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Bäcklund Transformations: a Link Between Diffusion Models and Hydrodynamic Equations

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Abstract: This work presents a new analytical method to transform exact solutions of linear diffusion equations into exact ones for nonlinear advection-diffusion models. The proposed formulation, based on Bäcklund transformations, is employed to obtain velocity fields for the unsteady two-dimensional Helmholtz equation, starting from analytical solutions of a heat conduction type model.

Keywords: exact solutions, nonlinear partial differential equations, Navier-Stokes equation, Helmholtz equation, Bäcklund transformation, linearization without approximations, diffusion model in pressure field, vector field plots, symbolical packages, low time processing.

1 Introduction

The Bäcklund transformations allow finding exact solutions of nonlinear PDEs by solving auxiliary linear ones. Although this application fully justify the relevance of all methods based on Cole-Hopf, Darboux and Bäcklund transformations [Zwillinger (1997); Polyanin and Zaitsev (2004)], there are some underlying principles behind these procedures, which seems to be even more important.

When one defines material derivatives in order to account for advection terms in transport equations, it is implicitly assumed that there exists a path followed by each molecule of the fluid along time, which is described by parametric equations. This point of view often induces to choose some specific variables as candidates for solutions to a given problem. It occurs that some of these choices eventually generates nonlinearities which otherwise would not necessarily appear in alternative formulations.

For instance, the Helmholtz equation can be viewed as an advection diffusion model where the solid interfaces acts as sources of vorticity. However, If the kinetic energy is chosen as the unknown variable instead of the vorticity function,

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the interfaces would be considered as sinks, so the physical interpretation of the corresponding scenario would be essentially analogous. Nevertheless, from the operational point of view, the last interpretation is advantageous, because advection terms are not expected to arise in a hydrodynamic model based on kinetic energy. Consequently, the resulting equation should be a linear model whose solutions could be mapped into ones of the original problem by applying nonlinear operators.

The only practical limitation of this approach is that it ever produces only particular solutions of the original problem. However, this is not a serious limitation, once the subspace of solutions can be easily generalized using symmetries admitted by the own target equation.

This work shows that Bäcklund-type transformations are more than mapping procedures. Behind these transformations arises a systematic method to obtain new dependent variables, which furnishes a useful point of view for simplifying the way of reasoning about modeling and solving nonlinear problems. In what follows it will be showed that exact solutions of the Helmholtz equation can be obtained by factorization and mapping into a linear diffusion model, whose auxiliary dependent variable represents a function of the kinetic energy.

2 Bäcklund transformations for the Helmholtz equation

The unsteady two-dimensional Helmholtz equation is given by

$$\frac{\partial \omega}{\partial x} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial x^2} \right) \tag{1}$$

where ω is the vorticity, u and v are the components of the velocity vector and ν is the kinematic viscosity. This equation can be factorized into the following system of first order PDEs:

$$\frac{\partial v}{\partial t} + u\omega = \nu \frac{\partial \omega}{\partial x} + \frac{\partial q}{\partial y} \tag{2}$$

$$-\frac{\partial u}{\partial t} + v\omega = \nu \frac{\partial \omega}{\partial y} - \frac{\partial q}{\partial x} \tag{3}$$

where q(x,y,t) is an unknown function. Indeed, differentiating (2) respect to x and (3) respect to y, adding the resulting equations and rearranging terms it yields

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)\omega = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial x^2} \right) \tag{4}$$

After applying the continuity equation for incompressible flows the derivatives in the last term of the left hand side cancel, so equation (1) is obtained by recognizing that the first term in (4) is the time derivative of the vorticity function. Naturally, this is equivalent to apply the curl over the Navier-Stokes equations, provided that the derivatives of the arbitrary function q, which absorbs the pressure field, belongs to the null space of the divergent operator. Differentiating now equation (3) respect to x and (2) respect to y, subtracting the results and cancelling the cross derivatives, a differential constraint is achieved:

$$-\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\omega+v\frac{\partial \omega}{\partial x}-u\frac{\partial \omega}{\partial y}=-\frac{\partial^2 q}{\partial x^2}-\frac{\partial^2 q}{\partial x^2} \tag{5}$$

