

## How to Select the Value of the Convergence Parameter in the Adomian Decomposition Method

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**Abstract:** In this paper, we investigate the problem of selecting of the convergence parameter  $c$  in the Adomian decomposition method. Through the curves of the  $n$ -term approximations  $\phi_n(t; c)$  versus  $c$  for different specified values of  $n$  and  $t$ , we demonstrate how to determine the value of  $c$  such that the decomposition series has a larger effective region of convergence.

**Keywords:** Nonlinear differential equation, Solution continuation, Adomian decomposition method, Adomian polynomials.

### 1 Introduction

With the development of computer technology and computational software, analytic approximate solutions with high accuracy for nonlinear problems become feasible by the analytic approximate methods such as the Adomian decomposition method (ADM) [Adomian (1986, 1989, 1994)], collocation method [Dai, Schnoor, and Atluri (2012)], variational iteration method [Wazwaz (2009)] and perturbation method [Hinch (1991)], etc.

The ADM [Adomian and Rach (1983); Adomian (1983, 1986, 1989, 1994); Wazwaz (2009, 2011); Serrano (2011); Lai, Chen, and Hsu (2008); Duan, Rach, Baleanu, and Wazwaz (2012)] is a practical technique for solving nonlinear functional equations, including ordinary differential equations (ODEs), partial differential equations (PDEs), integral equations, integro-differential equations, etc. The ADM provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering without unphysical restrictive assumptions such as required by linearization and perturbation. It is convenient for implementation in computational software [Duan, Rach, and Wazwaz (2013)] and new algorithms for the Adomian polynomials increase its

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computing efficiency [Duan (2011)]. The accuracy of the obtained analytic approximate solutions can be verified by direct substitution into the original equation [Fu, Wang, and Duan (2013); Duan, Rach, Wazwaz, Chaolu, and Wang (2013)].

First, we demonstrate the procedure of the ADM by solving a first-order nonlinear differential equation

$$\frac{d}{dt}u(t) + \alpha(t)u(t) + f(t, u(t)) = g(t), \quad u(t_0) = C_0, \quad (1)$$

where the functions  $\alpha, g$  and  $f$  are analytic.

We rewrite Eq. (1) in Adomian's operator-theoretic form

$$Lu = g - Ru - Nu, \quad (2)$$

where  $L = \frac{d}{dt}(\cdot)$ ,  $Ru = \alpha(t)u(t)$  and  $Nu = f(t, u(t))$ .

Applying the integral operator  $L^{-1}(\cdot) = \int_{t_0}^t (\cdot) dt$  to both sides of Eq. (2) yields

$$u(t) = \Phi + L^{-1}g - L^{-1}[Ru + Nu], \quad (3)$$

where  $\Phi = u(t_0) = C_0$  for the case of a first-order ODE.

In the ADM, the solution  $u(t)$  is represented by the Adomian decomposition series

$$u(t) = \sum_{n=0}^{\infty} u_n(t) \quad (4)$$

and the nonlinearity comprises the Adomian polynomials

$$Nu = \sum_{n=0}^{\infty} A_n(t), \quad (5)$$

where the Adomian polynomials  $A_n(t)$  [Adomian and Rach (1983)] are defined as

$$A_n(t) = A_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} f(t, \sum_{k=0}^{\infty} \lambda^k u_k(t)) \Big|_{\lambda=0}. \quad (6)$$

Various algorithms for the Adomian polynomials have been developed by Rach (1984, 2008), Wazwaz (2000), Abdelwahid (2003) and several others [Abbaoui, Cherruault, and Seng (1995); Zhu, Chang, and Wu (2005); Biazar, Ilie, and Khoshkenar (2006); Azreg-Aïnou (2009)]. Recently, new algorithms and subroutines written in MATHEMATICA for fast generation of the Adomian polynomials to high orders have been developed by Duan (2010a,b, 2011) and Duan and Guo (2010).

Substituting Eqs. (4) and (5) into Eq. (3) yields

$$\sum_{n=0}^{\infty} u_n(t) = \Phi + L^{-1}g - L^{-1}[R \sum_{n=0}^{\infty} u_n(t) + \sum_{n=0}^{\infty} A_n(t)]. \quad (7)$$

From Eq. (7), the solution components are determined by the Adomian recursion scheme

$$u_0(t) = \Phi + L^{-1}g, \quad (8)$$

$$u_{n+1} = -L^{-1}[Ru_n(t) + A_n(t)], \quad n \geq 0, \quad (9)$$

where the  $n$ -term approximation is given as  $\phi_n(t) = \sum_{k=0}^{n-1} u_k(t)$ .

We remark that the convergence of the Adomian decomposition series has previously been proven by several investigators [Abbaoui and Cherruault (1994, 1995); Abdelrazec and Pelinovsky (2011); Rach (2008)]. For example, Abdelrazec and Pelinovsky (2011) have published a rigorous proof of convergence for the ADM under the aegis of the Cauchy-Kovalevskaya theorem. In point of fact, the Adomian decomposition series is found to be a computationally advantageous rearrangement of the Banach-space analog of the Taylor expansion series about the initial solution component function.

