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Numerical Solution of Fractional Fredholm-Volterra Integro-Differential Equations by Means of Generalized Hat Functions Method

Baofeng Li[1]

Abstract: In this paper, operational matrix method based on the generalized hat functions is introduced for the approximate solutions of linear and nonlinear fractional integro-differential equations. The fractional order generalized hat functions operational matrix of integration is also introduced. The linear and nonlinear fractional integro-differential equations are transformed into a system of algebraic equations. In addition, the method is presented with error analysis. Numerical examples are included to demonstrate the validity and applicability of the approach.

Keywords: Fractional integro-differential equations, generalized hat functions, operational matrix, error analysis, numerical solution.

1 Introduction

Fractional calculus has been known for more than 300 years. These fractional phenomena allow us to describe a real object more accurately than the classical integer order methods. As we all know, the nature of real objects is fractional. However, for many of them the fractionality is very low. The fractional system describes many typical examples, such us the voltage current relation of a semi-infinite lossy transmission line [Wang (1987)], the diffusion of heat through a semi-infinite solid, where heat flow is equal to the half derivative of the temperature [Westerlund (2002)]. In recent years, there are a lot of methods for approximation of fractional derivatives and integrals can be used in wide filed of applications. Fractional order calculus plays an important roles in electrical engineering [Nakagava and Sorimachi (1992)], physics [Valdes-Parada; Ochoa-Tapia; Alvarez-Ramirez (2007)], signal processing [Vinagre and Chen(2003); Tseng (2007)], robotics [Maria da Graca Marcos, Duarte, Tenreiro Machado (2008)], chemistry [Oldham and Spanier (1974)], chaos [Tavazoei and Haeri (2008)], and so on. In general, it is difficult

[1] Department of Mathematics and Information Science, Tangshan Normal University, Tangshan, Hebei, China.

to derive the analytical solutions to most of the fractional differential equations. Therefore, it is important to develop some reliable and efficient techniques to solve fractional differential equations [Chen, Yi, Chen and Yu (2012); Yi and Chen (2012); Chen, Sun, Li and Fu (2013)]. The numerical solutions of fractional differential equations have attracted considerable attention from many researchers. The most commonly used methods are Variational Iteration Method [Zaid M. Odibat (2010)], Adomian Decomposition Method [EI-Kalla (2008) and Hosseini (2006)], and Generalized Differential Transform Method [Shaher and Zaid (2007); Zaid and Shaher (2008)]. Wavelet basis approach has also been successfully employed to solve the factional differential equations.

The motivation of this paper is to extend the application of generalized hat functions to provide approximate solution of linear and nonlinear integro-differential equations of fractional order. The linear and nonlinear integro- differential equations of fractional order can be solved by many numerical methods. Saeedi and Moghadam [Saeedi and Moghadam (2011)] applied CAS wavelets method to solve the numerical solution of nonlinear Volterra integro-differential equations of fractional order and nonlinear Fredholm integro- differential equations of fractional order. In Refs.[Zhu and Fan (2013), Zhu and Fan (2012)], the authors solved the same integro-differential equations by using the second kind Chebyshev wavelets[Babolian and Mordad (2011)].

The structure of this paper is as follows: In Section 2, the generalized hat functions are introduced. The generalized hat functions operational matrix of fractional integration is also introduced and the error analysis of generalized hat functions is given in Section 3. In Section 4, we summarize the application of generalized hat functions operational matrix method to the solution of the fractional integro-differential equation. Four numerical examples are provided to clarify the approach in Section 5. The conclusion is given in Section 6.

2 Generalized hat functions and their properties

The interval $[0,T]$ is divided into n subintervals $[ih,(i+1)h],\ i=0,1,2,\ldots,n-1$, of equal lengths h where $h=\frac{T}{n}$. The generalized hat functions' family of first $n+1$ hat functions is defined as follows [Podlubny (1999)]

$$\psi_0(x)=\begin{cases}\frac{h-x}{h}, & 0\le x<h,\\ 0, & otherwise,\end{cases} \tag{1}$$

$$\psi_i(x)=\begin{cases}\frac{x-(i-1)h}{h}, & (i-1)h\le x<ih,\\ \frac{(i+1)h-x}{h}, & ih\le x<(i+1)h,\\ 0, & otherwise,\end{cases} \qquad i=1,2,\ldots,n-1 \tag{2}$$

$$\psi_n(x) = \begin{cases} \frac{x-(T-h)}{h}, & T-h \le x \le T, \\ 0, & otherwise. \end{cases} \tag{3}$$

