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A RIM-based Time-domain Boundary Element Method for Three-Dimensional Non-homogeneous Wave Propagations

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Abstract: This paper presents a three-dimensional (3-D) boundary element method (BEM) scheme based on the Radial Integration Method (RIM) for wave propagation analysis of continuously non-homogeneous problems. The Kelvin fundamental solutions are adopted to derive the boundary-domain integral equation (BDIE). The RIM proposed by Gao (Engineering Analysis with Boundary Elements 2002; 26(10):905-916) is implemented to treat the domain integrals in the BDIE so that only boundary discretization is required. After boundary discretization, a set of second-order ordinary differential equations with respect to time variable are derived, which are solved using the Wilson-θ method. Main advantages of the proposed method are that 1) it can treat wave propagations in non-homogeneous domains with only boundary mesh required, and that 2) coefficient matrices arising from the BEM are evaluated and stored only once so that solving large-scale problems with huge time steps is possible. In the numerical examples, the present method is tested in terms of accuracy, capacity to treat non-homogeneous problems and large-scale potentials.

Keywords: Boundary element method, Radial integration method, Time domain, Non-homogeneous problems, Wave propagation.

1 Introduction

The boundary element method (BEM) offers an efficient way to solve wave propagation problems. When establishing the integral equations, the dynamic fundamental solutions, transformed-domain fundamental solutions or static fundamental solutions can be used. The first approach, generally represented as TD-BEM

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(TD for time-domain), can preserve the causality condition. Since the radiation condition is satisfied automatically, it is suitable for infinite and semi-infinite problems, see Beskos (1987, 1997), Manolis and Beskos (1988). However, the storage of matrices generated at each time-step should be mentioned as one of the main bottlenecks against implementing this method to solve large-scale problems. In order to improve its efficiency, several fast algorithms have been implemented in this field. Of particular interest are the fast multipole method (FMM) [Rokhlin (1985); Greengard and Rokhlin (1997a, 1997b)] and the adaptive cross approximation method (ACA) [Bebendorf and Rjasanow (2003); Kolk, Weber and Kuhn (2005); Kolk and Kuhn (2006); Benedetti, Aliabadi and Dav (2008); Benedetti, Milazzo and Aliabadi (2009)]. Furthermore, the conventional dynamic fundamental solutions employed in TD-BEM are limited on linear elastic homogeneous problems [Sohrabi-Bidar, Kamalian and Jafari (2010); Wang and Yao (2013); Galvin and Romero (2014a)] or layered homogeneous problems [Birgisson and Crouch (1998)], while specific fundamental solutions for other types of problems, for example, half-space problems [Johnson (1974); Galvin and Romero (2014b)], poroelastic problems [Chen (1994a, 1994b); Schanz (2001a, 2001b)], non-homogeneous problems [Sanchez-Sesma, Madariaga and Irikura (2001); Luzon, Sanchez-Sesma, Perez-Ruiz, Ramirez-Guman and Pech (2009)], etc., are still unavailable in closed form.

The second approach can be performed either in the Fourier-domain [Dominguez and Roesset (1978); Mansur, Abreu and Carrer (2004)] or in the Laplace-domain [Cruse and Rizzo (1968)]. Tadeu and his co-workers proposed an iterative procedure coupling the normal derivative of the integral equation (TBEM) and the method of fundamental solutions (MFS) to solve the transient acoustic [Tadeu, António and Ferreira (2013)] and elastic [António, Tadeu and Ferreira (2013)] wave propagation problems in the presence of multiple inclusions in frequency domain. In order to obtain the time-domain results, the inverse transformation is required. A new approach combining time and Laplace domains is to use the convolution quadrature method (CQM) developed by Lubich (1988a, 1988b), for example, Schanz and Antes (1997), Schanz (2001b), Abreu, Carrer and Mansur (2003), Antes, Schanz and Alvermann (2004) and Abreu, Mansur and Carrer (2006). It utilizes Laplace-domain fundamental solutions and provides solutions directly in time-domain. Li, Zhang, Xie, Zheng and Guo (2014) developed a general method to tackle arbitrary, non-null initial conditions in the analysis of three-dimensional elastodynamic problems using CQM. The main merits of CQM are that only the Laplace-domain fundamental solutions are needed so that it can be applied to problems where time-domain fundamental solutions are not available [Carrer and Mansur (2010)] and that it shows better stability behavior [Banjai and Schanz (2012)].

The third approach is based on the static fundamental solutions (Kelvin fundamental solutions). In this case, domain integrals occur in the boundary-domain integration equation (BDIE). Generally, domain discretization is required to evaluate these domain integrals (D-BEM) [Carrer and Mansur (2010); Dong, Zhang, Xie, Lu, Han and Wang (2015); Dong, Zhang, Xie, Lu, Li, Han and Li (2015)]. Such treatment may offset the main advantage of BEM that only boundary discretization is required. In order to avoid domain discretization, various techniques have been developed to transform these domain integrals into boundary integrals, for example, dual reciprocity method (DRM) [Albuquerque, Sollero and Aliabadi (2002); Vanegas, Patino and Vargas (2014); Useche and Albuquerque (2015); Vanegas and Patino (2015)] introduced in Ref. [Nardini and Brebbia (1983)] and explained in detail by Partridge, Brebbia and Wrobel (1992), and Radial Integration Method (RIM) proposed by Gao (2002a, 2002b). In DRM, the integrands are approximated with a series of prescribed basis functions and particular solutions are derived based on these basis functions and the differential operator of the problem. Then the domain integrals are transformed to the boundary employing the particular solutions. The main challenge to implement DRM is that it may be difficult to obtain the particular solutions for some 3D problems. The RIM can transform any domain integrals into a boundary integral and a radial integral which is independent of geometry. It has been employed in nonlinear and nonhomogeneous elastic problems [Gao, Zhang and Guo (2007)], fracture analysis [Gao, Zhang, Sladek and Sladek (2008)] and transient heat conduction problems [Yao, Yu, Gao and Gao (2014)].