We remark that the domain of convergence for the Adomian decomposition series may not always be sufficiently large for engineering purposes. In order to cope with such occurrences some authors have applied one of several well-know convergence acceleration techniques such as the diagonal Padé approximants [Adomian (1994); Wazwaz (2009)] or the iterated Shanks transform [Adomian (1994); Duan, Chaolu, Rach, and Lu (2013)].

Wazwaz (1999) and Wazwaz and El-Sayed (2001) have proposed different modified decomposition methods, where the sum  $\Phi + L^{-1}g$  was partitioned or decomposed and then its components were distributed to subsequent solution components in order to suppress the occasional phenomenon of noisy convergence as well as to facilitate the calculation of integrals.

In [Duan (2010a); Duan, Rach, and Wang (2013)], the parametrized recursion scheme, which embeds a convergence parameter  $c$  into the recursion scheme, was proposed in order to obtain decomposition solutions with larger effective regions of convergence.

By introducing a parameter  $c$  with a specified decomposition  $c = \sum_{n=1}^{\infty} c_n$  into the recursion scheme (8) and (9), we deduce the parametrized recursion scheme

$$u_0(t) = \Phi + L^{-1}g - c, \quad (10)$$

$$u_{n+1} = c_{n+1} - L^{-1}[Ru_n(t) + A_n(t)], \quad n \geq 0. \quad (11)$$

Similarly, we can introduce the convergence parameter  $c$  into the modified recursion schemes [Wazwaz (1999); Wazwaz and El-Sayed (2001)].

For sake of discussion, we list two specific decompositions of the convergence parameter  $c$  and their corresponding parametrized recursion schemes as

$$c = \sum_{n=1}^{\infty} c_n, \quad c_n = \frac{c}{2^n}, \quad (12)$$

$$u_0(t) = \Phi + L^{-1}g - c, \quad (13)$$

$$u_{n+1}(t) = \frac{c}{2^{n+1}} - L^{-1}Ru_n(t) - L^{-1}A_n(t), \quad n \geq 0, \quad (14)$$

and

$$c = \sum_{n=1}^{\infty} c_n, \quad c_n = (1-c)c^n, \quad |c| < 1, \quad (15)$$

$$u_0(t) = \Phi + L^{-1}g - c, \quad (16)$$

$$u_{n+1}(t) = (1-c)c^{n+1} - L^{-1}Ru_n(t) - L^{-1}A_n(t), \quad n \geq 0, \quad (17)$$

where the  $n$ -term approximation as parametrized by  $c$  is thus  $\phi_n(t; c) = \sum_{k=0}^{n-1} u_k(t)$ .

In this paper, we shall present a method to determine the value of the convergence parameter  $c$  through examination of the curves of  $\phi_n(t; c)$  versus  $c$  for different values of  $n$  and  $t$  such that the parametrized decomposition series has a larger effective region of convergence.

## 2 Illustration of the proposed method

For a specified  $t$ ,  $\phi_n(t; c)$  denotes the analytic approximations of the solution  $u(t)$  such that the curves of  $\phi_n(t; c)$  versus  $c$  become horizontal over the effective field of  $c$ . By virtue of this property, we can efficiently determine the value of the convergence parameter  $c$  such that the decomposition series has a larger effective region of convergence. We illustrate the effectiveness of our method through the following four examples.

**Example 1.** Consider the Riccati equation

$$\frac{du}{dt} + u^2 = 0, \quad u(0) = 1, \quad (18)$$

where the exact solution is  $u^*(t) = \frac{1}{t+1}$  for  $t > -1$ .

Applying the integral operator  $L^{-1} = \int_0^t (\cdot) dt$  to both sides of Eq. (18) yields  $u = 1 - L^{-1}u^2$ .

Next, we decompose the solution  $u = \sum_{n=0}^{\infty} u_n$  and the nonlinearity  $f(u) = u^2 = \sum_{n=0}^{\infty} A_n$ , where the Adomian polynomials  $A_n$  are

$$A_0 = u_0^2, A_1 = 2u_0u_1, A_2 = 2u_0u_2 + u_1^2, \dots, A_n = \sum_{k=0}^n u_k u_{n-k}. \tag{19}$$

Applying the parametrized recursion scheme (12)–(14)

$$u_0 = 1 - c, u_n = \frac{c}{2^n} - L^{-1}A_{n-1}, n \geq 1, \tag{20}$$

we calculate the parametrized solution components as

$$\begin{aligned} u_1 &= \frac{c}{2} - (1 - c)^2 t, \\ u_2 &= \frac{c}{4} + (-c + c^2) t + (1 - 3c + 3c^2 - c^3) t^2, \\ u_3 &= \frac{c}{8} + \left(-\frac{c}{2} + \frac{c^2}{4}\right) t + \left(\frac{3c}{2} - 3c^2 + \frac{3c^3}{2}\right) t^2 + (-1 + 4c - 6c^2 + 4c^3 - c^4) t^3, \\ &\dots, \end{aligned}$$

where the  $n$ -term parametrized approximation is  $\phi_n(t; c) = \sum_{k=0}^{n-1} u_k(t; c)$ .

In Figs. 1(a)–1(d), we plot the curves of  $\phi_{10}(t, c), \phi_{15}(t, c), \phi_{20}(t, c)$  versus  $c$  for  $t = 1, t = 1.5, t = 2$  and  $t = 2.5$ , respectively. We observe in each subfigure that there is an interval, where the horizontal segments overlap one another, and the corresponding fields of  $c$  contract as the values of  $t$  increase. Thus we observe that  $\phi_n(t; c)$  converges for a larger field of  $t$  when  $c$  equals about 0.6.

