

Meshless Local Petrov-Galerkin Mixed Collocation Method for Solving Cauchy Inverse Problems of Steady-State Heat Transfer

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Abstract: In this article, the Meshless Local Petrov-Galerkin (MLPG) Mixed Collocation Method is developed to solve the Cauchy inverse problems of Steady-State Heat Transfer. In the MLPG mixed collocation method, the mixed scheme is applied to independently interpolate temperature as well as heat flux using the same meshless basis functions. The balance and compatibility equations are satisfied at each node in a strong sense using the collocation method. The boundary conditions are also enforced using the collocation method, allowing temperature and heat flux to be over-specified at the same portion of the boundary. For the inverse problems where noise is present in the measurement, Tikhonov regularization method is used, to mitigate the inherent ill-posed nature of inverse problem, with its regularization parameter determined by the L-Curve method. Several numerical examples are given, wherein both temperature as well as heat flux are prescribed at part of the boundary, and the data at the other part of the boundary and in the domain have to be solved for. Through these numerical examples, we investigate the accuracy, convergence, and stability of the proposed MLPG mixed collocation method for solving inverse problems of Heat Transfer.

Keywords: MLPG, Collocation, heat transfer, inverse problem.

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1 Introduction

Computational modeling of solid/fluid mechanics, heat transfer, electromagnetics, and other physical, chemical & biological sciences have experienced an intense development in the past several decades. Tremendous efforts have been devoted to solving the so-called direct problems, where the boundary conditions are generally of the Dirichlet, Neumann, or Robin type. Existence, uniqueness, and stability of the solutions have been established for many of these direct problems. Numerical methods such as finite elements, boundary elements, finite volume, meshless methods etc., have been successfully developed and available in many off-the shelf commercial softwares, see [Atluri (2005)]. On the other hand, inverse problems, although being more difficult to tackle and being less studied, have equal, if not greater importance in the applications of engineering and sciences, such as in structural health monitoring, electrocardiography, etc.

One of the many types of inverse problems is to identify the unknown boundary fields when conditions are over-specified on only a part of the boundary, i.e. the Cauchy problem. Take steady-state heat transfer problem as an example. The governing differential equations can be expressed in terms of the primitive variable-temperatures:

$$(-kT_{,i})_{,i} = 0 \quad \text{in } \Omega \tag{1}$$

For direct problems, temperatures $T = \bar{T}$ are prescribed on a part of the boundary S_T , and heat fluxes $q_n = \bar{q}_n = -kn_i u_{,i}$ are prescribed on the other part of the boundary S_q . S_T and S_q should be a complete division of $\partial\Omega$, which means $S_T \cup S_q = \partial\Omega, S_T \cap S_q = \emptyset$. On the other hand, if both the temperatures as well as heat fluxes are specified or known only on a small portion of the boundary S_C , the inverse Cauchy problem is to determine the temperatures and heat fluxes in the domain as well as on the other part of the boundary.

In spite of the popularity of FEM for direct problems, it is essentially very unsuitable for solving inverse problems. This is because the traditional primal FEM are based on the global Symmetric Galerkin Weak Form of equation (1):

$$\int_{\Omega} kT_{,i}v_{,i}d\Omega - \int_{\partial\Omega} kn_i T_{,i}v dS = 0 \tag{2}$$

where v are test functions, and both the trial functions T and the test functions v are required to be continuous and differentiable. It is immediately apparent from equation (2) that the symmetric weak form [on which the primal finite element methods are based] does not allow for the simultaneous prescription of both the heat fluxes $q_n[\equiv T_{,i}]$ as well as temperatures T at the same segment of the boundary, $\partial\Omega$. Therefore, in order to solve the inverse problem using FEM, one has

to first ignore the over-specified boundary conditions, guess the missing boundary conditions, so that one can iteratively solve a direct problem, and minimize the difference between the solution and over-prescribed boundary conditions by adjusting the guessed boundary fields, see [Kozlov, Maz'ya and Fomin (1991); Cimetiere, Delvare, Jaoua and Pons (2001)] for example. This procedure is cumbersome and expensive, and in many cases highly-dependent on the initial guess of the boundary fields.

Recently, simple non-iterative methods have been under development for solving inverse problems without using the primal symmetric weak-form: with global RBF as the trial function, collocation of the differential equation and boundary conditions leads to the global primal RBF collocation method [Cheng and Cabral (2005)]; with Kelvin's solutions as trial function, collocation of the boundary conditions leads to the method of fundamental solutions [Marin and Lesnic (2004)]; with non-singular general solutions as trial function, collocation of the boundary conditions leads to the boundary particle method [Chen and Fu (2009)]; with Trefftz trial functions, collocation of the boundary conditions leads to Trefftz collocation method [Yeih, Liu, Kuo and Atluri (2010); Dong and Atluri (2012)]. The common idea they share is that the collocation method is used to satisfy either the differential equations and/or the boundary conditions at discrete points. Moreover, collocation method is also more suitable for inverse problems because measurements are most often made at discrete locations.

However, the above-mentioned direct collocation methods are mostly limited to simple geometries, simple constitutive relations, and text-book problems, because: (1) these methods are based on global trial functions, and lead to a fully-populated coefficient matrix of the system of equations; (2) the general solutions and particular solutions cannot be easily found for general nonlinear problems, and problems with arbitrary boundary conditions; (3) it is difficult to derive general solutions that are complete for arbitrarily shaped domains, within a reasonable computational burden. With this understanding, more suitable ways of constructing the trial functions should be explored.

One of the most simple and flexible ways is to construct the trial functions through meshless interpolations. Meshless interpolations have been combined with the global Symmetric Galerkin Weak Form to develop the so-called Element-Free Galerkin (EFG) method, see [Belytschko, Lu, and Gu (1994)]. However, as shown in the Weak Form (2), because temperatures and heat fluxes cannot be prescribed at the same location, cumbersome iterative guessing and optimization will also be necessary if EFG is used to solve inverse problems. Thus EFG is not suitable for solving inverse problems, for the same reason why FEM is not suitable for solving inverse problems.

Instead of using the global Symmetric Galerkin Weak-Form, the Meshless Local Petrov-Galerkin (MLPG) method by [Atluri and Zhu (1998)] proposed to construct both the trial and test functions in a local subdomain, and write local weak-forms instead of global ones. Various versions of MLPG method have been developed in [Atluri and Shen (2002a, b)], with different trial functions (Moving Least Squares, Local Radial Basis Function, Shepard Function, Partition of Unity methods, etc.), and different test functions (Weight Function, Shape Function, Heaviside Function, Delta Function, Fundamental Solution, etc.). These methods are primal methods, in the sense that all the local weak forms are developed from the governing equation with primary variables. For this reason, the primal MLPG collocation method, which involves direct second-order differentiation of the temperature fields, as shown in equation(1), requires higher-order continuous basis functions, and is reported to be very sensitive to the locations of the collocation points.

