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The Lie-group Shooting Method for Radial Symmetric Solutions of the Yamabe Equation

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Abstract: We transform the Yamabe equation on a ball of arbitrary dimension greater than two into a nonlinear singularly boundary value problem on the unit interval $[0,1]$. Then we apply Lie-group shooting method (LGSM) to search a missing initial condition of slope through a weighting factor $r \in (0,1)$. The best r is determined by matching the right-end boundary condition. When the initial slope is available we can apply the group preserving scheme (GPS) to calculate the solution, which is highly accurate. By LGSM we obtain precise radial symmetric solutions of the Yamabe equation. These results are useful in demonstrating the utility of Lie-group based numerical approaches to solving nonlinear differential equations.

Keywords: Yamabe equation, nonlinear singularly boundary value problem, group preserving scheme, Lie-group shooting method.

1 Introduction

The Yamabe equation is a nonlinear differential equation arising in geometry and related areas of physics [S.Y. Alice Chang, Z.C. Han and P. Yang (2005); L. Andersson, P.T. Chrusciel and H. Friedrich (1992); A.H. Bhrawy, A.S. Alofi, R.A. Van Gorder (2014); S. Brendle (2008); R.A. Van Gorder (2012)]. We shall consider the Yamabe equation on the unit ball $\mathbf{B}^m$ in $\mathbb{R}^m$, where $m = 3,4,5,...$ is the dimension of the space. We can define $\mathbf{B}^m$ by $\mathbf{B}^m = \left\{ \mathbf{z} \in \mathbb{R}^m \middle| \|\mathbf{z}\| \leq 1 \right\}$. Similarly, we define the boundary of $\mathbf{B}^m$ to be the manifold $\mathbf{S}^{m-1} = \left\{ \mathbf{z} \in \mathbb{R}^{m-1} \middle| \|\mathbf{z}\| = 1 \right\}$.

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We then consider the Yamabe boundary value problem

$$-\Delta y = y - \lambda y^{1+4/(m-2)}, \tag{1}$$

$$y(\mathbf{z}) = 1 \quad \text{for} \quad \mathbf{z} \in \mathbf{S}^{m-1}, \tag{2}$$

where Δ is the Laplacian operator and λ is a parameter. In recent years, various methods for solving nonlinear equations and systems have been considered in literature; for example, you can see [T.A. Elgohary, L. Dong, J.L. Junkins, S.N. Atluri (2014a); T.A. Elgohary, L. Dong, J.L. Junkins, S.N. Atluri (2014b); T. Elgohary, D. Kim, J. Turner, J. Junkins (2014)]. Equation (1) is strongly nonlinear partial differential equation for any choice of m greater than two. To simplify the situation, we introduce a radially symmetric function and radial variable by

$$y(\mathbf{z}) = v(x) \quad \text{and} \quad x = \sqrt{z_1^2 + z_2^2 + \ldots + z_m^2}. \tag{3}$$

Let us also pick the constant

$$n = 1 + \frac{4}{m-2}$$

so that for $m \geq 3$ we have $n \in (1,5]$. Then the boundary value problem (1)-(2) is put into the form

$$v''(x) + \frac{m-1}{x} v'(x) - v(x) + \lambda v^n(x) = 0, \tag{4}$$

$$v'(0) = 0, \qquad v(1) = 1. \tag{5}$$

We should remark that (4) is similar in form to the Lane-Emden equation of the first kind which has been considered in the literature. Note that (4) has an extra term, and that the form of the conditions is different (the relevant Lane-Emden problem is an initial value problem, not a boundary value problem). Both λ and n are parameters. We can take λ to be a real number, whereas n depends on m: If $m = 3$, $n = 5$; if $m = 4$, $n = 3$; if $m = 5$, $n = 7/3$; and so on. As m tends to infinity, n tends to one, so the problem is linear in this limit and the problem can be solved exactly. In this paper we solve eqs. (4) and (5) that has been less attention for various λ and m via the Lie group shooting method and obtain radial symmetric solutions of the Yamabe equation. we show agreement between numerical solution and the exact solution in the large m case.

The Lie group shooting method is based on the group-preserving scheme (GPS), created previously by [C.-S. Liu (2001)] for solving the IVP of ODEs. Liu [C.-S. Liu (2006a); C.-S. Liu (2006b); C.-S. Liu (2006c); C.-S. Liu (2012a)] has developed the GPS to solve the BVPs. In the formation of the Lie-group method for the solutions of BVPs, Liu [C.-S. Liu (2006b)] has presented the concept of one-step GPS by utilizing the closure property of the Lie-group, and hence, the new shooting method had been named the Lie-group shooting method (LGSM). Because the Lie-group method has certain advantages than other numerical methods due to its Lie-group structure, is shown to be a powerful technique to solve the various problems [S. Abbasbandy, M.S. Hashemi , C.-S. Liu (2011); C.W. Chang, J.R. Chang, C.-S. Liu (2006); C.W. Chang, J.R. Chang, C.-S. Liu (2008); C.-S. Liu (2006a); C.-S. Liu (2006b); C.-S. Liu (2006c); C.-S. Liu (2006d); C.-S. Liu (2008); C.-S. Liu (2009); C.-S. Liu (2010); C.-S. Liu (2011); C.-S. Liu (2012a); C.-S. Liu (2012b); C.-S. Liu (2012c); C.-S. Liu, (2012d); C.-S. Liu (2013a); C.-S. Liu (2013b); C.-S. Liu, (2013c); C.-S. Liu, C.W. Chang, J.R. Chang (2008); C.-S. Liu, J.R. Chang (2008); C.-S. Liu, (2014)].

This paper is organized as follows. In section 2 we give a short sketch of the GPS for ODEs, explain the making of one-step GPS by using the closure property of the Lie-group, and find out it through a single-parameter Lie-group element in terms of a parameter r and through a universal one-step Lie-group element. In section 3 we derive a Lie-group shooting method to solve BVPs, where a missing initial condition is derived in a closed-form in terms of r in a range of $r \in (0,1)$ and we obtain r by matching the right boundary condition. In section 4 we convert the Yamabe equation to an equation with equal value boundary conditions that is very important in LGSM and present the numerical results of the LGSM on the Yamabe problem for different values of λ and m that represents solutions with good accuracy.

