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Numerical Solution for the Variable Order Time Fractional Diffusion Equation with Bernstein Polynomials

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Abstract: In this paper, Bernstein polynomials method is proposed for the numerical solution of a class of variable order time fractional diffusion equation. Coimbra variable order fractional operator is adopted, as it is the most appropriate and desirable definition for physical modeling. The Coimbra variable order fractional operator can also be regarded as a Caputo-type definition. The main characteristic behind this approach in this paper is that we derive two kinds of operational matrixes of Bernstein polynomials. With the operational matrixes, the equation is transformed into the products of several dependent matrixes which can also be viewed as the system of linear equations after dispersing the variable. By solving the linear equations, the numerical solutions are acquired. Only a small number of Bernstein polynomials are needed to obtain a satisfactory result. Numerical examples are provided to show that the method is computationally efficient.

Keywords: Bernstein polynomials; variable order time fractional diffusion equation; operational matrix; numerical solution.

1 Introduction

In recent years, fractional calculus has attracted many researchers successfully in different disciplines of science and engineering. One of the main advantages of the fractional calculus is that the fractional derivatives provide a superior approach for the description of memory and hereditary properties of various materials and processes [Leda, Kalla and Al-Saqabi (2007)]. Many numerical methods using different kinds of fractional derivative operators for solving fractional differential equations have been proposed. The most commonly used ones are Adomian decomposition method (ADM) [EI-Sayed (1998); EI-Kalla (2011)], Variational iteration method (VIM) [Odibat (2010)], generalized differential transform method (GDTM) [Momani and Odibat (2007); Odibat and Momani(2008); Odibat and Momani(2008)], generalized block pulse operational matrix method [Li and Sun

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(2011)] and wavelet method [Wang (2012); Yi (2012); Wei, Chen, Li and Yi (2012); Zhou, Wang Wang and Liu (2011)] and so on.

Recently, more and more researchers are finding that numerous important dynamical problems exhibit fractional order behavior which may vary with space and time. This fact illustrates that variable order calculus is a natural candidate to provide an effective mathematical framework for the description of complex dynamical problems. The concept of the variable order operator is a much more recent development, which is a new orientation in science. Lorenzo and Hartley [Lorenzo and Hartley (1998); Lorenzo and Hartley (2002)] explored the concept of variable order integration and differentiation more deeply and searched out the relationship between the mathematical definitions and physical processes. Samko and Ross [Samko and Ross (1993); Samko (1995)] directly generalized the Riemann-liouvile and Marchaud fractional integration and differentiation of the case of variable order, and supplied some properties and formulas. Different authors have proposed different concepts of variable order differential operators, each of these has a specific meaning to suit desired goals. The variable order operator definitions recently proposed in the science include the Riemann-Liouvile definition, Caputo definition, Marchaud definition, Coimbra definition and Grünwald definition [Samko and Ross (1993); Lorenzo and Hartley (1998); Coimbra (2003)]. Coimbra [Coimbra (2003)] utilized the Laplace transform of Caputo's fractional derivative definition as the first point to suggest a new definition for the variable order differential operator. With its perfect physical interpretation, Coimbra's definition is more desirable and appropriate for modeling physical problems.

Since the kernel of the variable order operators is too complex for having a variable-exponent, the numerical solutions of variable order fractional differential equations are quite difficult to obtain, and have not attracted much attention. Therefore, the development of numerical techniques to solve variable order fractional differential equations has not taken off. Coimbra [Coimbra (2003)] employed a consistent approximation with first-order accurate for the solution of variable order differential equations. Soon [Soon and Coimbra (2005)] proposed a second-order Runge–Kutta method which is consisting of an explicit Euler predictor step followed by an implicit Euler corrector step to numerically integrate the variable order differential equation. Sun [Sun and Chen (2009)] introduced a classification of the variable-order fractional diffusion models to the diffusion curve of the variable order differential operator model based on the possible physical origins which motivated the variable-order, and developed the Crank–Nicholson scheme. Lin [Lin, Liu, Anh and Turner (2009)] studied the stability and the convergence of an explicit finite-difference approximation for the variable-order fractional diffusion equation with a nonlinear source term. Zhuang [Zhuang, Liu, Anh and Turner (2009)] obtained explicit and implicit Euler approximations for the variable-order fractional advection–diffusion nonlinear equation. Chen [Chen, Sun and Chen (2009)] studied a variable-order anomalous subdiffusion equation and employed two numerical schemes, one with fourth order spatial accuracy and first order temporal accuracy, the other with fourth order spatial accuracy and second order temporal accuracy. However, as far as we know, no one has attempted to seek the numerical solutions of the variable order fractional equations.