Instead of the primal methods, MLPG mixed finite volume and collocation method were developed in [Atluri, Han and Rajendran (2004); Atluri, Liu and Han (2006)]. The mixed MLPG approaches independently interpolate the primary and secondary fields, such as temperatures and heat fluxes, using the same meshless basis functions. The compatibility between primary and secondary fields is enforced through a collocation method at each node. Through these efforts, the continuity requirement on the trial functions is reduced by one order, and the complicated second derivatives of the shape function are avoided. Successful applications of the MLPG mixed finite volume and collocation methods were made in nonlinear and large deformation problems [Han, Rajendran and Atluri (2005)]; impact and penetration problems [Han, Liu, Rajendran and Atluri (2006); Liu, Han, Rajendran and Atluri (2006)], topology optimization problems [Li and Atluri (2008a, b)]; inverse problems of linear isotropic/anisotropic elasticity [Zhang, Dong, Alotaibi and Atluri (2013)]. A thorough review of the applications of MLPG method is given in [Sladek, Stanak, Han, Sladek, Atluri (2013)].

This paper is devoted to numerical solution of the inverse Cauchy problems of steady-heat transfer. Both temperature and heat flux boundary conditions are prescribed only on part of the boundary of the solution domain, whilst no information is available on the remaining part of the boundary. To solve the dilemma that global-weak-form-based methods (such as FEM, BEM and EFG) which do not allow the primal and dual fields to be prescribed at the same part of the boundary, the MLPG mixed collocation method is developed for inverse Cauchy problem of heat transfer. The moving least-squares approximation is used to construct the shape function. The nodal heat fluxes are expressed in terms of nodal temperatures by enforcing the relation between heat flux and temperatures at each nodal point. The governing equations for steady-state heat transfer problems are satisfied at each

node using collocation method. The temperature and heat flux boundary conditions are also enforced by collocation method at each measurement location along the boundary. The proposed method is conceptually simple, numerically accurate, and can directly solve the inverse problem without using any iterative optimization. The outline of this paper is as follows: we start in section 2 by introducing the meshless interpolation method with emphasis on the Moving Least Squares interpolation. In section 3, the detailed algorithm of the MLPG mixed collocation method for inverse heat transfer problem is given. In section 4, several numerical examples are given to demonstrate the effectiveness of the current method involving direct and inverse heat transfer problems. Finally, we present some conclusions in section 5.

2 Meshless Interpolation

Among the available meshless approximation schemes, the Moving Least Squares (MLS) is generally considered to be one of the best methods to interpolate random data with a reasonable accuracy, because of its locality, completeness, robustness and continuity. The MLS is adopted in the current MLPG collocation formulation, while the implementation of other meshless interpolation schemes is straightforward within the present framework. For completeness, the MLS formulation is briefly reviewed here, while more detailed discussions on the MLS can be found in [Atluri (2004)]

The MLS method starts by expressing the variable $T(\mathbf{x})$ as polynomials:

$$T(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{3}$$

where $\mathbf{p}^T(\mathbf{x})$ is the monomial basis. In this study, we use first-order interpolation, so that $\mathbf{p}^T(\mathbf{x}) = [1, x, y]$ for two-dimensional problems. $\mathbf{a}(\mathbf{x})$ is a vector containing the coefficients of each monomial basis, which can be determined by minimizing the following weighted least square objective function, defined as:

$$\begin{aligned} J(\mathbf{a}(\mathbf{x})) &= \sum_{I=1}^m w^I(\mathbf{x}) [\mathbf{p}^T(\mathbf{x}^I)\mathbf{a}(\mathbf{x}) - \hat{T}^I]^2 \\ &= [\mathbf{Pa}(\mathbf{x}) - \hat{\mathbf{T}}]^T \mathbf{W} [\mathbf{Pa}(\mathbf{x}) - \hat{\mathbf{T}}] \end{aligned} \tag{4}$$

where, $\mathbf{x}^I, I = 1, 2, \dots, m$ is a group of discrete nodes within the influence range of node \mathbf{x} , \hat{T}^I is the fictitious nodal value, $w^I(\mathbf{x})$ is the weight function. A fourth order

spline weight function is used here:

$$w^I(\mathbf{x}) = \begin{cases} 1 - 6r^2 + 8r^3 - 3r^4 & r \leq 1 \\ 0 & r > 1 \end{cases} \tag{5}$$

$$r = \frac{\|\mathbf{x} - \mathbf{x}^I\|}{r^I}$$

where, r^I stands for the radius of the support domain $\Omega_{\mathbf{x}}$.

Substituting $\mathbf{a}(\mathbf{x})$ into equation (3), we can obtain the approximate expression as:

$$T(\mathbf{x}) = \mathbf{p}^T \mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \hat{\mathbf{T}} = \Phi^T(\mathbf{x}) \hat{\mathbf{T}} = \sum_{I=1}^m \Phi^I(\mathbf{x}) \hat{T}^I \tag{6}$$

where, matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are defined by:

$$\mathbf{A}(\mathbf{x}) = \mathbf{P}^T \mathbf{W} \mathbf{P} \quad \mathbf{B}(\mathbf{x}) = \mathbf{P}^T \mathbf{W} \tag{7}$$

$\Phi^I(\mathbf{x})$ is named as the MLS basis function for node I , and it is used to interpolated both temperatures and heat fluxes, as discussed in section 3.2.

3 MLPG Mixed Collocation Method for Inverse Cauchy Problem of Heat Transder

3.1 Inverse Cauchy Problem of Heat Transder

Consider a domain Ω wherein the steady-state heat transfer problem, without internal heat sources, is posed. The governing differential equation is expressed in terms of temperature in equation (1). It can also be expressed in a mixed form, in terms of both the temperature and the heat flux fields:

$$q_i = -kT_{,i} \quad \text{in } \Omega \tag{8}$$

$$q_{i,i} = 0 \quad \text{in } \Omega \tag{9}$$

For inverse problems, we consider that both heat flux and temperature are prescribed at a portion of the boundary, denoted as S_C :

$$\begin{aligned} T &= \bar{T} \quad \text{at } S_C \\ -n_i k T_{,i} &= q_n = \bar{q}_n \quad \text{at } S_C \end{aligned} \tag{10}$$

The inverse problem is thus defined as, with the measured heat fluxes as well as temperatures at S_C , which is only a portion of the boundary of the whole domain, can we determine heat fluxes as well as temperatures in the other part of the boundary as well as in the whole domain? A MLPG mixed collocation method is developed to solve this problem, and is discussed in detail in the following two subsections.

3.2 *MLPG Mixed Collocation Method*

We start by interpolating the temperature as well as the heat flux fields, using the same MLS shape function, as discussed in section 2:

$$T(\mathbf{x}) = \sum_{J=1}^m \Phi^J(\mathbf{x}) \hat{T}^J \tag{11}$$

$$q_i(\mathbf{x}) = \sum_{J=1}^m \Phi^J(\mathbf{x}) \hat{q}_i^J \tag{12}$$

where, \hat{T}^J and \hat{q}_i^J are the fictitious temperatures and heat fluxes at node J .