2 One-step group-preserving scheme

2.1 The GPS

Many physical systems can be writte as

$$\mathbf{u}' = \mathbf{f}(\mathbf{u},x), \tag{6}$$

$$u_1(\alpha) = c, \quad u_2(\beta) = c, \tag{7}$$

where

$$\mathbf{u} := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathbf{f} := \begin{bmatrix} u_2 \\ f(x,u_1,u_2) \end{bmatrix}. \tag{8}$$

It is very important in the Lie group shooting method that we replace the originally un-equal boundary conditions by the boundary conditions with an equal value. Liu

[C.-S. Liu (2001)] has embedded Eq. (6) into an augmented differential system:

$$\mathbf{X}' := \frac{d}{dx}\begin{bmatrix}\mathbf{u}\\ \|\mathbf{u}\|\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{2\times 2} & \dfrac{\mathbf{f}(x,\mathbf{u})}{\|\mathbf{u}\|}\\ \dfrac{\mathbf{f}^T(x,\mathbf{u})}{\|\mathbf{u}\|} & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \|\mathbf{u}\|\end{bmatrix} := \mathbf{AX}, \tag{9}$$

where $\mathbf{A}$ is an element of the Lie-algebra $so(2,1)$ satisfying

$$\mathbf{A}^T\mathbf{g} + \mathbf{gA} = \mathbf{0}, \tag{10}$$

with

$$\mathbf{g} = \begin{bmatrix}\mathbf{I}_2 & \mathbf{0}_{2\times 1}\\ \mathbf{0}_{1\times 2} & -1\end{bmatrix} \tag{11}$$

a Minkowski metric. Here, $\mathbf{I}_2$ is the identity matrix, and the superscript T stands for the transpose. The augmented variable $\mathbf{X}$ automatically satisfies the cone condition:

$$\mathbf{X}^T\mathbf{gX} = \mathbf{u}\cdot\mathbf{u} - \|\mathbf{u}\|^2 = 0. \tag{12}$$

Accordingly, Liu [C.-S. Liu (2001)] has developed a group-preserving scheme (GPS) given as follows:

$$\mathbf{X}_{k+1} = \mathbf{G}(k)\mathbf{X}_k \tag{13}$$

where $\mathbf{X}_k$ denotes the numerical value of $\mathbf{X}$ at the discrete x_k, and $\mathbf{G}(k) \in SO_o(2,1)$ satisfies

$$\mathbf{G}^T\mathbf{gG} = \mathbf{g}, \tag{14}$$

$$\det \mathbf{G} = 1, \tag{15}$$

$$\mathrm{G}_0^0 > 0, \tag{16}$$

where $\mathrm{G}_0^0 > 0$ is the 00th component of $\mathbf{G}$ and $SO_o(2,1)$ is the 3-dimensional Lorentz group.

2.2 Generalized mid-point rule

Applying scheme (13) to Eq. (9) with a specified initial condition $\mathbf{X}_0$, we can compute the solution $\mathbf{X}(x)$ by GPS. Assuming that the stepsize used in GPS is $\Delta x = (\beta - \alpha)/k$ and starting from an initial augmented condition $\mathbf{X}_0 = (\mathbf{u}^T(\alpha), \|\mathbf{u}(\alpha)\|)^T$ we want to calculate the value $\mathbf{X}_f = (\mathbf{u}^T(\beta), \|\mathbf{u}(\beta)\|)^T$ at $x = \beta$. By applying Eq. (13) step-by-step we can obtain

$$\mathbf{X}_f = \mathbf{G}_k(\Delta x)\cdots\mathbf{G}_1(\Delta x)\mathbf{X}_0, \tag{17}$$

where $\mathbf{X}_f$ approximates the exact $\mathbf{X}_f$ with a certain accuracy depending on Δx. However, let us recall that each $\mathbf{G}_i$, $i = 1,2,...,k$ is an element of the Lie group $SO_o(2,1)$ and by the closure property of Lie group $\mathbf{G}_k(\Delta x)...\mathbf{G}_1(\Delta x)$, is also a Lie group denoted by $\mathbf{G}$. Hence, we have

$$\mathbf{X}_f = \mathbf{G}\mathbf{X}_0. \tag{18}$$

This is a one-step transformation from $\mathbf{X}_0$ to $\mathbf{X}_f$.

Theoretically, such a one-step $\mathbf{G}$ exists, and the remaining problem is how to determine $\mathbf{G}$. While an exact solution of $\mathbf{G}$ is not available, we can calculate $\mathbf{G}$ through a numerical method by a generalized mid-point rule, which is obtained from an exponential mapping of $\mathbf{A}$ by taking the values of the argument variables of $\mathbf{A}$ at a generalized mid-point. The Lie group generated form $\mathbf{A} \in so(2,1)$ is known as a proper orthochronous Lorentz group, which admits a closed-form representation:

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_2 + \frac{a-1}{\|\hat{f}\|^2}\hat{f}\hat{f}^T & \frac{b\hat{f}}{\|\hat{f}\|} \\ \\ \frac{b\hat{f}^T}{\|\hat{f}\|} & a \end{bmatrix}, \tag{19}$$

where

$$\hat{\mathbf{u}} = r\mathbf{u}_0 + (1-r)\mathbf{u}_f, \tag{20}$$

$$\hat{f} = f(\hat{x}, \hat{\mathbf{u}}), \tag{21}$$

$$a = \cosh\left((\beta-\alpha)\frac{\|\hat{f}\|}{\|\hat{\mathbf{u}}\|}\right), \tag{22}$$

$$b = \sinh\left((\beta-\alpha)\frac{\|\hat{f}\|}{\|\hat{\mathbf{u}}\|}\right). \tag{23}$$

Here, we use the initial $\mathbf{u}_0 = (u_1(\alpha), u_2(\alpha))^T$ and the final $\mathbf{u}_f = (u_1(\beta), u_2(\beta))^T$ through a suitable weighting factor r to calculate $\mathbf{G}$, where $0 < r < 1$ is a parameter and $\hat{x} = r\alpha + (1-r)\beta$. The above method employed a generalized mid-point rule to calculate $\mathbf{G}$, and the resultant is a single-parameter Lie group element $\mathbf{G}(r)$. In section 3 we will describe a process to find a suitable $r \in (0,1)$.

