considering analytical methods, although their implementation can be difficult for complicated problems. On the contrary, numerical methods provide approximate solutions within given tolerance, even in cases where the analytical solution is not known. A novel approach for solving a class of BVPs in two dimensions, namely the *unified transform* has been introduced by Fokas, [Fokas (1997); Fokas (2002)]. The solution of the B-

[1] Department of Electrical and Computer Engineering, School of Engineering, Democritus University of Thrace, University Campus, Kimmeria, GR 67100 Xanthi, Greece.

VP is initially expressed in integral form, in the complex k-plane, forming the so-called *global relation*. This relation couples the known and the unknown values of the solution and its derivatives on the boundary and additionally, it is valid for all values of the complex parameter k. The global relation can be solved analytically for a class of BVPs, however, for general problems numerical techniques should be considered. Recently, a collocation-type method has been introduced, [Fulton, Fokas and Xenophntos (2004); Sifalakis, Fokas, Fulton and Saridakis (2008)], for the numerical solution of the global relation resulting in solving a dense linear system, deriving an approximate solution to the unknown boundary values. The unified transform has been used on several polygonal domains for the solution of the Laplace equation, [Saridakis, Sifalakis and Papadopoulou (2008); Sifalakis, Fokas, Fulton and Saridakis (2008)]. Furthermore, the condition number of the resulting coefficient matrix can be improved by an appropriate choice of collocation points, [Sifalakis, Fokas, Fulton and Saridakis (2008)]. Moreover, the characteristics of the coefficient matrix, the convergence behavior and the computed error of the method are affected by the choice of the basis functions [Saridakis, Sifalakis and Papadopoulou (2008); Sifalakis, Fokas, Fulton and Saridakis (2008)]. Especially, it was shown in [Sifalakis, Fokas, Fulton and Saridakis (2008)] that better computed errors were obtained by choosing the Chebyshev basis functions, compared to the sine basis functions. Hence, the unified transform method has been considered for the solution of the Laplace equation in two space variables, on various irregular polygonal domains, choosing the sine as well as the Chebyshev basis functions. The resulting dense linear system is solved by the Explicit Preconditioned Generalized Minimum Residual restarted (EPGMRES(m)), method, [Saad and Schultz (1986)], in conjunction with the Modified Generic Factored Approximate Sparse Inverse (MGenFAspI) matrix, [Filelis-Papadopoulos and Gravvanis (2015)].

There exist several other methods for the solution of the Laplace equation, namely the Finite Difference Method (FDM), the Finite Element Method (FEM), the Boundary Element Method (BEM), the Trefftz method, the Method of Fundamental Solutions (MFS), [Golberg and Chen (1999)], the Meshless Local Petrov-Galerkin method, [Atluri and Zhu (1998)] and many more. There are plenty of advantages as well as limitations that characterize each of the above methods. Particularly, both the FDM and the FEM, although well established, suffer from the computational costs associated with the generation of the computational mesh, while the FDM presents difficulties for handling complicated geometries. On the other hand, the BEM is a mesh reduction method, decreasing the dimensions of the considered BVP by one and consequently, lowering the overall computational costs. However the BEM makes use of a fundamental solution of the PDE, which is not always available. Additionally, the computation of singular integrals is often required. The

MFS uses a fundamental solution of the considered PDE as a basis function for the approximation of the solution. Again, the success of this method lies on the availability of a fundamental solution, which is not always guaranteed for complicated problems. The Trefftz method can be classified as a boundary-type procedure, where the solution is approximated by the superposition of T-complete functions, which satisfy the corresponding PDE. It presents several advantages over the above methods such as the avoidance of computing singular integrals, the high accuracy due to the use of particular solutions as well as its high efficiency. However, the major drawbacks of the Trefftz method are the lack of explicit particular solutions for complicated PDEs, the large condition number of the associated coefficient matrix and the limited theoretical analysis compared to other methods (such as BEM). A detailed review of Trefftz and other boundary-type methods can be found in [Li, Lu, Huang and Cheng (2007)]. In the last years efforts have been made towards the development of more efficient, modified schemes, in order to overcome the aforementioned difficulties. In [Liu (2007b), (2007c)], a modified Trefftz method was developed, by considering the domain's characteristic length. This method can be applied to more complicated domains and it is highly accurate. In [Liu (2007a)], a modified Trefftz method was developed for the solution of the mixed boundary value problem of the Laplace equation, overcoming the ill-conditioning of the regular Trefftz method. Furthermore, in [Liu, Yeih and Atluri (2009)], preconditioning techniques were developed for tackling the problem of ill-conditioned matrices arising in Trefftz boundary-collocation methods, presenting computational efficiency and high accuracy. Besides the above methods, it has been realized that suitable coupling techniques can be considered for more complicated PDEs. In this direction the Hybrid-Trefftz Finite Element, [Jirousek and Guex (1986)], method has become an efficient computational tool for the solution of complex BVPs. Recently, a novel method namely T-Trefftz Voronoi Cell Finite Elements (VCFEM), [Dong and Atluri (2011a), (2011b)] has been developed. This method utilizes the characteristic length presented in [Liu (2007b)], avoids domain integrations and does not suffer from LBB conditions.