We rewrite equations (11) and (12) in matrix-vector form:

$$\mathbf{T} = \mathbf{\Phi} \hat{\mathbf{T}} \tag{13}$$

$$\mathbf{q} = \mathbf{\Phi} \hat{\mathbf{q}} \tag{14}$$

With the heat flux – temperature gradient relation as shown in equation (8), the heat fluxes at node I can be expressed as:

$$q_i(\mathbf{x}^I) = -k T_{,i}(\mathbf{x}^I) = -k \sum_{J=1}^m \Phi_{,i}^J(\mathbf{x}^I) \hat{T}^J; \quad I = 1, 2, \dots, N \tag{15}$$

where N is the total number of nodes in the domain.

This allows us to relate nodal heat fluxes to nodal temperatures, which is written here in matrix-vector form:

$$\mathbf{q} = \mathbf{K}_a \hat{\mathbf{T}} \tag{16}$$

And the balance equation of heat transfer is independently enforced at each node, as:

$$\sum_{J=1}^m \Phi_{,x}^J(\mathbf{x}^I) \hat{q}_x^J + \sum_{J=1}^m \Phi_{,y}^J(\mathbf{x}^I) \hat{q}_y^J = 0; \quad I = 1, 2, \dots, N \tag{17}$$

or, in an equivalent Matrix-Vector from:

$$\mathbf{K}_S \hat{\mathbf{q}} = \mathbf{0} \tag{18}$$

By substituting equation (16) and (14) into equation(18), we can obtain a discretized system of equations in term of nodal temperatures:

$$\mathbf{K}_{eq} \hat{\mathbf{T}} = \mathbf{0} \tag{19}$$

From equation (15) and (17), we see that both the heat transfer balance equation, and the heat flux temperature-gradient relation are enforced by the collocation method, at each node of the MLS interpolation. In the following subsection, the same collocation method will be carried out to enforce the boundary conditions of the inverse problem.

3.3 Over-Specified Boundary conditions in a Cauchy Inverse Heat transfer Problem

In most applications of inverse problems, the measurements are only available at discrete locations at a small portion of the boundary. In this study, we consider that both temperatures \hat{T}^J as well as heat fluxes \hat{q}_n^J are prescribed at discrete points $\mathbf{x}^I, I = 1, 2, 3, \dots, M$ on the same segment of the boundary. We use collocation method to enforce such boundary conditions:

$$\begin{aligned} \sum_{J=1}^m \Phi^J(\mathbf{x}^I) \hat{T}^J &= \bar{T}(\mathbf{x}^I) \\ -k \sum_{J=1}^m \Phi_{,n}^J(\mathbf{x}^I) \hat{T}^J &= \bar{q}_n(\mathbf{x}^I) \end{aligned} \tag{20}$$

or, in matrix-vector form:

$$\begin{aligned} \mathbf{K}_T \hat{\mathbf{T}} &= \mathbf{f}_T \\ \mathbf{K}_q \hat{\mathbf{T}} &= \mathbf{f}_q \end{aligned} \tag{21}$$

3.4 Regularization for Noisy Measurements

Equation (19) and (21) can be rewritten as:

$$\mathbf{K} \hat{\mathbf{T}} = \mathbf{f}, \quad \mathbf{K} = \begin{pmatrix} \mathbf{K}_{eq} \\ \mathbf{K}_T \\ \mathbf{K}_q \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} 0 \\ \mathbf{f}_T \\ \mathbf{f}_q \end{pmatrix} \tag{22}$$

This gives a complete, discretized system of equations of the governing differential equations as well as the over-specified boundary conditions. It can be directly solved using least square method without iterative optimization.

However, it is well-known that the inverse problems are ill-posed. A very small perturbation of the measured data can lead to a significant change of the solution. In order to mitigate the ill-posedness of the inverse problem, regularization techniques can be used. For example, following the work of Tikhonov and Arsenin [Tikhonov and Arsenin (1977)], many regularization techniques were developed. [Hansen and O’Leary (1993)] has given an explanation that the Tikhonov regularization of ill-posed linear algebra equations is a trade-off between the size of the regularized solution, and the quality to fit the given data. With a positive regularization parameter, which is determined by the L-curve method, the solution is determined as:

$$\min \left(\|\mathbf{K} \hat{\mathbf{T}} - \mathbf{f}\|^2 + \gamma \|\hat{\mathbf{T}}\|^2 \right) \tag{23}$$

This leads to the regularized solution:

$$\mathbf{T} = (\mathbf{K}^T \mathbf{K} + \gamma \mathbf{I})^{-1} \mathbf{K}^T \mathbf{f} \tag{24}$$

4 Numerical Examples

In this section, we firstly apply the proposed method to solve a direct problem with an analytical solution, in order to verify the accuracy and efficiency of the method. Then we apply the proposed method to solve three inverse Cauchy problems with noisy measurements, in order to explore the accuracy, stability, and converge of the MLPG mixed collocation method for solving inverse problems of heat transfer.

4.1 MLPG mixed collocation method for the direct heat transfer problem

Example 1: Patch Test

In this case, we consider a rectangular domain $\Omega = \{(x,y) | 0 \leq x \leq a, 0 \leq y \leq b\}$, as shown in Figure 1. Its left boundary is maintained at the temperature $T = 0^\circ\text{C}$, and the right boundary is prescribed with a temperature distribution as $T = Ay^\circ\text{C}$. The upper and lower boundaries are adiabatic. There is no heat source in the domain. The thermal conductivity is $k = 1\text{w}/(m \cdot ^\circ\text{C})$. The analytical solution is:

$$T(x,y) = \frac{Abx}{2a} + \sum_{n=1}^{\infty} \frac{2Ab[\cos(n\pi) - 1]}{n^2 \pi^2} \frac{\sinh[n\pi x/b] \cos[n\pi y/b]}{\sinh[an\pi/b]} \tag{25}$$

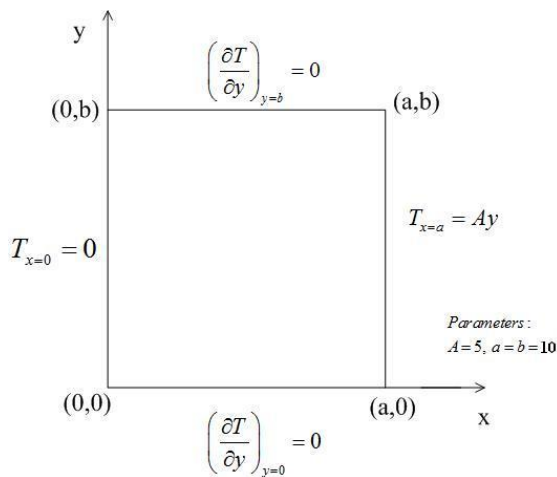


Figure 1: Heat conduction in a rectangular domain.