After space discretization, a set of second-order ordinary differential equations with respect to time variable are derived. These equations can be solved using time-stepping algorithms such as the Wilson-θ method, the Newmark method and the Houbolt method. For a long time the Houbolt method was probably the most popular and reliable time integration method used in the D-BEM [Hatzigeorgiou and Beskos (2002); Soares, Carrer and Mansur (2005); Pereira, Karam, Carrer and Mansur (2012)] or DR-BEM formulations [Albuquerque, Sollero and Aliabadi (2002, 2004); Useche and Albuquerque (2015)]; mainly, due to the fact that unwanted complex higher frequencies that cause numerical instability are damped out. However, it provides numerical results with considerable numerical damping and one has to select a small time step to obtain accurate results. The Newmark method [Newmark (1959)] has been widely used in FEM formulations, presenting over the Houbolt method a better control of the stability and accuracy, according to the values of the parameters β and γ , see Bathe (1996) and Cook (2002). Alternative time-marching schemes are also developed, for example, Carrer and Mansur (2004) and Souza, Carrer and Martins (2004).

To the best knowledge of the authors, few BEM work is found for analyzing transient response of 3-D non-homogeneous body in time domain. In this paper, a RIM-based time-domain BEM using static fundamental solutions is developed to treat this problem. The entire domain is assumed to be continuously non-homogeneous and the Kelvin fundamental solutions are employed to form the boundary-domain integration equation. The domain integrals resulting from the inertia term and the non-homogeneity term are transformed to boundary integrals by RIM. Consequently, only the boundary is discretized and 8-node discontinuous elements from Mi and Aliabadi (1992) are employed. The Wilson-θ method is implemented to perform the marching in time and its performance is tested in terms of accuracy and stability.

The paper is organized as follows: Firstly, the BDIE for 3D non-homogeneous domain utilizing Kelvin fundamental solutions is derived. Secondly, the domain integrals in the BDIE are converted to boundary integrals with RIM, resulting in a pure boundary integration equation (BIE). Then, a set of second-order ordinary differential equations are obtained by numerical implementation of the aforementioned BIE and solved by the Wilson-θ method. Finally, numerical examples are given in order to demonstrate accuracy, capacity to treat non-homogeneity and large-scale potential of the proposed method.

2 The boundary-domain integration equation for 3-D non-homogeneous problems

We consider a 3D domain V bounded by boundary S. In the absence of body force, the governing equation for linear elastodynamic problems can be written as

$$\sigma_{ij,i}(x,t) - \rho(x)\ddot{u}_j(x,t) = 0, \tag{1}$$

where u_j is displacement at point x and time t; σ_{ij} is stress and ρ is density. It is assumed that the shear modulus $\mu(x)$ and density $\rho(x)$ vary with Cartesian coordinates while Poisson's ratio ν is constant. In this case, the elasticity tensor $C_{ijkl}(x)$ can be written as

$$C_{ijkl}(x) = \mu(x)C^0_{ijkl}, \tag{2}$$

with

$$C^0_{ijkl} = \frac{2\nu}{1-2\nu}\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \tag{3}$$

According to the generalized Hooke's law, the relationship between σ_{ij} and displacement gradients $u_{k,l}$ is

$$\sigma_{ij} = \mu C^0_{ijkl}u_{k,l} \tag{4}$$

A weak form of Eq. (1) can be written as

$$\int_V U_{mj}\left(\sigma_{ij,i} - \rho \ddot{u}_j\right) \mathrm{d}V = 0 \tag{5}$$

where U_{mj} is the weight or test function. Herein, the elastic displacement fundamental solution is chosen as the weight function, which satisfies the following partial differential equation:

$$\mu_0 C^0_{ijkl} U_{mj,il} = -\delta_{mk} \Delta(x,y) \tag{6}$$

where $\Delta(x,y)$ is the Dirac delta function; μ_0 is the shear modulus of a homogeneous domain. Since the value of μ_0 will not affect the result, it is set to be 1. U_{ij} is written as

$$U_{ij}(y,x) = \frac{1}{16\pi\mu_0(1-\nu)r}\left[(3-4\nu)\delta_{ij} + r_{,i}r_{,j}\right] \tag{7}$$

where r is the distance between the source point y and field point x. Substitution of Eq. (4) into Eq. (5) and application of Gauss's divergence theorem yield

$$\begin{aligned} &\int_S U_{mj} t_j \mathrm{d}S - \int_S T_{mj} \frac{\mu}{\mu_0} u_j \mathrm{d}S + \int_V C^0_{ijkl} U_{mj,il} \mu u_k \mathrm{d}V \\ &+ \int_V H_{mj} u_j \mathrm{d}V - \int_V U_{mj} \rho \ddot{u}_j \mathrm{d}V = 0 \end{aligned} \tag{8}$$

where $t_j = \sigma_{ij} n_j$ is boundary traction, n_j is the outward unit normal to the boundary S, and

$$T_{mj} = \mu_0 C^0_{ikjl} U_{mk,i} n_l \tag{9}$$

$$H_{mj} = C^0_{ikjl} U_{mk,i} \mu_{,l} \tag{10}$$

Substituting Eq. (6) into Eq. (8), we have,

$$\begin{aligned} &\mu u_m(y) + \int_S T_{mj}(x,y) \mu u_j(x,t) \mathrm{d}S - \int_S U_{mj}(x,y) t_j(x,t) \mathrm{d}S \\ &= -\int_V U_{mj}(x,y) \rho \ddot{u}_j(x,t) \mathrm{d}V + \int_V H_{mj}(x,y) u_j(x,t) \mathrm{d}V \end{aligned} \tag{11}$$

The boundary integration equation can be obtained by taking the source point y to the boundary as,

$$\begin{aligned} &c(y) \mu u_m(y) + \int_S T_{mj}(x,y) \mu u_j(x,t) \mathrm{d}S - \int_S U_{mj}(x,y) t_j(x,t) \mathrm{d}S \\ &= -\int_V U_{mj}(x,y) \rho \ddot{u}_j(x,t) \mathrm{d}V + \int_V H_{mj}(x,y) u_j(x,t) \mathrm{d}V \end{aligned} \tag{12}$$

where $c(y)$=0.5 if y is on the smooth boundary.