Using the definition of generalized hat functions, we can obtain

$$\psi_i(kh) = \begin{cases} 1, & i = k, \\ 0, & i \neq k \end{cases} \tag{4}$$

and

$$\psi_i(x)\psi_j(x) = 0, \quad |i-j| \ge 2 \tag{5}$$

An arbitrary function $u \in L^2[0,T]$ is approximated in vector form as

$$u(x) \approx \sum_{i=0}^{n} u_i \psi_i(x) = U_{n+1}^T \Psi_{n+1}(x) \tag{6}$$

where $U_{n+1} = [u_0, u_1, \ldots, u_n]^T$ and $\Psi_{n+1}(x) = [\psi_0(x), \psi_1(x), \ldots, \psi_n(x)]^T$. Substituting Eq.(1)-(3) into the Eq. (6), we get the coefficients in Eq.(6) as following

$$u_i = u(ih), \quad i = 0, 1, 2, \ldots, n \tag{7}$$

3 Operational matrix of the integration for generalized hat functions

3.1 Fractional calculus

Before we introduce the generalized hat functions operational matrix of the fractional integration, we first review some basic definitions of fractional calculus, which have been given in [Li and Sun (2011)].

Definition 1. The Riemann-Liouville fractional integral of order α is given by

$$J^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} u(\tau) d\tau, \quad \alpha > 0 \tag{8}$$

$$J^0 u(x) = u(x) \tag{9}$$

Definition 2. The Caputo definition of fractional differential operator is given by

$$D^\alpha u(x) = \begin{cases} \frac{d^r u(x)}{dx^r}, & \alpha = r \in N; \\ \frac{1}{\Gamma(r-\alpha)} \int_0^x \frac{u^{(r)}(\tau)}{(x-\tau)^{\alpha-r+1}} d\tau, & 0 \le r-1 < \alpha < r. \end{cases} \tag{10}$$

The Caputo fractional derivatives of order α is also defined as $D^\alpha u(x) = J^{r-\alpha} D^r u(x)$, where D^r is the usual integer differential operator of order r. The relation between the Riemann- Liouville operator and Caputo operator is given by the following expressions:

$$D^{\alpha}J^{\alpha}u(x) = u(x) \tag{11}$$

$$J^{\alpha}D^{\alpha}u(x) = u(x) - \sum_{k=0}^{r-1} u^{(k)}(0^+)\frac{x^k}{k!}, \qquad x > 0 \tag{12}$$

3.2 Fractional order generalized hat functions operational matrix of integration.

If J^{α} is fractional integration operator of generalized hat functions, we can get:

$$J^{\alpha}\Psi_{n+1}(x) \approx P^{\alpha}_{n+1}\Psi_{n+1}(x) \tag{13}$$

where

$$P^{\alpha}_{n+1} = \frac{h^{\alpha}}{\Gamma(\alpha+2)} \begin{bmatrix} 0 & \zeta_1 & \zeta_2 & \zeta_3 & \cdots & \zeta_n \\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{n-1} \\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{n-2} \\ 0 & 0 & 0 & 1 & \cdots & \xi_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{(n+1)\times(n+1)} \tag{14}$$

where

$$\zeta_k = k^{\alpha}(\alpha - k + 1) + (k-1)^{\alpha+1}, \;\; k = 1, 2, \ldots, n \tag{15}$$

and

$$\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}, \quad k = 1, 2, \ldots, n-1 \tag{16}$$

P^{α}_{n+1} is called the generalized hat functions operational matrix of fractional integration.

Apart from the generalized hat functions, we consider another basis set of block pulse functions. The set of these functions, over the interval $[0, T)$, is defined as

$$b_i(x) = \begin{cases} 1, & ih \le x < (i+1)h \quad i = 0, 1, 2, \ldots, n-1 \\ 0, & otherwise, \end{cases} \tag{17}$$

with a positive integer value for n and $h = \frac{T}{n}$.

The following properties of block pulse functions will be used in this paper

$$b_i(x)b_j(x) = \begin{cases} 0, & i \ne j \\ b_i(x), & i = j \end{cases} \tag{18}$$

$$\int_0^T b_i(x)b_j(x)dx = \begin{cases} 0, & i \neq j \\ \frac{T}{n}, & i = j \end{cases} \tag{19}$$

Let $B_n(x) = [b_0(x), b_1(x), \ldots, b_{n-1}(x)]^T$. Suppose $J^\alpha(B_n(x)) \approx F_n^\alpha B_n(x)$, then F_n^α is called the block pulse operational matrix of fractional integration [21], here

$$F_n^\alpha = h^\alpha \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{n-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{n-2} \\ 0 & 0 & 1 & \cdots & \xi_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}, k = 1, 2, \ldots, n-1.$

There is a relation between the block pulse functions and generalized hat functions, namely

$$\Psi_{n+1}(x) = \Phi B_n(x) \tag{20}$$

where

$$\Psi_{n+1}(x) = [\psi_0(x), \psi_1(x), \ldots, \psi_n(x)]^T,$$

$$\Phi = \begin{bmatrix} 1/2 & 0 & 0 & \cdots & 0 \\ 1/2 & 1/2 & 0 & \cdots & 0 \\ 0 & 1/2 & 1/2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1/2 & 1/2 \\ 0 & \cdots & \cdots & 0 & 1/2 \end{bmatrix}_{(n+1)\times n}.$$