In Fig. 2, we plot the exact solution  $u^*(t)$  and the 16-term approximations  $\phi_{16}(t; c)$  for  $c = 0.4, 0.6, 0.8$ . We observe by comparing the three values of  $c$  that the decomposition series for  $c = 0.6$  has the largest effective region of convergence.

We note that if the convergence parameter  $c$  was not introduced, which corresponds to the case of  $c = 0$ , then the  $n$ -term approximation would be the Maclaurin polynomials of the exact solution, i.e.  $\phi_n(t) = 1 - t + t^2 - \dots + (-t)^{n-1}$ , which converges only on the interval  $-1 < t < 1$ .

Next, we apply the parametrized recursion scheme (15)–(17) to this example,

$$u_0 = 1 - c, u_n = (1 - c)c^n - L^{-1}A_{n-1}, n \geq 1, \tag{21}$$

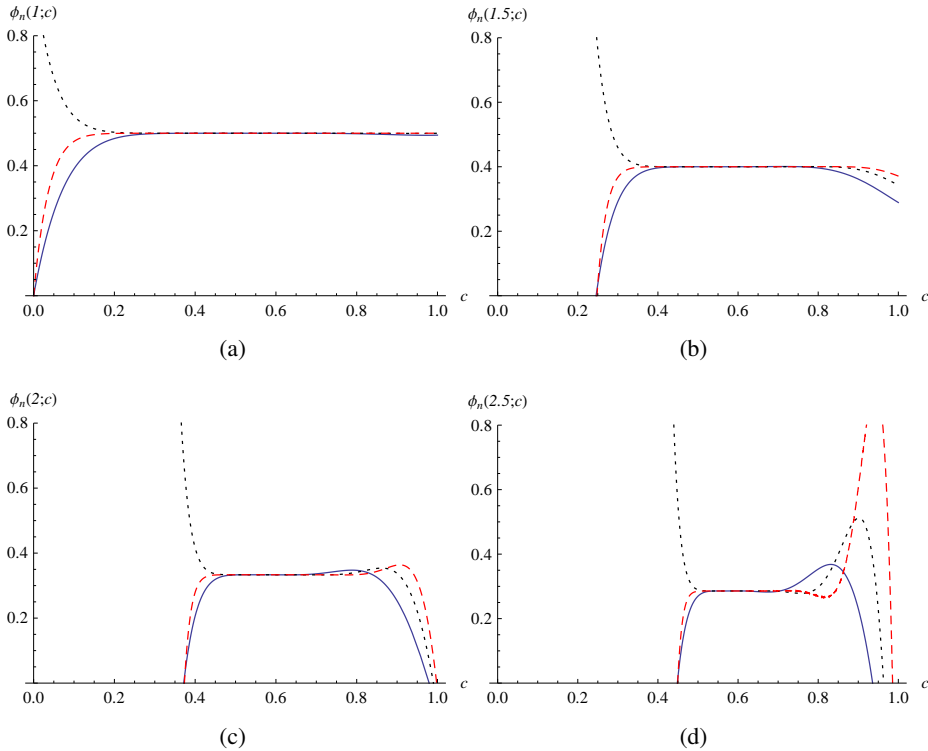


Figure 1: Curves of  $\phi_n(t; c)$  versus  $c$  for (a)  $t = 1$ , (b)  $t = 1.5$ , (c)  $t = 2$ , (d)  $t = 2.5$  and for  $n = 10$  (solid line),  $n = 15$  (dot line),  $n = 20$  (dash line).

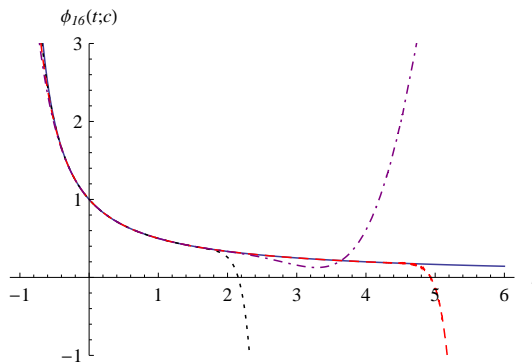


Figure 2: The exact solution  $u^*(t)$  (solid line) and the 16-term approximations  $\phi_{16}(t; 0.4)$  (dot line),  $\phi_{16}(t; 0.6)$  (dash line) and  $\phi_{16}(t; 0.8)$  (dot-dash line).

