 CMES, vol.109-110, no.3, pp.247-262, 2015

The Finite Points Approximation to the PDE Problems in Multi-Asset Options

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Abstract: In this paper we present a meshless collocation method based on the moving least squares (MLS) approximation for numerical solution of the multi-asset (d-dimensional) American option in financial mathematics. This problem is modeled by the Black-Scholes equation with moving boundary conditions. A penalty approach is applied to convert the original problem to one in a fixed domain. In finite parts, boundary conditions satisfy in associated $(d-1)$-dimensional Black-Scholes equations while in infinity they approach to zero. All equations are treated by the proposed meshless approximation method where the method of lines is employed for handling the time variable. Numerical examples for single- and two-asset options are illustrated.

Keywords: Meshless methods, MLS approximation, finite points method, American options, Black-Scholes equation.

1 Introduction

Option is one of the interesting financial instruments which is the right (but not the obligation) to buy (call option) or sell (put option) a risky asset at a prescribed fixed price (exercise or strike price) on (European option) or before (American option) a given date called expiry date. Options are mostly modeled by partial or stochastic differential equations. The price of an American option is governed by a linear complementarity problem involving the Black-Scholes differential operator and a constraint (or obstacle) on the value of the option. For details see Huang and Pang (1998); Jaillet, Lamberton, and Lapeyre (1990); Wilmott, Dewynne, and Howison (1993).

To remove the free boundary conditions in an American option, a method which

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adds a *penalty* term to the Black-Scholes equation can be employed. Using the penalty term is not quite new, for example, penalty method and front fixing method together with the finite difference method are discussed in Wu and Kwok (1997); Nielsen, Skavhaug, and Tveito (2002, 2008); Pantazopoulos, Houstis, and Kortesis (1998).

There are several methods for numerical solution of European and American options. A lattice method, as the first numerical method to Black-Scholes equation proposed in Cax, Ross, and Rubinstein (1979). Finite difference methods (FDM) combined with PSOR algorithm (Yves and O. Pironneau (2005)) and operator splitting method in (Dronen and Toivanen (2004)) are other approaches. Space-time adaptive finite difference methods for European multi-asset options are proposed in Persson and Von Sydow (2007); Lötstedt, Persson, Von Sydow, and Tysk (2007). In spite of the method of Persson and Von Sydow (2007) which uses second-order formulas in combination with adaptivity, in Linde, Persson, and Von Sydow (2009) a high accurate adaptive finite difference solver is introduced for the Black-Scholes equation. However, due to the discontinuity of the first derivative of the final condition, these discretizations yield only second-order accuracy locally. To circumvent this problem, authors use an extra (fine) grid around the discontinuity in a limited space- and time-domain. In Persson and Von Sydow (2010) a numerical method for pricing American options, in two cases, constant volatility and stochastic volatility, is proposed which uses a space-time adaptive finite difference method. Lee and Kim (2012), proposed a local differential quadrature (LDQ) method to solve the option-pricing models with stochastic volatilities.

Finite volume methods are also used for pricing American/European option with constant or time-dependent volatility. For example Wang (2004); Wang, Yang, and Teo (2006) have proposed a fitted finite volume method for spatial discretization and an implicit time stepping technique for temporal discretization which is combined with power penalty method for option pricing.

Finite element methods are more flexible for mesh adaptivity. See Yves and O. Pironneau (2005); Memon (2012); Young, Sun, and Shen (2009). In this scheme the analysis is performed within the framework of the vertical method of lines, where the spatial discretization is formulated as a Galerkin method with trial space spanned by proper basis functions like piecewise polynomials.

Recently, some papers have been published which consider the meshfree *radial basis functions* for solution of Black-Scholes equation for financial options. As examples see Fasshauer, Khaliq, and Voss (2004); Hon and Mao (1999); Hon (2002); Pettersson, Larsson, Marcusson, and Persson (2005). In Fasshauer, Khaliq, and Voss (2004) a penalty method is presented to handle the moving boundary in American options. This technique will be used in the present paper. It is known

that RBF techniques suffer from the ill-conditioned interpolation matrix. The condition numbers are sometimes increased exponentially. To overcome this drawback some precautions like preconditioning should be applied. We refer the reader to Beatson, Cherrie, and Mouat (1999); Brown, Ling, Kansa, and Levesley (2005); Fornberg and Piret (2007); Fornberg, Larsson, and Flyer (2011); Fornberg, Lehto, and Powell (2013) for details.

