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A Solution Procedure for a Vibro-Impact Problem under Fully Correlated Gaussian White Noises

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Abstract: This study is concerned with a solution procedure to obtain the probability density function (PDF) of a vibro-impact Duffing oscillator under fully correlated external and parametric Gaussian white noises. The proposed solution procedure consists of three steps. In the first step, the Zhuravlev non-smooth coordinate transformation is adopted to introduce an additional impulsive damping term, in which the original vibro-impact oscillator is converted into a new oscillator without any barrier. After that, the PDF of the new oscillator is obtained by solving the Fokker-Planck equation with the exponential-polynomial closure method. Last, the PDF of the original oscillator is formulated in terms of the methodology on seeking the PDF of a function of random variables. A numerical analysis on four different cases is conducted to examine the effectiveness of the proposed solution procedure. Comparison with the simulated result shows that the proposed solution procedure can provide a satisfactory PDF solution for the four cases. The tail region of the PDF solution is also approximated well. The numerical analysis also shows that the change of parametric excitation has a significant effect on the PDF distributions of displacement and velocity.

Keywords: vibro-impact, random vibration, probability density function, correlated white noises, Duffing oscillator.

1 Introduction

Random vibration of structural and mechanical systems widely exists in the field of engineering, such as civil engineering, mechanical engineering, aerospace engineering, and ocean engineering [Lutes and Sarkani (2004)]. In such a case, these dynamical systems are excited by random forces which are modeled as stochastic processes, and the associated dynamical response is evaluated using its statistical moments or probability density functions (PDFs). In particular, the case of ran-

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dom forces being modeled by Gaussian white noises has been extensively investigated analytically and numerically in the past few decades. It is a conventional way that the response of a system excited by Gaussian white noises can be modeled by a Markov process. Consequently the PDF of the response is governed by the Fokker-Planck (FP) equation. However, when the systems have a complicated functional form, the corresponding FP equation is too complicated in its expression to be solved. Only a few exact stationary PDF solutions were obtained in some very restricted cases [Caughey and Ma (1982); Dimentberg (1982); Jing and Sheu (1990); Wang and Zhang (2000); Zhu and Huang (2001)]. Therefore, approximate or numerical methods have to be developed to solve the problem of random vibration by an approximate way in the past few decades [Dunne and Ghanbari (1997)], such as finite element method [Langley (1985)], weighted residual methods [Soize (1988)], path integration (cell mapping) [Sun and Hsu (1988)], finite difference method [Roberts (1986)], variational method [Langley (1988)]. In particular, stochastic averaging is a widely used approximate method to solve the PDF solution [Roberts and Spanos (1986)]. The stochastic averaging method is suitable for the case of lightly damping and weak wide band excitation, in which a Markovian approximation of the appropriate envelop of the oscillator response (e.g., the envelope of the total energy) is easily formulated. Besides, another simple and versatile technique is Monte Carlo simulation in structural dynamics [Shinozuka (1972); Proppe, Pradlwarter, and Schuëller (2003)]. The equation of motion of the systems is numerically integrated over time in a straightforward manner to obtain the time series of the response. After that, the statistical moments and the PDF solutions are evaluated in terms of these obtained time series. However, when either a large-scale system or the tail region of the PDF solution is studied, the computational efforts are dramatically increased.

Compared with the PDF solution, the statistical moments of the response are much more accessible. Equivalent linearization (EQL) is a simple and versatile method to fulfill this target [Caughey (1963); Spanos (1981); Roberts and Spanos (2003); Elishakoff (2000); Socha (2005)]. The EQL method mainly relies on the two important assumptions. One assumption is that the obtained solutions are Gaussian. However, this assumption cannot be satisfied by nonlinear systems in most cases. The other assumption is that the original system can be adequately represented by an equivalent one. This assumption can be fulfilled in the case that both system nonlinearity and excitation intensity are generally small. In a similar manner, Gaussian closure method was also developed and restricted to the above assumption [Iyengar and Dash (1978); Wu (1987)]. The Gaussian closure method is a special case of cumulant-neglect closure technique by truncating the cumulant equations beyond the second cumulant [Wu and Lin (1984); Sun and Hsu (1987)]. Furthermore, the cumulant-neglect closure technique is usually conducted up to the fourth order closure to evaluate the non-Gaussian behavior of nonlinear systems. The sixth-order cumulant closure technique becomes much more tedious in its solution procedure.