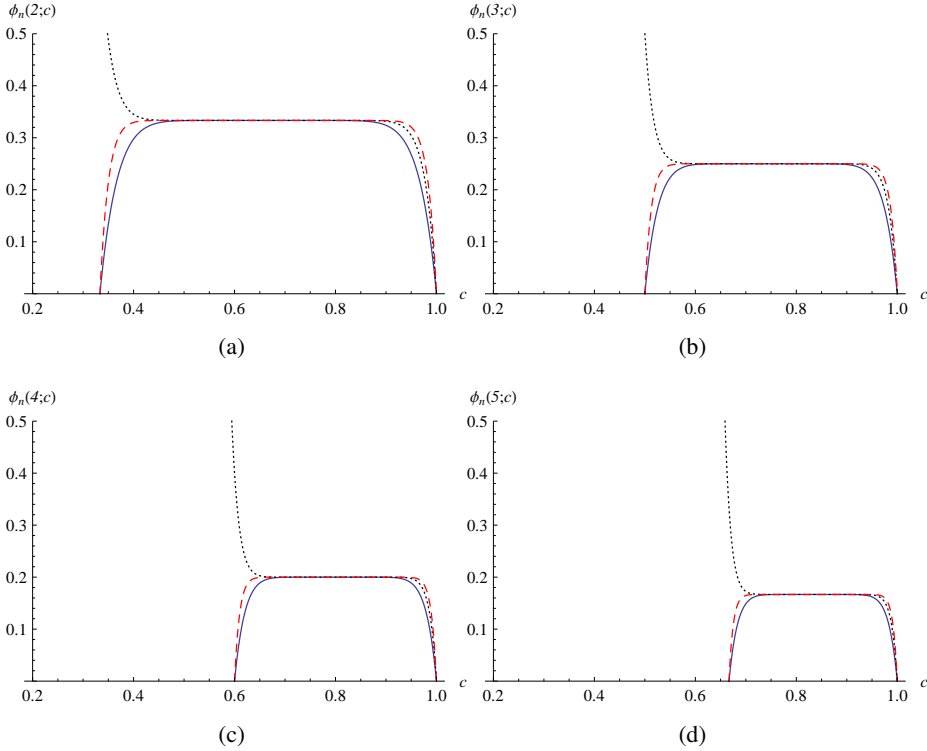


Figure 3: Curves of  $\phi_n(t; c)$  versus  $c$  for (a)  $t = 2$ , (b)  $t = 3$ , (c)  $t = 4$ , (d)  $t = 5$  and for  $n = 10$  (solid line),  $n = 15$  (dot line),  $n = 20$  (dash line).

where  $|c| < 1$ . We calculate the parametrized solution components as

$$\begin{aligned}
 u_1 &= (1 - c)(c - t + ct), \\
 u_2 &= (1 - c)(c - t + ct)^2, \\
 u_3 &= (1 - c)(c - t + ct)^3, \\
 u_4 &= (1 - c)(c - t + ct)^4, \\
 &\dots,
 \end{aligned}$$

where the  $n$ -term parametrized approximation is  $\phi_n(t; c) = \sum_{k=0}^{n-1} u_k(t; c)$ . In Figs. 3(a)–3(d), we plot the curves of  $\phi_{10}(t, c)$ ,  $\phi_{15}(t, c)$ ,  $\phi_{20}(t, c)$  versus  $c$  for  $t = 2$ ,  $t = 3$ ,  $t = 4$  and  $t = 5$ , respectively. As  $t$  increases, the horizontal segments contract to a neighborhood of  $c = 0.85$ .

In Fig. 4, we plot the exact solution  $u^*(t)$  and the 18-term approximations  $\phi_{18}(t; c)$  for  $c = 0.4, 0.6, 0.8$ . We observe for the three values of  $c$  that the decomposition

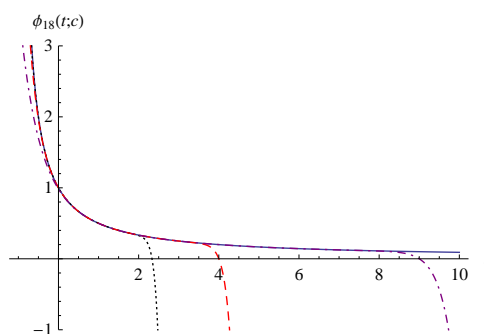


Figure 4: The exact solution  $u^*(t)$  (solid line) and the 18-term approximations  $\phi_{18}(t; 0.4)$  (dot line),  $\phi_{18}(t; 0.6)$  (dash line) and  $\phi_{18}(t; 0.8)$  (dot-dash line).

series for  $c = 0.8$  has the largest effective region of convergence.

In fact, the  $n$ -term approximation is

$$\phi_n(t; c) = \sum_{k=0}^{n-1} (1-c)(c-t+ct)^k, \quad (22)$$

which converges for  $t$  such that  $|c-t+ct| < 1$ , i.e. the interval  $-1 < t < \frac{1+c}{1-c}$ .

**Example 2.** Consider the nonlinear differential equation with a negative power nonlinearity

$$\frac{du}{dt} - \frac{1}{2u} = 0, \quad u(0) = 1, \quad (23)$$

where the exact solution is  $u^*(t) = \sqrt{1+t}$ .