3 Transforming domain integrals to the boundary with RIM

As can be seen from Eq. (12), two domain integrals caused by inertia term and non-homogeneity term respectively are included. A conventional way to treat such domain integrals is to discretize the domain into cells, which to some extent offsets the advantage of BEM. In order to keep only boundary discretization, the Radial Integration Method (RIM) proposed by Gao (2002a, 2002b) is adopted herein to transform the domain integrals to equivalent boundary integrals.

The basic idea of RIM is that, the domain integral of a general known function $f(x)$ with x representing the Cartesian coordinates,

$$I = \int_V f(x)\,\mathrm{d}V \tag{13}$$

can be transformed into a boundary integral and a radial integral which is independent of the geometry:

$$\begin{aligned} I &= \int_S \frac{r_{,i} n_i}{r^2(z)} F(z)\,\mathrm{d}S(z) \\ F(z) &= \int_0^{r(z)} f(x)\,\xi^2 \mathrm{d}\xi \end{aligned} \tag{14}$$

where $r(z)$ is the distance between the origin and boundary point z.

As for the unknown displacements u_j and acceleration $\ddot{u}_j$ at field point x, they are approximated by a series of prescribed basis functions in the first place so that RIM can be implemented. It is recommended that a combination of the radial basis functions (RBFs) and the polynomials in terms of global coordinates can give promising results [Golberg, Chen and Bowman (1999)]:

$$\begin{aligned} u_j &= \sum_{A=1}^{N_A} \alpha_j^A \phi^A(R) + a_j^i x_i + a_j^0 \\ \ddot{u}_j &= \sum_{A=1}^{N_A} \beta_j^A \phi^A(R) + b_j^i x_i + b_j^0 \end{aligned} \tag{15}$$

In the above, $\phi^A(R)$ is the RBF; $R = r\left(x, x^A\right)$ is the distance from the A-th Application Point x^A to the field point x. N_A is the total number of Application Points. $\alpha_j^A, a_j^i, a_j^0, \beta_j^A, b_j^i, b_j^0\,(A = 1, \cdots, N_A, i = 1,2,3)$ are undetermined coefficients which are independent of coordinates. In this paper, the compactly supported 4^{th}-order s-pline RBF is adopted, i.e.,

$$\phi^A(R) = \begin{cases} 1 - 6\left(\frac{R}{d_A}\right)^2 + 8\left(\frac{R}{d_A}\right)^3 - 3\left(\frac{R}{d_A}\right)^4, 0 \le R < d_A \\ 0, R \ge d_A \end{cases} \tag{16}$$

in which d_A is the radius of the support region at the A-th Application Point.

Substituting Eq. (15) into the domain integrals in Eq. (12) and transforming them into boundary integrals, we have

$$\begin{aligned}
&\int_V H_{mj}(x,y)\,u_j(x,t)\,\mathrm{d}V \\
&= \sum_{A=1}^{N_A} \alpha_j^A \int_S \frac{r_{,i}n_i}{r(y,z)^2} F_\alpha^A(y,z)\,\mathrm{d}S(z) \\
&+ a_j^i \left[\int_S \frac{r_{,i}}{r(y,z)^2} r_{,k}n_k F_\alpha^b(y,z)\,\mathrm{d}S(z) + \int_S \frac{y_i}{r(y,z)^2} r_{,k}n_k F_\alpha^c(y,z)\,\mathrm{d}S(z) \right] \\
&+ a_j^0 \int_S \frac{1}{r(y,z)^2} r_{,k}n_k F_\alpha^c(y,z)\,\mathrm{d}S(z)
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\int_V U_{mj}(x,y)\,\rho\ddot{u}_j(x,t)\,\mathrm{d}V \\
&= \sum_{A=1}^{N_A} \beta_j^A \int_S \frac{r_{,i}n_i}{r(y,z)^2} F_\beta^A(y,z)\,\mathrm{d}S(z) \\
&+ b_j^i \left[\int_S \frac{r_{,i}}{r(y,z)^2} r_{,k}n_k F_\beta^b(y,z)\,\mathrm{d}S(z) + \int_S \frac{y_i}{r(y,z)^2} r_{,k}n_k F_\beta^c(y,z)\,\mathrm{d}S(z) \right] \\
&+ b_j^0 \int_S \frac{1}{r(y,z)^2} r_{,k}n_k F_\beta^c(y,z)\,\mathrm{d}S(z)
\end{aligned} \tag{18}$$

where $r(y,z)$ is the distance from source point y to boundary point z; and

$$\begin{aligned}
F_\alpha^A(y,z) &= \int_0^{r(y,z)} H_{mj}(x,y)\,\phi^A(R)\,\xi^2\mathrm{d}\xi \\
F_\alpha^b(y,z) &= \int_0^{r(y,z)} H_{mj}(x,y)\,\xi^3\mathrm{d}\xi \\
F_\alpha^c(y,z) &= \int_0^{r(y,z)} H_{mj}(x,y)\,\xi^2\mathrm{d}\xi
\end{aligned} \tag{19}$$

$$\begin{aligned}
F_\beta^A(y,z) &= \int_0^{r(y,z)} U_{mj}(x,y)\,\rho(x)\,\phi^A(R)\,\xi^2\mathrm{d}\xi \\
F_\beta^b(y,z) &= \int_0^{r(y,z)} U_{mj}(x,y)\,\rho(x)\,\xi^3\mathrm{d}\xi \\
F_\beta^c(y,z) &= \int_0^{r(y,z)} U_{mj}(x,y)\,\rho(x)\,\xi^2\mathrm{d}\xi
\end{aligned} \tag{20}$$