So in this paper, we introduce Bernstein polynomials to seek the numerical solution of the variable order time fractional diffusion equation. With the simple structure and perfect properties, Bernstein polynomials play an important role in various areas of engineering and mathematics. Those polynomials have been widely used in solving fractional integral equations and fractional differential equations [Maleknejad (2011); Doha (2011); Mandal (2007); Yousefi (2010); Maleknejad (2011)]. In recent years, various polynomials have been applied to seek the numerical solution of integral differential equations, fractional integral equations and fractional differential equations.

In this paper, by making use of the perfect properties of Bernstein polynomials, we consider the following variable order time fractional diffusion equation (VOTFDE):

$$ {}_0\mathbb{D}_t^{q(x,t)} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) \; (x,t) \in \Omega = [0,1] \times [0,1] \tag{1} $$

with initial and boundary conditions

$$ \begin{aligned} u(x,0) &= g(x), \qquad x \in [0,1] \\ u(0,t) &= h(t), \qquad t \in [0,1] \end{aligned} \tag{2} $$

where $0 < q(x,t) < 1$ and ${}_0\mathbb{D}_t^{q(x,t)} u(x,t)$ denotes the variable order time fractional derivate defined by Coimbra [Coimbra (2003)]:

$$ \begin{aligned} {}_0\mathrm{D}_t^{q(x,t)} u(x,t) = & \frac{1}{\Gamma(1-q(x,t))} \int_0^t (t-\sigma)^{-q(x,t)} \frac{\partial u(x,\sigma)}{\partial \sigma} d\sigma \\ & + \frac{[u(x,0^+) - u(x,0^-)]\, t^{-q(x,t)}}{\Gamma(1-q(x,t))} \end{aligned} \tag{3} $$

The definition (3) is especially useful for the solution of well-posed physical problems. In addition, the differential operator (3) requires only one initial condition. We adopt the Coimbra-definition variable order operator in this work. For the sake of simplicity, assuming $u(x,0^+) = u(x,0^-)$, then the Coimbra definition can be regarded as the following Caputo-type definition [Lorenzo and Hartley (1998)].

$$ {}_0D_t^{q(x,t)} u(x,t) = \frac{1}{\Gamma(1-q(x,t))} \int_0^t (t-\sigma)^{-q(x,t)} \frac{\partial u(x,\sigma)}{\partial \sigma} d\sigma \tag{4} $$

The reminder of the paper is organized as follows: Section 2 is preparative, the basic definitions of Bernstein polynomials, function approximation and convergence analysis in Bernstein polynomials are given in Section 2. In Section 3, two kinds of operational matrixes of Bernstein polynomials are derived and we applied the operational matrixes and the definition of variable order fractional derivative to solve the equation as given at beginning. In Section 4, we present some numerical examples to demonstrate the efficiency of the method. We end the paper with a few concluding remarks in Section 5.

2 The basic definition and properties of Bernstein polynomials

2.1 The definition of Bernstein Polynomials basis [Maleknejad (2011)]

Bernstein Polynomials of degree$n$are defined by

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i} \tag{5}$$

By using the binomial expansion of $(1-x)^{n-i}$, Eq. (5) can be expressed as:

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i} = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i} \binom{n-i}{k} x^{i+k} \tag{6}$$

Now, we define:

$$\Phi(x) = [B_{0,n}(x), B_{1,n}(x), \cdots, B_{n,n}(x)]^T \tag{7}$$

then

$$\Phi(x) = AT_n(x) \tag{8}$$

where

$$A = \begin{bmatrix} (-1)^0 \binom{n}{0} & (-1)^1 \binom{n}{0}\binom{n-0}{1} & \cdots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} \\ 0 & (-1)^0 \binom{n}{1}\binom{n-1}{0} & \cdots & (-1)^{n-1}\binom{n}{1}\binom{n-1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^0\binom{n}{n} \end{bmatrix},$$

$$T_n(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^n \end{bmatrix}$$

$$(9)$$

It is obvious that

$$T_n(x) = A^{-1}\Phi(x) \tag{10}$$

2.2 function approximation

A function $f(x) \in L^2[0,1]$ can be expressed in terms of Bernstein Polynomials basis. In practice, only the first $(n+1)$ terms of Bernstein Polynomials are considered. In the usual way, (f,g) denotes the inner product. Hence

$$f(x) \cong \sum_{i=0}^{n} c_i B_{i,n}(x) = c^T \Phi(x) \tag{11}$$

where $c = [c_0, c_1, \cdots, c_n]^T$

Then we have

$$c = Q^{-1}(f, \Phi(x)) \tag{12}$$

where Q is an $(n+1)\times(n+1)$ matrix, which is called the dual matrix of $\Phi(x)$

$$\begin{aligned} Q &= \int_0^1 \Phi(x)\Phi^T(x)\,dx = \int_0^1 (AT_n(x))(AT_n(x))^T\,dx \\ &= A\left(\int_0^1 T_n(x)T_n^T(x)\,dx\right)A^T = AHA^T \end{aligned} \tag{13}$$

where H is a Hilbert matrix and displays as follows:

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n+1} \end{bmatrix} \tag{14}$$

We can also approximate the function $u(x,t) \in L^2([0,1]\times[0,1])$ as follows:

$$u(x,t) \cong \sum_{i=0}^{n}\sum_{j=0}^{n} u_{i,j} B_{i,n}(x) B_{j,n}(t) = \Phi^T(x)\, U\, \Phi(t) \tag{15}$$

where

$$U = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0n} \\ u_{10} & u_{11} & \cdots & u_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n0} & u_{n1} & \cdots & u_{nn} \end{bmatrix} \tag{16}$$

and U can be obtained as follows:

$$U = Q^{-1}(\Phi(x), (\Phi(t), u(x,t)))\, Q^{-1} \tag{17}$$

2.3 Convergence analysis

Suppose that the function $f:[0,1]\to \mathrm{R}$ is $m+1$ times continuously differentiable, namely $f\in C^{m+1}[0,1]$ and $\mathrm{Y}=Span\{B_{0,n},B_{1,n},B_{2,n}\cdots,B_{n,n}\}$. If $c^T\Phi(x)$ is the best approximation to f from Y, then the mean error bound is presented as follows:

$$\left\|f-c^T\Phi\right\|_2\leq \frac{\sqrt{2}MS^{\frac{2m+3}{2}}}{(m+1)!\sqrt{2m+3}} \tag{18}$$

where $M=\max_{x\in[0,1]}\left|f^{(m+1)}(x)\right|, S=\max\{1-x_0,x_0\}, 0\leq x_0<1$

Proof: We consider the Taylor polynomial

$$f_1(x)=f(x_0)+f'(x_0)(x-x_0)+f''(x_0)\frac{(x-x_0)^2}{2}+\cdots+f^{(m)}(x_0)\frac{(x-x_0)^m}{m!}$$

And which we know

$$|f(x)-f_1(x)|=\left|f^{(m+1)}(\varepsilon)\right|\frac{(x-x_0)^{m+1}}{(m+1)!}\quad \exists\varepsilon\in(0,1)$$

Since $c^T\Phi(x)$ is the best approximation to f out of Y so we have

$$\begin{aligned}\left\|f-c^T\Phi\right\|_2^2 &\leq \|f-f_1\|_2^2=\int_0^1(f(x)-f_1(x))^2dx\\ &=\int_0^1\left(\left|f^{(m+1)}(\varepsilon)\right|\tfrac{(x-x_0)^{m+1}}{(m+1)!}\right)^2dx\\ &\leq \tfrac{M^2}{[(m+1)!]^2}\int_0^1(x-x_0)^{2m+2}dx\\ &\leq \tfrac{2M^2S^{2m+3}}{[(m+1)!]^2(2m+3)}\end{aligned}$$

And by taking the square roots we have the above bound.

3 The operational matrix of Bernstein polynomials

3.1 The operational matrix of derivative of Bernstein polynomials [Yousefi (2010)]

The differentiation of vector $\Phi(x)$ in Eq. (8) can be expressed as:

$$\Phi'(x)=D\Phi(x) \tag{19}$$

where D is the $(n+1)\times(n+1)$ operational matrix of derivatives for Bernstein polynomials. Form Eq. (8) we have

$$\Phi'(x)=A\begin{bmatrix}0\\1\\ \vdots\\ nx^{n-1}\end{bmatrix} \tag{20}$$

Define the $(n+1)\times(n)$ matrix $V_{(n+1)\times n}$ and vector $T_n^*(x)$ as:

$$V_{(n+1)\times n}=\begin{bmatrix}0&0&\cdots&0\\1&0&\cdots&0\\0&2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&n\end{bmatrix},\quad T_n^*(x)=\begin{bmatrix}1\\x\\\vdots\\x^{n-1}\end{bmatrix}_{(n\times 1)} \tag{21}$$