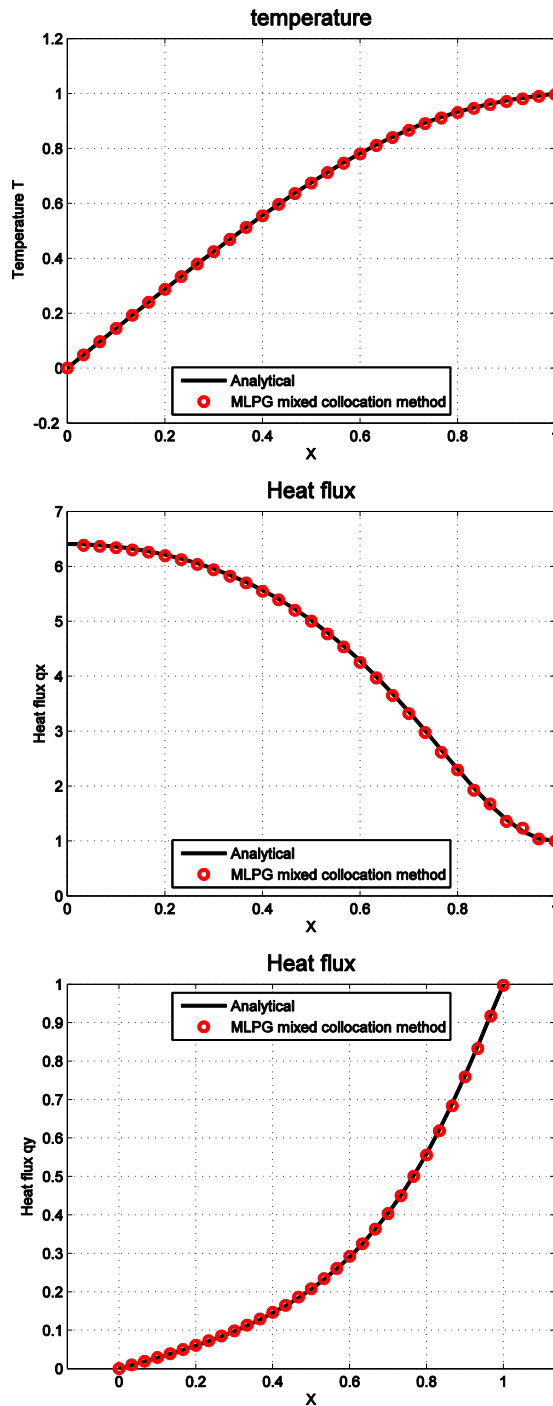


Figure 2: The normalized analytically and numerically solved T , q_x , q_y along the line $y = 3$ for the direct heat transfer problem of Example 1.

In this example, we consider that $a = b = 10$, $A = 5$, and use a uniform nodal configuration of 30×30 nodes. When the support domain is too small or too large, the relative computational error will become unacceptably large. It was found that $r = 2.5 - 3.0$ is an economical choice that gives good results without significantly increasing the computational burden, see [Wu, Shen, and Tao (2007)]. We select a support size of 25 times of the nodal distance, and use the first-order polynomial basis in the MLS approximations.

We solve this problem by using the MLPG mixed collocation method. Figure 2 gives the analytically and numerically solved temperature and heat fluxes, normalized to their maximum values. It can be seen that the computational results with MLPG mixed collocation method agrees well with the analytical solutions.

4.2 MLPG mixed collocation method for Cauchy inverse problem of heat transfer

Example 2: An L-shaped Domain

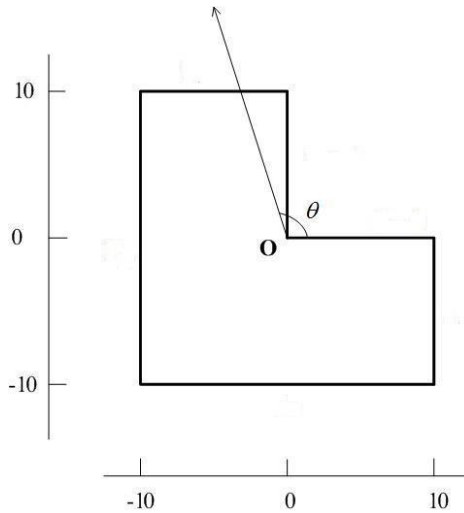


Figure 3: Heat transfer in an L-shape domain.

The second example is a Cauchy inverse heat transfer problem in the L-shaped domain as illustrated in Figure 3. The exact solution is given in the polar coordinates by:

$$T(r, \theta) = r^{\frac{2}{3}} \sin\left(\frac{2\theta - \pi}{3}\right) \tag{26}$$

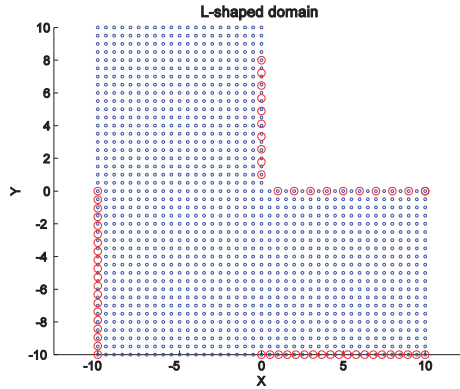


Figure 4: The nodal configuration and boundary collocation points of example 2.

$$q_r = -k \cdot \frac{2}{3} r^{-\frac{1}{3}} \sin\left(\frac{2\theta - \pi}{3}\right) \quad (27)$$

where k is the thermal conductivity, taken as 1 in this example. Equation (27) implies that the radial heat flux q_r is singular at the re-entrant corner O where $r = 0$. Node discretization of the L-shaped domain and the locations of temperature and heat flux measurements are shown in Figure 4. Temperatures and heat fluxes are overspecified at $S_C = \{(x, y) | 0 \leq x \leq 10, y = -10\} \cup \{(x, y) | -10 \leq y \leq 0, x = -10\} \cup \{(x, y) | 1 \leq x \leq 10, y = 0\} \cup \{(x, y) | 1 \leq y \leq 8, x = 0\}$. 1% white noise is added to the measured temperatures and heat fluxes. By MLPG mixed collocation method, we solve this inverse problem, and give the temperature and heat fluxes along the line $y = -1$. As can be seen from Figure 5 good agreements are given between the computed results and the analytical solution, even though the measurements are contaminated by noises.

Example 3: A semi-infinite domain

In this case, the steady-state heat transfer problem in a semi-infinite domain is considered, as shown in Figure 6. The half-space $\Omega = \{(x, y) | y \geq 0\}$ is insulated and kept at a temperature of zero at $\{(x, y) | y = 0, |x| > 1\}$. The line segment of $\{(x, y) | y = 0, |x| < 1\}$ is kept at a temperature of unity. The analytical solution [Brown and Churchill (2008)] of this example is given as:

$$T = \frac{1}{\pi} \arctan\left(\frac{2y}{x^2 + y^2 - 1}\right) \quad (0 \leq \arctan t \leq \pi) \quad (28)$$

This problem is solved using a truncated finite domain by MLPG mixed collocation method, in $\Omega = \{(x, y) | -2 \leq x \leq 2, 0 \leq y \leq 1\}$. The nodal configuration and collocation points are shown in Figure 7.