This paper is organized as follows: In section 2 the unified transform method leading to the derivation of the Dirichlet-to-Neumann map is reviewed. In section 3 the collocation-type method for the approximation of the derived Dirichlet-to-Neumann map is also reviewed. In section 4 the MGenFAspI preconditioning scheme is presented. In section 5 numerical results and discussions concerning the details of the proposed schemes are provided, as well as possible advantages over existing numerical methods.

2 The Unified Transform Method

Let us consider the 2D Laplace equation in complex coordinates, [Sifalakis, Fokas, Fulton and Saridakis (2008)], and the real-valued function $q(z,\bar{z})$, satisfying the Laplace equation in a simply bounded domain D with boundary ∂D. Then, considering the complex form of Green's theorem, the following global condition, [Fokas (2001)], can be derived:

$$\int_{\partial \mathrm{D}} \mathrm{e}^{-ikz}\frac{\partial \mathrm{q}}{\partial \mathrm{z}}\mathrm{dz}=0, \quad \mathrm{k}\in\mathbb{C}, \quad \Rightarrow \quad \sum_{j=1}^{n}\int_{S_j} e^{-ikz}\frac{\partial q}{\partial z}dz=0, \ k\in\mathbb{C}, \tag{1}$$

where S_j denotes the side of the polygon from vertex z_j to vertex z_{j+1}, j=1,2,...,n (the endpoints are not included).

If the real-valued function $q(z,\bar{z})$ satisfy the Laplace PDE in the interior D of a convex bounded polygon with vertices $\mathrm{z}_1, \mathrm{z}_2, \ldots, \mathrm{z}_n$, the following equations hold:

$$\begin{aligned} &g^{(j)}=\cos(\beta_j)q_s^{(j)}+\sin(\beta_j)q_n^{(j)}, \quad f^{(j)}= -\sin(\beta_j)q_s^{(j)}+\cos(\beta_j)q_n^{(j)}, \\ &z\in S_j, \ 1\le j\le n, \end{aligned} \tag{2}$$

where $q_s^{(j)}$ and $q_n^{(j)}$ denote the tangential and normal components of $\frac{\partial q}{\partial z}$ along the side S_j, respectively, $g^{(j)}$ denotes the derivative of the solution in the direction making an angle $\beta_{(j)}$, $0\le\beta_j\le\pi$, with the side S_j and $f^{(j)}$ denotes the derivative of the solution in the direction normal to the above direction.

The relation between $f^{(j)}$ and $g^{(j)}$ is represented by the generalized Dirichlet-Neumann map, which is given by the following equation, [Sifalakis, Fokas, Fulton and Saridakis (2008)]:

$$\begin{aligned} &\sum_{j=1}^{n}\frac{|h_j|}{|h_p|}e^{i(\beta_j-\beta_p)}e^{-(i\ell/h_p)(m_p-m_j)}\int_{-\pi}^{\pi}e^{i\ell(h_j/h_p)s}(f^{(j)}(s)-ig^{(j)}(s))ds=0, \\ &\ell\in\Re^+, p=1,...n, \end{aligned} \tag{3}$$

where

$$h_j:=\frac{1}{2\pi}(z_{j+1}-z_j), \quad m_j:=\frac{1}{2}(z_{j+1}+z_j), \quad z_{n+1}=z_1, \quad j=1,2,...,n, \tag{4}$$

and

$$k_p=-\frac{\ell}{h_p}, \quad \ell\in\Re^+, \quad p=1,2,...,n. \tag{5}$$

If the set $\left\{g^{(j)}\right\}_{j=1}^{n}$ is the known component of the derivative of the solution, the set $\left\{f^{(j)}\right\}_{j=1}^{n}$ is then valid for all values of the complex parameter k and can be approximated by collocation-type methods.