Eqs. (17) and (18) can be expressed in a simple form as:

$$\int_V H_{mj}(x,y)\,u_j(x,t)\,\mathrm{d}V = \left\{\tilde{H}_y\right\}^T \{\alpha\} \tag{21}$$

$$\int_V U_{mj}(x,y)\,\rho\ddot{u}_j(x,t)\,\mathrm{d}V = \left\{\tilde{C}_y\right\}^T\{\beta\} \tag{22}$$

where $\left\{\tilde{H}_y\right\}^T$, $\left\{\tilde{C}_y\right\}^T$, $\{\alpha\}$ and $\{\beta\}$ are arranged as follows:

$$\left\{\tilde{H}_y\right\}^T = \left\{\hat{F}_\alpha^1, \cdots, \hat{F}_\alpha^{N_A}, \hat{F}_a^1, \hat{F}_a^2, \hat{F}_a^3, \hat{F}_a^0\right\} \tag{23}$$

$$\left\{\tilde{C}_y\right\}^T = \left\{\hat{F}_\beta^1, \cdots, \hat{F}_\beta^{N_A}, \hat{F}_b^1, \hat{F}_b^2, \hat{F}_b^3, \hat{F}_b^0\right\} \tag{24}$$

$$\{\alpha\} = \left\{\left\{\alpha_j^1\right\}^T, \cdots, \left\{\alpha_j^{N_A}\right\}^T, \left\{a_j^1\right\}^T, \left\{a_j^2\right\}^T, \left\{a_j^3\right\}^T, \left\{a_j^0\right\}^T\right\}^T \tag{25}$$

$$\{\beta\} = \left\{\left\{\beta_j^1\right\}^T, \cdots, \left\{\beta_j^{N_A}\right\}^T, \left\{b_j^1\right\}^T, \left\{b_j^2\right\}^T, \left\{b_j^3\right\}^T, \left\{b_j^0\right\}^T\right\}^T \tag{26}$$

where

$$\hat{F}_\alpha^A = \int_S \frac{r_{,i}n_i}{r(y,z)^2} F_\alpha^A(y,z)\,\mathrm{d}S(z),\; A = 1, \cdots, N_A \tag{27}$$

$$\hat{F}_a^i = \int_S \frac{r_{,i}}{r(y,z)^2} r_{,k}n_k F_\alpha^b(y,z)\,\mathrm{d}S(z) + \int_S \frac{y_i}{r(y,z)^2} r_{,k}n_k F_\alpha^c(y,z)\,\mathrm{d}S(z),\; i = 1,2,3 \tag{28}$$

$$\hat{F}_a^0 = \int_S \frac{1}{r(y,z)^2} r_{,k}n_k F_\alpha^c(y,z)\,\mathrm{d}S(z) \tag{29}$$

$$\hat{F}_\beta^A = \int_S \frac{r_{,i}n_i}{r(y,z)^2} F_\beta^A(y,z)\,\mathrm{d}S(z),\; A = 1, \cdots, N_A \tag{30}$$

$$\hat{F}_b^i = \int_S \frac{r_{,i}}{r(y,z)^2} r_{,k}n_k F_\beta^b(y,z)\,\mathrm{d}S(z) + \int_S \frac{y_i}{r(y,z)^2} r_{,k}n_k F_\beta^c(y,z)\,\mathrm{d}S(z),\; i = 1,2,3 \tag{31}$$

$$\hat{F}_b^0 = \int_S \frac{1}{r(y,z)^2} r_{,k}n_k F_\beta^c(y,z)\,\mathrm{d}S(z) \tag{32}$$

$$\left\{\alpha_j^A\right\}^T = \left\{\alpha_1^A, \alpha_2^A, \alpha_3^A\right\}, \left\{\beta_j^A\right\}^T = \left\{\beta_1^A, \beta_2^A, \beta_3^A\right\},\; A = 1, \cdots, N_A \tag{33}$$

The following relationships are used in Eq. (17) and Eq. (18):

$$\begin{aligned} &\int_V H_{mj}a_j^i x_i \mathrm{d}V = \int_V H_{mj}a_j^i(r_i + y_i)\,\mathrm{d}V = \int_V H_{mj}a_j^i(rr_{,i} + y_i)\,\mathrm{d}V \\ &\int_V U_{mj}\rho b_j^i x_i \mathrm{d}V = \int_V U_{mj}\rho b_j^i(r_i + y_i)\,\mathrm{d}V = \int_V U_{mj}\rho b_j^i(rr_{,i} + y_i)\,\mathrm{d}V \end{aligned} \tag{34}$$

Substituting Eq. (17) and Eq. (18) into Eq. (12), a pure boundary integral equation can be obtained:

$$\begin{aligned} &c(y)\,\mu u_m(y) + \int_S T_{mj}(x,y)\,\mu u_j(x,t)\,\mathrm{d}S - \int_S U_{mj}(x,y)\,t_j(x,t)\,\mathrm{d}S \\ &= -\left\{\tilde{C}_y\right\}^T\{\beta\} + \left\{\tilde{H}_y\right\}^T\{\alpha\} \end{aligned} \tag{35}$$

4 Discretization of boundary integration equation

The boundary S is discretized into Ne boundary elements and Nb boundary nodes,

$$S = \sum_{e=1}^{Ne} S_e \tag{36}$$

Also, NI internal nodes are casted into the domain V, making the total number of nodes to be N.