A vibro-impact oscillator is a conventional nonlinear system in the field of science and engineering [Babitsky (1998); Ibrahim (2009)]. It can be used to model many physical phenomena such as vibration of beams or pipes with a stop, ship roll motion against icebergs, rotor-stator rubbing in rotating machinery, vibro-impact vibration of heat exchanger tubes to aerodynamic excitation, etc. These vibro-impact systems undergo vibration until they collide with barriers, during which velocity jumps occur with energy losses. Consequently, the systems show a strongly nonlinear behavior. Sometimes, these systems are driven by random excitations, such as wind loadings, sea waves, and seismic loads. Therefore, the investigation of vibro-impact systems under random excitations is a crucial issue in the scientific community. Furthermore, the response of the vibro-impact systems may be not Gaussian [Baratta (1990)]. As described above, the PDF solution of nonlinear systems under Gaussian white noise is govern by the FP equation. Although extensive efforts have been made on developing some analytical or approximate methods on solving the FP equation, the application of the analytical or approximate methods are quite limited in the case of vibro-impact systems. Some exact PDF solutions were obtained in some very special cases [Dimentberg and Iourtchenko (2004); Jing and Sheu (1990); Jing and Young (1990, 1991)]. Most problems of vibro-impact random vibration have to be solved by approximate methods. Stochastic averaging methods are widely applied to the problems of vibro-impact systems under random excitation. A stochastic averaging method is developed to solve the multi-degree-of-freedom vibro-impact problems with a Hertzian contact model [Huang, Liu, and Zhu (2004)]. The stochastic averaging method was also adopted to investigate the vibro-impact oscillator with two-sided barriers using the classical impact model with instantaneous velocity jumps [Sri Namachchivaya and Park (2005)]. In this classical impact model, the relation between rebound velocity and impact velocity is determined by the Newton's law using a restitution factor. The restitution factor is used to evaluate the degree of impact losses. This classical impact model can be applicable to the cases of both purely elastic barriers and inelastic ones. Using the similar classical impact model, the stochastic averaging method was proposed to study the response of vibro-impact Duffing oscillators [Feng, Xu, and Wang (2008)] and vibro-impact Duffing-Van der Pol oscillators [Feng, Xu, Rong, and Wang (2009)]. In these researches, the Zhuravlev non-smooth coordinate transformation was conducted to convert a vibro-impact oscillator into an oscillator without barriers such that the PDF solution of the converted system was easily handled by the FP equation using the stochastic averaging method [Ibrahim (2009); Dimentberg and Iourtchenko (2004); Zhuravlev (1976)]. More recently, this technique was extended to investigate the response probability density functions of a Duffing-Van der Pol vibro-impact system under correlated Gaussian white noise excitations [Li, Xu, Feng, and Wang (2013)]. In the case of the restitution factor being very close to unity (i.e., lightly inelastic impacts), the stochastic averaging methods worked well in the examined cases. Besides the stochastic averaging methods, a numerical path integration method was also developed together with the Zhuravlev-Ivanov coordinate transformation to examine the response PDF of stochastic vibro-impact systems with high energy losses at impacts [Dimentberg, Gaidai, and Naess (2009)]. Additionally, Monte Carlo simulation was also applied for analyzing the response of vibro-impact systems in the cases of either a one-sided barrier or two-sided barriers [Iourtchenko and Song (2006)]. Many computational efforts are spent on tracing impacts during the simulation process. The response interval also needs to be carefully selected to obtain an adequate PDF value.

As mentioned above, the stochastic response of vibro-impact systems is investigated mostly by stochastic averaging methods. Furthermore, the problem of the systems with corrected random excitations is less addressed in previous studies, either. In attempt to solve these limitations, an alternative solution procedure is developed in this paper for the PDF solution of a vibro-impact Duffing oscillator excited by fully corrected Gaussian white noises. The oscillator is restrained by a one-sided barrier locating at its equilibrium. The colliding between the oscillator and the barrier is modeled using the classical model with instantaneous lightly inelastic impacts. In the proposed solution procedure, the Zhuravlev non-smooth coordinate transformation is first is adopted to convert the original vibro-impact oscillator into a new oscillator without any barrier, but with an additional impulsive damping term. After that, the PDF of the new oscillator is handled by solving the F-P equation with the exponential-polynomial closure (EPC) method [Er (1998); Zhu, Er, Iu, and Kou (2010, 2012)]. Last, the PDF of the original oscillator is obtained using the methodology on seeking the PDF of a function of random variables. A numerical analysis on four different cases is conducted to show the effectiveness of the proposed solution procedure. The effects of the nonlinearity in displacement and the parametric excitation are further investigated on the PDF distributions of the vibro-impact oscillators. Comparison with the simulated result shows that the proposed solution procedure can provide a satisfactory approximate PDF solution, especially for the tail region of the PDFs.

2 Problem formulation

Equations 1 and 2 represent the equations of motion for a vibro-impact Duffing oscillator excited by fully correlated Gaussian white noises with a unilateral zerooffset barrier

$$\ddot{y}+c\dot{y}+ky+\mu y^3=\xi(t)+\varepsilon y\xi(t), \quad y>0 \tag{1}$$

$$\dot{y}_+=-r\dot{y}_-, \quad y=0, \quad 0<r\leq 1 \tag{2}$$

where $\ddot{y}$, $\dot{y}$, y are the acceleration, velocity and displacement of the oscillator, respectively; c is the damping coefficient; k is the linear stiffness coefficient; μ is the nonlinearity coefficient of displacement; $\xi(t)$ is a zero-mean Gaussian white noise. ε is the parametric excitation factor; r is the restitution factor; $\dot{y}_-$ and $\dot{y}_+$ are impact velocities before and after a impact, respectively.