Moving least squares (MLS) approximation is another class of meshfree approximation methods which has been widely used in numerical solutions of PDEs. Some applications can be found in Belytschko, Krongauz, Organ, Fleming, and Krysl (1996); Atluri and Zhu (1998); Atluri and Shen (2002); Atluri (2005); Sladek, Sladek, and Atluri (2004). MLS is a stable numerical approximation with polynomial order of accuracy. In contrast with the global RBFs, the final approximation matrix is often well-conditioned. In this paper, we use the MLS as trial approximation to build a collocation scheme for penalized Black-Scholes equation in option pricing. This method is usually called *finite points method*. No background mesh is required for numerical simulation because the method uses scattered points to represent the consideration domain.

It should be noted that Yongsik, Hyeong-Ohk, and Hyeng Keun (2014) applied a meshfree method based on diffuse approximation of derivatives to solve option pricing in the single-asset case for European, Asian and Double barrier options. The present paper is different from that of Yongsik, Hyeong-Ohk, and Hyeng Keun (2014) because it concerns multi-asset American options instead. Besides, the approximation method of this paper uses the standard derivatives of the MLS instead of diffuse derivatives. Diffuse derivatives can be obtained via a generalized moving least squares approximation as discussed in Mirzaei, Schaback, and Dehghan (2012) and can be easily applied to the problem of this paper, too.

The rest of paper is organized as below. In Section 2 the Black-Scholes model and the penalty approach are discussed. In Section 3 the MLS approximation is reviewed and in Section 4 the finite points method is developed for multi-asset American options. Finally, in Section 5 some numerical results are given.

2 American options and a penalty formulation

Following Fasshauer, Khaliq, and Voss (2004), assume that there are d assets whose prices at time t are denoted by $S(t) = (S_1(t), \ldots, S_d(t))$. We can determine the value of $f(S,t)$ of the American put option by solving the d-dimensional Black-Scholes equation accompanying with moving boundary conditions. An American option problem is a free boundary problem because of the possibility of early exercise at any point during its life. For more details see Wilmott (1998); Brandimarte

(2006). If the moving boundary is denoted by notation $\overline{S}(t) = (\overline{S}_1(t), \ldots, \overline{S}_d(t))$, the time of expiry time by T, the volatility of i-th underlying asset by σ_i, the risk free interest rate by r, the dividend paid by asset i by D_i and the correlation between assets i and j by ρ_{ij}, then the following *Black-Scholes* equation with moving boundary conditions and payoff function (terminal condition)

$$\begin{aligned} &\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 f}{\partial S_i \partial S_j} + \sum_{i=1}^{d}(r-D_i)S_i\frac{\partial f}{\partial S_i} - rf = 0, \\ &\quad S_i > \overline{S}_i(t), \quad i = 1,\ldots,d, \quad 0 \leqslant t < T, \\ &\quad f(S,T) = \max\Big\{E - \sum_{i=1}^{d}\alpha_i S_i, 0\Big\}, \end{aligned} \tag{1}$$

is known as a mathematical model for multi-asset American option problems. Here E, the exercise price of the option, and α_i are given.

In Fasshauer, Khaliq, and Voss (2004) following Nielsen, Skavhaug, and Tveito (2002, 2008) a penalty term of the form $\frac{\varepsilon C}{f_\varepsilon+\varepsilon+q}$ is added to the original Black-Scholes equation to convert the problem to one on a fixed domain. Here $0 < \varepsilon \ll 1$ is a small regularization parameter, $C \geq rE$ is a positive constant, and q is a barrier function of the form $q(S) = E - \sum_{i=1}^{d}\alpha_i S_i$. The reason for choosing this penalty term has been fully discussed in Nielsen, Skavhaug, and Tveito (2002) and Fasshauer, Khaliq, and Voss (2004).