By the ADM, we rewrite Eq. (23) as

$$u = 1 + \frac{1}{2}L^{-1}\frac{1}{u},$$

where  $L^{-1} = \int_0^t (\cdot) dt$ . The first several Adomian polynomials for the nonlinearity



$f(u) = 1/u$  are

$$\begin{aligned}
 A_0 &= \frac{1}{u_0}, \\
 A_1 &= -\frac{u_1}{u_0^2}, \\
 A_2 &= \frac{u_1^2}{u_0^3} - \frac{u_2}{u_0^2}, \\
 A_3 &= -\frac{u_1^3}{u_0^4} + \frac{2u_1u_2}{u_0^3} - \frac{u_3}{u_0^2}, \\
 A_4 &= \frac{u_1^4}{u_0^5} - \frac{3u_1^2u_2}{u_0^4} + \frac{u_2^2}{u_0^3} + \frac{2u_1u_3}{u_0^3} - \frac{u_4}{u_0^2}, \\
 &\dots
 \end{aligned}$$

By the parametrized recursion scheme (12)–(14), the solution components  $u_n$  are determined as

$$u_0 = 1 - c, \quad u_n = \frac{c}{2^n} + \frac{1}{2}L^{-1}A_{n-1}, \quad n \geq 1. \tag{24}$$

Then, we calculate the parametrized solution components in succession as

$$\begin{aligned}
 u_1 &= \frac{c}{2} - \frac{t}{2(-1+c)}, \\
 u_2 &= \frac{c}{4} - \frac{ct}{4(-1+c)^2} + \frac{t^2}{8(-1+c)^3}, \\
 u_3 &= \frac{c}{8} - \frac{c(-1+2c)t}{8(-1+c)^3} + \frac{3ct^2}{16(-1+c)^4} - \frac{t^3}{16(-1+c)^5}, \\
 &\dots
 \end{aligned}$$

In Figs. 5(a)–5(d), we plot the curves of  $\phi_{11}(t, c)$ ,  $\phi_{13}(t, c)$ ,  $\phi_{15}(t, c)$  versus  $c$  for  $t = 1, t = 5, t = 10$  and  $t = 15$ , respectively. The plots show as the parameter  $c$  decreases from 0 to  $-6$  that the effective region of convergence of the decomposition series gradually increases.

In Fig. 6, we plot the exact solution  $u^*(t)$  and the 15-term approximations  $\phi_{15}(t; c)$  for  $c = 0, -2, -4$ , where the gradual expansion of effective regions of convergence is obvious.

We note that if the parameter  $c$  was not introduced, i.e. the case of  $c = 0$ , then the  $n$ -term approximation  $\phi_n(t)$  would be the Maclaurin polynomials of the exact solution  $u^*(t)$ , which converges only on the interval  $-1 \leq t \leq 1$ .

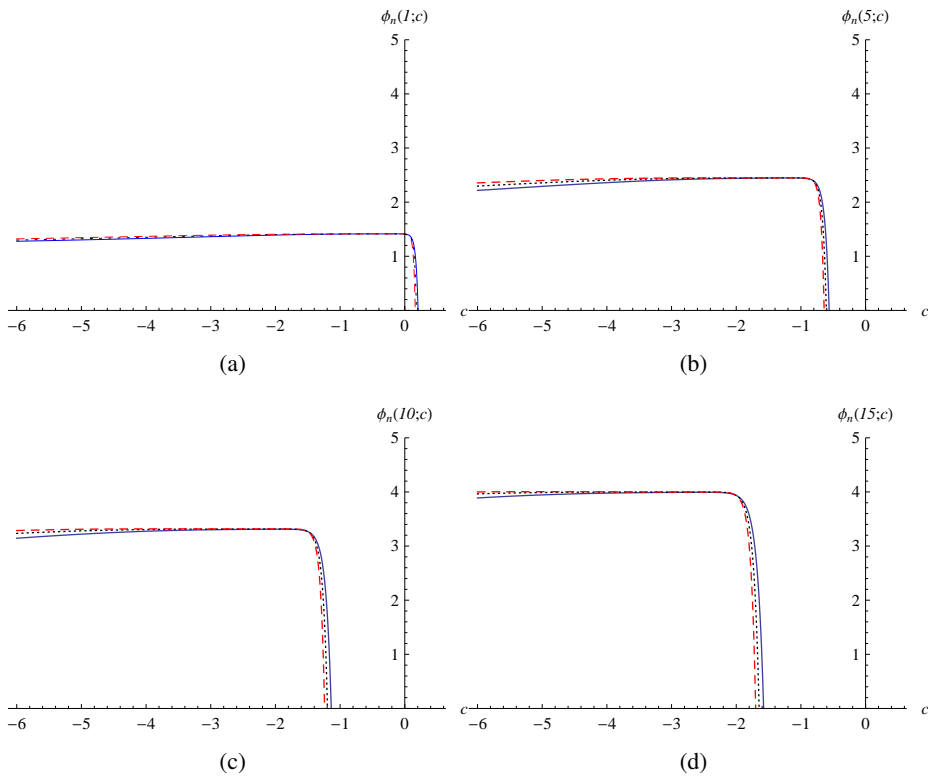


Figure 5: Curves of  $\phi_n(t; c)$  versus  $c$  for (a)  $t = 1$ , (b)  $t = 5$ , (c)  $t = 10$ , (d)  $t = 15$  and for  $n = 11$  (solid line),  $n = 13$  (dot line),  $n = 15$  (dash line).

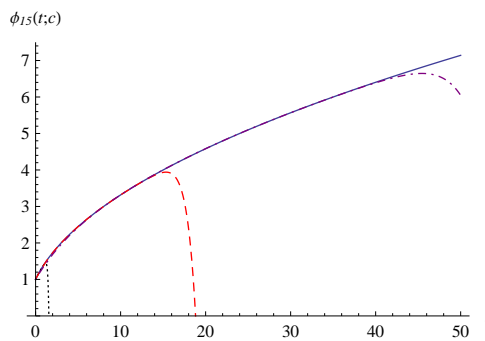


Figure 6: The exact solution  $u^*(t)$  (solid line) and the 15-term approximations  $\phi_{15}(t; 0)$  (dot line),  $\phi_{15}(t; -2)$  (dash line) and  $\phi_{15}(t; -4)$  (dot-dash line).