The mean and correlation function of $\xi(t)$are formulated below

$$E[\xi(t)]=0 \tag{3}$$

$$E[\xi(t)\xi(t+\tau)]=2\pi K\delta(\tau) \tag{4}$$

where $E[\bullet]$ is the expectation; $2\pi K$ is the excitation intensity of $\xi(t)$ and $\delta(\bullet)$ is the Dirac delta function

It is difficult to directly substitute Eqs. 1 and 2 into the FP equation for obtaining the PDF of the vibro-impact oscillator. Therefore, the Zhuravlev non-smooth coordinate transformation is adopted leading the above vibro-impact oscillator to be converted into an oscillator without barriers [Dimentberg and Iourtchenko (2004); Ibrahim (2009)]. The transformation procedure is as follows

$$y=|z|, \quad \dot{y}=\dot{z}sgn(z), \quad \ddot{y}=\ddot{z}sgn(z) \tag{5}$$

where $\ddot{z}$, $\dot{z}$, z are the acceleration, velocity and displacement of the converted oscillator, respectively and $sgn(\bullet)$ is the sign function. Equation 5 is formulated due to the fact: $y=|z|$ is equivalent to$y=zsgn(z)$; $z[d(sgn(z))/dt]=0$ because $sgn(z)$ is only taken as -1, 0, or 1.

Equation 2 gives the impact condition for each impact. In the following transformation procedure, the restitution factor r is assumed to be close to unity. In such a case, the response of the nonlinear oscillator may have much less significant discontinuities in its time derivative. That is, the response can be approximately treated as a continuous process. This assumption allows the vibro-impact problem to be solved by conventional approximate methods [Baratta (1990); Feng, Xu, and Wang (2008)], e.g., the stochastic averaging method and the FP equation method.

According to Eq. 5, Eq. 2 is converted to

$$\dot{z}_+=r\dot{z}_- \quad at \quad z=0 \tag{6}$$

where $\dot{z}_-$ and $\dot{z}_+$ are the impact velocities before and after a impact for the converted oscillator, respectively.

In terms of Eq. 6, the reduction of the converted velocity jump is evaluated by an amount proportional to $(1-r)$. The Dirac delta-function is used to introduce this velocity jump into the equation of motion as an additional impulsive damping term [Dimentberg and Iourtchenko (2004); Ibrahim (2009)]. By this way, the equation of motion and the impact condition are integrated into one equation. The additional damping term due to impacts is approximately expressed as

$$(\dot{z}_+ - \dot{z}_-)\delta(t-t_i) = (1-r)\dot{z}\delta(t-t_i) \quad \textit{given that} \quad |\dot{z}_+| < |\dot{z}| < |\dot{z}_-| \tag{7}$$

Because the response of the vibro-impact oscillator can be approximately treated as a continuous process, the following mathematical manipulation can be performed in the vicinity of each impact $z(t) = z(t_i) + \dot{z}(t_i)(t-t_i)$. Herein, $z(t_i)$ is the displacement at the time instant of impacts t_i and $z(t_i) = 0$. Therefore, $(t-t_i) = z(t)/\dot{z}(t_i)$. In a small interval of the vicinity of each impact, the Dirac delta function is applied as $\delta(t-t_i) = \delta(z(t)/\dot{z}(t_i))$. According to the fact that $\delta(t-t_i) = |\dot{z}|\,\delta(z)$, Equation 7 is further formulated as

$$(1-r)\dot{z}\delta(t-t_i) = (1-r)\dot{z}\,|\dot{z}|\,\delta(z) \tag{8}$$

By introducing the additional impulsive damping term of Eq. 8, Equations 1 and 2 finally are integrated into one equation as follows

$$\ddot{z} + c\dot{z} + kz + \mu z^3 + (1-r)\dot{z}\,|\dot{z}|\,\delta(z) = sgn(z)\xi(t) + \varepsilon z\xi(t) \tag{9}$$

Letting $x_1 = z$, $x_2 = \dot{z}$, Equation 9 can be reformulated in a set of first-order differential equations

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -cx_2 - kx_1 - \mu x_1^3 - (1-r)x_2\,|x_2|\,\delta(x_1) + sgn(x_1)\xi(t) + \varepsilon x_1\xi(t) \end{cases} \tag{10}$$

Because the restitution factor r is assumed to be close to unity in the present transformation procedure, $(1-r)$ can be treated as a small parameter. The response $\{x_1, x_2\}^T$ is approximately treated as a Markov process and its PDF is governed by the following FP equation

$$\begin{aligned} \frac{\partial p}{\partial t} = &-x_2\frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left[\left\{cx_2 + kx_1 + \mu x_1^3 + (1-r)x_2\,|x_2|\,\delta(x_1)\right\}p\right] \\ &+ \frac{1}{2}\cdot 2\pi K\cdot\{sgn(x_1) + \varepsilon x_1\}^2\frac{\partial^2 p}{\partial x_2^2} \end{aligned} \tag{11}$$

In this paper, the stationary PDF solution is considered and the term on the left side of Eq. 11 is zero. Therefore, Equation 11 is reduced as follows

$$
\begin{aligned}
&-x_2\frac{\partial p}{\partial x_1}+\{c+(1-r)\,|x_2|\,\delta(x_1)+(1-r)x_2 sgn(x_2)\delta(x_1)\}\,p\\
&+\left\{cx_2+kx_1+\mu x_1^3+(1-r)x_2\,|x_2|\,\delta(x_1)\right\}\frac{\partial p}{\partial x_2}\\
&+\pi K\{1+2\varepsilon x_1 sgn(x_1)+\varepsilon^2 x_1^2\}\frac{\partial^2 p}{\partial x_2^2}=0
\end{aligned}
\tag{12}
$$