3 The collocation-type method

Let us consider the generalized Dirichlet-Neumann map, defined by equation (3), and the known set $\left\{g^{(j)}\right\}_{j=1}^{n}$. The set $\left\{f^{(j)}\right\}_{j=1}^{n}$ is then approximated, [Sifalakis, Fokas, Fulton and Saridakis (2008)], by

$$f_N^{(j)}(s) = f_*^{(j)}(s) + \sum_{r=1}^{N} u_r^j \phi_r(s), \tag{6}$$

and

$$f_*^{(j)}(s) = \frac{1}{2\pi}\left[(s+\pi)f^{(j)}(\pi) - (s-\pi)f^{(j)}(-\pi)\right], \quad j = 1,\ldots,n, \tag{7}$$

where $\phi_r(s)$ represent the basis functions and N is an even integer.
The real coefficients u_r^j are approximated by the following set of equations:

$$\sum_{j=1}^{n} \frac{|h_j|}{|h_p|} e^{i(\beta_j-\beta_p)} e^{-(i\ell/h_p)(m_p-m_j)} \sum_{r=1}^{N} u_r^j F_r\left(\frac{\ell h_j}{h_p}\right) = G_p(\ell), \quad \ell \in \Re^+, p = 1,\ldots n, \tag{8}$$

$$G_p(\ell) = i\sum_{j=1}^{n} \frac{|h_j|}{|h_p|} e^{i(\beta_j-\beta_p)} e^{-(i\ell/h_p)(m_p-m_j)} \int_{-\pi}^{\pi} e^{i\ell(h_j/h_p)s}(g^{(j)}(s) + if_*^{(j)}(s))ds, \tag{9}$$

$\ell \in \Re^+, p = 1,\ldots n,$

$$F_r(\ell) = \int_{-\pi}^{\pi} e^{i\ell s}\phi_r(s)ds, \quad r = 1,\ldots,N. \tag{10}$$

For the imaginary part of equation (8), the parameter ℓ is chosen as $\ell = 1,2,\ldots,\frac{N}{2}$, while for the real part of equation (8), the parameter ℓ is chosen as $\ell = \frac{1}{2},\frac{3}{2}\ldots,\frac{N-1}{2}$. Moreover, the sine and Chebyshev basis functions are given in [Sifalakis, Fokas, Fulton and Saridakis (2008)].

The end values of the unknown functions $f^{(j)}$ can be derived by the continuity requirements, [Fulton, Fokas and Xenophonos (2004); Sifalakis, Fokas, Fulton and Saridakis (2008)], $q_z^{(j)}(z_j) = q_z^{(j-1)}(z_j)$. This is accomplished by considering the following identity, [Sifalakis, Fokas, Fulton and Saridakis (2008)]:

$$\frac{\partial q^{(j)}}{\partial z} = \frac{1}{2}e^{-ia_j}(q_s^{(j)} + iq_n^{(j)}) \quad \text{and} \quad \frac{\partial q^{(j)}}{\partial z} = \frac{1}{2}e^{-i(a_j-\beta_j)}(g^{(j)} + if^{(j)}). \tag{11}$$

Using the continuity requirements, the following expressions are derived:

$$f^{(j)}(\pi) = \frac{\cos(\delta_{j+1}-\delta_j)g^{(j)}(\pi) - g^{(j+1)}(-\pi)}{\sin(\delta_{j+1}-\delta_j)}, \tag{12}$$

$$f^{(j)}(-\pi)=\frac{g^{(j-1)}(\pi)-\cos(\delta_j-\delta_{j-1})g^{(j)}(-\pi)}{\sin(\delta_j-\delta_{j-1})}. \tag{13}$$