If we assume $\Omega = \{(S_1,\ldots,S_d) : S_i \geqslant 0,\ i = 1,\ldots,d\}$, by adding the penalty term to the original Black-Scholes equation we will have the following nonlinear PDE for $S \in \Omega$ and $0 \leqslant t < T$,

$$\frac{\partial f_\varepsilon}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 f_\varepsilon}{\partial S_i \partial S_j} + \sum_{i=1}^{d}(r-D_i)S_i\frac{\partial f_\varepsilon}{\partial S_i} - rf_\varepsilon + \frac{\varepsilon C}{f_\varepsilon+\varepsilon-q} = 0. \tag{2}$$

The terminal condition in time $t = T$ is enforced by

$$f_\varepsilon(S,T) = \max\Big\{E - \sum_{i=1}^{d}\alpha_i S_i, 0\Big\}, \tag{3}$$

and the boundary conditions are

$$f_\varepsilon(S,t) = g_i(S_i,t), \quad \text{where } S_j = 0, \quad j \neq i, \quad i,j = 1,\ldots d, \tag{4}$$

$$\lim_{S_i\to\infty} f_\varepsilon(S,t) = 0, \quad S \in \Omega, \quad i = 1,\ldots d, \tag{5}$$

where the functions g_i are the solutions of the associated $(d-1)$-dimensional Black-Scholes equations. (If one of the asset prices is zero at time t^*, then the asset will

be worthless for any $t > t^*$. Thus g_i should be the solutions of $(d-1)$-dimensional Black-Scholes equations). In numerical implementation the domain Ω is truncated and reduced to finite domain $\widehat{\Omega} = \{(S_1,\ldots,S_d) : 0 \leqslant S_i \leqslant S_\infty,\ i = 1,\ldots,d\}$ where S_∞ should be large enough to approximately reserve all properties of underlying problem. For the two-asset examples used in our numerical results is Section 5, the following single-asset problems for $i = 1,2$ should be solved to determine the boundary conditions

$$\frac{\partial g_i}{\partial t} + \frac{1}{2}\sigma_i^2 S_i^2 \frac{\partial^2 g_i}{\partial S_i^2} + (r - D_i) S_i \frac{\partial g_i}{\partial S_i} - r g_i + \frac{\varepsilon C}{g_i + \varepsilon - q(S_i)} = 0,$$
$$0 \leqslant S_i \leqslant S_\infty,\ 0 \leqslant t < T,$$
$$g_i(0,t) = \frac{E}{\alpha_i}, \quad g_i(S_\infty,t) = 0,$$
$$g_i(S_i,T) = \max\left\{E - \alpha_i S_i, 0\right\},$$

where $q(S_i) = E - \alpha_i S_i$.

3 MLS for trial approximtion

Meshless or meshfree methods write everything entirely in terms of *scattered points* and this is an alternative to the mesh-based approximation methods like as finite element and finite volume methods. The radial basis functions (RBFs) are frequently used as meshless approximations in recent years. Another family of mostly used meshfree methods are moving least squares (MLS) and related methods, introduced by Lancaster and Salkauskas (1981). Applications of MLS for numerical solution of PDEs have come to the picture by diffuse element method (see Nyroles, Touzot, and Villon (1992)), element-free Galerkin method (see Belytschko, Lu, and Gu (1994)), meshless local Petrov-Galerkin (MLPG) methods (see Atluri and Shen (2002); Atluri and Zhu (1998); Dong, Alotaibi, Mohiuddine and Atluri (2014); Zhang, Dong, Alotaibi and Atluri (2013)), etc.

Here we aim to apply the MLS collocation method for the nonlinear equation (2) with the terminal and boundary conditions (3)-(5). But before that we explain the MLS approximation itself. Suppose that Ω is a closed subset of $\mathbb{R}^d$ and let

$$X = \{x_1, x_2, \ldots, x_N\} \subset \Omega$$

are N points scattered in domain Ω. Suppose that $\mathbf{f} = (f(x_1),\ldots,f(x_N))^T$ is a vector containing the values of function $f \in C^m(\Omega)$ on X where m is a positive integer. Besides, let the space of d-variate polynomials of degree at most m is denoted by $\mathbb{P}_m^d$ and consider $\{p_1,\ldots,p_Q\}$ as a basis for this space, where $Q :=$

$\dim \mathbb{P}_m^d = \binom{m+d}{m}$. The MLS approximation $\hat{f}(x)$ is the best approximation of $f(x)$ out of span$\{p_1,\ldots,p_Q\}$ in respect to the following discrete and *moving* 2-norm

$$\|f\|_{2,X,w} := \sum_{j=1}^{N} w(x_j,y)[f(x_j)]^2, \quad y \in \Omega,$$

induced by the inner product

$$\langle f,g\rangle_{X,w} = \sum_{j=1}^{N} w(x_j,y) f(x_j) g(x_j), \quad y \in \Omega.$$