Considering $|x_2| = x_2 sgn(x_2)$, Equation 12 is finally given as

$$
\begin{aligned}
&-x_2\frac{\partial p}{\partial x_1}+\{c+2(1-r)\,|x_2|\,\delta(x_1)\}\,p\\
&+\left\{cx_2+kx_1+\mu x_1^3+(1-r)x_2\,|x_2|\,\delta(x_1)\right\}\frac{\partial p}{\partial x_2}\\
&+\pi K\{1+2\varepsilon x_1 sgn(x_1)+\varepsilon^2 x_1^2\}\frac{\partial^2 p}{\partial x_2^2}=0
\end{aligned}
\tag{13}
$$

It is too difficult to solve Eq. 13 exactly. The EPC method is employed herein. Considering the following properties for a conventional stationary PDF solution

$$
\begin{cases}
p\,(x_1,x_2)\geq 0,\ (x_1,x_2)\in R^2\\
\lim\limits_{x_i\to\pm\infty} p\,(x_1,x_2)=0,\ i=1,2\\
\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x_1,x_2)dx_1dx_2=1
\end{cases}
\tag{14}
$$

An approximate PDF $\tilde{p}(x_1,x_2;\mathbf{a})$ solution to Eq. 14 is formulated as an exponential-polynomial function as follows

$$
\tilde{p}(x_1,x_2;\mathbf{a})=Ce^{Q_n(x_1,x_2;\mathbf{a})}
\tag{15}
$$

where C is a normalization constant the polynomial $Q_n(x_1,x_2;\mathbf{a})$ is expressed as

$$
Q_n\,(x_1,x_2;\mathbf{a})=\sum_{i=1}^{n}\sum_{j=0}^{i} a_{ij}x_1^{i-j}x_2^{j}
\tag{16}
$$

which is a complete nth-degree polynomial in x_1 and x_2. To satisfy the requirements of Eq. 14, it is also assumed that

$$
\lim_{x_i\to\pm\infty} Q_n\,(x_1,x_2;\mathbf{a})=-\infty,\ i=1,\ 2
\tag{17}
$$

Because $\tilde{p}(x_1,x_2;\mathbf{a})$ is only an approximation solution of Eq. 13 and the number of unknown parameters N_p is always limited in practice, substituting $\tilde{p}(x_1,x_2;\mathbf{a})$ usually leads to the following residual error

$$\begin{aligned}
&\Delta(x_1,x_2;\mathbf{a}) = -x_2\frac{\partial \tilde{p}}{\partial x_1} + [c+2(1-r)\,|x_2|\,\delta(x_1)]\,\tilde{p} \\
&+ \left[cx_2+kx_1+\mu x_1^3+(1-r)x_2\,|x_2|\,\delta(x_1)\right]\frac{\partial \tilde{p}}{\partial x_2} \\
&+ \pi K\{1+2\varepsilon x_1 sgn(x_1)+\varepsilon^2x_1^2\}\frac{\partial^2 \tilde{p}}{\partial x_2^2} \\
&= \left\{-x_2\frac{\partial Q_n}{\partial x_1}+c+2(1-r)\,|x_2|\,\delta(x_1)\right. \\
&+ \{cx_2+kx_1+\mu x_1^3+(1-r)x_2\,|x_2|\,\delta(x_1)\}\frac{\partial Q_n}{\partial x_2} \\
&\left.+ \pi K\{1+2\varepsilon x_1 sgn(x_1)+\varepsilon^2x_1^2\}\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^2+\frac{\partial^2 Q_n}{\partial x_2^2}\right]\right\}\tilde{p}(x_1,x_2;\mathbf{a}) \\
&= F(x_1,x_2;\mathbf{a})\tilde{p}(x_1,x_2;\mathbf{a})
\end{aligned} \tag{18}$$

Due to the fact that $\tilde{p}(x_1,x_2;\mathbf{a})$ is a non-zero exponential function, the only possibility of the residual error being zero is that $F(x_1,x_2;\mathbf{a}) = 0$. However, $F(x_1,x_2;\mathbf{a})$ is not zero in most cases. Therefore, a special weighted residual procedure is adopted by introducing another set of mutually independent functions $H_s(x_1,x_2)$ spanning space R^{N_p} to make the projection of $F(x_1,x_2;\mathbf{a})$ on R^{N_p} vanish.