3.3 Error analysis

In this section, from Eq.(6), we suppose

$$u_n(x) = \sum_{i=0}^{n} u(ih)\psi_i(x) \tag{21}$$

And

$$J_n^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} u_n(\tau)d\tau \tag{22}$$

where $J_n^\alpha u(x)$ denotes the approximation of α order Riemann-Liouville fractional integral of $u(x)$. Let $\varepsilon_n(x) = |J^\alpha u(x) - J_n^\alpha u(x)|$, then we have the following theorem.

Theorem 3.1 If $u(x),\ x\in[0,T]$ is approximated by the Eq. (6), then
(i) $|u(jh)-u_n(jh)|=0$, for $j=0,1,2,\ldots,n$;
(ii) $|u(x)-u_n(x)|\leq\frac{1}{2n^2}|u''(jh)|+O\left(\frac{1}{n^3}\right)$, for $jh<x<(j+1)h,\ \ j=0,1,2,\ldots,n-1$;
(iii) For $jh<x<(j+1)h$, $\varepsilon_n(x)\leq\frac{MT^{2+\alpha}}{2n^2\Gamma(\alpha+1)}+O\left(\frac{1}{n^3}\right)$, where $|u''(jh)|\leq M, M>0$.

Proof (i) From Eq. (4), the value of $u_n(x)$ at jth point $x=jh,\ j=0,1,2,\ldots,n$ is given by $u_n(jh)=\sum\limits_{i=0}^{n}u(ih)\psi_i(jh)=u(jh)$, then $|u(jh)-u_n(jh)|=0$, for $j=0,1,2,\ldots,n$.

(ii) If $jh<x<(j+1)h,\ \ j=0,1,2,\ldots,n-1$, then from Eq.(1)-(3) and Eq.(21), we have

$$\begin{aligned}u_n(x)&=\sum_{i=0}^{n}u(ih)\psi_i(x)=u(jh)\psi_j(x)+u((j+1)h)\psi_{j+1}(x)\\&=u(jh)\left(\tfrac{(j+1)h-x}{h}\right)+u(jh+h)\left(\tfrac{x-jh}{h}\right)\\&=u(jh)-jh\left(\tfrac{u(jh+h)-u(jh)}{h}\right)+x\left(\tfrac{u(jh+h)-u(jh)}{h}\right)\\&=u(jh)+(x-jh)\left(\tfrac{u(jh+h)-u(jh)}{h}\right)\end{aligned}\tag{23}$$

when $h\to 0$, we obtain

$$u_n(x)\approx u(jh)+(x-jh)u'(jh)\tag{24}$$

Using the Taylor's series of $u(x)$, in the powers of $(x-jh)$, we have

$$u(x)=\sum_{k=0}^{\infty}\frac{(x-jh)^k}{k!}u^{(k)}(jh)\tag{25}$$

where $u^{(k)}$ denotes the kth order derivative of $u(x)$. From Eq. (24) and Eq. (25), we get

$$u(x)-u_n(x)=\sum_{k=2}^{\infty}\frac{(x-jh)^k}{k!}u^{(k)}(jh)=\frac{(x-jh)^2}{2}u''(jh)+O\left((x-jh)^3\right)\tag{26}$$

Because $(x-jh)<h$, $nh=T$, so we have

$$|u(x)-u_n(x)|\leq\frac{T^2}{2n^2}\left|u''(jh)\right|+O\left(\frac{1}{n^3}\right)\tag{27}$$

(iii) According to the definition of the absolute error $\varepsilon_n(x)$, we obtain

$$\varepsilon_n(x)=|J^{\alpha}u(x)-J_n^{\alpha}u(x)|\leq\frac{1}{\Gamma(\alpha)}\int_0^x(x-\tau)^{\alpha-1}|u(\tau)-u_n(\tau)|d\tau\tag{28}$$

For $jh < x < (j+1)h$, we get

$$\begin{aligned}
&\varepsilon_n(x) \le \tfrac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} |u(\tau)-u_n(\tau)| d\tau \\
&= \tfrac{1}{\Gamma(\alpha)} \left[\sum_{r=0}^{j-1} \int_{rh}^{(r+1)h} (x-\tau)^{\alpha-1} |u(\tau)-u_n(\tau)| d\tau + \int_{jh}^{x} (x-\tau)^{\alpha-1} |u(\tau)-u_n(\tau)| d\tau \right]
\end{aligned} \tag{29}$$