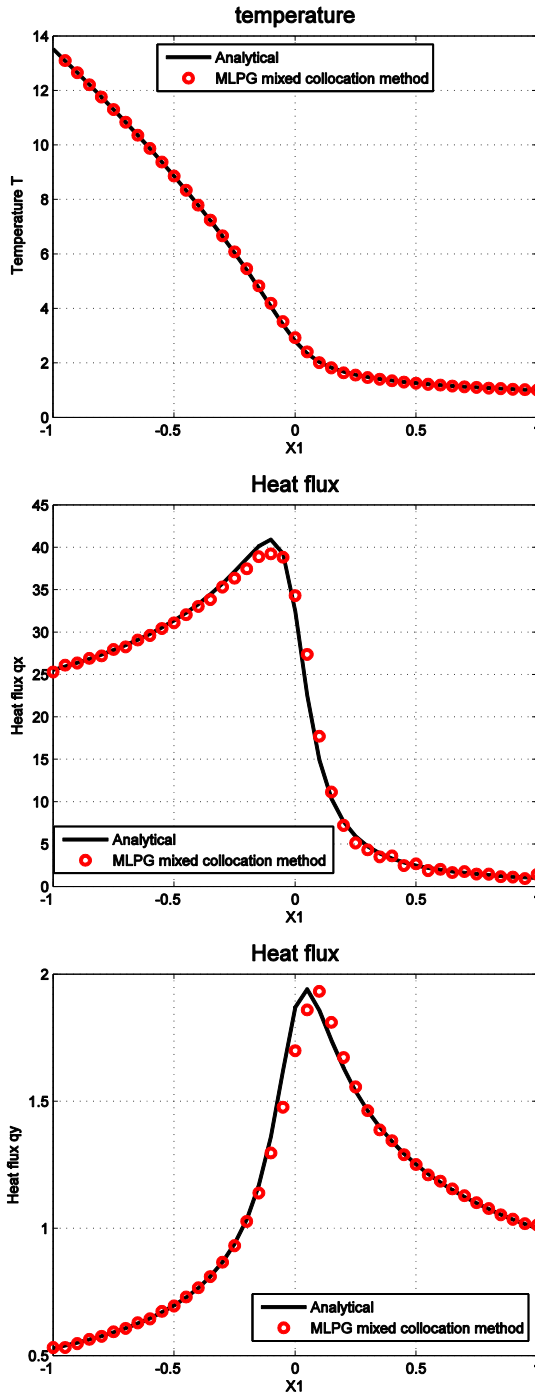


Figure 5: T , q_x , and q_y along the line $y = -1$, normalized to their maximum values, for the inverse problem of Example2.

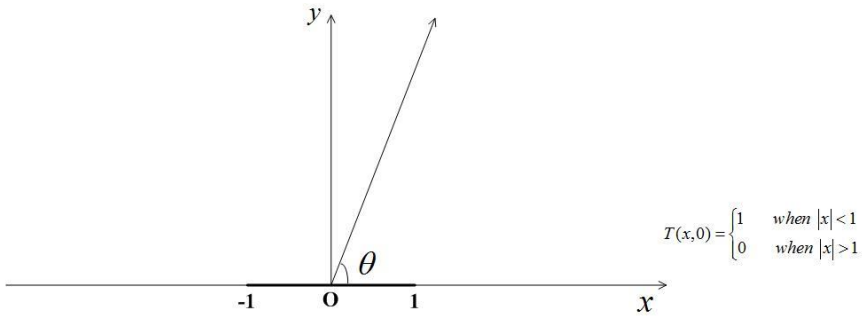


Figure 6: Heat conduction in a semi-infinite domain.

cation points are shown in figure 7, with temperature and heat fluxes measured at $S_C = \{(x,y) | -1 \leq x \leq 1, y = 2\}$, and polluted by 1% white noise. By MLPG mixed collocation method, we solve this inverse Cauchy problem of heat transfer, and plot the numerically identified temperature and heat fluxes along the line $y = 0.5$. Figure 8 gives the comparison between numerical and analytical solution, demonstrating the validity of the proposed MLPG mixed collocation method.

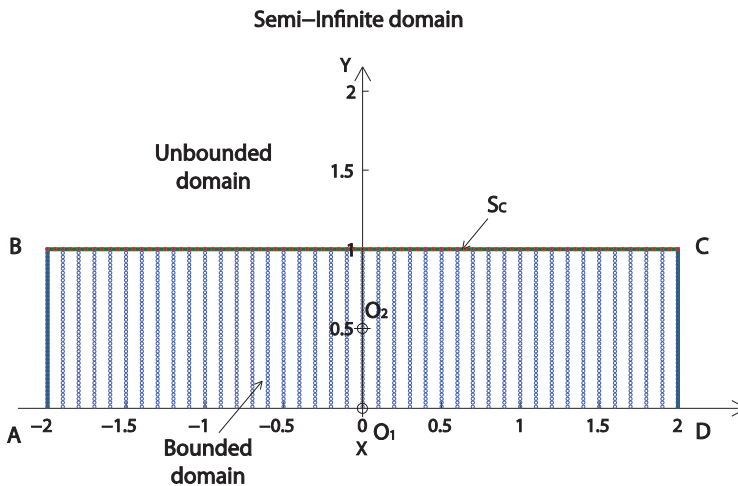


Figure 7: The finite truncated domain and the nodal configuration of the MLPG mixed collocation model for example 3.