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x_1,x_2;\ \mathbf{a})H_s(x_1,x_2)dx_1dx_2 = 0, \quad s = 1,2,...,N_p \tag{19}$$

where

$$H_s(x_1,x_2) = x_1^{l-m}x_2^m f_1(x_1) f_2(x_2) \tag{20}$$

where $l = 1,2,\ldots,n$; $m = 0,1,2,\ldots,l$. That is to say, Equation 13 is satisfied by the approximate PDF solution $\tilde{p}(x_1,x_2;\mathbf{a})$ in weak sense. Numerical experience shows that $f_1(x_1)$ and $f_2(x_2)$ can be adopted using the results given by equivalent linearization methods.

$$f_1(x_1) = \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{x_1^2}{2\sigma_1^2}\right\} \tag{21}$$

$$f_2(x_2) = \frac{1}{\sqrt{2\pi}\sigma_2}\exp\left\{-\frac{x_2^2}{2\sigma_2^2}\right\} \tag{22}$$

Finally, nonlinear algebraic equations are formulated for the unknown parameter **a**. The conventional Newton-Raphson method can be used to solve the nonlinear algebraic equations. The initial solution can be used with the result given by a standard EQL method. It is also noted that the conventional Newton-Raphson method sometimes suffers from an inadequate guess initial solution. Therefore, the scalar homotopy methods are recommended as a viable alternative to the classical Newton-Raphson method. [Dai, Paik and Atluri (2011a, 2011b); Dai, Schnoor and Atluri (2012); Liu, Yeih, Kuo and Atluri (2009)]. Although the scalar homotopy methods cannot match the Newton-Raphson method in terms of computing speed, they have a merit that they do not involve with inverting the Jacobian matrix of nonlinear algebraic equations. Moreover, they are robust to the initial guesses.

Once $\tilde{p}(x_1,x_2;\mathbf{a})$ is solved for $\tilde{p}(z,\dot{z})$ of the converted oscillator, the PDFs of the original oscillator, namely $\tilde{p}_Y(y)$ and $\tilde{p}_{\dot{Y}}(\dot{y})$, can be also accessed in terms of Eq. 5. The methodology for seeking the PDF distribution of a function of random variables is adopted following the approach introduced in Ref. [Lutes and Sarkani (2004)].

For seeking the PDF of y, namely $\tilde{p}_Y(y)$, $y$is a function of z with the relationship given in Eq. 5. The relationship between y and $z$can be simply expressed as $y = |z| = g(z)$ and $g(z)$ is a general function. Therefore,

$$\tilde{p}_Y(y) = \sum_j \frac{\tilde{p}_Z\left[g_j^{-1}(y)\right]}{\left|\frac{dg(u)}{du}\right|_{u=g_j^{-1}(y)}} \tag{23}$$

with the summation being over all inverse points $z = g_j^{-1}(y)$that map from z to y. $\tilde{p}_Z(z) = \int_{-\infty}^{+\infty} \tilde{p}(z,\dot{z})d\dot{z}$; $g^{-1}(\bullet)$ is the inverse function of $g(\bullet)$; $g_j(\bullet)$ is the j^{th} piecewise function of $g(\bullet)$ is given as

$$y = |z| = g(z) = \begin{cases} z, & z > 0 \\ 0, & z = 0 \\ -z, & z < 0 \end{cases} \tag{24}$$

In terms of Eqs. 23 and 24,

$$\tilde{p}_Y(y) = \tilde{p}_Z^+(y) + \tilde{p}_Z^-(-y), \quad y > 0 \tag{25}$$

where $\tilde{p}_Z^+(\bullet)$ and $\tilde{p}_Z^-(\bullet)$ are the PDFs locating at the positive domain and the negative domain of z, respectively. Furthermore, it is defined that $\tilde{p}_Y(0) = 2\tilde{p}_Z(0)$ because Eq. 23 is null for $z = 0$

For seeking the PDF of $\dot{y}$, namely $\tilde{p}_{\dot{Y}}(\dot{y})$, $\dot{y}$ is a function of multiple random variables with the relationship in Eq. 5. In a similar way, it is defined that $\dot{y} = \dot{z}sgn(z) = h(z,\dot{z})$ and $h(\bullet)$ is also a general function.

$\tilde{p}_{\dot{Y}}(\dot{y})$ can be obtained in terms of its cumulative distribution function $F_{\dot{Y}}(\dot{y})$

$$F_{\dot{Y}}(\dot{y}) = \iint_{h(z,\dot{z})\leq \dot{y}} \tilde{p}(z,\dot{z})\,dz d\dot{z} \tag{26}$$

The cumulative distribution function can be further formulated in a piecewise integral form

$$F_{\dot{Y}}(\dot{y}) = \int_{0}^{+\infty} dz \int_{-\infty}^{\dot{y}} \tilde{p}(z,\dot{z})\,d\dot{z} + \int_{-\infty}^{0} dz \int_{-\dot{y}}^{+\infty} \tilde{p}(z,\dot{z})\,d\dot{z} \tag{27}$$

Taking the derivative with respect to $\dot{y}$ on Eq. 27 gives the formulation of $\tilde{p}_{\dot{Y}}(\dot{y})$ as

$$\tilde{p}_{\dot{Y}}(\dot{y}) = \int_{0}^{+\infty} \tilde{p}(z,\dot{y})\,dz + \int_{-\infty}^{0} \tilde{p}(z,-\dot{y})\,dz \tag{28}$$