Eq. (20) can be restated as

$$\Phi'(x)=AV_{(n+1)\times n}T_n^*(x) \tag{22}$$

We expand vector $T_n^*(x)$ in terms of $\Phi(x)$. Form Eq. (1), we get

$$T_n^*(x)=B^*\Phi(x) \tag{23}$$

where

$$B^*=\begin{bmatrix}A_{[1]}^{-1}\\A_{[2]}^{-1}\\\vdots\\A_{[n]}^{-1}\end{bmatrix} \tag{24}$$

$A_{[k]}^{-1}$ is the kth row of $A^{-1}k=1,2,\cdots,n$.
Then we have

$$\Phi'(x)=AV_{(n+1)\times n}B^*\Phi(x) \tag{25}$$

Therefore we have the operational matrix of the derivative as:

$$D=AV_{(n+1)\times n}B^* \tag{26}$$

3.1.1 Proposed method for the numerical solution of the variable order time fractional diffusion equation

According to the Caputo-type definition, the variable order fractional formula of the polynomials $x^m t^n$ in Caputo sense is as follows:

$${}_0D_t^{q(x,t)}(x^m t^n)=\begin{cases}\dfrac{\Gamma(n+1)}{\Gamma(n+1-q(x,t))}x^m t^{n-q(x,t)},n=1,2,\cdots\\0,n=0\end{cases} \tag{27}$$

Proof:

$$
\begin{aligned}
D_t^{q(x,t)} x^m t^n = & \frac{x^m n}{\Gamma(1-q(x,t))} \int_0^t (t-\sigma)^{-q(x,t)} \sigma^{n-1} d\sigma \qquad (let \ \ \sigma\text{=ty}) \\
= & \frac{x^m n}{\Gamma(1-q(x,t))} \int_0^1 (t-ty)^{-q(x,t)} (ty)^{n-1} t dy \\
= & \frac{x^m n}{\Gamma(1-q(x,t))} \int_0^1 (t)^{-q(x,t)} (t)^n (1-y)^{-q(x,t)} y^{n-1} dy \\
= & \frac{x^m n t^{n-q(x,t)}}{\Gamma(1-q(x,t))} \int_0^1 (1-y)^{-q(x,t)} y^{n-1} dy \\
= & \frac{x^m n t^{n-q(x,t)}}{\Gamma(1-q(x,t))} B(n, 1-q(x,t)) \\
= & \frac{x^m n t^{n-q(x,t)}}{\Gamma(1-q(x,t))} \frac{\Gamma(n)\Gamma(1-q(x,t))}{\Gamma(1-q(x,t)+n)} \\
= & \frac{\Gamma(n+1)}{\Gamma(n+1-q(x,t))} x^m t^{n-q(x,t)}
\end{aligned}
$$

Especially, when $m=0$, the above formula is displayed as follows:

$$
{}_0D_t^{q(x,t)} t^n = \begin{cases} \dfrac{\Gamma(n+1)}{\Gamma(n+1-q(x,t))} t^{n-q(x,t)}, n=1,2,\cdots \\ 0, n=0 \end{cases} \tag{28}
$$

Now, we consider Eq. (1) and Eq. (2).

$$
\begin{aligned}
& {}_0\mathbb{D}_t^{q(x,t)} u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t)\, (x,t) \in \Omega = [0,1] \times [0,1] \\
& u(x,0) = g(x) \\
& u(0,t) = h(t)
\end{aligned}
$$

Firstly, if we approximate the function $u(x,t)$ with Bernstein polynomials, it can be written as Eq. (15), where U is unknown. With the Eq. (28), we get

$$\begin{aligned}
{}_0\mathrm{D}_t^{q(x,t)}u(x,t) &= {}_0D_t^{q(x,t)}u(x,t) \\
&\simeq {}_0D_t^{q(x,t)}\Phi^T(x)U\Phi(t) \\
&= \Phi^T(x)UA_0D_t^{q(x,t)}T_n(t) \\
&= \Phi^T(x)UA_0D_t^{q(x,t)}\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} \\
&= \Phi^T(x)UA\begin{bmatrix} 0 \\ \frac{\Gamma(2)}{\Gamma(2-q(x,t))}t^{1-q(x,t)} \\ \vdots \\ \frac{\Gamma(n+1)}{\Gamma(n+1-q(x,t))}t^{n-q(x,t)} \end{bmatrix} \\
&= \Phi^T(x)UA\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-q(x,t))}t^{-q(x,t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-q(x,t))}t^{-q(x,t)} \end{bmatrix}\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} \\
&= \Phi^T(x)UAMA^{-1}\Phi(t)
\end{aligned} \tag{29}$$

where

$$M = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-q(x,t))}t^{-q(x,t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-q(x,t))}t^{-q(x,t)} \end{bmatrix} \tag{30}$$