The inner product depends on moving point $y \in \Omega$. This finally leads to a local approximation if y is chosen properly. If for $x \in \Omega$ we define

$$\hat{f}(x) := \sum_{k=1}^{Q} \alpha_k(y) p_k(x) = \mathbf{p}^T(x)\alpha(x), \quad y \in \Omega,$$

then from the theory of best approximation in pre-Hilbert spaces, $\alpha(y) = (\alpha_1(y),\ldots,\alpha_Q(y))^T$ should be chosen such that

$$\langle f - \hat{f}, p_\ell\rangle_{X,w} = 0, \quad \ell = 1,2,\ldots Q.$$

This leads to the normal system

$$A(y)\alpha(y) = \mathbf{b}(y)$$

where

$$A(y)_{k\ell} = \langle p_k, p_\ell\rangle_{X,w} = \sum_{j=1}^{N} w(x_j,y) p_k(x_j) p_\ell(x_j), \quad k,\ell = 1,2,\ldots,Q,$$

$$\mathbf{b}(y)_\ell = \langle f, p_\ell\rangle_{X,w} = \sum_{j=1}^{N} w(x_j,y) p_\ell(x_j) f(x_j), \quad \ell = 1,2,\ldots,Q.$$

We noted that y should selected properly. In MLS to have a local approximation we put $y := x$ where x is current evaluation point, and the weight function $w : \Omega \times \Omega \to \mathbb{R}_+ \cup \{0\}$ is defined such that it becomes smaller the further away its arguments are from each other. Ideally, w vanishes for arguments $x, x_j \in \Omega$ with $\|x - x_j\|_2$ greater than a certain threshold, say δ. Such a behavior can be modeled by using a translation-invariant weight function. This means that w is of the form

$$w(x_j,x) = \varphi(r), \; r = \frac{\|x - x_j\|_2}{\delta},$$

where δ is the scaling parameter and $\varphi(r) > 0$ for $r \in [0,1]$ and $\varphi(r) = 0$ for $r > 1$. Let $I(x)$ is the set of active indices in x, i.e. $I(x) = \{j : \|x - x_j\| \leqslant \delta\}$ and let $X_x = \{x_j, : j \in I(x)\}$. If we define

$$P = P(x) = \big(p_k(x_j)\big) \in \mathbb{R}^{|I(x)| \times Q},$$
$$W = W(x) = \operatorname{diag}\{w(x_j, x)\} \in \mathbb{R}^{|I(x)| \times |I(x)|},$$
$$\mathbf{f} = \mathbf{f}(x) = \big(f(x_j)\big) \in \mathbb{R}^{|I(x)|}$$

where $|I(x)|$ denotes the size of $I(x)$, then we clearly have $A(x) = P^T W P$ and $\mathbf{b}(x) = P^T W \mathbf{f}$. According to Wendland (2005) if X_x is $\mathbb{P}_m^d$-unisolvent then $A(x)$ is positive definite and the MLS approximation is well-defined at x. Solving the normal equation gives

$$\alpha(x) = (P^T W P)^{-1} P^T W \mathbf{f},$$

which leads to

$$\hat{f}(x) = \mathbf{p}^T(x)\alpha(x) = \mathbf{p}^T(x)(P^T W P)^{-1} P^T W \mathbf{f} = \mathbf{a}(x)\mathbf{f},$$

where $\mathbf{a}(x) = \mathbf{p}^T(x)(P^T W P)^{-1} P^T W$ is called *shape function* vector at x. The extended form of the above equation can be written as

$$\hat{f}(x) = \sum_{j \in I(x)} a_j(x) f(x_j). \tag{6}$$

This equation shows that the approximant $\hat{f}$ is written in terms of values of f at scattered points x_j. The smoothness of a_j and thus $\hat{f}$ is directly connected to the smoothness of weight function w. Indeed if $w \in C^k(\Omega)$ then $\hat{f} \in C^k(\Omega)$. See Wendland (2005) for proof.

The MLS approximation is known to be a stable numerical algorithm, because as it was proven in Wendland (2005)

$$\sum_{j \in I(x)} |a_j(x)| \leqslant C_L,$$

where C_L is a small constant independent of X. This means that the *Lebesgue functions* are uniformly bounded by C_L. The reader should compare this with the known results on polynomial interpolation. MLS is computationally attractive because it solves for every point x a locally weighted least squares problem which is turn out to be a very efficient method because it is not necessary to set up and solve a large system of equations.

