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Dynamic Response Analysis of the Fractional-Order System of MEMS Viscometer

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Abstract: This paper presented dynamic response analysis for an MEMS viscometer. The responses are governed by a set of differential equations containing fractional derivatives. The memory-free Yuan-Agrawal's approach was extended to solve fractional differential equations containing arbitrary fractional order derivative and then a simple yet efficient numerical scheme was constructed. Numerical examples show that the proposed method can provide very accurate results and computational efforts can be significantly saved. Moreover, the numerical scheme was extended to solve problems with a nonlinear spring. The influences of the nonlinear parameters on the dynamic responses were also efficiently analyzed. The dependence of the angular frequency on damping parameters was also revealed. The presented method can provide us a new perspective to measure the fluid viscosity.

Keywords: MEMS viscometer; Memory-free scheme; Fractional derivative.

1 Introduction

The measure the viscosity of the fluid is of fundamental importance for the industry of oil well exploration. Ronaldson, Fitt, and Goodwin (2006) and Fitt, Goodwin, and Ronaldson (2009) developed a mathematical model, respectively, for a transversely oscillating micro-electro-mechanical system (MEMS) viscometer. The mode of operation employed a sort of plucking mechanism to measure the decay. The motions of the machine were modeled by a fractional differential equation. Therefore, we have to predict the response of the governing equation in order to measure the viscosity. Well-known, it is difficult to obtain the analytical

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solution of the fractional differential equation. Alternatively, an efficient numerical method is urgently needed.

In recent decades, fractional derivatives (FDs) were widely investigated because such mathematical models are useful in describing the behavior of some real systems. For example, FDs have been used successfully to model frequency dependent damping behavior of some viscoelastic materials. Bagley and Torvik (1979, 1984) applied the FDs to describe the frequency dependence materials. After that, excellent performance of the FDs in modeling viscoelastic materials has attracted more and more attention [Song and Jiang (1998); Shukla, Tamsir, Srivastava, and Kumar (2014); Chakraverty and Tapaswini (2014)]. Besides, there are a large number of other applications of FDs in the areas of physics, cybernetics, mechanics, biologies, economics, etc. [Podlubny (1998); Chen and Moore (2002)].

Many numerical solution techniques were developed to obtain analytical or numerical solutions of fractional differential equations, for example, the finite difference method [Meerschaert and Tadjeran (2004)], predictor-corrector approach [Diethelm, Neville, and Freed (2002)], variational iteration method [Sweilam, Khader, and Al-Bar (2007)], homotopy analysis method [Song and Zhang (2007)], to mention a few. Due to the non-local character of the fractional derivative, storing the past responses requires a large amount of computer memory. Accordingly, it is cumbersome to search even numerical solutions of fractional dynamic systems in a long time duration. In order to eliminate the drawback of long memory requirement, Yuan and Agrawal proposed a memory-free approach [Yuan and Agrawal (2002)], i.e., the Yuan-Agrawal's (YA) approach. In this scheme, the fractional differential equation can be converted to a set of first order ordinary differential equations. The YA approach was further extended by Diethelm, Neville, and Freed (2002); Trinks and Ruge (2002), etc. The computational accuracy is improved by Agrawal (2009).

In this paper, we presented a simple yet efficient scheme, based on the YA approach, to solve the dynamic responses of the MEMS viscometer. The YA approach was extended to dynamic systems with arbitrary fractional order. A widely used method, the predictor-corrector (PC) method, was utilized to validate the given scheme. Very accurate numerical results can be provided. Moreover, it is much more efficient than the PC method as the presented scheme is inherently memory-free.