Substituting Eq.(27) into Eq.(29), we have

$$\begin{aligned}
\varepsilon_n(x) \le \tfrac{1}{\Gamma(\alpha)} &\left[\sum_{r=0}^{j-1} \int_{rh}^{(r+1)h} (x-\tau)^{\alpha-1} \left(\tfrac{T^2}{2n^2} |u''(rh)| + O\left(\tfrac{1}{n^3}\right)\right) d\tau \right. \\
&\left. + \int_{jh}^{x} (x-\tau)^{\alpha-1} \left(\tfrac{T^2}{2n^2} |u''(rh)| + O\left(\tfrac{1}{n^3}\right)\right) d\tau \right]
\end{aligned} \tag{30}$$

If $Max|u''(kh)| \le M$, $k = 0,1,2,\ldots,j$, then we obtain

$$\begin{aligned}
&\varepsilon_n(x) \le \tfrac{1}{\Gamma(\alpha)} \left(\tfrac{MT^2}{2n^2} + O\left(\tfrac{1}{n^3}\right) \right) \left[\left[\sum_{r=0}^{j-1} \int_{rh}^{(r+1)h} (x-\tau)^{\alpha-1} d\tau + \int_{jh}^{x} (x-\tau)^{\alpha-1} d\tau \right] \right. \\
&\quad = \tfrac{x^\alpha}{\Gamma(\alpha+1)} \left(\tfrac{MT^2}{2n^2} \right) + O\left(\tfrac{1}{n^3}\right) \\
&\quad \le \tfrac{(j+1)^\alpha}{\Gamma(\alpha+1)} \left(\tfrac{MT^2}{2n^{2+\alpha}} \right) + O\left(\tfrac{1}{n^3}\right) \le \tfrac{MT^{2+\alpha}}{2n^2\Gamma(\alpha+1)} + O\left(\tfrac{1}{n^3}\right)
\end{aligned} \tag{31}$$

This completes the proof.

The fractional order integration of the function t was selected to verify the correctness of matrix P^{α}_{n+1}. The fractional order integration of the function $u(t) = t$ is easily obtained as follows

$$J^{\alpha} u(t) = \frac{\Gamma(2)}{\Gamma(\alpha+2)} t^{\alpha+1} \tag{32}$$

When $\alpha = 0.5$, $m = 32$, the comparison results for the fractional integration is shown in Figure 1

4 The algorithm for finding numerical solution of fractional integro-differential equations

4.1 Linear fractional integro-differential equations

Consider the linear fractional integro-differential equations

$$D^{\alpha} u(x) = \lambda_1 \int_0^x k_1(x,t) u(t) dt + \lambda_2 \int_0^1 k_2(x,t) u(t) dt + f(x) \tag{33}$$

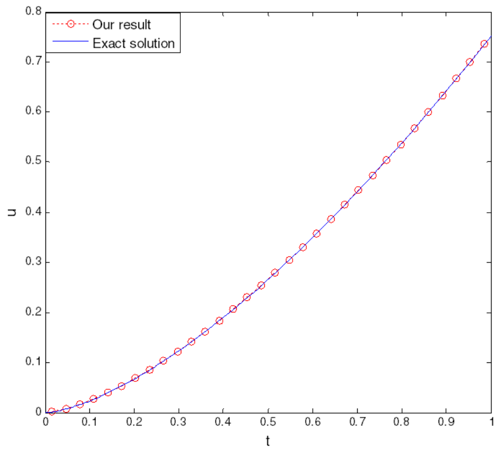

Figure 1: 0.5-order integration of the function $u(t) = t$.

subject to initial conditions

$$u^{(s)}(0) = 0, \quad s = 0,1,\ldots,r-1, \quad r-1 < \alpha \le r, \quad r \in N \tag{34}$$

where $u^{(s)}(x)$ stands for the sth-order derivative of $u(x)$, $D^{\alpha}(\cdot)$ denotes the Caputo fractional order derivative of order α, $f(x)$ is input term and $u(x)$ is the output response. $k_1(x,t), k_2(x,t)$ are given functions. λ_1, λ_2 are real constants.

Now we approximate $D^{\alpha}u(x)$, $k_1(x,t), k_2(x,t)$ and $f(x)$ in terms of generalized hat functions as follows

$$\begin{aligned} D^{\alpha}u(x) \approx U_{n+1}^T \Psi_{n+1}(x),\ k_1(x,t) \approx \Psi_{n+1}^T(x) K_1 \Psi_{n+1}(t), k_2(x,t) \\ \approx \Psi_{n+1}^T(x) K_2 \Psi_{n+1}(t) \end{aligned} \tag{35}$$

and

$$f(x) \approx F_{n+1}^T \Psi_{n+1}(x) \tag{36}$$

where $K_1 = [k_{ij}^1]_{(n+1)\times(n+1)}$, $K_2 = [k_{ij}^2]_{(n+1)\times(n+1)}$ and $F_{n+1} = [f_0, f_1, \ldots f_n]^T$.
Now using Eq. (35) and Eq.(12), we obtain

$$u(x) = J^{\alpha} D^{\alpha} u(x) \approx J^{\alpha}(U_{n+1}^T \Psi_{n+1}(x)) = U_{n+1}^T P_{n+1}^{\alpha} \Psi_{n+1}(x) \tag{37}$$