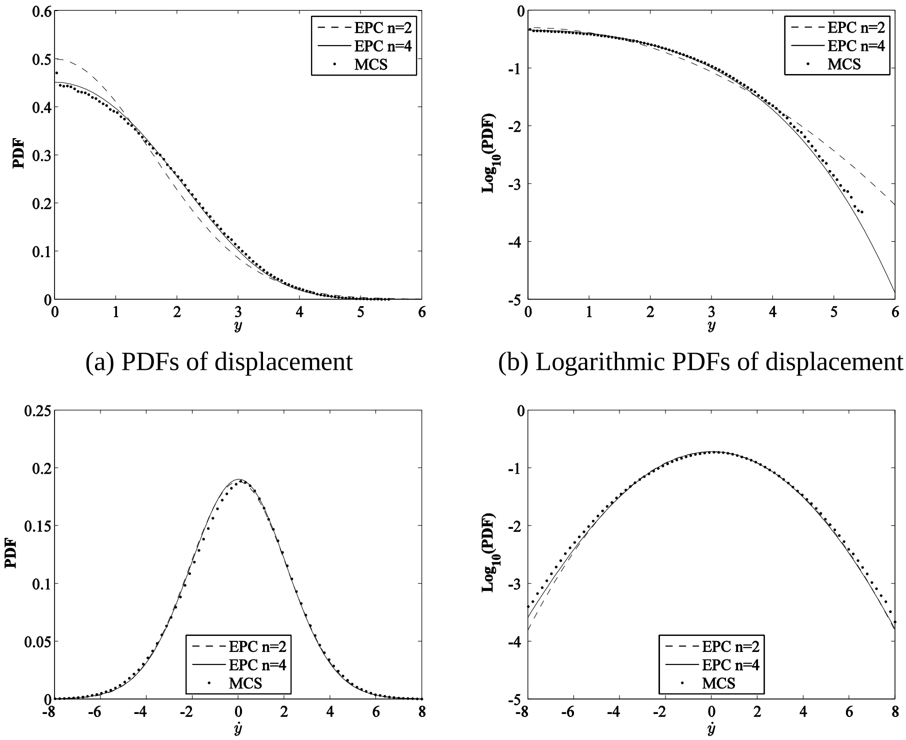
3 Numerical analysis

In this paper, a numerical analysis is further studied to show the effectiveness of the proposed solution procedure. According to Eqs. 1 and 2, the parameter values are given as $c = 0.1$, $k = 1$, $2\pi K = 1$ and $r = 0.95$ in the following cases. Different values of the nonlinearity coefficient μ and the parametric excitation factor ε are used to investigate their effects on the PDF distributions of displacement and velocity, which is listed in Tab. 1. Either weak nonlinearity or strong nonlinearity in displacement is considered in each case. The sign of parametric excitation factor is also taken to be positive or negative respectively to show the effects of the parametric excitation. In addition, Monte Carlo simulation with 1×10^{7} samples is conducted to provide an adequate PDF for comparison for each case The Monte Carlo simulation is conducted according to Eqs. 1 and 2. The simulation procedure follows the methodology introduced in Ref. [Iourtchenko and Song (2006)].

Item	μ	ε	Remarks
Case 1	0.1	0.1	Weak nonlinearity + positive parametric excitation factor
Case 2	0.1	-0.1	Weak nonlinearity + negative parametric excitation factor
Case 3	1	0.1	Strong nonlinearity + positive parametric excitation factor
Case 4	1	-0.1	Strong nonlinearity + negative parametric excitation factor

3.1 Case 1: Weak nonlinearity + a positive parametric excitation factor

First, the case of weak nonlinearity ($\mu = 0.1$) is considered with a positive parametric excitation factor ($\varepsilon = 0.1$). Figure 1 provides the comparison on the PDFs obtained with each method.



(a) PDFs of displacement (b) Logarithmic PDFs of displacement

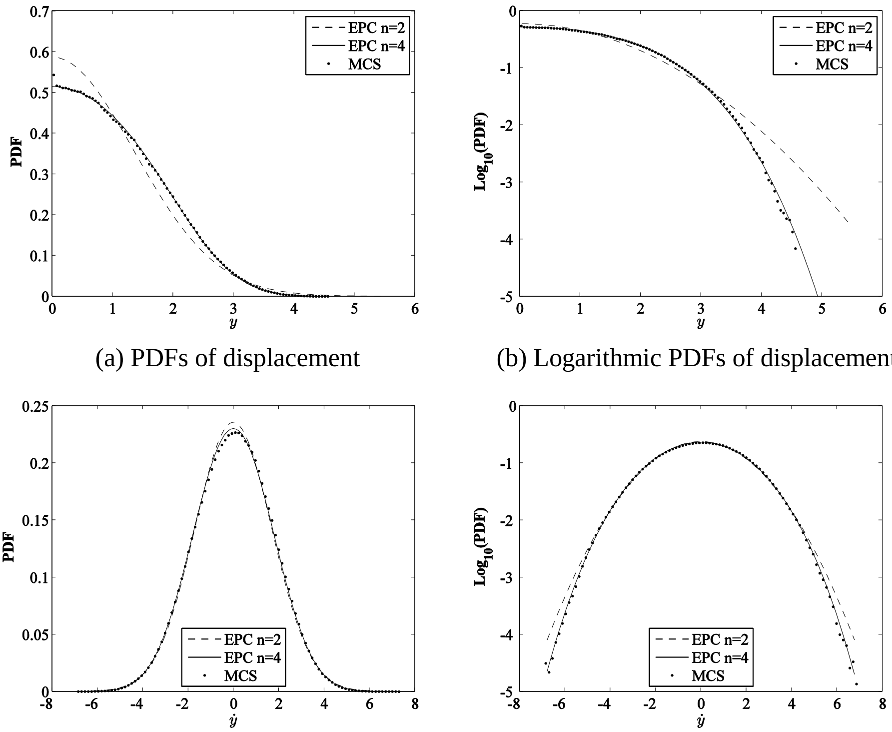
(c) PDFs of velocity (d) Logarithmic PDFs of velocity