2 Equations of motions

The schematic of the spider MEMS viscometer is shown in Fig. 1. A large number of legs give rise to its informal name: the spider. Also, it edge-clamped plate has

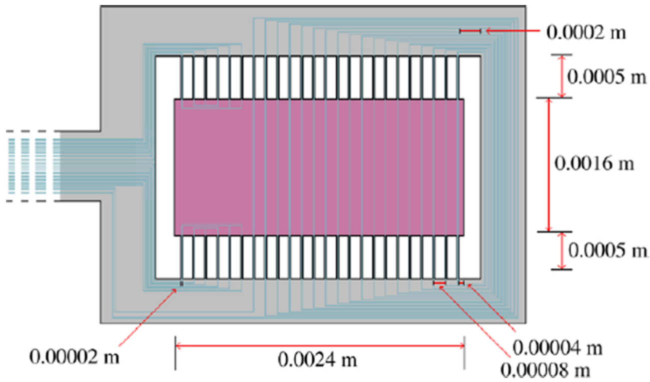

Figure 1: The sensor of the MEMS spider viscometry device [Fitt, Goodwin, and Ronaldson (2009)].

been used to determine the density and viscosity of Newtonian fluids, which flow is modeled by the Navier-Stokes equations. More details of modeling the device are given in Ronaldson, Fitt, and Goodwin (2006); and Fitt, Goodwin, and Ronaldson (2009).

There are two modes for this viscometer, i.e., the forced mode and the plucked mode [Fitt, Goodwin, and Ronaldson (2009)]. This study is restricted to the plunked mode, i.e.; the device is released from an initial displacement and its subsequent decaying oscillations are measured. It is found that the plucked mode gives rise to a fractional differential equation, such that the non-dimensional governing equations can be described as

$$D^2x+\eta Dx+x=\beta\int_0^t \frac{Dx(\tau)}{(t-\tau)^{3/2}}d\tau \quad x(0)=1, Dx(0)=0 \tag{1}$$

Where the Caputo derivative is given as . The parameters are $\eta=r/(k\sqrt{\rho_s Bda})$, and $\beta=(\mu\rho/k\pi)^{1/2}(Ba/\rho_s^3 d^3)^{1/4}$ is the elastic damping provided by the legs of devise, $k=\sqrt{\omega_0^2\rho_s}$, ω_0 is the frequency of the plate, ρ_s is the density of the plate material, B is the plate width, a is the plate length, d is the plate depth, and μ is the fluid viscosity.

3 A memory-free algorithm

Rewrite Eq. (1) in a generalized way as

$$D^2x+\beta\sqrt{\pi}D^\gamma x+\eta Dx+x=0 \quad x(0)=1, \quad Dx(0)=0 \tag{2}$$

The term, $D_*^\gamma x(t)(1<\gamma<2)$, is the Caputo derivative of order γ. Rewrite Eq. (2) as follow

$$\begin{aligned} &Dx_1(t) = x_2(t) \\ &Dx_2(t) = -\beta\sqrt{\pi}D_*^\gamma x_1(t) - \eta x_2(t) - x_1(t) \end{aligned} \tag{3}$$

fractional derivatives comply the law of exponent [Yuan and Agrawal (2002)]

$$D_*^{\gamma_1+\gamma_2}x(t) = D_*^{\gamma_1}D_*^{\gamma_2}x(t) \tag{4}$$

with γ_1 and γ_2 are positive real constants. Equation (4) can be rewritten as

$$\begin{aligned} &Dx_1(t) = x_2(t) \\ &Dx_2(t) = -\beta\sqrt{\pi}D_*^{\gamma-1}x_2(t) - \eta x_2(t) - x_1(t) \end{aligned} \tag{5}$$

by denoting the $\gamma - 1 = \alpha$ with $0<\alpha<1$ and

$$D_*^\alpha x_2(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{Dx_2(\tau)}{(t-\tau)^\alpha}d\tau, \quad 0<\alpha<1 \tag{6}$$

in which

$$\Gamma(\alpha) = \int_0^\infty e^{-z}z^{\alpha-1}dz \tag{7}$$

represents the Gamma function, and $Dx_2(t)$ represents the first order derivative with respect to t. Using the following relationship

$$\frac{1}{\Gamma(1-\alpha)} = \Gamma(\alpha)\frac{\sin\pi\alpha}{\pi} \tag{8}$$

Substituting Eq. (3) and (4) into Eq. (2), we can obtain

$$D_*^\alpha x_2(t) = \frac{\sin\pi\alpha}{\pi}\int_0^t [\int_0^\infty e^{-z}z^{\alpha-1}dz]\frac{Dx_2(\tau)}{(t-\tau)^\alpha}d\tau, \quad 0<\alpha<1 \tag{9}$$