Substituting Eq. (20) into Eq.(37), we have

$$u(x) \approx U_{n+1}^T P_{n+1}^{\alpha} \Phi B_n(x) \tag{38}$$

Let $E = [e_0, e_1, \ldots, e_{n-1}] = U_{n+1}^T P_{n+1}^{\alpha} \Phi$, then

$$\begin{aligned}
\int_0^x k_1(x,t)y(t)dt &= \int_0^x \Psi_{n+1}^T(x) K_1 \Psi_{n+1}(t) \Psi_{n+1}^T(t) [U_{n+1}^T P_{n+1}^{\alpha}]^T dt \\
&= \Psi_{n+1}^T(x) K_1 \int_0^x \Phi B_n(t) B_n^T(t) [U_{n+1}^T P_{n+1}^{\alpha} \Phi]^T dt \\
&= \Psi_{n+1}^T(x) K_1 \Phi \int_0^x B_n(t) B_n^T(t) E^T dt \\
&= \Psi_{n+1}^T(x) K_1 \Phi \int_0^x diag(E) B_n(t) dt \\
&= \Psi_{n+1}^T(x) K_1 \Phi diag(E) F_n^1 B_n(x) \\
&= B_n^T(x) \Phi^T K_1 \Phi diag(E) F_n^1 B_n(x) \\
&= \tilde{Q}^T B_n(x)
\end{aligned} \tag{39}$$

where $\tilde{Q}$ is a n-vector with elements equal to the diagonal entries of the following matrix

$$Q = \Phi^T K_1 \Phi diag(E) F_n^1 \tag{40}$$

and

$$\begin{aligned}
\int_0^1 k_2(x,t)y(t)dt &= \int_0^1 \Psi_{n+1}^T(x) K_2 \Psi_{n+1}(t) \Psi_{n+1}^T(t) [U_{n+1}^T P_{n+1}^{\alpha}]^T dt \\
&= \Psi_{n+1}^T(x) K_2 \Phi \int_0^1 B_n(t) B_n^T(t) [U_{n+1}^T P_{n+1}^{\alpha} \Phi]^T dt \\
&= \Psi_{n+1}^T(x) K_2 \Phi \int_0^1 B_n(t) B_n^T(t) dt E^T \\
&= \frac{1}{n} \Psi_{n+1}^T(x) K_2 \Phi E^T \\
&= \frac{1}{n} B_n^T(x) \Phi^T K_2 \Phi E^T \\
&= \frac{1}{n} E \Phi^T K_2^T \Phi B_n(x)
\end{aligned} \tag{41}$$

Substituting the above equations into Eq. (33), we have

$$U_{n+1}^T \Phi B_n(x) = \lambda_1 \tilde{Q}^T B_n(x) + \frac{\lambda_2}{n} E \Phi^T K_2^T \Phi B_n(x) + F_{n+1}^T \Phi B_n(x) \tag{42}$$

Dispersing Eq. (42), we obtain

$$U_{n+1}^T \Phi = \lambda_1 \tilde{Q}^T + \frac{\lambda_2}{n} E \Phi^T K_2^T \Phi + F_{n+1}^T \Phi \tag{43}$$

which is a linear system of algebraic equations. By solving this system we can obtain the approximation of Eq. (37).

4.2 Nonlinear fractional integro-differential equations

In this section we deal with nonlinear fractional integro-differential equation of the form

$$D^{\alpha}u(x) = \lambda_1 \int_0^x k_1(x,t)[u(t)]^p dt + \lambda_2 \int_0^1 k_2(x,t)[u(t)]^q dt + f(x) \tag{44}$$

subject to initial conditions

$$u^{(s)}(0) = 0$$

where $p,q \in N$, and the other parameters and variables are the same as the section 4.1. While dealing with such a situation, the same procedure (as in linear case) of expansion of fractional order derivatives via generalized hat functions is adopted with exception at the term containing $[u(t)]^p$, $[u(t)]^q$.