Figure 1: Comparison on the PDFs in Case 1

The results show that when a complete second order polynomial is used in the EPC method, the result (i.e., EPC $n = 2$) is the same as that obtained with a standard equivalent linearization method. Because the result of the equivalent linearization method is Gaussian, EPC ($n = 2$) denotes a PDF obtained with a Gaussian PDF. Therefore, it is reasonable to define EPC ($n = 2$) as a special Gaussian PDF in the case of vibro-impact problems. Figures 1(a) and 1(b) show the PDFs of displacement. The maximum value of PDF is located at the barrier (i.e., $y = 0$) and the PDF becomes smaller when approaching the positive infinite boundary. EPC ($n = 2$) differs significantly from the simulated results (MCS). This difference is much more significant in the tail region as shown in Fig. 1(b). When a complete fourth order polynomial is used, the result (i.e., EPC $n = 4$) agrees well with MCS, even in the tail region. For velocity in Figs. 1(c) and 1(d), both EPC ($n = 2$) and EPC ($n = 4$) are close to MCS and EPC ($n = 4$) gives an improved PDF. The PDF of velocity has a slightly non-symmetric shape in Fig. 1(d)

3.2 Case 2: Weak nonlinearity + a negative parametric excitation factor

Figure 2 shows the effect of parametric excitation by changing ε from 0.1 to -0.1



(a) PDFs of displacement (b) Logarithmic PDFs of displacement

(c) PDFs of velocity (d) Logarithmic PDFs of velocity

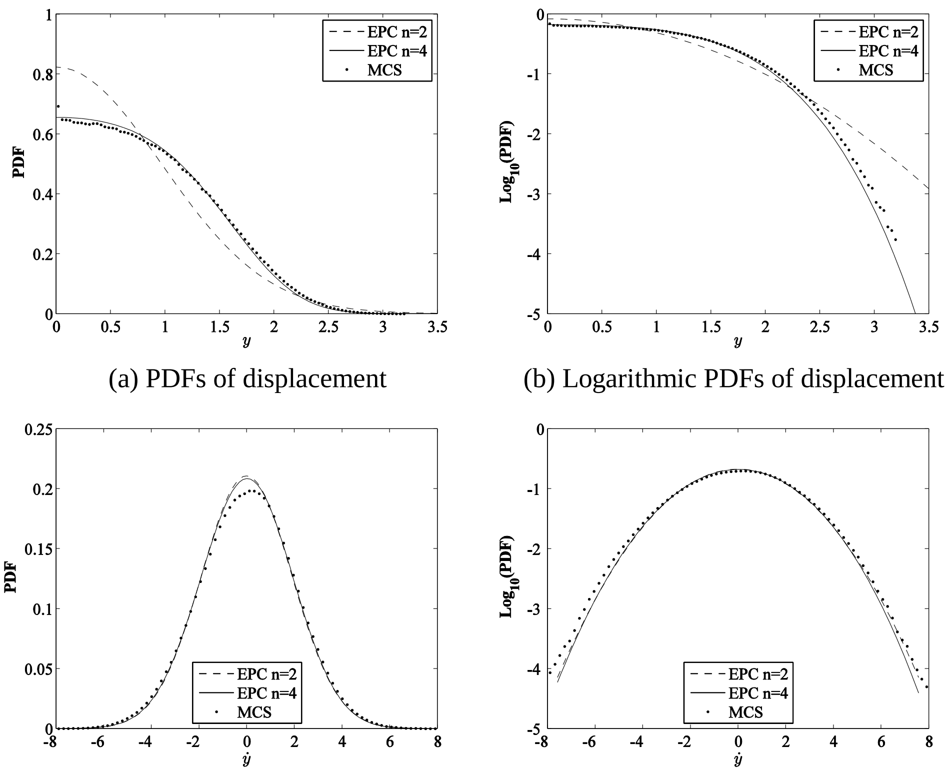
Figure 2: Comparison on the PDFs in Case 2

The similar conclusion to Case 1 can be made. The PDF of displacement differs far from being Gaussian as shown in Figs. 2(a) and 2(b) because EPC ($n = 2$) denotes the result obtained with a Gaussian PDF. EPC ($n = 4$) agrees very well with the simulated result. For the case of velocity in Figs. 2(c) and 2(d), both EPC ($n = 2$) and EPC ($n = 4$) are close to MCS and EPC ($n = 4$) gives an improved PDF.

Compared with Case 1, the PDF distributions of displacement and velocity are significantly affected by the sign change of the parametric excitation factor. Both the maximum PDF and the tail region of displacement are changed as shown in Figs. 1(a), 1(b), 2(a) and 2(b) correspondingly. In the case of velocity, the PDF of velocity in Case 1 is a little non-symmetric and softening in tail region. In contrast, the PDF of velocity in this case shows a hardening behavior in tail region. This difference is formulated due to the fact that the oscillator is excited by different correlated Gaussian white noises.