Under the transformation

$$z = (t-\tau)y^{1/\alpha} \tag{10}$$

they further obtained

$$D_*^\alpha x_2(t) = 2\frac{\sin\pi\alpha}{\pi}\int_0^\infty\int_0^t e^{-(t-\tau)y^{1/\alpha}}Dx_2(\tau)d\tau dy \tag{11}$$

Denote $u = \sin \pi\alpha/(\pi\alpha)$ and $\Phi(y,t) = \int_0^t e^{-(t-\tau)y^{1/\alpha}} Dx_2(\tau)d\tau$, one can rewrite Eq. (1) as

$$
\begin{aligned}
&Dx_1(t) = x_2(t) \\
&Dx_2(t) = -\beta\sqrt{\pi}u\int_0^\infty \Phi(y,t)dy - \eta x_2(t) - x_1(t)
\end{aligned}
\tag{12}
$$

The unbounded integral is calculated numerically using Laguerre integration which gives

$$
\int_0^\infty \Phi(y,t)dy = \sum_{i=1}^n w_i^{(n)} e^{y_i^{(n)}} \Phi(y_i^{(n)},t) \tag{13}
$$

Where n represents the number of Laguerre points, $w_i^{(n)}$'s are the wrights and $y_i^{(n)}$'s are the Laguerre point abscissae. Observe that

$$
D\Phi(y,t) = Dx_2(t) - y^{1/\alpha}\Phi(y,t) \tag{14}
$$

one can finally rewrite Eq. (1) as a set of first-order differential equations

$$
\boldsymbol{A}D\boldsymbol{r}(t) = \boldsymbol{B}\boldsymbol{r}(t) \tag{15}
$$

where $\boldsymbol{r}(t) = [x_1(t)x_2(t)\Phi(y_1^{(n)},t)\Phi(y_2^{(n)},t)\cdots\Phi(y_n^{(n)},t)]^T$. The coefficient matrixes are listed as follows:

$$
A = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & -1 & 0 & \cdots & 1
\end{bmatrix}
$$

$$
\mathrm{B} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
-1 & -\eta & -\beta\sqrt{\pi}uw_1^{(n)}\exp(y_1^{(n)}) & \cdots & -\beta\sqrt{\pi}uw_n^{(n)}\exp(y_n^{(n)}) \\
0 & 0 & -(y_1^{(n)})^2 & \vdots & 0 \\
\vdots & \vdots & \cdots & \ddots & 0 \\
0 & 0 & \cdots & 0 & -(y_n^{(n)})^2
\end{bmatrix}
$$

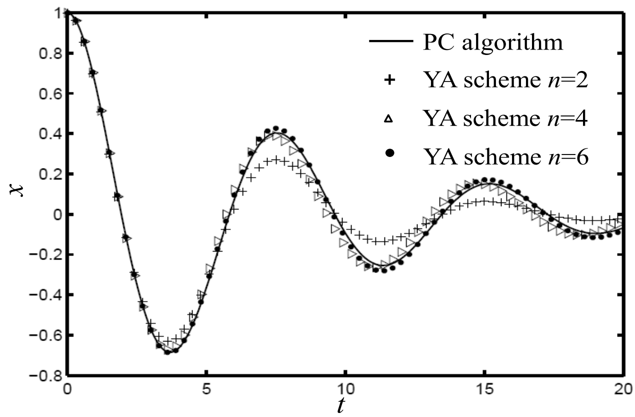

Figure 2: Comparison of numerical solution between YA scheme with different Laguerre node points and PC method. Parameters are $\eta = 0$ and $\beta = 0.3$.