In order to replace the undetermined coefficient vectors $\{\alpha\}$ and $\{\beta\}$ in Eq. (35)with $\{u_j\}$ and $\{\ddot{u}_j\}$, the Application Points in Eq. (15) are collocated at all nodes, which means $N_A = N$. Then the displacement and acceleration of k-th node can be expressed as

$$\begin{aligned} u_j^k &= \sum_{A=1}^{N} \alpha_j^A \phi^A(R) + a_j^i x_i^k + a_j^0 \\ \ddot{u}_j^k &= \sum_{A=1}^{N} \beta_j^A \phi^A(R) + b_j^i x_i^k + b_j^0, k = 1, \cdots, N \end{aligned} \tag{37}$$

where x_i^k is the coordinate of the k-th node.

The number of coefficients α_j^A, a_j^i, a_j^0 and β_j^A, b_j^i, b_j^0 is larger than the number of equations in Eq. (37). In order to determine these coefficients, the following equilibrium conditions have to be satisfied:

$$\begin{aligned} &\sum_{A=1}^{N} \alpha_j^A = 0, \sum_{A=1}^{N} \alpha_j^A x_i^A = 0 \\ &\sum_{A=1}^{N} \beta_j^A = 0, \sum_{A=1}^{N} \beta_j^A x_i^A = 0 \end{aligned} \tag{38}$$

where x_i^A is the Cartesian coordinates of the A-th node.

Rewrite Eq. (37) and Eq. (38) in matrix form,

$$\begin{aligned} \{\hat{u}_j\} &= [\phi]\{\alpha\} \\ \{\hat{\ddot{u}}_j\} &= [\phi]\{\beta\} \end{aligned} \tag{39}$$

where

$$\begin{aligned} \{\hat{u}_j\} &= \left\{ \{u_j^1\}^T, \{u_j^2\}^T, \cdots, \{u_j^N\}^T, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0} \right\}_{3N+12}^T = \left\{ \{u_j\}^T, \mathbf{0} \right\}^T \\ \{\hat{\ddot{u}}_j\} &= \left\{ \{\ddot{u}_j^1\}^T, \{\ddot{u}_j^2\}^T, \cdots, \{\ddot{u}_j^N\}^T, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0} \right\}_{3N+12}^T = \left\{ \{\ddot{u}_j\}^T, \mathbf{0} \right\}^T \end{aligned} \tag{40}$$

$$[\phi]=\begin{bmatrix}\phi_{11}I & \cdots & \phi_{1N}I & x_1^1 I & x_2^1 I & x_3^1 I & I\\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \phi_{11}I & \cdots & \phi_{NN}I & x_1^N I & x_2^N I & x_3^N I & I\\ I & \cdots & I & 0 & 0 & 0 & 0\\ x_1^1 I & \cdots & x_1^N I & 0 & 0 & 0 & 0\\ x_2^1 I & \cdots & x_2^N I & 0 & 0 & 0 & 0\\ x_3^1 I & \cdots & x_3^N I & 0 & 0 & 0 & 0\end{bmatrix} \tag{41}$$

where I is the 3×3 unit matrix.

If no two nodes share the same coordinates, the matrix $[\phi]$ is invertible and thereby

$$\begin{aligned}\{\alpha\} &= [\phi]^{-1}\{\hat{u}_j\}\\ \{\beta\} &= [\phi]^{-1}\{\hat{\ddot{u}}_j\}\end{aligned} \tag{42}$$

According to Eq. (40), the matrix $[\phi]^{-1}$ can be expressed in the block form as

$$[\phi]^{-1}=\left[[\phi_1]_{(3N+12)\times 3N},[\phi_2]_{(3N+12)\times 12}\right] \tag{43}$$

Then Eq. (42) can be rewritten as

$$\begin{aligned}\{\alpha\} &= [\phi_1]\{u_j\}\\ \{\beta\} &= [\phi_1]\{\ddot{u}_j\}\end{aligned} \tag{44}$$

Substitution of Eq. (44) into Eq. (35) leads to

$$\begin{aligned}&c(y)\,\mu u_m(y)+\int_S T_{mj}(x,y)\,\mu u_j(x,t)\,\mathrm{d}S-\int_S U_{mj}(x,y)\,t_j(x,t)\,\mathrm{d}S\\ &=-\{C_y\}^T\{\ddot{u}_j\}+\{H_y\}^T\{u_j\}\end{aligned} \tag{45}$$

where

$$\{C_y\}^T=\{\tilde{C}_y\}^T[\phi_1],\{H_y\}=\{\tilde{H}_y\}^T[\phi_1] \tag{46}$$

After collocating the source point y at every boundary and internal node, a set of second-order partial differential equations are formed, which can be expressed in matrix form:

$$\begin{bmatrix}C_{b1,b1} & C_{b1,b2} & C_{b1,I}\\ C_{b2,b1} & C_{b2,b2} & C_{b2,I}\\ C_{I,b1} & C_{I,b2} & C_{I,I}\end{bmatrix}\begin{Bmatrix}\ddot{u}_{b1}\\ \ddot{u}_{b2}\\ \ddot{u}_I\end{Bmatrix}+\begin{bmatrix}H_{b1,b1} & H_{b1,b2} & H_{b1,I}\\ H_{b2,b1} & H_{b2,b2} & H_{b2,I}\\ H_{I,b1} & H_{I,b2} & H_{I,I}\end{bmatrix}\begin{Bmatrix}u_{b1}\\ u_{b2}\\ u_I\end{Bmatrix}=\begin{bmatrix}U_{b1,b1} & U_{b1,b2}\\ U_{b2,b1} & U_{b2,b2}\\ U_{I,b1} & U_{I,b2}\end{bmatrix}\begin{Bmatrix}t_{b1}\\ t_{b2}\end{Bmatrix} \tag{47}$$

The subscripts $b1$ and $b2$ mean that the variables are on known displacement boundary and known traction boundary respectively. The subscript I means the variables belong to internal nodes.