From Eq. (38), we have $u(x) \approx EB_n(x)$ and hence

$$[u(t)]^p \approx [EB_n(t)]^p = [e_0^p, e_1^p, \ldots, e_{n-1}^p]B_n(t) = E_pB_n(t) \tag{45}$$

and

$$[u(t)]^q \approx [EB_n(t)]^q = [e_0^q, e_1^q, \ldots, e_{n-1}^q]B_n(t) = E_qB_n(t) \tag{46}$$

Following the procedure of section 4.1 and using the Eq.(45) and Eq.(46), the Eq.(44) is transformed into a nonlinear system of algebraic equations

$$U_{n+1}^T\Phi = \lambda_1\tilde{W}^T + \frac{\lambda_2}{n}E_q\Phi^T K_2^T\Phi + F_{n+1}^T\Phi \tag{47}$$

where $\tilde{W}$ is a n-vector with elements equal to the diagonal entries of the following matrix.

$$W = \Phi^T K_1 \Phi diag(E_p)F_n^1 \tag{48}$$

Solving the system of equations given by Eq. (47), the approximate numerical solution $u(x)$ is obtained. The Eq. (47) can be solved by iterative numerical technique such as Newton's method. Also the Matlab function "fsolve" is available to deal with such a nonlinear system of algebraic equations.

5 Numerical examples

In order to illustrate the effectiveness of the proposed method, we consider numerical examples of linear and nonlinear nature.

Example 5.1 Consider this equation:

$$D^{2.3}y(x) = \frac{1}{4}\int_0^x (x-t)y(t)dt + \frac{1}{2}\int_0^1 xt\cdot y(t)dt + f(x) \tag{49}$$

where $f(x) = \frac{\Gamma(4.5)}{\Gamma(2.2)}x^{1.2} - \frac{x^{5.5}}{99} - \frac{x}{11}$, such that $y''(0) = y'(0) = y(0) = 0$, the exact solution is $y(x) = x^{3.5}$. The numerical results for $n = 8$, 16, 32, 64 are shown in Figs. 2-5. From the Figs. 2-5, we can find easily that the numerical solutions are in good agreement with the exact solutions. Table 1 shows the absolute errors obtained by generalized hat functions method and CAS wavelets method (CASW) for different n, respectively. Through Table 1, we can also see that the errors are smaller and smaller when n increases. Comparing with the absolute errors obtained by CAS wavelets method, generalized hat functions method can reach higher degree of accuracy.

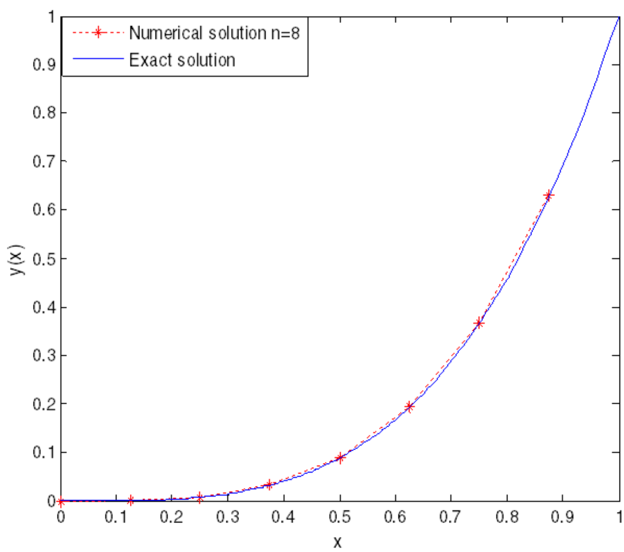

Figure 2: Comparison of Num. sol. and Exa. Sol. of $n = 8$.

Example 5.2 Consider the following nonlinear equation:

$$D^{2.2}y(x) = \frac{1}{3}\int_0^x (x+t)[y(t)]^2 dt + \frac{1}{4}\int_0^1 (x-t)[y(t)]^3 dt + f(x) \tag{50}$$

such that $y''(0) = y'(0) = y(0) = 0$.The exact solution of the equation is $y(x) = x^3$, where$f(x) = \frac{\Gamma(4)}{\Gamma(1.8)}x^{0.8} - \frac{5x^8}{56} - \frac{x}{40} + \frac{1}{44}$.

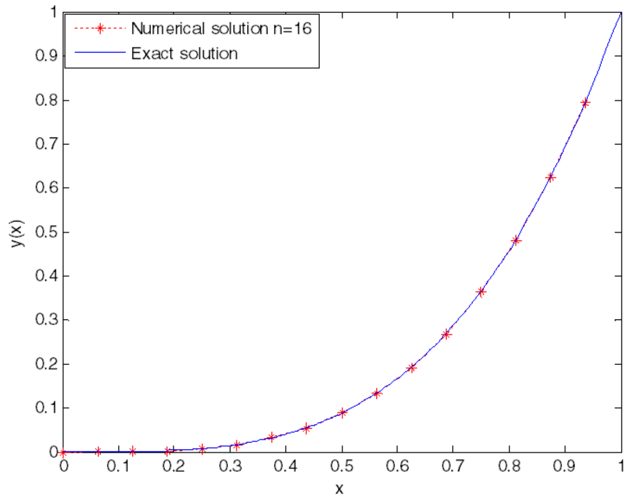

Figure 3: Comparison of Num. sol. and Exa. Sol. of $n = 16$.