Let us now consider the following equation:

$$A_{jrp}(\ell)=\frac{|h_j|}{|h_p|}e^{i(\beta_j-\beta_p)}e^{-(i\ell/h_p)(m_p-m_j)}F_r\left(\frac{\ell h_j}{h_p}\right)\ j,r=1,...,N,\ p=1,...,n\ \ell\in\Re \tag{14}$$

Then equations (8), (9) and (10) can be rewritten in the following form:

$$\sum_{j=1}^{n}\sum_{r=1}^{N}A_{jrp}(\ell)u_r^j=G_p(\ell)\ \Rightarrow\ Au=s \tag{15}$$

Equation (15), represents a dense linear system Au=s, where A is the coefficient matrix, u is the solution vector (coefficients of the collocation method) and s is the right hand side [Saridakis, Sifalakis and Papadopoulou (2012)].

4 Modified Generic Factored Approximate Sparse Inverse Preconditioning

Let us consider the Incomplete LU factorization, A=LU+E, of the coefficient matrix A, where L and U are the lower and upper factors respectively, while E is the error matrix. It should be stated that, the U factor retains the diagonal elements. Computing the GenFAspI (M=GH) matrix requires the "a priori", [Chow (2000); Filelis-Papadopoulos (2014)], knowledge of the sparsity patterns of the upper triangular, G and lower triangular, H factors. The sparsity patterns are obtained by the sparsification of the L and U factors, using a prescribed drop tolerance (droptol) and are then raised to a predetermined power (level of fill), i.e. lfill. The process for computing the GenFAspI matrix is explained as follows:

$$A=LU\Leftrightarrow \mathrm{A}^{-1}=U^{-1}L^{-1}\Leftrightarrow M=GH, \tag{16}$$

where$M=A^{-1}$, $G=U^{-1}$ and $H=L^{-1}$. The GenFAspI matrix is then computed by solving the systems:

$$M=G_{droptol}^{lfill}H_{droptol}^{lfill}\Leftrightarrow\begin{cases}UG_{droptol}^{lfill}=I\\ H_{droptol}^{lfill}L=0\end{cases}, \tag{17}$$

where $G_{droptol}^{lfill}$ is the upper triangular sparse inverse matrix factor and $\mathrm{H}_{droptol}^{lfill}$ is the lower triangular sparse inverse matrix factor. The process for computing the

GenFAspI matrix can be modified to improve performance, based on a decoupled column-wise approach, as follows:

$$M = G^{lfill}_{droptol} H^{lfill}_{droptol} \Leftrightarrow \begin{cases} Ug_{:,j} = e_{:,j} \\ Lh_{:,j} = e_{:,j} \end{cases} ,0 \leq j < m, \tag{18}$$

where $g_{:,j}$ are the elements of G, $h_{:,j}$ are the elements of H, $e_{:,j}$ are the elements of the identity matrix and m is the order of the coefficient matrix of the linear system. The Modified GenFAspI (MGenFAspI) matrix, [Filelis-Papadopoulos and Gravvanis (2015)], is used to compute each column of the factors of the approximate inverse, independently, by a restricted solution process only for the elements included in the respective sparsity patterns of the factors, [Filelis-Papadopoulos and Gravvanis (2015)]. The elements that do not belong to the sparsity patterns of either G or H are set to zero while solving the linear systems, explicitly.

This modified scheme is used to avoid searches for elements of the factors G and H or L and U, required during the computation of an approximate inverse by the GenFAspI matrix, [Filelis-Papadopoulos and Gravvanis (2015)], thus enhancing performance. The complexity of the MGenFAspI matrix is O((nnz(L)nnz(H)+nnz(U)nnz (G))/m), where nnz(:) operator denotes the nonzero elements of a matrix and m is the matrix dimension, [Filelis-Papadopoulos and Gravvanis (2015)]. The algorithms for computing the Generic Factored Approximate Sparse Inverse matrix (GenFAspI) as well as the Modified Generic Factored Approximate Sparse Inverse matrix (MGenFAspI) are given in [Filelis-Papadopoulos and Gravvanis (2015)].