5 Time-marching scheme

Eq. (47) can be expressed in a more general form:

$$M\ddot{U} + KU = R \tag{48}$$

In order to solve Eq. (48), the Wilson-θ method is implemented. A brief review of this method is given in the following and readers are referred to Ref. [Bathe (1996)] for detailed descriptions. In Wilson-θ method, a linear variation of acceleration is assumed during the time interval $[t, t+\theta\Delta t]$, where $\theta \geq 1.0$. Let $\tau, 0 \leq \tau \leq \theta\Delta t$ denote the increase in time; then from time t to $t+\theta\Delta t$,

$$\ddot{U}^{t+\tau} = \ddot{U}^{t} + \frac{\tau}{\theta\Delta t}\left(\ddot{U}^{t+\theta\Delta t} - \ddot{U}^{t}\right) \tag{49}$$

For unconditional stability, $\theta \geq 1.37$ is required in FEM and usually $\theta = 1.4$ is employed. In the following section, the sensitivity to this parameter will be examined.

6 Numerical Examples

A C++ code has been developed based on the proposed method. In the code, each entry is calculated and stored as an *eight-byte* value. The program runs on a laptop with Intel Core(TM) Duo T9550 processor and 4GB physical memory. In order to demonstrate accuracy, capacity to treat non-homogeneity and large-scale potential of the proposed method, a number of numerical tests are carried out, including a stability analysis on the time-step size and parameter θ in Wilson-θ method, an accuracy analysis on the number of internal nodes and a comparison of storage between the presented method and conventional TD-BEM.

6.1 Test 1: Stability analysis on the time-step size and parameter θ.

In this example, the classical 1D numerical model is considered. This model consists of a one-dimensional bar of length L under Heaviside-type load $p_z = p_0 H(t-0)$, where $p_0 = 5$. Since the scheme presented herein is derived for 3D case, a cuboid domain in dimension $1\times1\times2$ is analyzed. Fig. 1 shows the definition of the model and boundary conditions.

The boundary mesh is illustrated in Fig. 2, consisting of 40 boundary elements and 63 ($3\times3\times7$) internal nodes. Eight-node discontinuous elements from Mi and Aliabadi (1992) as shown in Fig. 3 are adopted to treat edge and corner nodes, where the positioning parameter $0 < \lambda < 1$ of collocation nodes stands for the degree of continuity. It is taken to be 2/3 in the analyses. Poisson's ratio is set to be 0. In order to represent non-homogeneity, the shear modulus and mass density are

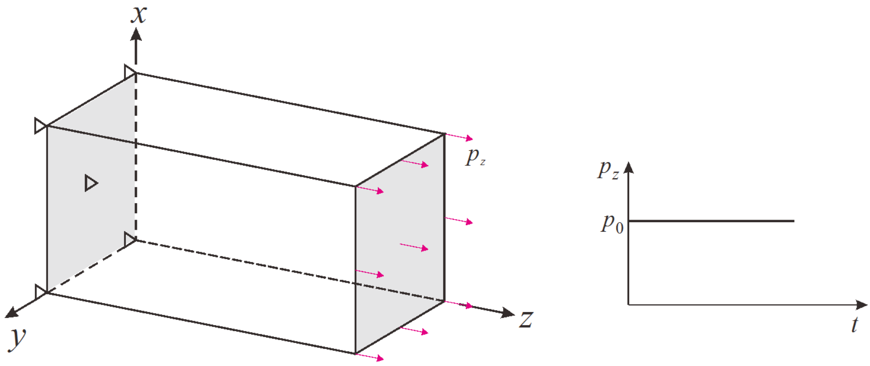

Figure 1: One dimensional bar: geometry and loading definition.

assumed to vary quadratically and linearly along z direction respectively:

$$\mu = \mu_0 \left(a_\mu \frac{z}{L} + 1\right)^2, \\ \rho = \rho_0 \left(a_\rho \frac{z}{L} + 1\right)^1 \tag{50}$$

where $\mu_0 = 1000, a_\mu = 2, \rho_0 = 1000, a_\rho = 3$. This leads to a wave velocity that goes from $\sqrt{2}$ to $1.5\sqrt{2}$.

In order to measure the time-step size Δt, a dimensionless variable, say $\beta_{\Delta t}$, is usually adopted [Mansur (1983); Junior (2007); Oyarzun, Loureiro, Carrer and Mansur (2011)],

$$\beta_{\Delta t} = c_d \Delta t / l \tag{51}$$

where l is the smallest boundary element length and c_d is the wave velocity. S-ince the wave velocity is a variable in this case, the maximum of c_d is chosen to determine the time-step.

In order to study the influence of time-step size and parameter θ on stability, various combinations listed in Tab.1 are chosen in the analyses. The BEM results are compared with graded finite element method proposed by Santare, Thamburaj and Gazonas (2003) in 1D.