If the approximant $\hat{f}$ is smooth enough, the derivatives of $\hat{f}$ can be used to approximate the derivatives of f. Indeed

$$D^{\beta} f(x) \approx D^{\beta} \hat{f}(x) = \sum_{j \in I(x)} D^{\beta} a_j(x) f(x_j),$$

where β is a multi-index. It remains to show that how derivatives of a_j can be calculated. More details may be found in MLS literatures and specially in Belytschko, Krongauz, Organ, Fleming, and Krysl (1996).

Here we use the C^0 Gaussian weight function

$$\varphi(r) = \begin{cases} \frac{e^{-(\varepsilon r)^2} - e^{-\varepsilon^2}}{1 - e^{-\varepsilon^2}}, & 0 \leqslant r \leqslant 1, \\ 0, & r > 1, \end{cases}$$

where ε is a shape parameter, and/or the C^2 spline weight function

$$\varphi(r) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4, & 0 \leqslant r \leqslant 1 \\ 0 & r > 1. \end{cases}$$

4 Numerical algorithm

Assume that the set of *trial* points is $X = \{s_1, \ldots, s_N\} \subset \widehat{\Omega}$, where we change x to s to adjust the notation for Black-Scholes equation. A new set of points should be selected as *test* points where some of them should scatter over the boundary $\Gamma = \partial\widehat{\Omega}$ to impose the boundary conditions. We use the same trial and test points in numerical experiments. Without lose of generality, suppose that $N = N_I + N_B$ where the first N_I points are located inside and the remaining N_B points are located on the boundary of domain $\widehat{\Omega}$.

The vector $\mathbf{f}_{\varepsilon}(t) = [f_{\varepsilon}(s_1,t), \ldots, f_{\varepsilon}(s_N,t)]$ contains the values of unknown function $f_{\varepsilon} = f_{\varepsilon}(S,t)$, the solution of the boundary value problem (2) with initial and boundary conditions (3)-(5). In our numerical strategy, we replace the exact function f_{ε} by the approximation function $\hat{f}_{\varepsilon}$ constructed from the MLS method in (6). Thus equation (2) can be approximated by

$$\begin{aligned} &\sum_{\ell=1}^{N} a_{\ell}(S) \frac{\partial f_{\varepsilon}(s_{\ell},t)}{\partial t} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \rho_{ij} \sigma_i \sigma_j S_i S_j \sum_{\ell=1}^{N} \frac{\partial^2 a_{\ell}(S)}{\partial S_i \partial S_j} f_{\varepsilon}(s_{\ell},t) \\ &+ \sum_{i=1}^{d} (r - D_i) S_i \sum_{\ell=1}^{N} \frac{\partial a_{\ell}(S)}{\partial S_i} f_{\varepsilon}(s_{\ell},t) - r \sum_{\ell=1}^{N} a_{\ell}(S) f_{\varepsilon}(s_{\ell},t) \\ &+ \frac{\varepsilon C}{\sum_{\ell=1}^{N} a_{\ell}(S) f_{\varepsilon}(s_{\ell},t) + \varepsilon - \left(E - \sum_{i=1}^{d} \alpha_i S_i\right)} = 0. \end{aligned}$$

If we apply the above equation at internal test points $S = s_k$, $k = 1,\ldots,N_I$, in a vector form we have

$$\begin{aligned} &A\frac{\partial \mathbf{f}_\varepsilon(t)}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\rho_{ij}\sigma_i\sigma_j\mathbf{s}_i\mathbf{s}_j A^{(i,j)}\mathbf{f}_\varepsilon(t) + \sum_{i=1}^{d}(r-D_i)\mathbf{s}_i A^{(i)}\mathbf{f}_\varepsilon(t) \\ &- rA\mathbf{f}_\varepsilon(t) + \frac{\varepsilon C}{A\mathbf{f}_\varepsilon(t) + \varepsilon - \left(E - \sum_{i=1}^{d}\alpha_i\mathbf{s}_i\right)} = 0, \end{aligned} \tag{7}$$

where we introduced the following notations

$$\begin{aligned} &A_{k,\ell} = a_\ell(s_k), \quad A^{(i)}_{k\ell} = \frac{\partial a_\ell(s_k)}{\partial S_i}, \quad A^{(i,j)}_{k\ell} = \frac{\partial^2 a_\ell(s_k)}{\partial S_i \partial S_j}, \\ &\quad \ell = 1,\ldots,N, \quad k = N_b+1,\ldots,N, \\ &\mathbf{s}_i = \mathrm{diag}(s_{1,i},\ldots,s_{N_I,i}). \end{aligned}$$