5 Numerical Results

In this section the applicability of the proposed schemes is presented. The unified transform method was used for the solution of the Laplace equation in two space variables, for several convex irregular polygonal domains. The considered polygons were arbitrarily selected and their vertices lie on the ellipse defined by the following equation, [Fokas (2008)]:

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \tag{19}$$

The polygons are rotated by an angle of 1/5 to avoid alignment with the coordinate axes, [Fokas (2008)].

For the collocation-type numerical method, both the sine and the Chebyshev basis functions were considered. It should be noted that, the same boundary conditions (Dirichlet: β=0, Neumann: $\beta = \pi/2$, Mixed: $\beta = \pi/3$) were imposed on every side of each polygon. The unknown coefficients u_r^j are computed by an ((N×n)×(N×n))

dense linear system which is formed by considering equations (8)-(10). The N parameter represents the number of collocation points while the n parameter is the number of the sides of the polygon. The error is computed with reference to the exact solution of the Laplace equation, q(x,y)=sinh(3x)sin(3y), [Saridakis, Sifalakis and Papadopoulou (2008); Sifalakis, Fokas, Fulton and Saridakis (2008)]. The exact values of $\left\{f^{(j)}\right\}_{j=1}^{n}$, $\left\{g^{(j)}\right\}_{j=1}^{n}$ can be derived by considering the exact solution along with the equation (11). Furthermore, the relative error is given by $E_\infty = \|f - f_{appr}\|_\infty / \|f\|_\infty$. Additionally, the Gauss-Kronrod, [Kronrod (1965)], quadrature method was used for the computation of the right hand side of the dense linear system.

In Figure 1, various shapes of irregular polygons used are given. In Table 1, the convergence behavior, the computed errors and the number of nonzeros, nnz(), of the G and H factors of the MGenFAspI matrix are given for various values of the parameters N, ILUTfill, lfill and droptol with ILUTtol=0.001, for several irregular polygons, using the sine basis functions and various boundary conditions. In Table 2, the convergence behavior, the computed errors and the number of nonzeros, nnz(), of the G and H factors of the MGenFAspI matrix are given for various values of the parameters N, ILUTfill, lfill and droptol with ILUTtol=0.001, for several irregular polygons, using the Chebyshev basis functions and various boundary conditions. The parameters ILUTfill and ILUTtol refer to the level of fill-in and magnitude of discarded elements respectively, during the ILU factorization. In both cases the EPGMRES(10) method in conjunction with the MGenFAspI matrix was used.

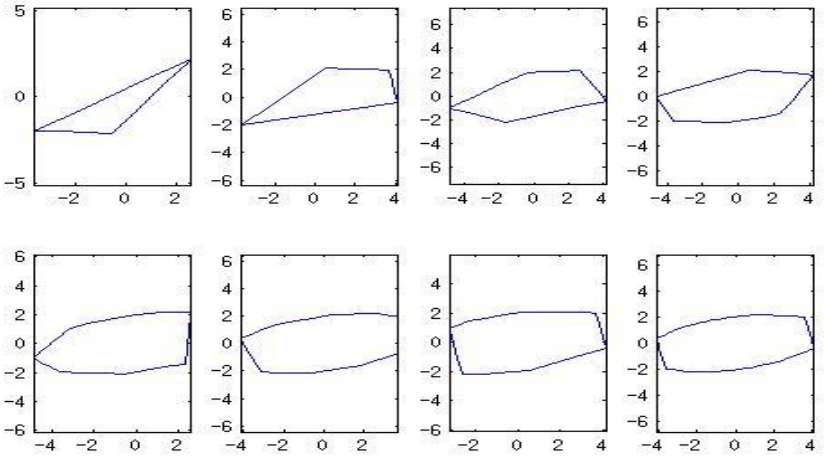

Figure 1: Various shapes of irregular polygons used.