Figs. 4-6 show the displacement u_z in the middle of the bar and the traction t_z at the fixed end under different parameters. As can be seen from Fig. 4 (case 1, 2 and 3) that results become unstable when $\theta = 1.3$ while $\theta = 1.4$ and 1.8 give good stability within the time range considered. This is in accordance with the conclusion in FEM that $\theta \geq 1.37$ is recommended. Fig. 5 shows that only the smallest time step length

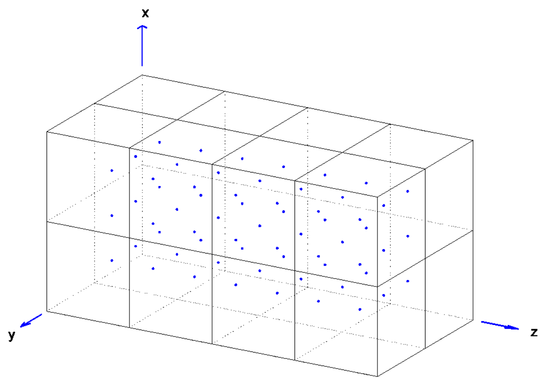

Figure 2: One dimensional bar: boundary discretization and internal nodes.

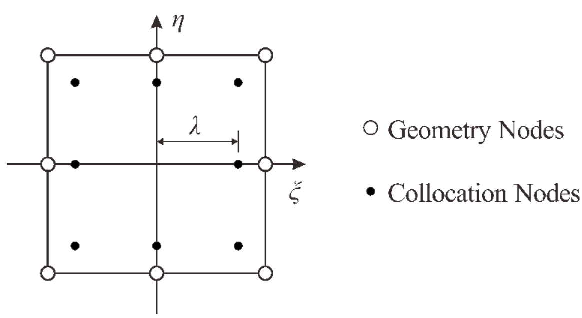

Figure 3: Eight-node discontinuous element.

Table 1: Parameters for Wilson-θ method.

CASE	1	2	3	4	5	6	7
θ	1.8	1.4	1.3	1.4	1.4	1.8	1.8
$\beta_{\Delta t}$	0.3	0.3	0.3	0.2	0.5	0.2	0.5

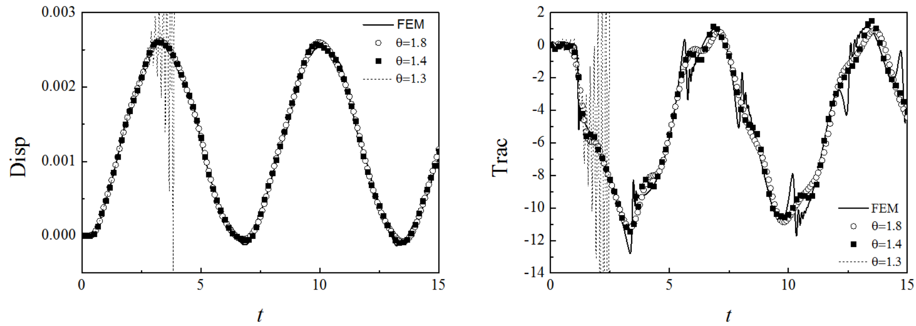

Figure 4: Displacement component u_z at middle point and traction component t_z at fixed end: Wilson-θ analysis with $\beta_{\Delta t} = 0.3$.

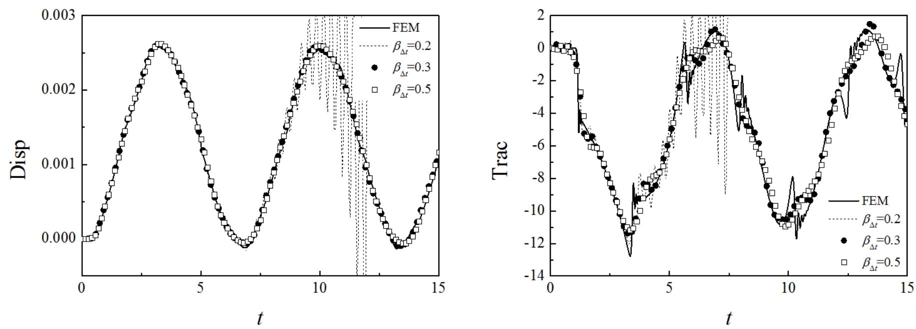

Figure 5: Displacement component u_z at middle point and traction component t_z at fixed end: Wilson-θ analysis with $\theta = 1.4$.

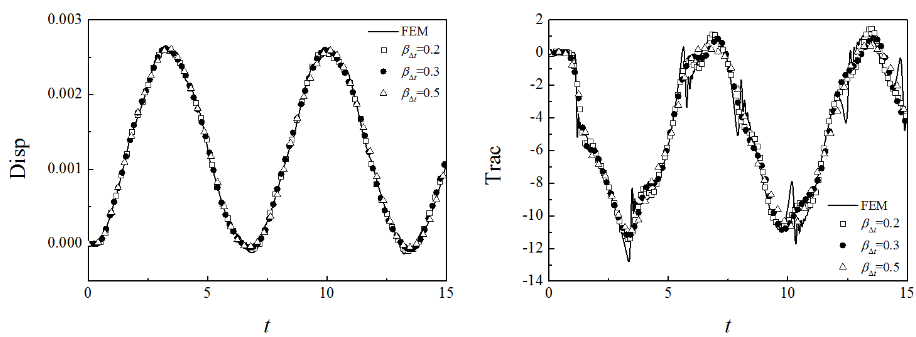

Figure 6: Displacement component u_z at middle point and traction component t_z at fixed end: Wilson-θ analysis with $\theta = 1.8$.

($\beta_{\Delta t} = 0.2$ in case 4) leads to instability when $\theta = 1.4$. Fig. 6 shows that all three values of $\beta_{\Delta t}$ give satisfactory stabilities when $\theta = 1.8$. In general values of 0.3 and 0.5 is recommended for $\beta_{\Delta t}$.