The approach of Eq. (19) can be realized alternatively by using

$$\mathbf{G}' = \mathbf{A}(x, \mathbf{u})\mathbf{G}. \tag{24}$$

Integrating the above equation and using the mean-value theorem we obtain

$$\mathbf{G} = \exp\left[\int_\alpha^\beta \mathbf{A}(x,\mathbf{u})dx\right] = \exp[(\beta-\alpha)\mathbf{A}(\hat{x}, \hat{\mathbf{u}})]. \tag{25}$$

Inserting Eq. (9) for A and calculating the exponential we can derive Eqs. (19)-(23) again.

2.3 A universal Lie-group mapping between two points on the cone

Now we define a new vector

$$\mathbf{F} := \frac{\hat{f}}{\|\hat{\mathbf{u}}\|}, \tag{26}$$

such that Eqs. (19), (22) and (23) can also be expressed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_2 + \frac{a-1}{\|\mathbf{F}\|^2}\mathbf{F}\mathbf{F}^T & \frac{b\mathbf{F}}{\|\mathbf{F}\|} \\ \\ \frac{b\mathbf{F}^T}{\|\mathbf{F}\|} & a \end{bmatrix}, \tag{27}$$

$$a = \cosh\left[(\beta-\alpha)\|\mathbf{F}\|\right], \tag{28}$$

$$b = \sinh\left[(\beta-\alpha)\|\mathbf{F}\|\right]. \tag{29}$$

From Eqs. (18) and (27) the one-step Lie-group transformation is written as

$$\mathbf{u}_f = \mathbf{u}_0 + \eta\mathbf{F}, \tag{30}$$

$$\|\mathbf{u}_f\| = a\|\mathbf{u}_0\| + b\frac{\mathbf{F}.\mathbf{u}_0}{\|\mathbf{F}\|}, \tag{31}$$

where

$$\eta := \frac{(a-1)\mathbf{F}.\mathbf{u}_0 + b\|\mathbf{u}_0\|\|\mathbf{F}\|}{\|\mathbf{F}\|^2}. \tag{32}$$

Substituting

$$\mathbf{F} = \frac{1}{\eta}(\mathbf{u}_f - \mathbf{u}_0), \tag{33}$$

into Eq. (31) and dividing both the sides by $\|\mathbf{u}_0\|$, we can obtain

$$\frac{\|\mathbf{u}_f\|}{\|\mathbf{u}_0\|} = a + b\frac{(\mathbf{u}_f-\mathbf{u}_0).\mathbf{u}_0}{\|\mathbf{u}_f-\mathbf{u}_0\|\|\mathbf{u}_0\|}, \tag{34}$$

where

$$a = \cosh\left(\frac{(\beta-\alpha)\|\mathbf{u}_f-\mathbf{u}_0\|}{\eta}\right), \tag{35}$$

$$b = \sinh\left(\frac{(\beta-\alpha)\|\mathbf{u}_f-\mathbf{u}_0\|}{\eta}\right), \tag{36}$$

are obtained by inserting Eq. (33) for **F** into Eqs. (28) and (29). Let

$$\cos\theta := \frac{(\mathbf{u}_f - \mathbf{u}_0).\mathbf{u}_0}{\|\mathbf{u}_f - \mathbf{u}_0\|\|\mathbf{u}_0\|}, \tag{37}$$

and

$$\gamma := (\beta - \alpha)\|\mathbf{u}_f - \mathbf{u}_0\|, \tag{38}$$

where $0 \leq \theta \leq \pi$ is the intersection angle between vectors $\mathbf{u}_f - \mathbf{u}_0$ and $\mathbf{u}_0$, and thus from Eqs. (34)-(36) it follows that

$$\frac{\|\mathbf{u}_f\|}{\|\mathbf{u}_0\|} = \cosh\left(\frac{\gamma}{\eta}\right) + \cos\theta\,\sinh\left(\frac{\gamma}{\eta}\right). \tag{39}$$

Upon defining

$$Z := \exp\left(\frac{\gamma}{\eta}\right), \tag{40}$$

from Eq. (39) we obtain a quadratic equation for Z:

$$(1+\cos\theta)Z^2 - \frac{2\|\mathbf{u}_f\|}{\|\mathbf{u}_0\|}Z + 1 - \cos\theta = 0. \tag{41}$$

The solution is found to be

$$Z = \frac{\frac{\|\mathbf{u}_f\|}{\|\mathbf{u}_0\|} + \sqrt{\left(\frac{\|\mathbf{u}_f\|}{\|\mathbf{u}_0\|}\right)^2 - 1 + \cos^2\theta}}{1+\cos\theta}, \tag{42}$$

and thus from Eqs. (40) and (38) we can compute η by

$$\eta = \frac{(\beta - \alpha)\|\mathbf{u}_f - \mathbf{u}_0\|}{\ln Z}. \tag{43}$$

Therefore, between any two points $(\mathbf{u}_0, \|\mathbf{u}_0\|)$ and $(\mathbf{u}_f\|\mathbf{u}_f\|)$ on the cone, there exists a Lie-group element $\mathbf{G} \in SO_0(2n,1)$ mapping $(\mathbf{u}_0, \|\mathbf{u}_0\|)$ onto $(\mathbf{u}_f, \|\mathbf{u}_f\|)$:

$$\begin{bmatrix} \mathbf{u}_f \\ \|\mathbf{u}_f\| \end{bmatrix} = \mathbf{G}\begin{bmatrix} \mathbf{u}_0 \\ \|\mathbf{u}_0\| \end{bmatrix}, \tag{44}$$

where $\mathbf{G}$ is uniquely determined by $\mathbf{u}_0$ and $\mathbf{u}_f$ through Eqs. (27), (28), (29), (33), (37) ,(42) and (43) . We write this $\mathbf{G}$ to be $\mathbf{G}(\mathbf{u}_0, \mathbf{u}_f)$, in order to emphasize it as being a Lie-group mapping between the quantities of $\mathbf{u}_0$ and $\mathbf{u}_f$, which are the values of $\mathbf{u}$ occurred at two ends of a whole interval of $x \in [\alpha, \beta]$.