In a short notation, equation (7) can be rewritten as

$$A\frac{\partial \mathbf{f}_\varepsilon(t)}{\partial t} + H\mathbf{f}_\varepsilon(t) + \frac{\varepsilon C}{A\mathbf{f}_\varepsilon(t) + \varepsilon - Q} = 0. \tag{8}$$

Note that, in the above formulation, constants C, ε and E are interpreted as constant vectors of size N_I, and the division and multiplication in the third term are element-by-element operators.

The terminal condition (3) provides the primary vector solution at $t = T$, i.e.

$$\mathbf{f}_\varepsilon(T) = \max\{E - \sum_{i=1}^{d}\alpha_i\mathbf{s}'_i, 0\}, \quad \mathbf{s}'_i = \mathrm{diag}(s_{1,i},\ldots,s_{N,i}). \tag{9}$$

To enforce the boundary conditions, we should solve some one dimensional problems ($d = 1$). In this case $f_\varepsilon(0,t) = E$ and $f_\varepsilon(S_\infty,t) = 0$. If MLS is applied, we have

$$\sum_{\ell=1}^{N} a_\ell(0) f_\varepsilon(s_\ell,t) = E, \quad \text{and} \quad \sum_{\ell=1}^{N} a_\ell(S_\infty) f_\varepsilon(s_\ell,t) = 0.$$

In vector form,

$$B\mathbf{f}_\varepsilon(t) = \begin{bmatrix} E \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} a_1(0) & \cdots & a_N(0) \\ a_1(S_\infty) & \cdots & a_N(S_\infty) \end{bmatrix} \in \mathbf{R}^{2\times N}. \tag{10}$$

Equations (8)-(10) form a system of Differential Algebraic Equations (DAE). For a d-dimensional problem, due to (4), the corresponding $(d-1)$-dimensional problem

should be solved to provide the values at half part of the boundary, say Γ_1. The boundary conditions at the remaining parts (say Γ_2) are zero due to (5). Thus the subroutines of lower dimensional problems should be called, inductively. Let $\mathbf{g}(t)$ be the vector solution of corresponding $(d-1)$-dimensional problem over boundary Γ_1. Then

$$\sum_{\ell=1}^{N} a_\ell(s_k) f_\varepsilon(s_\ell,t) = g_k(t), \quad s_k \in \Gamma_1, \quad \text{and} \quad \sum_{\ell=1}^{N} a_\ell(s_k) f_\varepsilon(s_\ell,t) = 0, \quad s_k \in \Gamma_2,$$

or,

$$B\mathbf{f}_\varepsilon(t) = \begin{bmatrix} \mathbf{g}(t) \\ \mathbf{0} \end{bmatrix}, \quad B_{k\ell} = a_\ell(s_k), \quad k = N_I+1,\ldots,N. \tag{11}$$

Now, equations (8), (9) and (11) form a system of DAE. There are some possibilities to find the approximate solution of DAE which are discussed here. It could, for instance, be solved like any other linear first–order implicit DAE system by invoking an ODE solver on it which would be an instance of the *Method of Lines*. The conventional finite difference methods, such as a Crank-Nicolson scheme, could also be applied to discretize the time variable t. In this case, the nonlinearity in the third term of (8) is linearized by evaluating the unknown quantities from the previous time level. In fact, we replace $\mathbf{f}_\varepsilon(t)$ by $\tilde{\mathbf{f}}_\varepsilon(t)$ in the last term of left-hand side of (8) and for $\Delta t = T/F$ and $t \in [(n-1)\Delta t, n\Delta t]$, $n = F, F-1,\ldots,1$, we set