We have checked by using the specific decomposition (15) of the parameter  $c$  that the decomposition series for  $c = -0.35$  has a larger effective region of convergence than that for  $c = 0$ .

**Example 3.** Consider the Lane-Emden equation

$$\frac{d^2u}{dt^2} + \frac{2}{t} \frac{du}{dt} + u^5 = 0, \quad u(0) = 1, u'(0) = 0, \tag{25}$$

where the exact solution is  $u^*(t) = (1 + \frac{t^2}{3})^{-1/2}$ . Let

$$L = t^{-2} \frac{d}{dt} (t^2 \frac{d}{dt} (.)), \tag{26}$$

then Eq. (25) becomes

$$Lu = -u^5. \tag{27}$$

Applying the inverse operator  $L^{-1} = \int_0^t t^{-2} \int_0^t t^2 (. ) dt dt$  to both sides of Eq. (27) yields

$$u = 1 - L^{-1}u^5. \tag{28}$$

The first several Adomian polynomials for the quintic nonlinearity  $f(u) = u^5$  are

$$\begin{aligned} A_0 &= u_0^5, \\ A_1 &= 5u_0^4u_1, \\ A_2 &= 10u_0^3u_1^2 + 5u_0^4u_2, \\ A_3 &= 10u_0^2u_1^3 + 20u_0^3u_1u_2 + 5u_0^4u_3, \\ A_4 &= 5u_0u_1^4 + 30u_0^2u_1^2u_2 + 10u_0^3u_2^2 + 20u_0^3u_1u_3 + 5u_0^4u_4, \\ &\dots \end{aligned}$$

The components of the solution  $u = \sum_{n=0}^{\infty} u_n$  are determined by the parametrized recursion scheme (12)–(14)

$$u_0 = 1 - c, \quad u_n = \frac{c}{2^n} - L^{-1}A_{n-1}, \quad n \geq 1. \tag{29}$$

We obtain the parametrized solution components as

$$\begin{aligned} u_1 &= \frac{c}{2} + \frac{1}{6}(-1 + c)^5t^2, \\ u_2 &= \frac{c}{4} - \frac{5}{12}(-1 + c)^4ct^2 - \frac{1}{24}(-1 + c)^9t^4, \\ u_3 &= \frac{c}{8} + \frac{5}{24}(-1 + c)^3c(1 + c)t^2 + \frac{3}{16}(-1 + c)^8ct^4 + \frac{5}{432}(-1 + c)^{13}t^6, \\ &\dots, \end{aligned}$$

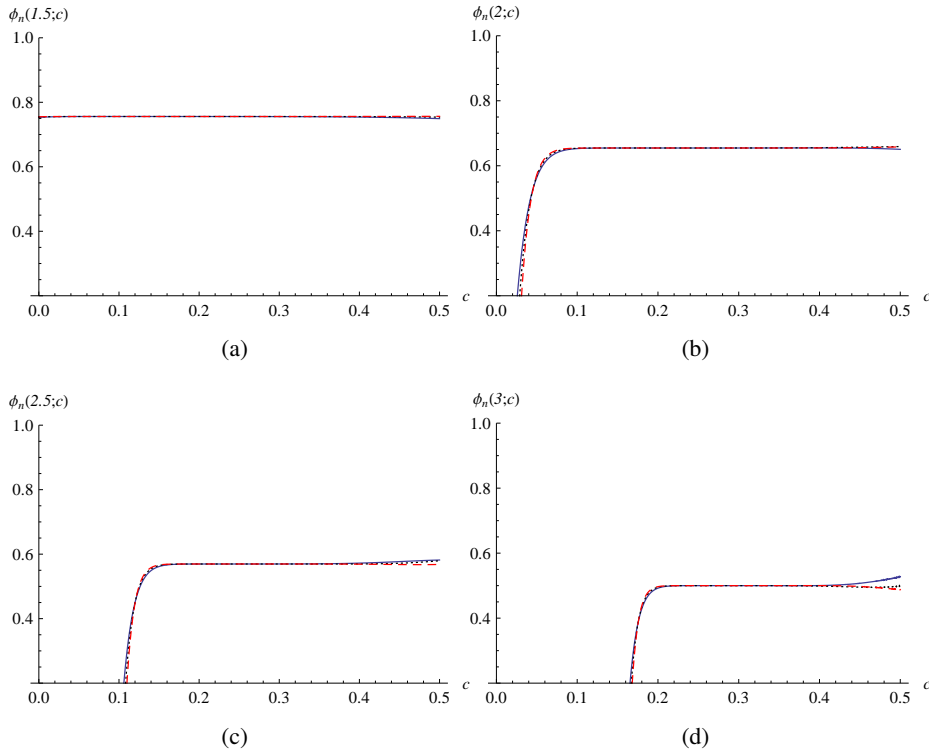


Figure 7: Curves of  $\phi_n(t; c)$  versus  $c$  for (a)  $t = 1.5$ , (b)  $t = 2$ , (c)  $t = 2.5$ , (d)  $t = 3$  and for  $n = 12$  (solid line),  $n = 14$  (dot line),  $n = 16$  (dash line).

where the  $n$ -term parametrized approximation is  $\phi_n(t; c) = \sum_{k=0}^{n-1} u_k(t; c)$ .