Table 1: Convergence behavior, number of non-zero elements and computed error of the EPGMRES(8) in conjunction with MGenFAspI, using sine basis functions, for several irregular polygons and various boundary conditions.

n	N	ILUTfill	lfill	droptol	Iter.	nnz(G) +nnz(H)	Error	Boundary Conditions
3	16	1	1	0.1	3	61+62	4.9743E-02	Dirichlet
4	16	1	1	0.1	2	74+69	1.5231E-02	Dirichlet
5	16	1	1	0.1	3	94+95	2.3403E-03	Neumann
6	16	1	1	0.1	3	122+131	8.0101E-03	Mixed
8	16	1	1	0.1	5	212+192	1.2186E-03	Dirichlet
10	16	3	1	0.1	6	598+551	2.500E-03	Mixed
12	16	19	2	0.1	13	3724+5153	7.500E-03	Mixed
16	16	28	2	0.1	22	9733+9358	4.600E-03	Neumann

Table 2: Convergence behavior, number of non-zero elements and computed error of the EPGMRES(8) in conjunction with MGenFAspI, using Chebyshev basis functions, for several irregular polygons and various boundary conditions.

n	N	ILUTfill	lfill	droptol	Iter.	nnz(G)+nnz(H)	Error	Boundary Conditions
3	16	2	0	0	8	141+141	8.2083E-06	Dirichlet
4	16	2	0	0	6	189+189	2.0168E-06	Dirichlet
5	16	2	0	0	7	237+237	1.0401E-07	Neumann
6	16	2	0	0	8	285+285	2.6915E-08	Mixed
8	16	8	1	0.1	8	1423+1699	1.0021E-10	Dirichlet
10	16	16	1	0.1	6	2654+3759	1.4334E-10	Mixed
12	16	50	2	0.1	8	6419+9550	1.9922E-08	Mixed
16	16	59	3	0.1	10	13235+19345	6.5279E-11	Neumann

It should be stated that, the computed errors are much better when choosing the Chebyshev basis functions compared to the sine basis functions, [Sifalakis, Fokas, Fulton and Saridakis (2008)]. Furthermore, the number of iterations is not increased noticeably for a larger number of polygonal sides independently of the basis functions used. When choosing the Chebyshev basis functions and while $n<8$, better results were obtained by keeping lfill=0 and droptol=0. In all cases and for $n<16$, it should be mentioned that, a good sparsity pattern is achieved for the G and H factors minimizing the memory requirements. For larger values of n, the factors G and H tend to be denser because of the larger value of the parameter lfill required in order to obtain small errors. It should be noted that, in all cases,

the convergence behavior as well as the computed errors are comparably better to those presented in [Sifalakis, Fokas, Fulton and Saridakis (2008); Sifalakis, Fulton, Papadopoulou and Saridakis (2009)].

In Figure 2, the condition number of the coefficient matrix is given for both the sine and the Chebyshev basis functions, for various irregular polygons. It should be stated that, for matrices of the same order, the condition number grows considerably for a higher value of n, compared to a higher value of N. Near a vertex z_j the behavior of the solution depends on the interior angle ω_j, [Fulton, Fokas and Xenophontos (2004)], and the problem is singular when $\omega_j > \pi$. As $n \to \infty$ then $\omega_j \to \pi$, reaching the borderline case where the problem is ill-posed, thus increasing the condition number of the coefficient matrix. When $\omega_j = \pi$, the end values $f^{(j)}(\pi)$, $f^{(j)}(-\pi)$ cannot be computed using the continuity requirements (12), (13), since $\delta_{j+1} = \delta_j$ and $\sin(\delta_{j+1} - \delta_j)=0$. In this case the end values must be coupled with the unknown coefficients u_r^j, [Fulton, Fokas and Xenophontos (2004)]. A numerical treatment for the case of corner singularities can be found in [Fornberg and Flyer (2011)]. It should be noted that, the convergence behavior of the proposed preconditioning scheme is in qualitative agreement with theoretical estimates, [Filelis-Papadopoulos and Gravvanis (2015)]. Moreover, the memory requirements are minimized because of the low number of nonzero elements kept with the G and H factors. For the choice of the basis functions, it should be mentioned that, the Chebyshev basis function leads to better computed errors with additional computational work and memory requirements, in comparison to the sine basis.

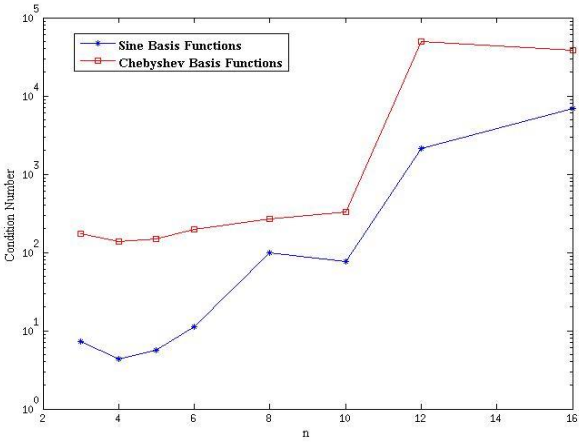

Figure 2: Condition number of the coefficient matrix for both the sine and Chebyshev basis functions, with various values of n (irregular polygons).