And with the Eq. (25), we get

$$\frac{\partial^2\Phi^T(x)U\Phi(t)}{\partial x} = \Phi^T(x)\left(\left(AV_{(n+1)\times n}B^*\right)^T\right)^2U\Phi(t) \tag{31}$$

Substituting Eq. (29) and Eq. (31) into Eq. (1), we have

$$\Phi^T(x)UAMA^{-1}\Phi(t) = \Phi^T(x)\left(\left(AV_{(n+1)\times n}B^*\right)^T\right)^2U\Phi(t) + f(x,t) \tag{32}$$

Dispersing Eq. (32) with (x_i,t_j), $(i = 1,2,\cdots,n; j = 1,2,\cdots,n)$, by using Mathematica 9.0, we are able to obtain U.

4 Numerical examples

To demonstrate the practicability and efficiency of the proposed method based on Bernstein polynomials, we give the following examples

Example 1: Consider the following variable order time fractional diffusion equation

$$
\begin{aligned}
&{}_0\mathbb{D}_t^{q(x,t)}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),\ \ (x,t)\in\Omega=[0,1]\times[0,1]\\
&u(x,0) = 10x^2(1-x)\\
&u(0,t) = u(1,t) = 0
\end{aligned}
$$

where

$$
\begin{aligned}
&f(x,t) = 20x^2(1-x)\left[\frac{t^{2-q(x,t)}}{\Gamma(3-q(x,t))} + \frac{t^{1-q(x,t)}}{\Gamma(2-q(x,t))}\right] - 20(t+1)^2(1-3x),\\
&q(x,t) = \frac{2+\sin(xt)}{4}.
\end{aligned}
$$

The exact solution of the above equation is $u(x,t) = 10x^2(1-x)(1+t)^2$.

We solve the problem by adopting of the technique described in section 3 and by making use of Mathematica.

Taking $n=2$ dispersing $x_i = \frac{k_i}{3}-\frac{1}{6}$, $t_j = \frac{k_j}{3}-\frac{1}{6}$ $(k_i = 1,2,3; k_j = 1,2,3)$, we can obtain the matrix U as follows:

$$
U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6.896 & 9.255 \\ 0 & 0 & 0 \end{bmatrix}
$$

Taking $n=3$, dispersing $x_i = \frac{k_i}{3}-\frac{1}{6}$, $t_j = \frac{k_j}{3}-\frac{1}{6}$ $(k_i = 1,2,3; k_j = 1,2,3)$, the matrix U is displayed as follows:

$$
U = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{10}{3} & 5.556 & 8.889 & 13.333 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

Taking $n=4$, dispersing $x_i = \frac{k_i}{5}-\frac{1}{10}$, $t_j = \frac{k_j}{5}-\frac{1}{10}$ $(k_i = 1,2,\cdots,5; k_j = 1,2\cdots,5)$, we can obtain the matrix U as follows:

$$
U = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{5}{3} & 2.500 & 3.611 & 5.000 & 6.667 \\ \frac{5}{2} & 3.750 & 5.417 & 7.500 & 10.000 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

We can obtain the absolute errors for $t = 1/4s, t = 1/2s, t = 3/4s$ respectively, as is shown in **Tables 1-3**

Table 1: Absolute error for $t = 1/4s$, and different values of$n$for Example 1.

x	$n = 2$	$n = 3$	$n = 4$
0.1	0.4290	1.1380e-15	3.6082e-16
0.2	0.5127	1.8874e-15	4.4409e-16
0.3	0.3448	2.3315e-15	5.5511e-16
0.4	0.0190	2.4425e-15	6.6613e-16
0.5	0.3708	2.2205e-15	6.6613e-16
0.6	0.7320	1.3323e-15	8.8818e-16
0.7	0.9677	1.3323e-15	8.8818e-16
0.8	0.9873	8.8818e-16	8.8818e-16
0.9	0.6960	2.2204e-16	4.4409e-16

Table 2: Absolute error for $t = 1/2s$, and different values of n for Example 1.

x	$n = 2$	$n = 3$	$n = 4$
0.1	0.8347	1.5266e-15	5.5511e-17
0.2	1.1239	2.6645e-15	
0.3	1.0026	3.9968e-15	6.6613e-16
0.4	0.6058	3.9968e-15	4.4409e-16
0.5	0.0685	4.4409e-15	1.3323e-15
0.6	0.4742	3.5527e-15	1.7767e-15
0.7	0.8874	3.5527e-15	1.7764e-15
0.8	1.0361	3.1086e-15	1.7764e-15
0.9	0.7853	2.4425e-15	8.8818e-16