The above $\mathbf{G}(\mathbf{u}_0, \mathbf{u}_f)$ is different from the one in Eq. (19). These two Lie-group elements $\mathbf{G}(r)$ and $\mathbf{G}(\mathbf{u}_0, \mathbf{u}_f)$ are constructed by different manners. When the former is an approximation by using the generalized mid-point rule, the latter is a universal mapping between $(\mathbf{u}_0, \|\mathbf{u}_0\|)$ and $(\mathbf{u}_f, \|\mathbf{u}_f\|)$ independent to the vector field $\mathbf{f}$ and the parameter r, which means that such a mapping is applicable to all ODEs systems.

3 The Lie-group shooting method

It is interesting that by putting $\mathbf{G}(r) = \mathbf{G}(u_0, u_f)$ we can conclude the required equations for finding the missing initial condition. From Eqs. (6)-(8) it follows that

$$\dot{u}_1 = u_2, \tag{45}$$

$$\dot{u}_2 = f(x, u_1, u_2), \tag{46}$$

$$u_1(\alpha) = c, \quad u_1(\beta) = c, \tag{47}$$

$$u_2(\alpha) = A, \quad u_2(\beta) = B, \tag{48}$$

where A and B are two unknown constants, and c is a given positive constant determined by the user. From Eqs. (33), (47) and (48) it follows that

$$\mathbf{F} := \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \frac{1}{\eta} \begin{bmatrix} 0 \\ B-A \end{bmatrix}. \tag{49}$$

From Eqs. (43), (42) and (37) by inserting Eq. (8) for $\mathbf{u}$ and noting that

$$\mathbf{u}_0 = \begin{bmatrix} u_1(\alpha) \\ u_2(\alpha) \end{bmatrix} = \begin{bmatrix} c \\ A \end{bmatrix}, \quad \mathbf{u}_f = \begin{bmatrix} u_1(\beta) \\ u_2(\beta) \end{bmatrix} = \begin{bmatrix} c \\ B \end{bmatrix}, \tag{50}$$

we obtain

$$\eta = \frac{(\beta-\alpha)\sqrt{(A-B)^2}}{\ln Z}, \tag{51}$$

where

$$Z = \frac{\frac{\sqrt{c^2+B^2}}{\sqrt{c^2+A^2}} + \sqrt{\frac{c^2+B^2}{c^2+A^2} - 1 + \cos^2\theta}}{1+\cos\theta}, \tag{52}$$

$$\cos\theta = \frac{A(B-A)}{\sqrt{(A-B)^2}\sqrt{c^2+A^2}}. \tag{53}$$

By comparing Eq. (49) with Eq. (26), with the aid of Eqs. (20), (21), (47) and (48) we obtain

$$rA+(1-r)B=0, \tag{54}$$

$$A-B+\frac{\eta}{\xi}\hat{f}=0, \tag{55}$$

where

$$\hat{f}:=f(r\alpha+(1-r)\beta, rc+(1-r)c, rA+(1-r)B), \tag{56}$$

$$\xi:=\sqrt{c^2+[rA+(1-r)B]^2}. \tag{57}$$

The above derivation of the governing equations (54)- (57) is based on by equating the two F in Eqs. (49) and (26). It also means that the two Lie group elements defined by Eqs. (19) and (27) are equal. In this sense we may call our shooting technique a Lie-group shooting method.

3.1 The solution of A

Firstly, Liu [C.-S. Liu (2006b)] analytically solved A for general second-order BVPs. Remarkably, Eqs.(54) and (55) can be used to solve A exactly. From Eqs. (54) and (57) it follows that

$$\xi=c, \tag{58}$$

which is a positive constant. This equation is very important to reduce our following works to conclude a closed-form solution of A in terms of r. This is the reason that we replace the originally un-equal boundary conditions by the boundary conditions with an equal value in Eq. (7), which lends to $rA+(1-r)B=0$ as shown in Eq. (54), and hence ξ defined by Eq. (57) becomes a constant as shown in Eq. (58). Hence, from Eq. (55) with the aid of Eqs. (51)-(54), (56) and (58) we can obtain a single algebraic equation for solving the unknown variable A:

$$Ac+\eta_0\hat{f}=0, \tag{59}$$

where

$$\hat{f}=f(r\alpha+(1-r)\beta, c, 0), \tag{60}$$

$$Z=\frac{\sqrt{c^2+B^2}+\sqrt{B^2}}{\sqrt{c^2+A^2}-\sqrt{A^2}}, \tag{61}$$

$$\eta_0=\frac{(\beta-\alpha)\sqrt{A^2}}{\ln Z}, \tag{62}$$

and $B=rA/(r-1)$ has a different sign with A due to $r\in(0,1)$. Eq. (59) can be used to solve A for a given r. If A is available, we can return to Eqs. (45)-(48) and integrate them by a suitable IVP solver.