In Figs. 7(a)–7(d), we plot the curves of  $\phi_{12}(t, c)$ ,  $\phi_{14}(t, c)$ ,  $\phi_{16}(t, c)$  versus  $c$  for  $t = 1.5$ ,  $t = 2$ ,  $t = 2.5$  and  $t = 3$ , respectively. We observe in each of the subfigures that there is an interval, where the horizontal segments overlap one another, and the corresponding fields of  $c$  contract as the values of  $t$  increase. Thus we observe that  $\phi_n(t; c)$  converges for a larger field of  $t$  when  $c$  equals about 0.3.

In Fig. 8, we plot the exact solution  $u^*(t)$  and the 16-term approximations  $\phi_{16}(t; c)$  for  $c = 0, 0.15, 0.3$ . We observe for the three values of  $c$  that the series solution has the largest effective region of convergence for  $c = 0.3$ .

We note that if the parameter  $c$  was not introduced, i.e. the case of  $c = 0$ , then the  $n$ -term approximation  $\phi_n(t)$  would be the Maclaurin polynomials of the exact solution  $u^*(t)$ , and converge only on the interval  $-\sqrt{3} \leq t \leq \sqrt{3}$ .

We have checked by using the specific decomposition (15) of the parameter  $c$  that

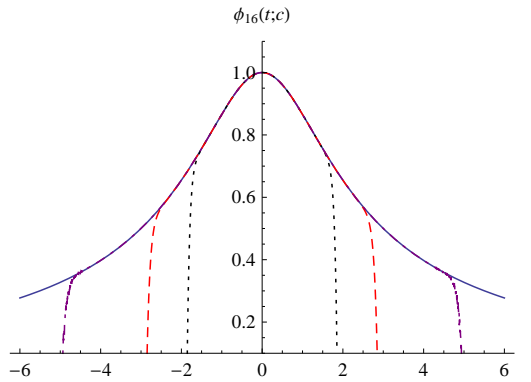


Figure 8: The exact solution  $u^*(t)$  (solid line) and the 16-term approximations  $\phi_{16}(t;0)$  (dot line),  $\phi_{16}(t;0.15)$  (dash line) and  $\phi_{16}(t;0.3)$  (dot-dash line).

the decomposition series for  $c = 0.2$  has a larger effective region of convergence than that for  $c = 0$ .

**Example 4.** Consider the nonlinear differential equation with a logarithmic non-linearity

$$\frac{du}{dt} = t + \frac{1}{t} \ln(u), \quad u(1) = 2, \tag{30}$$

where this initial value problem does not have an exact analytic solution.

Applying the integral operator  $L^{-1} = \int_1^t (\cdot) dt$  to both sides of Eq. (30) yields

$$u = 2 + \frac{1}{2}(t^2 - 1) + L^{-1} \frac{1}{t} \ln(u). \tag{31}$$

We compute the components of the solution  $u = \sum_{n=0}^{\infty} u_n$  by the parametrized modified recursion scheme as

$$u_0 = 2 - c, \quad u_1 = \frac{c}{2} + \frac{1}{2}(t^2 - 1) + L^{-1} \frac{1}{t} A_0, \quad u_n = \frac{c}{2^n} + L^{-1} \frac{1}{t} A_{n-1}, \quad n \geq 2, \tag{32}$$

where the first several Adomian polynomials  $A_n$  for the logarithmic nonlinearity

$f(u) = \ln(u)$  are

$$\begin{aligned} A_0 &= \ln(u_0), \\ A_1 &= \frac{u_1}{u_0}, \\ A_2 &= -\frac{u_1^2 - 2u_0u_2}{2u_0^2}, \\ A_3 &= \frac{u_1^3 - 3u_0u_1u_2 + 3u_0^2u_3}{3u_0^3}, \\ A_4 &= -\frac{u_1^4 - 4u_0u_1^2u_2 + 2u_0^2(u_2^2 + 2u_1u_3) - 4u_0^3u_4}{4u_0^4}, \\ &\dots \end{aligned}$$

The parametrized solution components are computed as

$$\begin{aligned} u_1 &= \frac{c}{2} + \frac{1}{2}(-1 + t^2) + \ln(2 - c)\ln(t), \\ u_2 &= \frac{c}{4} - \frac{-1 + t^2 + 2\ln(t)(-1 + c + \ln(2 - c)\ln(t))}{4(-2 + c)}, \\ &\dots, \end{aligned}$$

where the  $n$ -term parametrized approximation is  $\phi_n(t; c) = \sum_{k=0}^{n-1} u_k(t; c)$ .

In Figs. 9(a)–9(d), we plot the curves of  $\phi_8(t, c)$ ,  $\phi_{10}(t, c)$ ,  $\phi_{12}(t, c)$  versus  $c$  for  $t = 1.5$ ,  $t = 2$ ,  $t = 2.5$  and  $t = 3$ , respectively. The curves display that the effective region of convergence of  $\phi_n(t; c)$  gradually increases as  $c$  decreases from 0 to  $-6$ .