$$\frac{\partial \mathbf{f}_\varepsilon(t)}{\partial t} \approx \frac{1}{\Delta t}\big(\mathbf{f}_\varepsilon^{(n)} - \mathbf{f}_\varepsilon^{(n-1)}\big), \quad \mathbf{f}_\varepsilon(t) \approx \frac{1}{2}\big(\mathbf{f}_\varepsilon^{(n)} + \mathbf{f}_\varepsilon^{(n-1)}\big),$$
$$\tilde{\mathbf{f}}_\varepsilon(t) \approx \mathbf{f}_\varepsilon^{(n)}, \quad \mathbf{g}(t) \approx \frac{1}{2}\big(\mathbf{g}^{(n)} + \mathbf{g}^{(n-1)}\big),$$

where $\mathbf{f}_\varepsilon^{(n)} = \mathbf{f}_\varepsilon(n\Delta t)$. Iterations start by $n = F$, where the terminal condition (9) is used for $\mathbf{f}_\varepsilon^{(F)} = \mathbf{f}_\varepsilon(T)$, and continue by decreasing n to 1. If the time step Δt remains unchanged, then a single LU decomposition of the final stiffness matrix is provided once, and corresponding backward and forward substitutions are enough to obtain the solution at any time level.

5 Numerical results

Here some numerical experiments are given for single and multi-asset options. To compare the results with the results of finite difference and RBF interpolation methods of Fasshauer, Khaliq, and Voss (2004) we consider only the case $\varepsilon = 0.01$. The effect of this penalty parameter are extensively studied in Nielsen, Skavhaug, and Tveito (2008, 2002). The Gaussian and spline weight functions for MLS approximation and different values for degree of polynomial basis functions (m) are

considered. In all cases regular node distributions with mesh-size h are used, although the method can always work for scattered data. The size support δ of weight function w should be large enough to ensure the regularity of matrix $A(x)$ in MLS approximation. Indeed δ should depend on the density of nodes and the degree of polynomial basis functions. Here $\delta = 2mh$ is used. The shape parameter ε in Gaussian weight function effects the numerical results, and unfortunately there is no optimal value available for that. Experiments show that $m \leqslant \varepsilon \leqslant 3m$ produce accurate results. Here $\varepsilon = 2m$ is used.

Table 1: Comparing finite difference, RBF and finite points solutions

S	FD 1001	RBF 41	RBF 101	MLS 41	MLS 101
0.6	0.4000037	0.4000012	0.4000036	0.4000069	0.4000042
0.7	0.3001161	0.3001120	0.3001159	0.3001481	0.3001213
0.8	0.2020397	0.2020191	0.2020368	0.2020869	0.2020467
0.9	0.1169591	0.1168706	0.1169460	0.1165985	0.1169025
1.0	0.0602833	0.0601659	0.0602888	0.0597755	0.0602273
1.1	0.0293272	0.0291898	0.0293064	0.0289271	0.0292648
1.2	0.0140864	0.0139888	0.0140717	0.0138714	0.0140527
1.3	0.0070408	0.0069832	0.0070328	0.0069509	0.0070266
1.4	0.0038609	0.0038313	0.0038584	0.0038321	0.0038564
RMSE		7.794e–5	1.015e–5	8.653e–5	1.200e–5

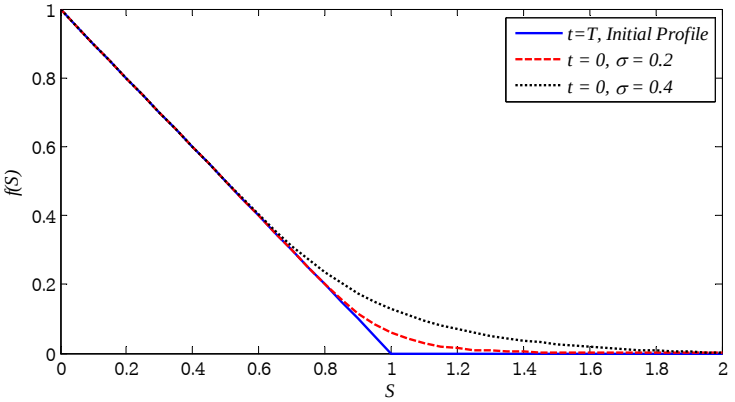

Figure 1: Initial ($t = T = 1$), and final ($t = 0$) profiles for two values of σ.