From **Tables 1-3**, we can see that the absolute error is very small and only a small number of Bernstein polynomials are needed when $n \geq 3$. When $n = 2$, it is not surprising that the absolute error is very big. As it is impossible to get satisfactory results by using the polynomials of 2th degree to approximate the polynomials of 3th

Also when n is definite, the more points we take, the more accurate solution we get. The **figures 1-3** explain the fact. ($n_{x,t}$ is the number of the x_i, t_j)

Table 3: Absolute error for $t = 3/4s$, and different values of n for Example 1.

x	$n=2$	$n=3$	$n=4$
0.1	1.1270	1.2768e-15	4.996e-16
0.2	1.5135	2.5535e-15	4.9909e-16
0.3	1.3434	3.9968e-15	1.3323e-15
0.4	0.8003	4.4409e-15	8.8818e-16
0.5	0.0680	5.3291e-15	2.2205e-15
0.6	0.6697	4.4409e-15	1.7764e-16
0.7	1.2291	5.3291e-15	1.7764e-16
0.8	1.4265	4.8850e-15	4.9909e-16
0.9	1.0780	4.8850e-15	4.9909e-16

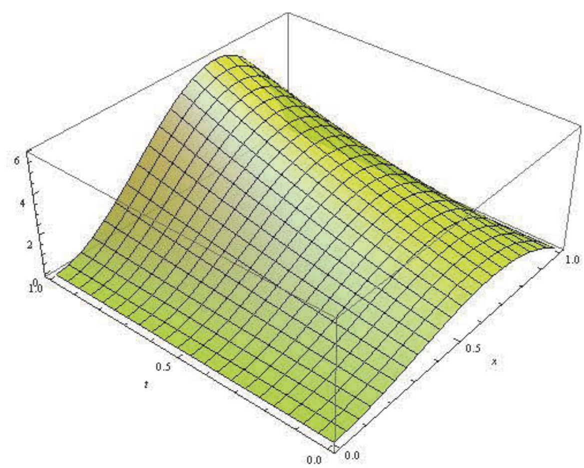

Figure 1: Numerical solution of $n_{x,t} = 3$

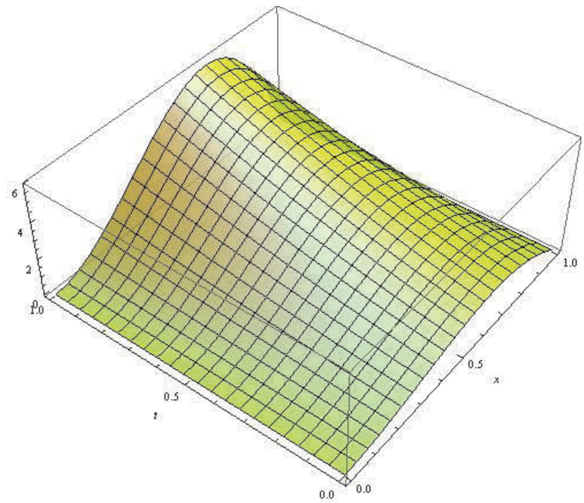

Figure 2: Numerical solution of $n_{x,t} = 6$

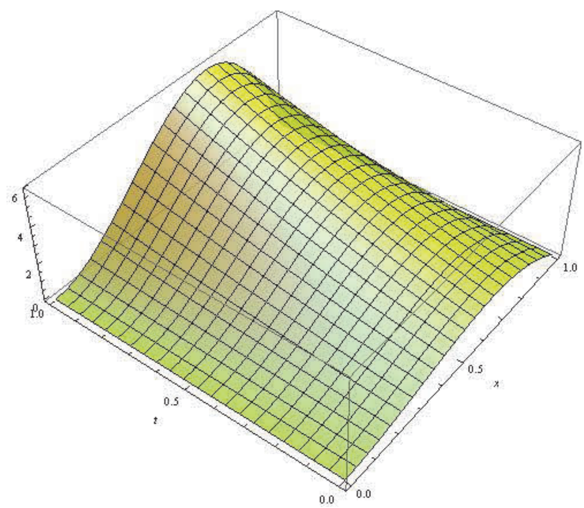

Figure 3: Exact solution for Example 1

Example 2: Consider the below variable order time fractional diffusion equation

$$
\begin{aligned}
&{}_0\mathbb{D}_t^{q(x,t)}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \ \ (x,t) \in \Omega = [0,1]\times[0,1] \\
&u(x,0) = 3x^3+1 \\
&u(0,t) = t^4-3t+1
\end{aligned}
$$