3.2 The case of $A > 0$

Here we first consider the case of $A > 0$. Inserting Eq. (62) for η_0 into Eq. (59) we obtain

$$\ln Z = -\frac{(\beta-\alpha)\hat{f}}{c}. \tag{63}$$

Defining

$$g_1 := \exp\left(-\frac{(\beta-\alpha)\hat{f}}{c}\right), \tag{64}$$

and substituting Eq. (61) for Z into Eq. (63) we obtain

$$\frac{\sqrt{c^2+B^2}+\sqrt{B^2}}{\sqrt{c^2+A^2}-\sqrt{A^2}} = g_1. \tag{65}$$

By using $A > 0$ and $B < 0$, Eq. (65) can be written as

$$g_1 A - B = g_1\sqrt{c^2+A^2} - \sqrt{c^2+B^2}. \tag{66}$$

Squaring the above equation and cancelling out the common terms we can rearrange it to

$$2g_1\sqrt{c^2+A^2}\sqrt{c^2+B^2} = (1+g_1^2)c^2 + 2g_1 AB. \tag{67}$$

Squaring again and cancelling out the common term and factor we can obtain

$$4g_1^2(A^2+B^2) - 4g_1(1+g_1^2)AB = (1-g_1^2)^2c^2. \tag{68}$$

Inserting $B = rA/(r-1)$ and through some algebraic manipulations we eventually obtain

$$\frac{4g_1}{(r-1)^2}[g_1+(1-g_1)^2r-(1-g_1)^2r^2]A^2 = (1-g_1^2)^2c^2. \tag{69}$$

If the following condition holds

$$\Psi_1(r) := g_1+(1-g_1)^2r-(1-g_1)^2r^2 > 0, \tag{70}$$

then A has a positive solution:

$$A = \sqrt{\frac{(r-1)^2(1-g_1^2)^2c^2}{4\Psi_1 g_1}}. \tag{71}$$

The condition (70) can be used to identify the range where r is permitted.

3.3 *The case of* $A < 0$

Next we consider the case of $A < 0$. Inserting Eq. (62) for η_0 into Eq. (59) we obtain

$$\ln Z = \frac{(\beta - \alpha)\hat{f}}{c}. \tag{72}$$

Defining

$$g_2 := \exp\left(\frac{(\beta - \alpha)\hat{f}}{c}\right), \tag{73}$$

and substituting Eq. (61) for Z into Eq. (72) we obtain

$$\frac{\sqrt{c^2 + B^2} + \sqrt{B^2}}{\sqrt{c^2 + A^2} - \sqrt{A^2}} = g_2. \tag{74}$$

By using $A < 0$ and $B > 0$, Eq. (74) can be written as

$$g_2 A - B = \sqrt{c^2 + B^2} - g_2\sqrt{c^2 + A^2}. \tag{75}$$

Squaring the above equation and cancelling out the common terms we can rearrange it to

$$2g_2\sqrt{c^2 + B^2}\sqrt{c^2 + A^2} = (1 + g_2^2)c^2 + 2g_2 AB. \tag{76}$$

Squaring again and cancelling out the common term and factor we can obtain

$$4g_2^2(A^2 + B^2) - 4g_2(1 + g_2^2)AB = (1 - g_2^2)^2 c^2. \tag{77}$$

Inserting $B = rA/(r-1)$ and through some algebraic manipulations we eventually obtain

$$\frac{4g_2}{(r-1)^2}[g_2 + (1 - g_2)^2 r - (1 - g_2)^2 r^2]A^2 = (1 - g_2^2)^2 c^2. \tag{78}$$

If the following condition holds

$$\Psi_2(r) := g_2 + (1 - g_2)^2 r - (1 - g_2)^2 r^2 > 0, \tag{79}$$

then A has a negative solution:

$$A = -\sqrt{\frac{(r-1)^2(1 - g_2^2)^2 c^2}{4\Psi_2 g_2}}. \tag{80}$$

3.4 Adjusting the slope A

In the previous two subsections we have derived two closed-form formulae to calculate the slope A for each r in its admissible range. If A is available, we can apply the GPS to integrate the $(\mathbf{u},x)$-IVP in Eqs. (45)-(48). Up to this point we should note that the Lie-group shooting method is an exactly solving technique for the second-order nonlinear BVPs without making any assumption or the approximation in derivations of all required formulas.

Now, in order to determine a correct r and thus a correct A, we need a numerical integration of the nonlinear ODEs in Eqs. (45)-(48) via a shooting technique. For a trial r in the admissible range, we can calculate A and then numerically integrate Eqs. (45)-(48) from $x=\alpha$ to $x=\beta$, and compare the end value of $u_1^r(\beta)$ with the exact one $u_1(\beta)=c$. If $|u_1^r(\beta)-c|$ is smaller than a given error tolerance ε, then the process of finding the solution of A is finished. Otherwise, we need to calculate the end values of $u_1(\beta)$ corresponding to different $r_1<r$ and $r_2>r$, which are denoted by $u_1^{r_1}(\beta)$ and $u_1^{r_2}(\beta)$, respectively. If $[u_1^{r_1}(\beta)-c][u_1^r(\beta)-c]<0$, then there exists one root between r_1 and r; otherwise, the root is located between (r,r_2). Continuing this process we can quickly select a suitable r to satisfy the criterion of $|u_1^r(\beta)-c|\leq\varepsilon$.

3.5 The GPS

We have derived the closed-form solutions to calculate the slope A for each r in its admissible range, and thus we can integrate the $(\mathbf{u},x)$-IVP in Eqs. (45)-(48) by the GPS method. The Lie group generated from $\mathbf{A}\in so(2,1)$ is known as a proper orthochronous Lorentz group. An exponential mapping of $\mathbf{A}(n)$ admits the closed-form representation:

$$\exp[\Delta x\mathbf{A}(n)]=\begin{bmatrix}\mathbf{I}_2+\frac{(\alpha_n-1)}{\|f_n\|^2}f_nf_n^T & \frac{\beta_nf_n}{\|f_n\|}\\ \frac{\beta_nf_n^T}{\|f_n\|} & \alpha_n\end{bmatrix} \tag{81}$$

where

$$\mathbf{f}_n=\mathbf{f}(x_n,\mathbf{u}_n), \tag{82}$$

$$\alpha_n=\cosh\left(\frac{\Delta x\|\mathbf{f}_n\|}{\|\mathbf{u}_n\|}\right), \tag{83}$$

$$\beta_n=\sinh\left(\frac{\Delta x\|\mathbf{f}_n\|}{\|\mathbf{u}_n\|}\right). \tag{84}$$

Substituting the above $\exp[\Delta x \mathbf{A}(n)]$ for $\mathbf{G}$ into Eq. (18) and taking its first row, we obtain

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \frac{(\alpha_n - 1)\mathbf{f}_n.\mathbf{u}_n + \beta_n \|\mathbf{u}_n\| \|\mathbf{f}_n\|}{\|\mathbf{f}_n\|^2} \mathbf{f}_n. \tag{85}$$

The numerical scheme (85) was first derived by Liu [C.-S. Liu (2001)]. We use from (85) to integrate IVPs. The major difference between GPS and the traditional numerical methods is that those schemes are all formulated directly in the usual Euclidean space $\mathbb{R}^n$; none of them are considered in the Minkowski space $\mathbb{M}^{n+1}$. One of the benefits of GPS in the augmented Minkowski space is that the resulting schemes can avoid the spurious solutions and ghost fixed points.