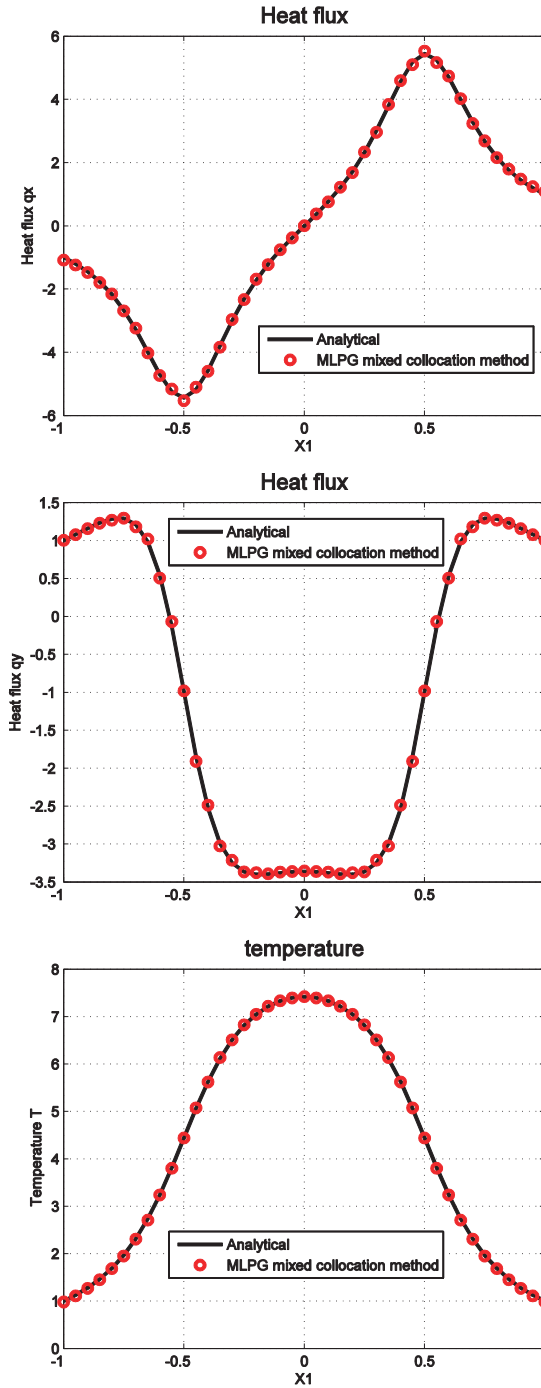


Figure 8: T , q_x , and q_y along the line $y = 0.5$, normalized to their maximum values, for the inverse problem of Example 3.

Example 4: Patch with different levels of white noise, different numbers of collocation points, and different sizes of S_c

We reconsider the heat transfer problem in a rectangular domain $\Omega = \{(x,y) | 0 \leq x \leq a, 0 \leq y \leq b\}$, as shown in Figure 1. But for this case, an inverse problem is solved instead of a direct one. Different levels of white noise, different numbers of collocation points, and different sizes of S_c are considered to investigate the stability, convergence, and sensitivity of the proposed MLPG mixed collocation method for inverse Cauchy problems.

a. Numerical stability

We consider that various percentages of random noises, p_T and p_q are added into the measured \bar{T} and \bar{q}_n , at 40 points uniformly distributed along $S_C = \{(x,y) | 0 \leq y \leq b, x = 0\} \cup \{(x,y) | 0 \leq y \leq b, x = a\}$. We analyze the numerical solutions with three levels of noise (1%, 3% and 5%) added to: (i) the Dirichlet data (temperatures); (ii) the Neumann data (heat fluxes); and (iii) the Cauchy data (temperatures and heat fluxes), respectively.

Figure 9-11 present the heat fluxes q_x, q_y and temperature T at $y = 3$ numerically identified by using the MLPG mixed collocation method. It can be seen that, for each fixed level of noise, the numerical solutions are stable approximations to the corresponding exact solution, free of unbounded and rapid oscillations.

b. Numerical convergence

In this problem, we consider noisy measurements with $p_q = p_T = 5\%$ along $S_C = \{(x,y) | 0 \leq x \leq 10, y = 0\} \cup \{(x,y) | x = 0, 1 \leq y \leq 10\} \cup \{(x,y) | x = 10, 0 \leq y \leq 10\}$. Different numbers of uniformly distributed collocation points are used, i.e. $n_c = 2, 4, 10, 20, 40, 80, 160, 320, 640, 1280$, on each of the three sides of S_C .

In order to analyze the accuracy, we introduce the following root mean-square (RMS) errors:

$$\begin{aligned}
 E_T &= \sqrt{\frac{1}{N} \sum_{i=1}^N (T - \bar{T})^2} / \sqrt{\frac{1}{N} \sum_{i=1}^N \bar{T}^2} \\
 E_{q_x} &= \sqrt{\frac{1}{N} \sum_{i=1}^N (q_x - \bar{q}_x)^2} / \sqrt{\frac{1}{N} \sum_{i=1}^N \bar{q}_x^2} \\
 E_{q_y} &= \sqrt{\frac{1}{N} \sum_{i=1}^N (q_y - \bar{q}_y)^2} / \sqrt{\frac{1}{N} \sum_{i=1}^N \bar{q}_y^2}
 \end{aligned} \tag{29}$$

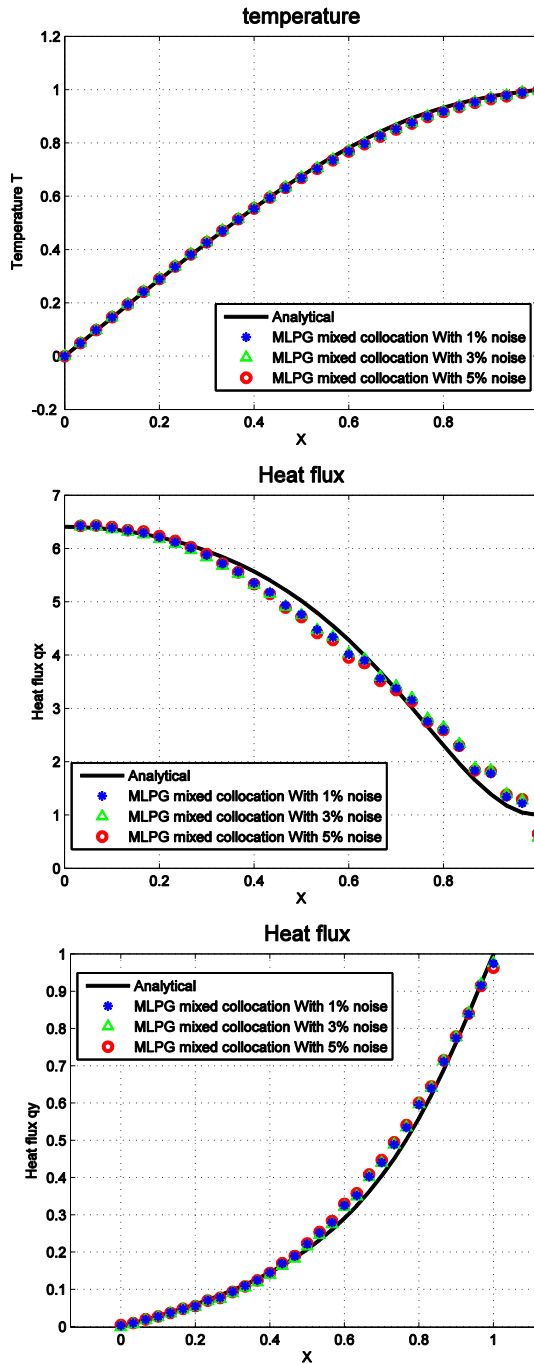


Figure 9: The numerically identified (a) T (b) q_x , (c) q_y along the line $y = 3$ with various levels of noise added into the prescribed temperatures (the Dirichlet data), i.e. $p_T \in \{1\%, 3\%, 5\%\}$ for the Cauchy problem given by example 4.