where

$$
\begin{aligned}
&q(x,t) = (2+x/3+t)/5 \\
&f(x,t) = \frac{45t^{\frac{1}{15}(9-3t-x)}\left(1+3x^3\right)\left(-1-\frac{27000t^3}{(-54+3t+x)(-39+3t+x)(-24+3t+x)}\right)}{(9-3t-x)\Gamma\left(1+\frac{1}{5}\left(-2-t-\frac{x}{3}\right)\right)} \\
&\qquad -18\left(1-3t+t^4\right)x
\end{aligned}
$$

The exact solution of the problem is $u(x,t) = \left(3x^3+1\right)\left(t^4-3t+1\right)$

when $n=4$, dispersing $x_i = \frac{k_i}{5}-\frac{1}{10}$, $t_j = \frac{k_j}{5}-\frac{1}{10}$ $(k_i = 1,2,\cdots,5; k_j = 1,2\cdots,5)$, we can get the matrix U as follows:

$$
U = \begin{bmatrix}
1 & 0.25 & -0.5 & -1.25 & -1 \\
1 & 0.25 & -0.5 & -1.25 & -1 \\
1 & 0.25 & -0.5 & -1.25 & -1 \\
1.75 & 0.4375 & -0.875 & -2.1875 & -1.75 \\
4 & 1 & -2 & -5 & -4
\end{bmatrix}
$$

The absolute error is displayed as follows:

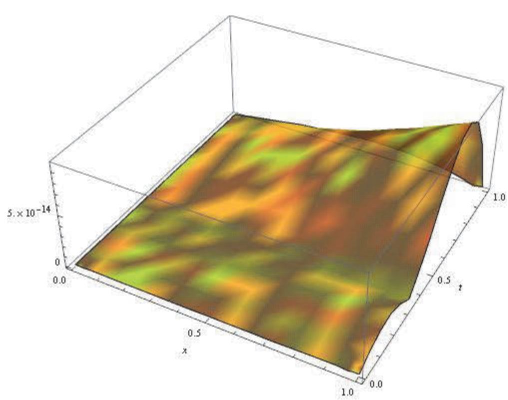

Figure 4: The absolute error for Example 2 when $n=4$

when $n=5$ dispersing $x_i=\frac{k_i}{6}-\frac{1}{12}$, $t_j=\frac{k_j}{6}-\frac{1}{12}$ $(k_i=1,2,\cdots,6;k_j=1,2\cdots,6)$ the matrix U is displayed as follows:

$$U=\begin{bmatrix} 1.00 & 0.40 & -0.20 & -0.8 & -1.20 & -1.00 \\ 1.00 & 0.40 & -0.20 & -0.8 & -1.20 & -1.00 \\ 1.00 & 0.40 & -0.20 & -0.8 & -1.20 & -1.00 \\ 1.30 & 0.52 & -0.26 & -1.04 & -1.56 & -1.30 \\ 2.20 & 0.88 & -0.44 & -1.76 & -2.64 & -2.20 \\ 4.00 & 1.60 & -0.80 & -3.20 & -4.80 & -4.00 \end{bmatrix}$$

The absolute error is displayed as follows:

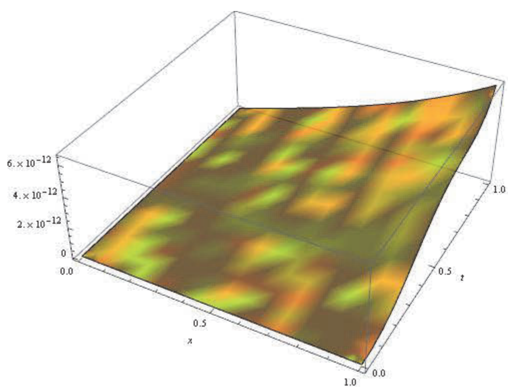

Figure 5: The absolute error for Example 2 when $n=5$

Example 3: consider this variable order time fractional diffusion equation

$$
\begin{aligned}
&{}_0\mathbb{D}_t^{q(x,t)}u(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t),\ \ (x,t)\in\Omega=[0,1]\times[0,1]\\
&u(x,0)=10x^2(1-x)\\
&u(0,t)=u(1,t)=0
\end{aligned}
$$

where

$$q(x,t)=(2+x/3+t)/5$$

$$f(x,t)=t^2\cos x+\frac{450t^{\frac{1}{15}(24-3t-x)}\cos x}{(-24+3t+x)(-9+3t+x)\Gamma\left(1+\frac{1}{5}\left(-2-t-\frac{x}{3}\right)\right)}$$

The exact solution of the above equation is $u(x,t)=t^2\cos x$

It is obvious that the exact solution is no longer the form of polynomials. However, we can still solve the problem by adopting of the technique described in section 3. With the increasing of$n$ the absolute error becomes more and more smaller. The figures below confirm the truth of our point.