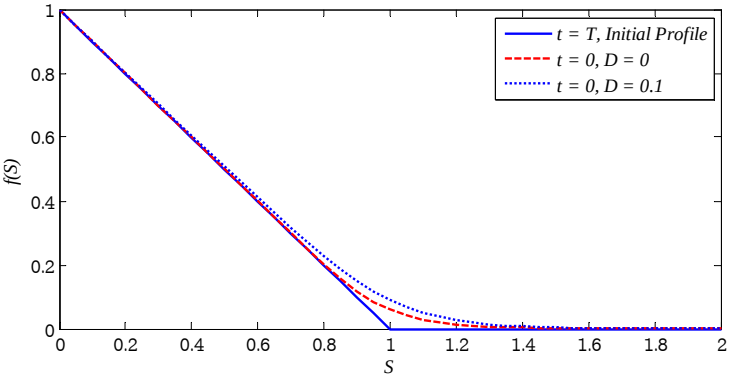

Figure 2: Initial ($t = T = 1$), and final ($t = 0$) profiles for two values of D.

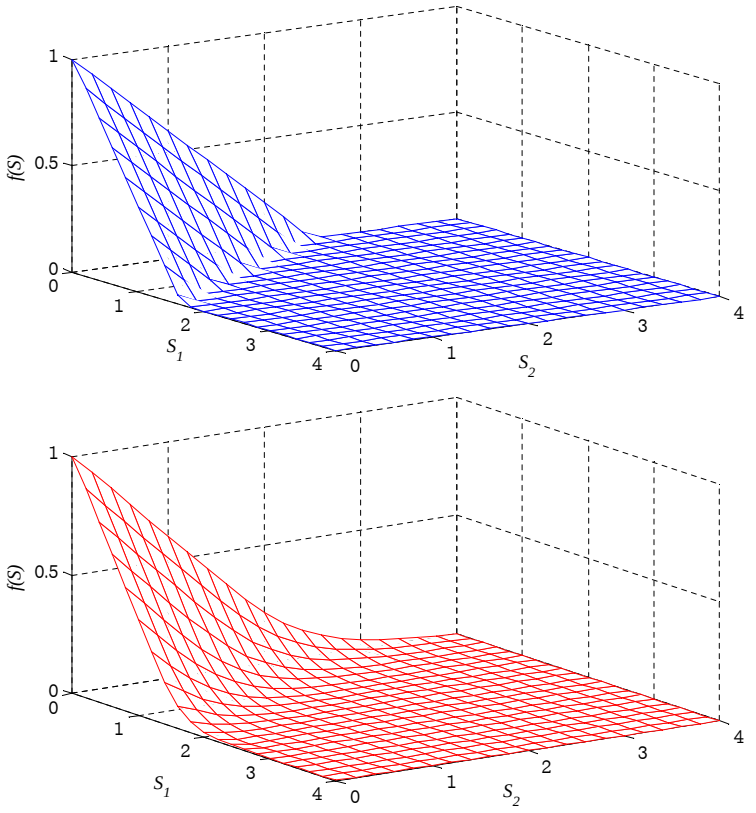

Figure 3: Initial ($t = T = 1$), and final ($t = 0$) profiles for correlated ($\rho_{12} = 0.5$) case.

For the single-asset option we set $r = 0.1$, $\sigma = 0.2$, $D = 0$, $E = 1$, $T = 1$, $S_0 = 0$, $S_\infty = 2$ and use the Crank-Nicolson scheme in time with $\Delta t = 0.01$. Results are presented in the last two columns of Table 1 and compared with the finite difference (with many points as a reference solution) and the radial basis function (RBF) solutions of Fasshauer, Khaliq, and Voss (2004). Numbers in the first row of Table 1 represent the number of points used in numerical simulation for each method.

For a two-asset problem we use $r = 0.1$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\alpha_1 = 0.6$, $\alpha_2 = 0.4$, $D_1 = 0.05$, $D_2 = 0.01$, $E = 1$, $T = 1$ and $\widehat{\Omega} = [0,4] \times [0,4]$. The time difference method with $\theta = 0.5$ is again employed. In Figure 3 the terminal profile and the profile at $t = 0$ are plotted in the case of correlated assets ($\sigma_{12} = 0.5$). Results are in excellent agreement with those given in Fasshauer, Khaliq, and Voss (2004).

In Figure 1 the profiles at $t = 0$ are plotted for different values of σ. Other parameters remain unchanged. In Figure 2 the profiles for $D = 0$ and $D = 0.1$ are compared.

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