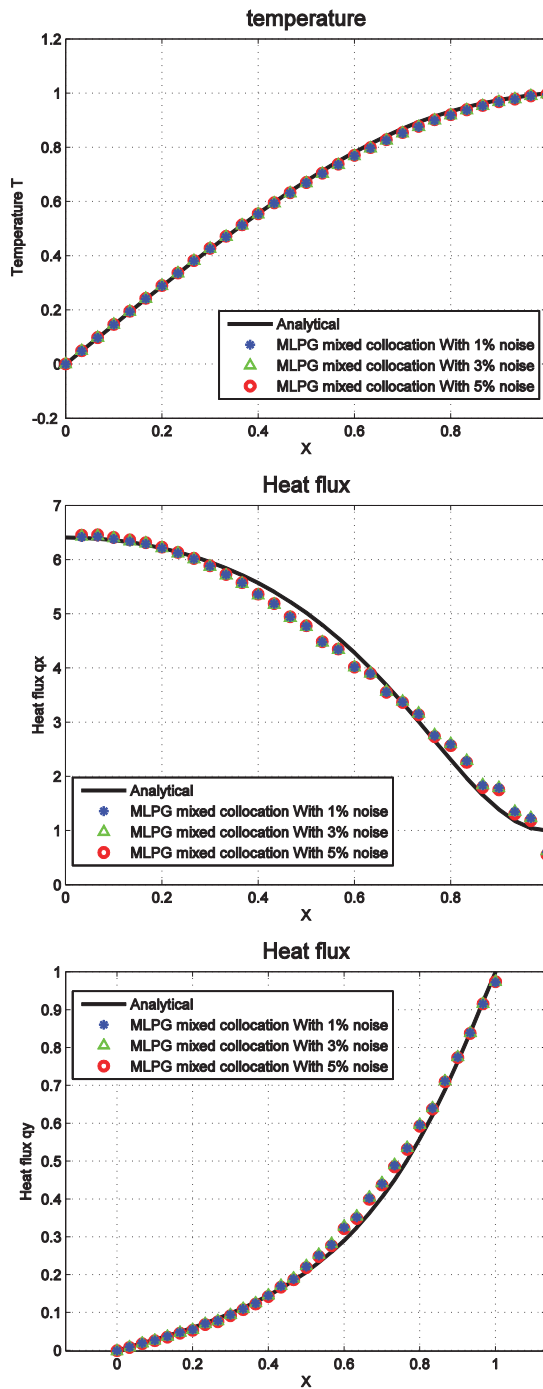


Figure 10: The numerically identified (a) T (b) q_x , (c) q_y along the line $y = 3$ with various levels of noise added into the prescribed fluxes (the Neumann data), i.e. $p_q \in \{1\%, 3\%, 5\%\}$ for the Cauchy problem given by example 4.

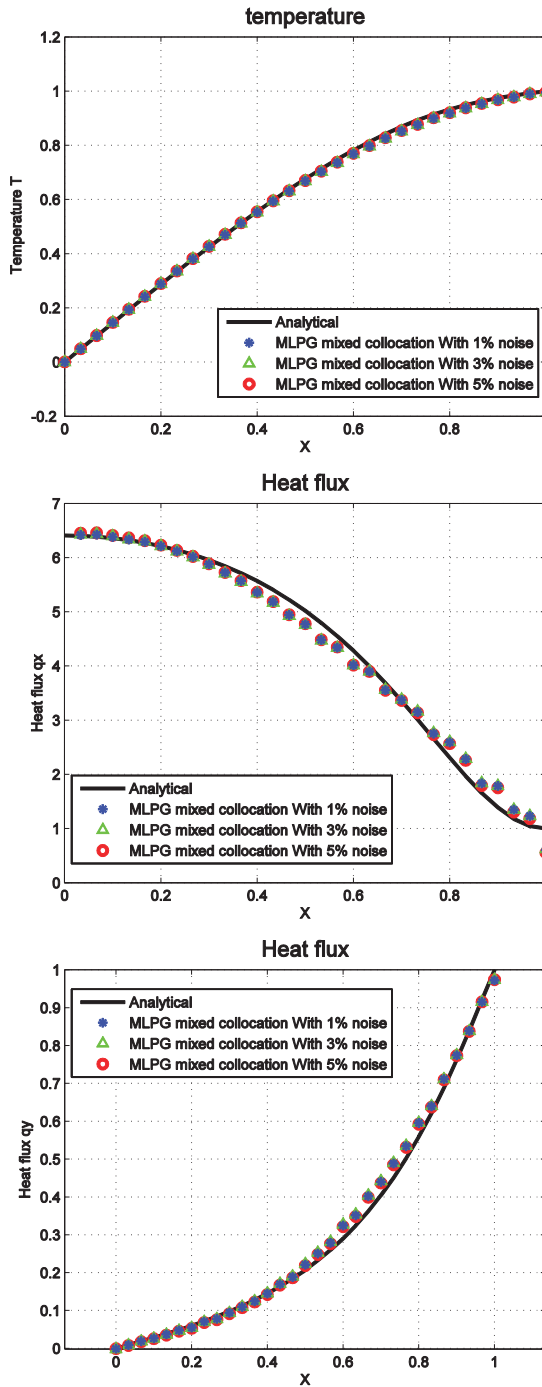


Figure 11: The numerically identified (a) T (b) q_x , (c) q_y along the line $y = 3$ with various levels of noise added into the prescribed temperatures and heat fluxes (the Cauchy data), i.e. $p_q = p_T \in \{1\%, 3\%, 5\%\}$ for the Cauchy problem given by example 4.

Table 1: The numerical errors based on MLPG mixed collocation method using different number of collocation points ($n_c = 2, 4, 10, 20, 40, 80, 160, 320, 640, 1280$) for example 4 with 5% noise.

n_c	E_T	E_{q_x}	E_{q_y}
2	$8.61 e-2$	$2.953 e-1$	$1.625 e-1$
4	$5.72 e-2$	$1.721 e-1$	$2.445 e-1$
10	$8.1 e-3$	$6.9 e-2$	$4.33 e-2$
2	$4.8 e3$	$39 e2$	$366 e-2$
40	$2.0 e-3$	$1.53 e-2$	$1.87 e-2$
80	$9.245 e-4$	$7.3 e-3$	$1.95 e-2$
160	$8.734 e-4$	$9.3 e-3$	$8.4 e-3$
320	$5.4098 e-4$	$5.4 e-3$	$7.0 e-3$
640	$4.5084 e-4$	$3.2 e-3$	$5.7 e-3$
1280	$4.3408 e-4$	$3.2 e-3$	$6.3 e-3$