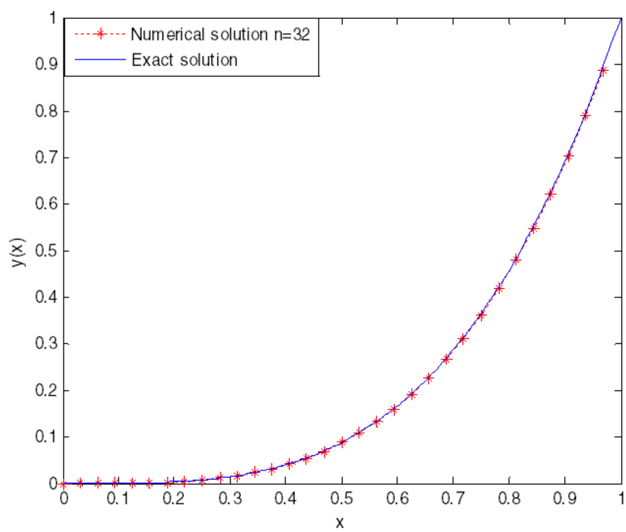

Figure 4: Comparison of Num. sol. and Exa. Sol. of $n = 32$.

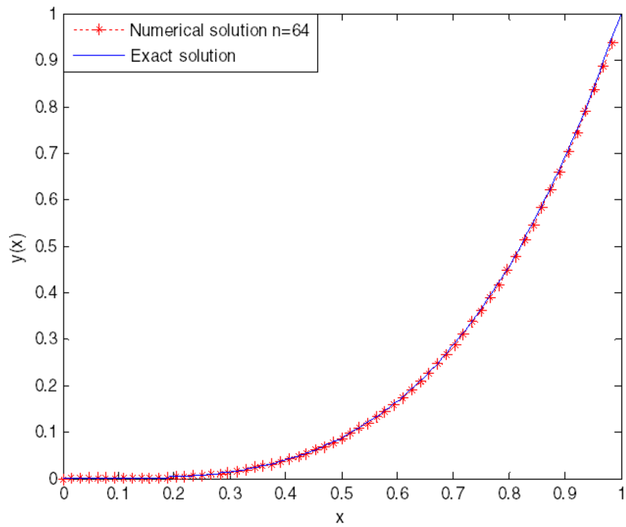

Figure 5: Comparison of Num. sol. and Exa. Sol. of $n = 64$.

Table 1: The absolute errors for different values of n.

x	$n=8$		$n=16$		$n=32$	
	Ours	CASW	Ours	CASW	Ours	CASW
0	0	2.7434e-005	0	5.2528e-006	0	4.2730e-007
1/8	1.2316e-004	7.2373e-003	3.2343e-005	2.1658e-004	4.2465e-006	1.0174e-004
2/8	3.1235e-004	6.2642e-003	5.8723e-005	5.2365e-004	5.6172e-005	4.1987e-004
3/8	4.6237e-004	3.1236e-003	6.1423e-005	8.2316e-004	6.2164e-005	9.2364e-004
4/8	2.8236e-003	5.2374e-002	2.2317e-004	2.4582e-003	6.9423e-005	4.1726e-003
5/8	4.1327e-003	3.6285e-002	4.4326e-004	7.0243e-003	3.2375e-004	8.1648e-003
6/8	5.2321e-003	2.2364e-002	6.4235e-003	4.4565e-002	4.5421e-004	2.3112e-002
7/8	6.0432e-003	9.1287e-001	7.2324e-003	8.2364e-002	6.2376e-004	8.0723e-002

Figures 6-9 show the numerical solutions and exact solution for $n = 16,\ 32,\ 64$. We can see that the numerical solutions are more and more close to the exact solution with the value of $n$becomes large by taking a closer look at Figures 6-8.

Example 5.3 Consider this equation:

$$D^{\alpha+1}y(x) = \int_0^x (e^t+1)[y(t)]^2dt + \int_0^1 xt[y(t)]^2dt + f(x) \tag{51}$$

where $f(x) = e^x - \frac{(e^x-x-1)^3}{3} - x(\frac{e^2}{4} - 2e + \frac{11}{3})$, with initial conditions $y'(0) = y(0) =$