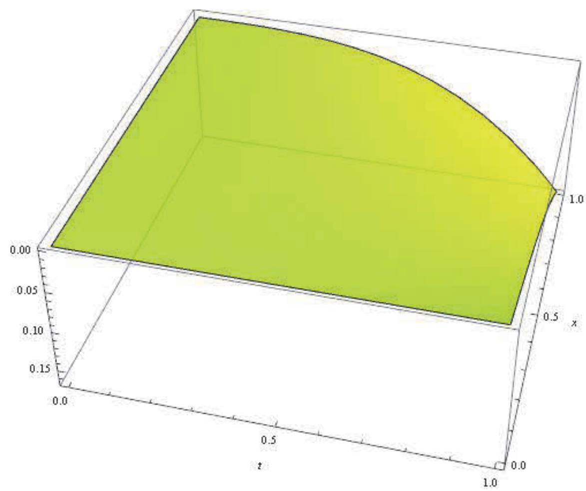

Figure 6: The absolute error for Example 3 when $n = 5$

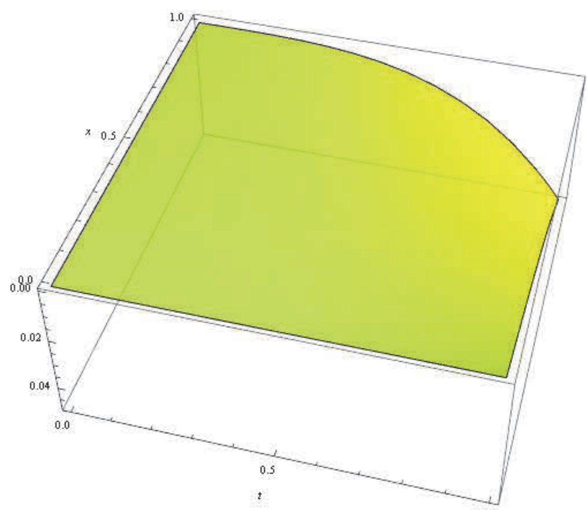

Figure 7: The absolute error for Example 3 when $n = 7$

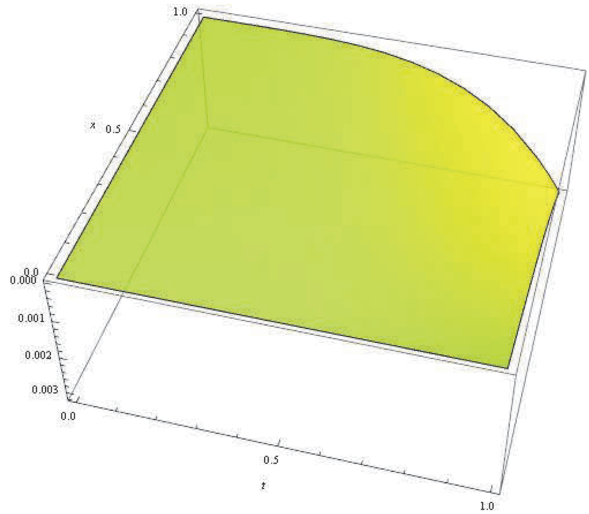

Figure 8: The absolute error for Example 3 when $n = 10$

From **Example1-3**, we can find that the method in this article can be effectively used in the numerical solution of the variable order fractional equation. At the same time the feasibility of the method is also proved. From the above results, the numerical solutions are in good agreement with the exact solution. We draw a conclusion that the method is not only effective to get the numerical solution with good coincidence, but also more convenient in computation than that in [Lin, Liu, Anh and Turner (2009)].

5 Conclusion

In this paper, we present a numerical method for the variable order time fractional diffusion equation through combining some properties of Bernstein polynomials and the variable order fractional derivatives effectively. Taking advantage of the definition of the variable order fractional derivatives and the simplicity of Bernstein polynomials, we transform the fractional diffusion equation into the product of some dependent matrixes which can be viewed as the system of linear equations after dispersing the variable and it can be solved easily by the least square method. Numerical examples show that the presented method is accurate and much convenient for solving the variable order fractional equation.

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