The unified transform is a hybrid method, combining analytical information in the complex Fourier plane instead of the physical plane, along with a numerical approximation. Thus, it provides the spectral analogue of the classical Green's representations. The main advantage of the method is the derivation of exact solutions for problems for which the usual methods fail, [Fokas and Kalimeris (2014)]. Furthermore, it has been realized that the automatic production of a generalized Dirichlet-to-Neumann map is not present in other boundary integral methods, [Fulton, Fokas and Xenophontos (2004)]. Additionally, the method does not make use of fundamental solutions, which are not always available or simple, [Fulton, Fokas and Xenophontos (2004)]. The unified transform is a boundary-type method, decreasing the dimensionality of the considered BVP, thus minimizing the computational cost. Moreover, in contrast to the BEM, the unified transform does not involve the computation of singular integrals. Future research will be focused on addressing various issues concerning the method. Firstly, the method needs to be extended to three dimensions by developing computationally efficient numerical schemes; a first attempt has been made in [Ambrose and Nicholls (2014)]. Furthermore, effective algorithms should be developed for the solution of more complicated PDEs with non-constant coefficients and source terms. As mentioned in [Davis and Fornberg (2014)], a comprehensive list of PDEs with their associated global relations should be created in order to facilitate further research. Additionally, coupling techniques with existing methods should be considered, in order to bring out the advantages of the new formulation, for solving more practical problems arising in engineering and science.

Finally, it should be noted that current research efforts are focused on an alternative approach, that is using the finite Fourier transform of Legendre polynomials, which can be expressed in terms of modified Bessel function, cf. [Fornberg and Flyer (2011); Hashemzadeh, Fokas and Smitheman (2015)] and efficient solvers are sought for the resulting system.

References

Ambrose, D. M.; Nicholls, D. P. (2014): Fokas integral equations for three dimensional layered- media scattering. *J. Comput. Phys.,* vol. 276, pp. 1–25.

Atluri, S. N.; Zhu, T. (1998): A new meshless local Petrov–Galerkin (MLPG) approach in computational mechanics. *Computational Mechanics,* vol. 22, pp. 117–127.

Chow, E. (2000): A priori sparsity patterns for parallel sparse approximate inverse preconditioners. *SIAM J. Sci. Comput.,* vol. 21, pp. 1804-1822.

Davis, C-I. R; Fornberg, B. (2014): A spectrally accurate numerical implementa-

tion of the Fokas transform method for Helmholtz-type PDEs. *Complex Variables and Elliptic Equations*, vol. 59, no 4, pp. 564–577.

Dong, L.; Atluri, S. N. (2011a): A Simple Procedure to Develop Efficient & Stable Hybrid/Mixed Elements, and Voronoi Cell Finite Elements for Macro- & Micromechanics. *CMC: Computers, Materials & Continua*, vol. 24, no 1, pp. 61-104.

Dong, L.; Atluri, S. N. (2011b): Development of T-Trefftz four-node quadrilateral and Voronoi Cell Finite Elements for macro- & micromechanical modeling of solids. *CMES: Computer Modeling in Engineering & Sciences*, vol. 81, no 1, pp. 69-118.

Filelis-Papadopoulos, C. K.; Gravvanis, G. A. (2015): A class of generic factored and multilevel recursive approximate inverse techniques for solving general sparse systems. *Engineering Computations*, to appear.

Filelis-Papadopoulos, C. K.: Gravvanis, G. A. (2014): Generic approximate sparse inverse matrix techniques. *Int. J. Comput. Methods*, vol. 11, no. 6, online.

Fokas, A. S. (1997): A unified transform method for solving linear and certain nonlinear PDEs. *Proc. Roy. Soc. London Ser.*, A 53, pp. 1411–1443.