4 The Yamabe equation

We consider eqs. (4) and (5) and use the following transformation:

$$v(x) = u(x) - c + 1. \tag{86}$$

Therefore, the Yamabe problem is transformed to the following problem:

$$u''(x) = \frac{1-m}{x} u'(x) + u(x) - c + 1 - \lambda (u(x) - c + 1)^n, \tag{87}$$

$$u'(0) = 0, \quad u(1) = c. \tag{88}$$

Then, we convert Eqs. (87) and (88) to the following system:

$$u_1'(x) = u_2(x), \tag{89}$$

$$u_2'(x) = f(x, u_1, u_2), \tag{90}$$

$$u_2(0) = 0, \quad u_1(1) = c, \tag{91}$$

where

$$f(x, u_1, u_2) = \frac{1-m}{x} u'(x) + u(x) - c + 1 - \lambda (u(x) - c + 1)^n. \tag{92}$$

Now, consider the following equation:

$$u_1'(x) = u_2(x), \tag{93}$$

$$u_2'(x) = F(x, u_1, u_2), \tag{94}$$

$$u_1(-1) = c, \quad u_1(1) = c, \tag{95}$$

where

$$F(x,u_1,u_2) = \begin{cases} f(x,u_1,u_2), & 0 \le x \le 1, \\ \\ f(-x,u_1,-u_2), & -1 \le x \le 0, \end{cases} \tag{96}$$

is a symmetric extension of Eq. (90). As explained in previous section we can apply LGSM for Eqs. (93)-(96) on $[-1,1]$, but in practical numerical calculations we only need to calculate the above equations from $x = -1$ to $x = 0$, where we adjust the slope $u_2(-1) = A$ by the method in 3.4. If the target equation $u'(0) = 0$ is satisfied then we obtain the numerical solution by merely mapping the solution into the interval of $0 \le x \le 1$.

4.1 Numerical results

The Yamabe equation has the exact solution $v(x) = 1$ for $\lambda = 1$. we consider $u_2(-1) = A < 0$ for cases $\lambda < 1$ and $u_2(-1) = A > 0$ for cases $\lambda > 1$ and apply the Lie group shooting method. In order to avoid overflow in computations we can select a suitable c for different λ. In the Tables Tab. 1 and Tab. 2 are shown the parameters used in the Lie group shooting method for $\lambda = 0.1$, $\lambda = -1$, $\lambda = 2$, $\lambda = 4$ and various m. In the Figure Fig. 1 we show two examples for error mis-maching plot respect to r. We show the solutions obtained for the Yamabe equation using the Lie group shooting method for $\lambda = 0.1$, $\lambda = -1$, $\lambda = 2$, $\lambda = 4$ and various m in the Figures Fig. 2 and Fig. 3. Figure Fig. 4 show the residual error of solution Eqs. (4) and (5) for $\lambda = 0.1$ and $\lambda = 2$ and different m . We should point out that we use of $v''(x_i) \simeq (v'(x_{i+1}) - v'(x_{i-1}))/2\Delta x$ in Eq. (4) for obtain the residual error that $v'(x_i)$ calculated by LGSM. In Figure Fig. 5 we show symmetric radial solutions of the Yamabe respect to $v(x) = 1$ and that whenever the parameter m becomes large the solution converges to $v(x) = 1$. Finally, in the Tables Tab. 3-Tab. 8 we compared results obtained for $v(x)$ and $v'(x)$ through LGSM for $\lambda = -1$ and $\lambda = 4$ with the MIDRICH (midpoint method with Richardson extrapolation [L. Lapidus and J.H. Seinfeld (1971)]) that is a powerful method for solving the singular problems. This comparison shows the accuracy of LGSM.

5 Conclusions

In this paper we obtain solutions for the Yamabe equation for different values of λ via the Lie group shooting method. Lie-group shooting method was developed to derive algebraic equations to find the missing initial condition. Furthermore, by adjusting the boundary conditions from the one of un-equal boundary-value to the one with equal boundary-value, the present approach can provide a closed-form formula to calculate the missing initial condition without need of any iteration.

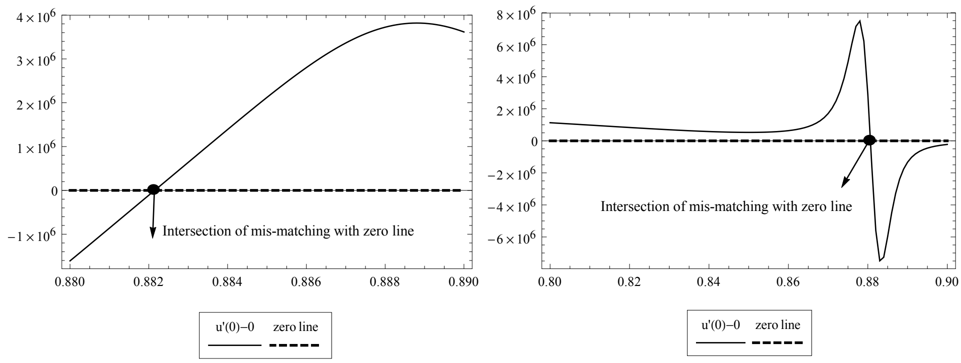


Figure 1: Plot of $u'(0)-0$ respect to r for $\lambda=0.1$ and $m=3$ (left), $\lambda=2$ and $m=4$ (right).