4 Results and discussions

In order to verify the presented scheme, we compared the results between the YA method and PC approach [Diethelm (2002)]. As shown in Fig. 2, we can see that the results by the presented method are converged to the solution by PC as the number of Laguerre node points increasing.

The time responses of x with different β are shown in Fig. 3. It is very interesting to observe that the time-dependent periods of Eq. (1) depend on β. Different from the linear integer-order system, the linear fractional-order system can be expected to exhibit nonlinear mechanics behaviour.

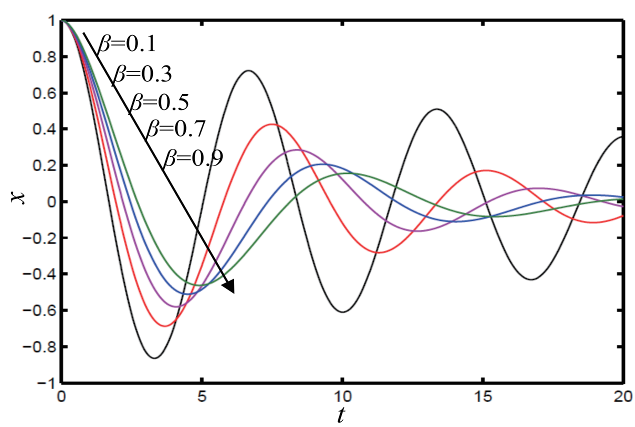

Figure 3: The dynamic response of Eq. (1) with different β and $\eta = 0$ obtained by YA scheme.

The effect of the β on the angular frequency of the Eq. (1) are shown as Fig. 4. The angular frequency linearly decreases first in the interval [0, 0.5], and then decreases more and more slowly as the β increasing.

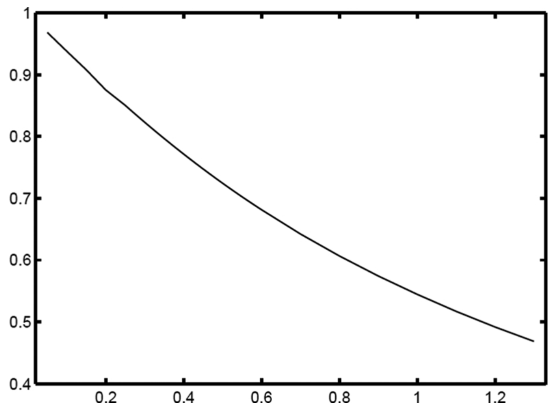

Figure 4: The correlation between the dynamic response angular frequency of Eq. (1) and parameter β obtained by YA scheme.

This phenomenon provides us a new perspective to determine the viscosity of the fluid. Due to the existence of a one-to-one correlation between the damping parameter β and angular frequency ω, thus we can measure the viscosity by measuring the frequency instead of measuring the decay [Fitt, Goodwin, and Ronaldson (2009)].

When β is very small, it needs a long time to decay for the dynamic responses. Well-known, due to the non-local nature of the fractional differential operators, the PC algorithm and/or finite difference method increase rapidly as the discretized time points (n) increases. Their required computational resources increases by a magnitude of $O(n^2)$. In the presented scheme, the fractional derivative term was transformed into a set of standard differential equations with no fractional derivative terms. As a result, the computation resources needed increases linearly at the magnitude of $O(n)$. It is suitable for seeking a solution during a long duration interval, such as the case of $\beta = 0.05$.

Figure 4 shows the ratios of the computing time of PC approach to that of YA scheme. It grows rapidly as the duration interval increases. Note that, the duration interval was discretized by n time points with time step $h = 0.01$. The presented approach is much more efficient than the PC method, especially when dynamic responses are required in a relatively long duration interval.

As shown in Fig. 4, roughly speaking, the ratio increases linearly versus duration interval. It is probably because the computational resources for the presented and the PC are at the order of magnitude of $O(n)$ and $O(n^2)$, respectively. Given an uniform time step h, the number of time points (n) is indirectly proportional to the duration interval, so is the ratio of respective computational resources.