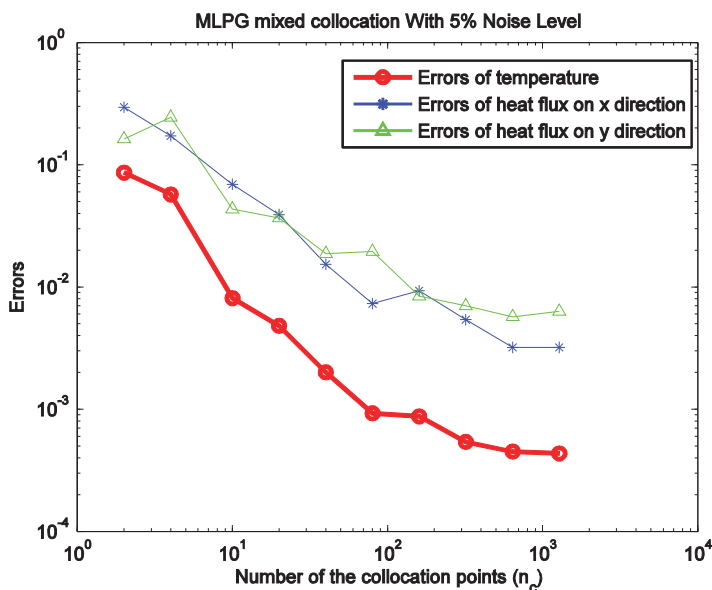


Figure 12: The errors as functions of the number of the collocation points for example 4.

where T_{q_x} , q_y and $\bar{T}\bar{q}_x, \bar{q}_y$ represent, respectively, the numerical and exact values.

Table 1 and figure 12 present the numerical error of identified temperatures and heat fluxes with different number of the collocation points, from where the numerical convergence can be observed. It can be seen that all errors E_{q_x} , E_{q_y} and E_T keep decreasing as the number of the collocation points increases. When n_c is more than 10^2 , the convergence rate of the aforementioned slows down, which is possibly due to the presence of noisy measurements.

c. Influence of the size of S_C

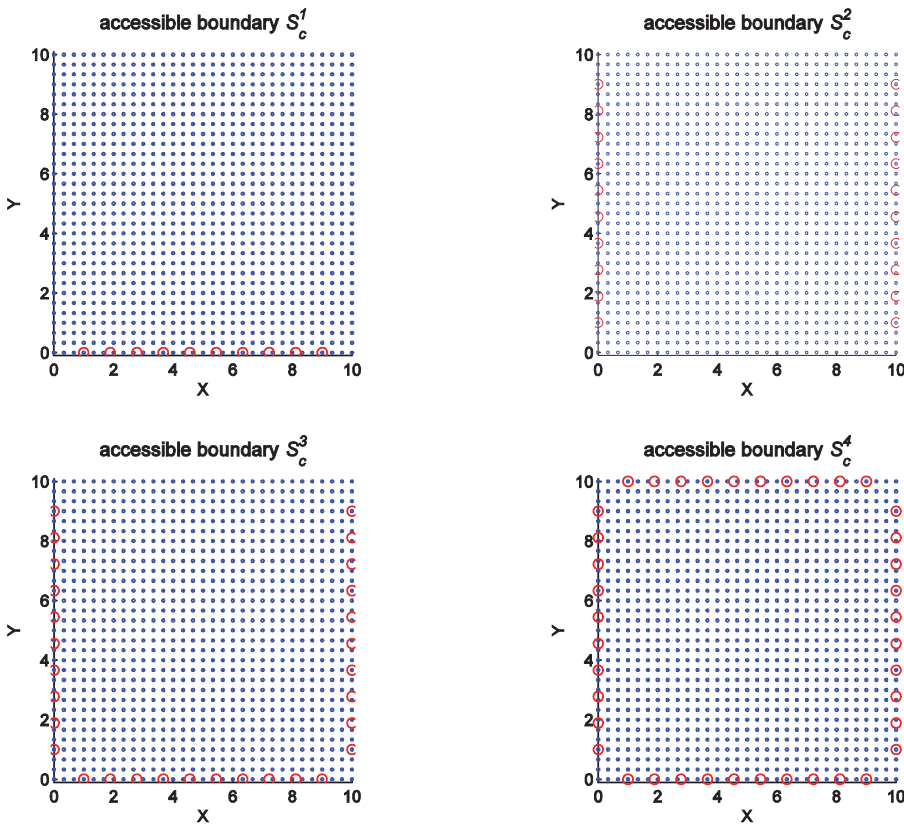


Figure 13: Four different sizes of accessible boundary ($S_c^1, S_c^2, S_c^3, S_c^4$), respectively.

In this case, we investigate how the size of S_C affects the accuracy of the numerical solution. For the given Cauchy problem, measured temperatures and heat fluxes are contaminated with with 5% random noise. Four different sizes of S_c are considered

and are illustrated in Figure 13, in which collocation points along S_c are shown as red circles. These four different accessible boundaries are defined as:

$$S_c^1 = \{(x,y) | y = 0, 1 \leq x \leq 9\}$$

$$S_c^2 = \{(x,y) | x = 0, 1 \leq y \leq 9\} \cup \{(x,y) | x = 10, 1 \leq y \leq 10\}$$

$$S_c^3 = \{(x,y) | y = 0, 1 \leq x \leq 9\} \cup \{(x,y) | x = 0, 1 \leq y \leq 9\}$$

$$\cup \{(x,y) | x = 10, 1 \leq y \leq 9\}$$

$$S_c^4 = \{(x,y) | y = 0, 1 \leq x \leq 9\} \cup \{(x,y) | x = 0, 1 \leq y \leq 9\}$$

$$\cup \{(x,y) | y = 10, 1 \leq x \leq 9\} \cup \{(x,y) | x = 10, 1 \leq y \leq 9\}$$

Table 2 presents the numerical accuracy of the MLPG mixed collocation method. It can be seen that the numerical accuracy improves with larger sizes of S_c , but it is still acceptable even with the smallest size of S_c

Table 2: The numerical errors of MLPG mixed collocation method using different sizes of S_c for example 4.

S_c	E_T	E_{q_x}	E_{q_y}
S_c^1	3.28 e2	0.13	0.10
S_c^2	4.9 e-3	3.52 e-2	1.92 e-2
S_c^3	1.2 e-3	1.51 e-2	1.34 e-2
S_c^4	4.2684 e-4	3.4 e-3	5.2 e-3

5 Conclusion

In this article, the MLPG mixed collocation method is applied to solve the inverse Cauchy problems of steady-state heat transfer. The temperature as well as the heat fluxes are interpolated independently using the same MLS basis functions. The balance and compatibility equations are satisfied at each node in a strong sense using the collocation method. The boundary conditions are also enforced using the collocation method, allowing temperature and heat flux to be over-specified at the same portion of the boundary. For the inverse problems where noise is present in the measurement, Tikhonov regularization method is used, to mitigate the inherent ill-posed nature of inverse problem with its regularization parameter determined by the L-Curve method. Through several numerical examples, we investigated the numerical accuracy, stability, and convergence of the MPLG mixed collocation

method. It is shown that the proposed method is simple, accurate, stable, and thus is suitable for solving inverse problems of heat transfer.

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