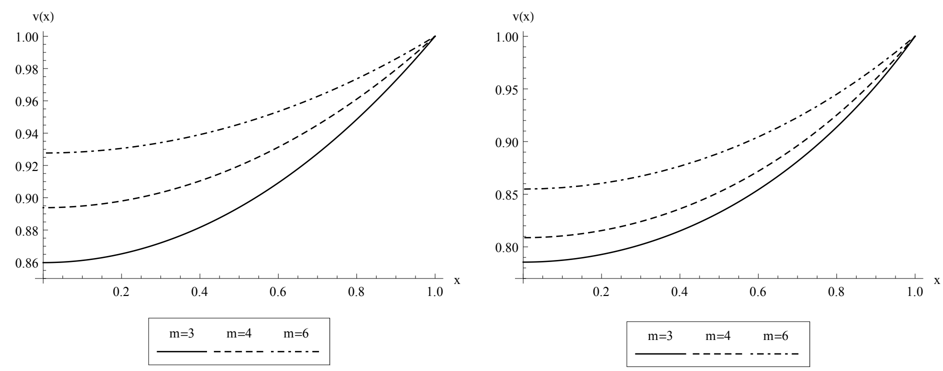


Figure 2: Numerical solutions obtained using LGSM for $\lambda=0.1$ (left) and $\lambda=-1$ (right).

Table 1: Parameters used for $\lambda = 0.1$ (up) and $\lambda = -1$ (down).

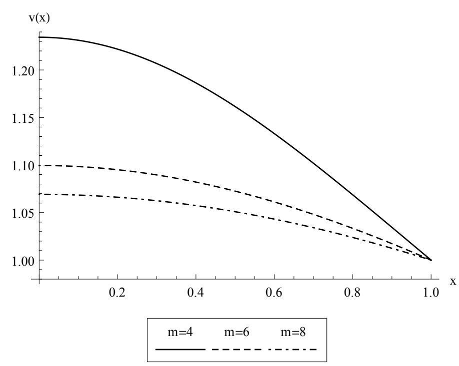
m	c	Δx	r	ε
3	1	0.0001	0.882147796469	2×10^{-9}
4	1	0.001	0.914317329532	3×10^{-8}
6	1	0.005	0.944591577888	7×10^{-8}

m	c	Δx	r	ε
3	10	0.0001	0.873285360792	6×10^{-8}
4	10	0.001	0.892809997425	4×10^{-8}
6	10	0.005	0.922926007352	2×10^{-7}

Table 2: Parameters used for $\lambda = 2$ (up) and $\lambda = 4$ (down).

m	c	Δx	r	ε
4	1	0.001	0.880650585250	6×10^{-10}
6	1	0.005	0.944044641065	4×10^{-8}
8	1	0.01	0.960964387295	3×10^{-6}

m	c	Δx	r	ε
6	10	0.005	0.903135356042	2×10^{-7}
8	10	0.01	0.935530255437	4×10^{-6}
10	10	0.01	0.950318540055	6×10^{-4}



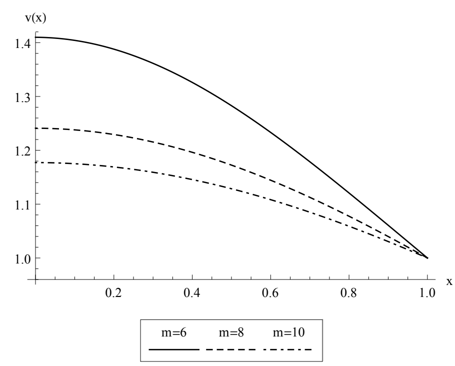


Figure 3: Numerical solutions obtained using LGSM for $\lambda = 2$ (left) and $\lambda = 4$ (right).

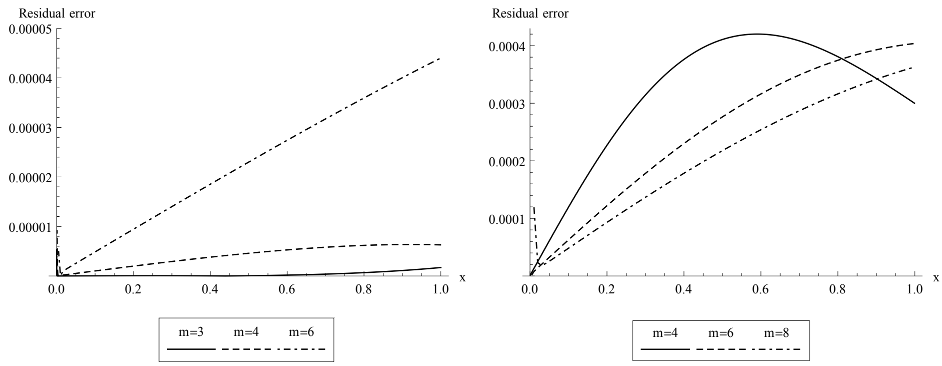


Figure 4: Residual error obtained using LGSM for $\lambda = 0.1$ (left) and $\lambda = 2$ (right).

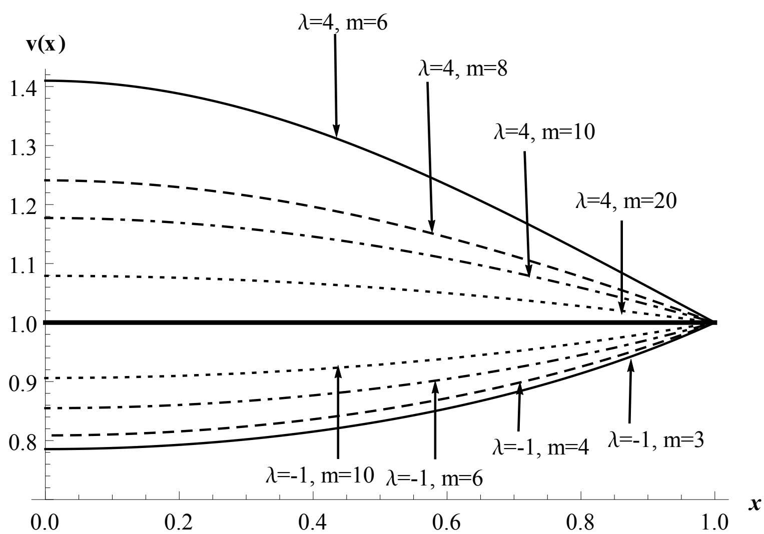


Figure 5: Plot of the radial symmetric solutions to the Yamabe equation for $\lambda = -1$ and $\lambda = 4$ and also $\lim_{m\to\infty} v_{m,\lambda}(x) = v_{m,1}(x) = 1$.