Fokas, A. S. (2001): Two-dimensional linear PDEs in a convex polygon. *Proc. Roy. Soc. London Ser.*, A 457, pp. 371–393.

Fokas, A. S. (2002): A new transform method for evolution PDEs. *IMA J. Appl. Math.,* vol. 67, pp. 559-590

Fokas, A. S. (2008): A Unified Approach to Boundary Value Problems. SIAM, Philadelphia.

Fokas, A. S.; Kalimeris, K. (2014): Eigenvalues for the Laplace operator in the interior of an equilateral triangle. *Comput. Methods Funct. Theory,* vol. 14, pp. 1-33.

Fornberg, B.; Flyer, N. (2011): A numerical implementation of Fokas boundary integral approach: Laplace's equation on a polygonal domain. *Proc. R. Soc.*, A 467, pp. 2083-3003.

Fulton, S.; Fokas, A. S.; Xenophontos, C. (2004): An analytical method for linear elliptic PDEs and its numerical implementation. *J. Comput. Appl. Math.,* vol. 167, pp. 465–483.

Golberg, M. A.; Chen, C. S. (1998): The method of fundamental solutions for potential, Hemholtz and diffusion problems. *Boundary Integral Methods: Numerical and Mathematical Aspects* Ed. MA. Golberg, WIT Press & Computational Mechanics Publications, pp. 103–176.

Hashemzadeh, P.; Fokas, A. S.; Smitheman, S. A. (2015): A numerical technique for linear elliptic partial differential equations in polygonal domains. *Proc. R. Soc.,*

A 471, DOI: 10.1098/rspa.2014.0747.

Jirousek, J.; Guex, Lan. (1986): The hybrid-Trefftz finite element model and its application to plate bending. *International Journal for Numerical Methods in Engineering,* vol.23, no 4, pp. 651-693.

Kronrod, A. S. (1965): Nodes and Weights of Quadrature Formulas. Consultants Bureau, New York.

Li, Z. C.; Lu, T. T.; Huang, H. T.; Cheng, A. H. D. (2007): Trefftz, collocation, and other boundary methods–A comparison. *Num. Meth. Par. Diff. Eq.,* vol. 23, pp. 93-144.

Liu, C. S. (2007a): A highly accurate solver for the mixed-boundary potential problem and singular problem in arbitrary plane domain. *CMES: Computer Modeling in Engineering & Sciences*, vol. 20, pp. 111-122.

Liu, C. S. (2007b): A modified Trefftz method for two-dimensional Laplace equation considering the domain's characteristic length. *CMES: Computer Modeling in Engineering & Sciences*, vol.21, no. 1, pp. 53-65.

Liu, C. S. (2007c): An effectively modified direct Trefftz method for 2D potential problems considering the domain's characteristic length. *Engineering Analysis with Boundary Elements,* vol. 31, no. 12, pp. 983-993.

Liu, C. S.; Yeih, W.; Atluri, S. N. (2009): On solving the ill-conditioned system Ax=b: general-purpose conditioners obtained from the boundary-collocation solution of the Laplace equation, using Trefftz expansions with multiple length scales. *CMES: Computer Modeling in Engineering & Sciences*, vol. 44, no. 3, pp. 281-312.

Saad, Y.; Schultz, M. H. (1986): GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Stat. Comput.,* vol. 7, pp. 856-869.

Saridakis, Y. G.; Sifalakis, A. G.; Papadopoulou, E. P. (2012): Efficient numerical solution of the generalized Dirichlet–Neumann Map for linear elliptic PDEs in regular polygon domains. *J. Comp. Appl. Math.*, vol. 236, no. 9, pp. 2515–2528.

Sifalakis, A. G.; Fokas, A. S.; Fulton, S.; Saridakis, Y. G. (2008): The generalized Dirichlet–Neumann map for linear elliptic PDEs and its numerical implementation. *J. Comput. Appl. Math.,* vol. 219, no. 1, pp. 9-34.

Sifalakis, A. G.; Fulton, S.; Papadopoulou, E. P.; Saridakis, Y. G. (2009): Direct and iterative solution of the generalized Dirichlet-Neumann map for elliptic PDEs on square domains. *J. Comput. Appl. Math.*, vol. 227, no. 1, pp. 171–184.