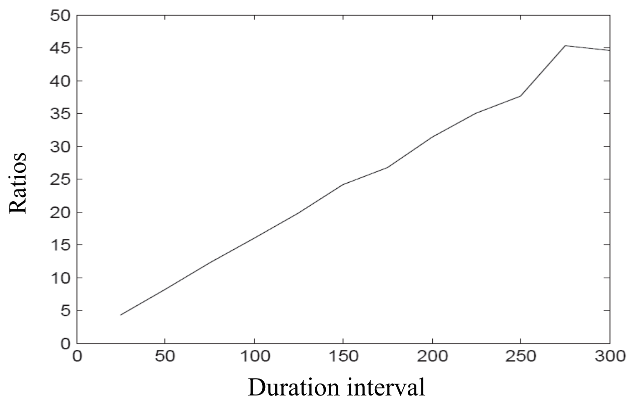

Figure 5: The ratios versus duration interval of the computing time of PC approach to that of YA scheme with the same step length $h = 0.01$.

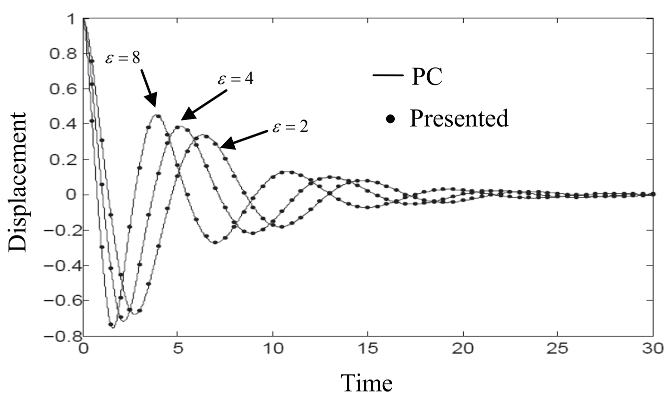

Figure 6: Comparison between YA scheme (dots) and PC method (solid line) for system (16) with $\alpha = 0, \beta = 0.5$ and $\varepsilon = 2,4,8$.

Consider the device as a nonlinear (Duffing-type) springs that is likely to be exactly true in practice [Fitt, Goodwin, and Ronaldson (2009)], where the governing equation becomes

$$D^2x + \beta\sqrt{\pi}D^{1.5}x + \eta Dx + x + \varepsilon x^3 = 0 \quad x(0) = 1, \quad Dx(0) = 0 \tag{16}$$

It is well known that there appears to be little hope of finding any close-form solutions to Eq. (16), the proposed numerical scheme can easily be modified to nonlinear equations.

In practice, $ADu(t) = Bu(t)$ is replaced by $ADu(t) = Bu(t) + f(t)$ with the nonlinear term being treated as an external excitation as $f(t) = [0, -x^3(t), 0, \cdots, 0]^T$. Figure 5 shows a comparison between the proposed scheme and PC approach with $\eta = 0$, $\beta = 0.5$ for $\varepsilon = 0, 2, 4, 8$. The dynamic response frequency, phase and amplitude increase slightly with ε increasing. It easy to observe the similarity between the dynamic response of nonlinear system (16) and the linear system (1), seen the figure (6) and (3).

5 Conclusions

Based on the several numerical experimental presented in this paper, we found that the proposed method can efficiently provide very accurate numerical solutions when compared with the PC algorithm. Different from the linear integer-order damping oscillator, the linear fractional-order damping oscillator is frequency dependent. This gives us a new perspective to determine the viscosity of the fluid. As the computational effort can be significantly saved compared with PC, the proposed method is especially suitable for analyzing the dynamic responses during a long time interval and/or multi-term fractional differential equation. More ever, it can be also extended to a nonlinear model, which does not increase in difficult in programming effort.

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