Table 3: Comparison of numerical results obtained by LGSM and MIDRICH for $\lambda = -1, m = 3$.

Nodes	LGSM		MIDRICH	
	$v(x)$	$v'(x)$	$v(x)$	$v'(x)$
0	0.785535	5.33891×10^{-8}	0.785556	0
0.1	0.787347	0.036260	0.787367	0.036262
0.2	0.792812	0.073161	0.792830	0.073164
0.3	0.802027	0.111385	0.802043	0.111389
0.4	0.815162	0.151704	0.815177	0.151709
0.5	0.832472	0.195044	0.832486	0.195049
0.6	0.854314	0.242567	0.854326	0.242573
0.7	0.881180	0.295805	0.881190	0.295810
0.8	0.913741	0.356857	0.913748	0.356861
0.9	0.952917	0.428724	0.952921	0.428723
1	1	0.515876	1	0.515868

Table 4: Comparison of numerical results obtained by LGSM and MIDRICH for $\lambda = -1, m = 4$.

Nodes	LGSM		MIDRICH	
	$v(x)$	$v'(x)$	$v(x)$	$v'(x)$
0	0.808748	3.6322×10^{-8}	0.808941	0
0.1	0.810439	0.033527	0.810616	0.033540
0.2	0.815504	0.067558	0.815666	0.067581
0.3	0.824019	0.102612	0.824167	0.102643
0.4	0.836114	0.139252	0.836247	0.139287
0.5	0.851980	0.178105	0.852098	0.178143
0.6	0.871873	0.219897	0.871974	0.219933
0.7	0.896129	0.265487	0.896211	0.265517
0.8	0.925179	0.315923	0.925238	0.315941
0.9	0.959571	0.372509	0.959603	0.372508
1	1	0.436909	1	0.436876

Table 5: Comparison of numerical results obtained by LGSM and MIDRICH for $\lambda = -1, m = 6$.

Nodes	LGSM		MIDRICH	
	$v(x)$	$v'(x)$	$v(x)$	$v'(x)$
0	0.854891	1.87684×10^{-7}	0.855624	0
0.1	0.856279	0.026477	0.856948	0.026506
0.2	0.860329	0.053231	0.860934	0.053284
0.3	0.867079	0.080539	0.867623	0.080608
0.4	0.876603	0.108688	0.877084	0.108767
0.5	0.888998	0.137987	0.889415	0.138067
0.6	0.904399	0.168766	0.904747	0.168842
0.7	0.922970	0.201395	0.923245	0.201455
0.8	0.944919	0.236283	0.945113	0.236318
0.9	0.970496	0.273897	0.970599	0.273893
1	1	0.314773	1	0.314714

Table 6: Comparison of numerical results obtained by LGSM and MIDRICH for $\lambda = 4, m = 6$.

Nodes	LGSM		MIDRICH	
	$v(x)$	$v'(x)$	$v(x)$	$v'(x)$
0	1.40991	-1.42957×10^{-7}	1.40842	0
0.1	1.40421	−0.108274	1.40300	−0.108075
0.2	1.38787	−0.212333	1.38694	−0.212051
0.3	1.36151	−0.308428	1.36085	−0.308166
0.4	1.32609	−0.393453	1.32568	−0.393291
0.5	1.28286	−0.465145	1.28264	−0.465135
0.6	1.23323	−0.522177	1.23314	−0.522336
0.7	1.17867	−0.564147	1.17867	−0.564463
0.8	1.12069	−0.591468	1.12074	−0.591906
0.9	1.06071	−0.605214	1.06075	−0.605726
1	1	−0.606924	1	−0.607454

Table 7: Comparison of numerical results obtained by LGSM and MIDRICH for $\lambda = 4, m = 8$.

Nodes	LGSM		MIDRICH	
	$v(x)$	$v'(x)$	$v(x)$	$v'(x)$
0	1.24105	-3.68056×10^{-6}	1.23887	0
0.1	1.23796	−0.055933	1.23608	−0.055779
0.2	1.22934	−0.110699	1.22775	−0.110447
0.3	1.21536	−0.163232	1.21406	−0.162939
0.4	1.19629	−0.212563	1.19527	−0.212279
0.5	1.17251	−0.257851	1.17174	−0.257618
0.6	1.14445	−0.298413	1.14390	−0.298262
0.7	1.11261	−0.333739	1.11226	−0.333691
0.8	1.07755	−0.363505	1.07735	−0.363567
0.9	1.03982	−0.387562	1.03974	−0.387732
1	1	−0.405933	1	−0.406199

Table 8: Comparison of numerical results obtained by LGSM and MIDRICH for $\lambda = 4, m = 10$.

Nodes	LGSM		MIDRICH	
	$v(x)$	$v'(x)$	$v(x)$	$v'(x)$
0	1.17726	-5.1712×10^{-4}	1.17559	0
0.1	1.17510	−0.039218	1.17363	−0.039139
0.2	1.16905	−0.077876	1.16778	−0.077743
0.3	1.15918	−0.115451	1.15812	−0.115289
0.4	1.14564	−0.151452	1.14478	−0.151286
0.5	1.12861	−0.185434	1.12793	−0.185283
0.6	1.10830	−0.217001	1.10780	−0.216885
0.7	1.08499	−0.245824	1.08464	−0.245756
0.8	1.05896	−0.271640	1.05875	−0.271630
0.9	1.03052	−0.294260	1.03042	−0.294312
1	1	−0.313565	1	−0.313680

Therefore, we can determine r very quickly through only a few trials. From the numerical results, it is clear that for values $\lambda < 1$ solutions are ascending and for values $\lambda > 1$ are descending but whenever the parameter m becomes large the solution converges to $v(x) = 1$ (exact solution for $\lambda = 1$). Compared with other numerical methods, the new approach is shown to be accurate.

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