3.3 Case 3: Strong nonlinearity + a positive parametric excitation factor

In the third case, the nonlinearity in displacement becomes strong by increasing μ from 0.1 to 1.0. The comparison on the obtained PDFs is shown in Fig. 3.



(a) PDFs of displacement

(b) Logarithmic PDFs of displacement

(c) PDFs of velocity

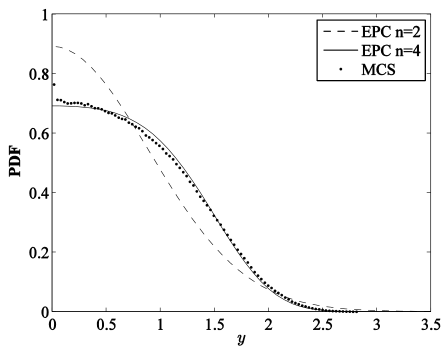
(d) Logarithmic PDFs of velocity

Figure 3: Comparison on the PDFs in Case 3

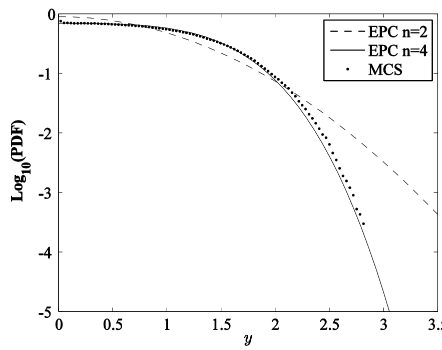
It is well known that the problem of strong nonlinearity has been a challenging topic in stochastic mechanics. As shown in Figs. 3(a) and 3(b), EPC ($n = 2$) differs significantly from MCS, showing that the PDF of displacement is highly non-Gaussian. In such a case, EPC ($n = 4$) can present a satisfactory approximate PDF with the simulated result, even in the tail region as shown in Fig. 3(b). In the case of velocity, Figs. 3(c) and 3(d) show a difference can be observed in the origin region and the tail region. However, both EPC ($n = 2$) and EPC ($n = 4$) are still close to MCS, showing the PDF of velocity is still nearly Gaussian.

3.4 Case 4: Strong nonlinearity + a negative parametric excitation factor

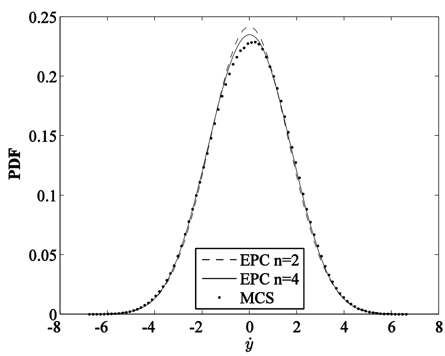
In the last case, the effect of parametric excitation is investigated in the case of strong nonlinearity by changing ε from 0.1 to -0.1. Figure 4 shows the comparison on the PDFs obtained with each method. As shown in Figs. 4(a) and 4(b), EPC ($n = 4$) agrees with the simulated result very well whereas EPC ($n = 2$) differs a lot from MCS. As shown in Figs. 4(c) and 4(d), the PDFs of velocity obtained with each method are close to each other and EPC ($n = 4$) can provide an improved result compared with EPC ($n = 2$).



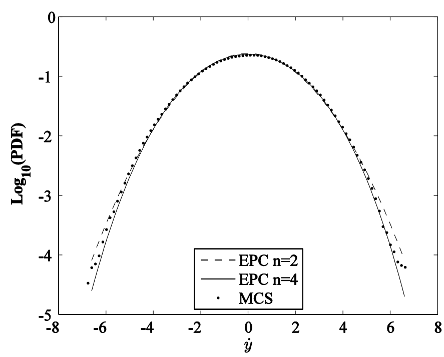
(a) PDFs of displacement



(b) Logarithmic PDFs of displacement



(c) PDFs of velocity



(d) Logarithmic PDFs of velocity

Figure 4: Comparison on the PDFs in Case 4

Comparison with Case 3 shows that the effect of parametric excitation is still significant although strong nonlinearity in displacement exists in the oscillator. The sign change of the parametric excitation factor leads to the significant changes on the PDF distribution of displacement and velocity in between Fig. 3 and Fig. 4. The change is significant in the tail region of the PDF of displacement comparing Fig. 3(b) with Fig. 4(b). The behavior of the tail region of velocity is also changed from being softening to being hardening as shown in Figs. 3(d) and 4(d).

4 Conclusions

This paper proposes a solution procedure to obtain the PDF solution of a vibro-impact Duffing oscillator under fully correlated external and parametric Gaussian white noises. The proposed solution procedure consists of three steps including the Zhuravlev non-smooth coordinate transformation, the EPC method and the methodology on seeking the PDF of a function of random variables. The study further investigates four different cases with different values of the nonlinearity coefficient and parametric excitation factor. Comparison with the simulated result shows that the proposed solution procedure is effective to obtain a satisfactory PDF solution. The tail region of the PDFs is also approximated well. The numerical analysis also shows that the change of parametric excitation has a significant effect on the PDF distributions of displacement and velocity.

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