6.2 Test 2: Sensitivity of accuracy to the number of internal nodes.

In this example, the influence of number of internal nodes on accuracy is studied. The model and boundary mesh as given in test 1 are used, while 0, 3 and 63 internal nodes are scattered into the domain respectively for comparison. For 0-internal-nodes case, no internal nodes are used. For 3-internal-nodes case, three internal nodes are equally spaced along the longitude axis of the rod. For 63-internal-nodes case, the pattern is shown in Fig. 2. The parameters for time marching scheme are chosen to be $\beta_{\Delta t} = 0.3$, $\theta = 1.4$, which have been verified in the first test.

The BEM results of displacement and traction in terms of various numbers of internal nodes are shown in Fig. 7 and compared with FEM. It is observed that the number of internal nodes has little impact on the accuracy and that the BEM results with all the three cases are in good agreement with the FEM results. This indicates that only a few or even no internal nodes are required in the proposed method for the 3-D non-homogeneous wave propagations.

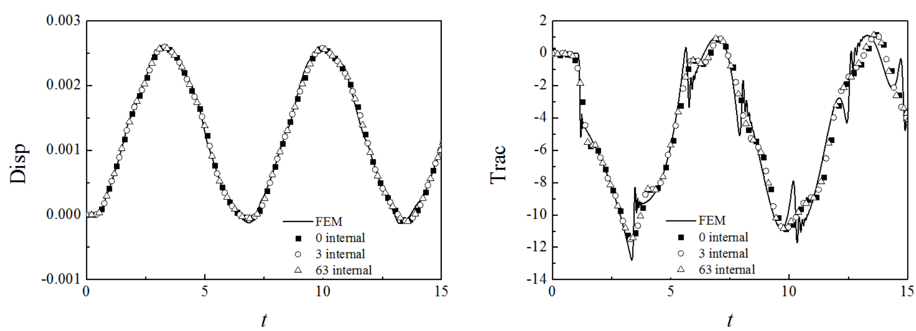

Figure 7: Displacement component u_z at middle point and traction component t_z at fixed end: Wilson-θ analysis.

6.3 Test 3: Storage analysis: a comparison with conventional TD-BEM.

In this example, the storage requirement versus the number of DOFs is studied. The model in Test 1 is meshed with different element sizes in order to generate different scales. The parameters for time marching scheme are chosen to be $\beta_{\Delta t} = 0.3, \theta = 1.4$. Total time length considered is set to be 15. The storage versus the number of DOFs by using the proposed RIM-based BEM is outputted as shown in

Fig. 8 and compared with that from the conventional TD-BEM (based on dynamic fundamental solutions and direct time stepping scheme, see Manolis and Beskos (1988)). As this is a comparison only on storage, for the conventional TD-BEM, homogeneous material is assumed with $\mu = 1000$ and $\rho = 1000$.

It is shown that the physical memory consumed by conventional TD-BEM grows dramatically with the increase of number of DOFs. When the number of DOF-s reaches 3840, the physical memory is projected to be 10.4 GB, which already exceeds the maximum physical memory of the computer. The physical memory needed for 6000 number of DOFs projects to 36.9 GB. This indicates that the conventional TD-BEM meets serious challenges when treating large-scale problems. The reason behind this is that the storage of conventional TD-BEM depends not only on the number of DOFs, but also on the number of coefficient matrices, which depends strongly on the total time steps required for a wave travelling through the entire domain.

By contrast, the physical memory used by the proposed method (herein presented as RIM-based BEM) just rises slightly with the increase of number of DOFs. Only about 1.5 GB is required for 6000 number of DOFs with 354 time steps. This is because the number of matrices arising from this method is time-independent and can be used for all time steps. This advantage makes this method potential for the wave propagation analysis of large-scale non-homogeneous problems.

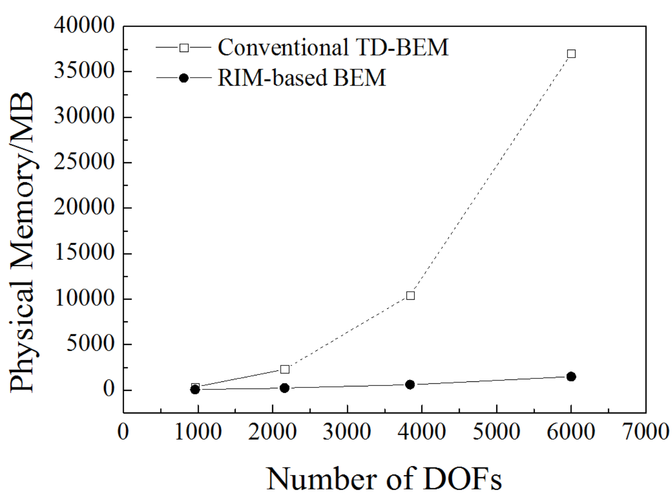

Figure 8: Storage requirement: a comparison.

7 Conclusions

In this paper, a RIM-based time domain BEM was proposed to solve transient dynamic problems in 3D non-homogeneous domains. The Kelvin fundamental solution was adopted to form the boundary-domain integration equations. The use of the Radial Integration Method (RIM) kept the advantage of BEM that the discretization and integration were restricted to the boundary. The Wilson-θ method was adopted as time-marching scheme. The main advantages of the proposed method are that it can treat 3-D non-homogeneous wave propagations with only boundary discretization and that the associated storage is independent of the number of time steps, making it suitable for large-scale problems. Numerical results demonstrated 1) good stability, 2) satisfactory accuracy and 3) large-scale potential of the proposed method.

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