In Fig. 10, we plot the MATHEMATICA numeric solution  $u^*(t)$ , and the 12-term approximations  $\phi_{12}(t; c)$  for  $c = 0$ ,  $-3$  and  $-6$ . We observe for the three values of  $c$  that the decomposition series for  $c = -6$  has a larger effective region of convergence than that for  $c = 0$ .

We have checked by using the decomposition (15) of the parameter  $c$  that the decomposition series for  $c = -0.7$  has a larger effective region of convergence than that for  $c = 0$ .

In summary, by introducing the convergence parameter  $c$  and its specified decomposition, we adjust each term of the decomposition series. For some field of  $c$ , the decomposition series has a larger effective region of convergence than for other fields of  $c$ . Through the curves of  $\phi_n(t; c)$  versus  $c$  for different values of  $n$  and  $t$ , we can determine the optimal field of  $c$ .

From our investigation, we observe that the decomposition of the convergence parameter  $c$  is nonunique. Here we have demonstrated two simple decompositions

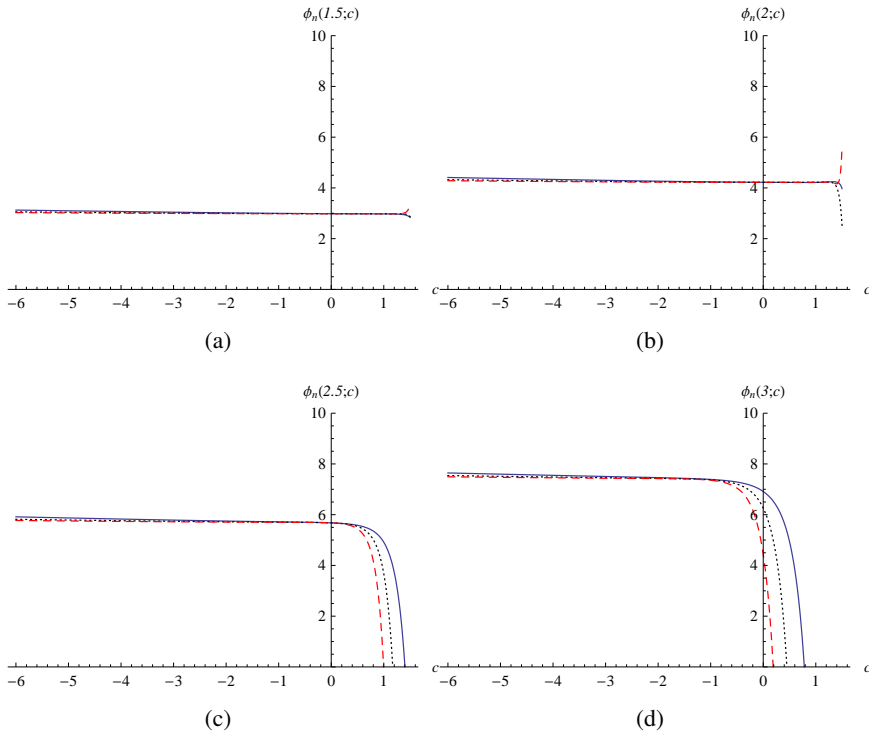


Figure 9: Curves of  $\phi_n(t; c)$  versus  $c$  for (a)  $t = 1.5$ , (b)  $t = 2$ , (c)  $t = 2.5$ , (d)  $t = 3$  and for  $n = 8$  (solid line),  $n = 10$  (dot line),  $n = 12$  (dash line).

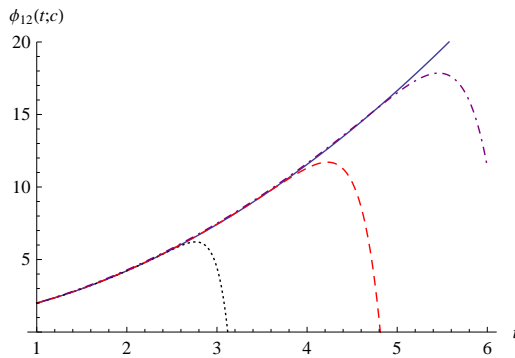


Figure 10: The MATHEMATICA numerical solution  $u^*(t)$  (solid line) and the 12-term approximations  $\phi_{12}(t; 0)$  (dot line),  $\phi_{12}(t; -3)$  (dash line) and  $\phi_{12}(t; -6)$  (dot-dash line).

of the parameter  $c$  in Eqs. (12) and (15) and their effects on the solutions by the Adomian decomposition series.

### 3 Conclusions

In this paper, we have presented a new method to optimize the value of the convergence parameter  $c$  in the ADM. Our proposed method examines the curves of  $\phi_n(t; c)$  versus  $c$  for different values of  $n$  and  $t$ . If the curves become horizontal in some region of  $c$ , then that region corresponds to an effective field of  $c$ . We can contract the effective field of  $c$  by increasing the values of the argument  $t$ . We can choose  $c$  through this method and thus expand the effective region of convergence of the Adomian decomposition series for solutions of nonlinear differential equations. We have investigated four examples of nonlinear differential equations to demonstrate how to practically expand the region of convergence for the solution by the Adomian decomposition series of nonlinear ODEs.

**Acknowledgement:** This work was supported by the Innovation Program of the Shanghai Municipal Education Commission (14ZZ161).

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