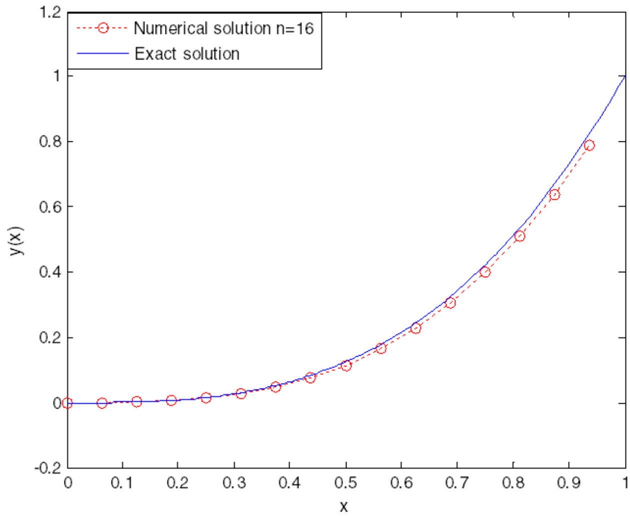

Figure 6: Comparison of Num. sol. and Exa. Sol. of $n = 16$ for Example 3.

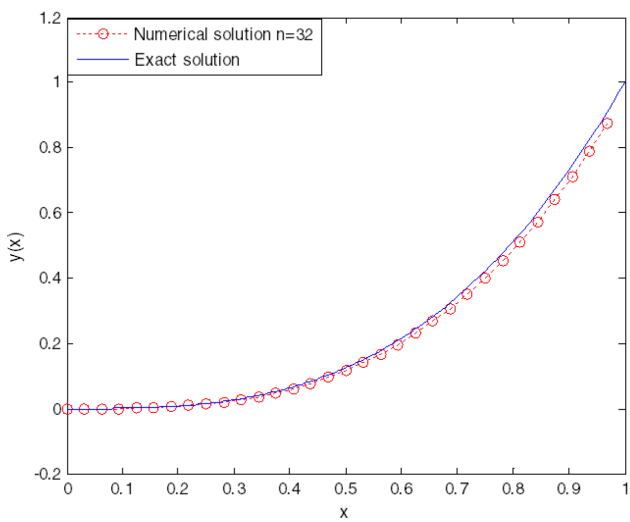

Figure 7: Comparison of Num. sol. and Exa. Sol. of $n = 32$for Example 3.

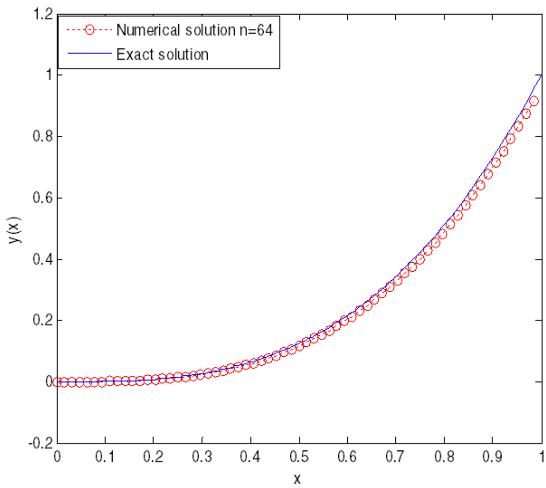

Figure 8: Comparison of Num. sol. and Exa. Sol. of $n = 64$for Example 3.

0. The exact solution of the problem for $\alpha = 1$ is $y(x) = e^x - x - 1$.
The comparison of numerical results for $\alpha = 0.7$, $\alpha = 0.8$, $\alpha = 0.9$, $\alpha = 1$ and the exact solution for $\alpha = 1$ are shown in Figure. 9.

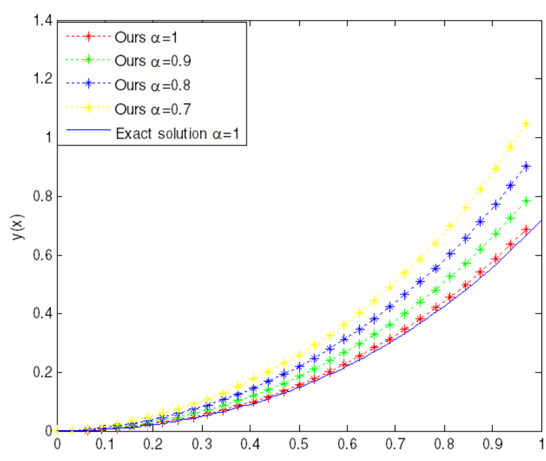

Figure 9: Numerical solution and exact solution of $\alpha = 1$.

From Figure 9, we can see clearly that the numerical solutions are in very good agreement with the exact solution when $\alpha = 1$. It is evident from the Figure 9 that, as α close to 1, the numerical solutions by the generalized hat functions converge to the exact solution.

6 Conclusion

In this work, we introduce the generalized hat functions and operational matrix of the fractional integration. Using the operational matrix to solve the fractional linear and nonlinear integro-differential equations numerically. By solving the linear and nonlinear system, numerical solutions are obtained. The error analysis of generalized hat functions is proposed. The numerical results show that the approximations are in very good coincidence with the exact solution

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