This constraint will be employed to specify q(x,y,t). The first term is null due to the continuity equation and the second is promptly recognized as ω^2. Therefore, rewriting (5) in terms of the stream function, it becomes possible to express q(x,y,t) as a function of this dependent variable:

$$\left(\nabla^2\Psi\right)^2-\frac{\partial\Psi}{\partial \mathrm{x}}\frac{\partial\omega}{\partial \mathrm{x}}-\frac{\partial\Psi}{\partial \mathrm{y}}\frac{\partial\omega}{\partial \mathrm{y}}=-\frac{\partial^2 \mathrm{q}}{\partial \mathrm{x}^2}-\frac{\partial^2 \mathrm{q}}{\partial \mathrm{x}^2} \tag{6}$$

Notice that (6) can be written as

$$\left(\nabla^2\Psi\right)^2+\nabla\Psi.\nabla\left(\nabla^2\Psi\right)=-\nabla^2\mathrm{q} \tag{7}$$

The left hand side of (7) is the divergence of a product, so equation (7) reduces to

$$\nabla.\left(\nabla^2\Psi\nabla\Psi\right)=-\nabla^2\mathrm{q} \tag{8}$$

Once the expression between brackets is identified as the gradient of the kinetic energy per unit mass, the left hand side is recognized as the laplacian of this new dependent variable. Thus, equation (8) becomes

$$\frac{1}{2}\nabla^2\left(\nabla\Psi.\nabla\Psi\right)=-\nabla^2\mathrm{q} \tag{9}$$

From this result the definition of q(x,y,t) is readily obtained:

$$\mathrm{q}=-\frac{1}{2}\nabla\Psi.\nabla\Psi+\mathrm{h} \tag{10}$$

In this equation h(x,y,t) is any harmonic function, e.g., an arbitrary solution of the two dimensional Laplace equation. In this case, the general solution of the Laplace equation is given by

$$h=\mathrm{a}\left(\mathrm{x}+\mathrm{i}\ \mathrm{y},\mathrm{t}\right)+\mathrm{b}\left(\mathrm{x}-\mathrm{i}\ \mathrm{y},\mathrm{t}\right) \tag{11}$$

Here a and b denote arbitrary functions. Replacing (10) in (2) and (3) it results

$$\frac{\partial v}{\partial t}+u\omega=\nu\frac{\partial \omega}{\partial x}-u\frac{\partial u}{\partial y}-v\frac{\partial v}{\partial y}+\frac{\partial h}{\partial y} \tag{12}$$

and

$$-\frac{\partial u}{\partial t}+u\omega=\nu\frac{\partial \omega}{\partial y}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}-\frac{\partial h}{\partial x} \tag{13}$$

Substituting the definition of the vorticity function it yields

$$\frac{\partial v}{\partial t}+u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\nu\nabla^2 v-u\frac{\partial u}{\partial y}-v\frac{\partial v}{\partial y}+\frac{\partial h}{\partial y} \tag{14}$$

$$-\frac{\partial u}{\partial t}+u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=-\nu\nabla^2 u+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}-\frac{\partial h}{\partial x} \tag{15}$$

Regrouping terms it becomes possible to identify the pressure field in the Navier-Stokes equations:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\nabla^2 u-\frac{\partial h}{\partial x} \tag{16}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\nu\nabla^2 v-\frac{\partial h}{\partial y} \tag{17}$$

erefore, prescribing the differential constraint (5) is equivalent to state that the pressure is a harmonic function. In what follows it will be showed that when the harmonic function is neglected in equation (10), a diffusion model for the kinetic energy per unit mass is obtained. This auxiliary model allows finding exact solutions to the Helmholtz equations which reproduce the main features of viscous flows.

3 Diffusion model for the kinetic energy

Equation (6) can be recast in terms of the kinetic energy per mass unit, defined as

$$\mathrm{f}=-\frac{1}{2}\nabla\Psi.\nabla\Psi=-\frac{1}{2}\left(u^2+v^2\right) \tag{18}$$

Isolating the first derivatives of the vorticity function from (2) and (3), namely

$$\frac{\partial \omega}{\partial x}=\frac{1}{\nu}\left[\frac{\partial v}{\partial t}+u\omega-\frac{\partial\left(f+h\right)}{\partial y}\right] \tag{19}$$

and

$$\frac{\partial \omega}{\partial y} = \frac{1}{\nu}\left[-\frac{\partial u}{\partial t} + v\omega + \frac{\partial (f+h)}{\partial x}\right] \tag{20}$$

and substituting these expressions in equation (6) it yields

$$\left(\nabla^2\Psi\right)^2 + \frac{v}{\nu}\left[\frac{\partial v}{\partial t} + u\omega - \frac{\partial (f+h)}{\partial y}\right] - \frac{u}{\nu}\left[-\frac{\partial u}{\partial t} + v\omega + \frac{\partial (f+h)}{\partial x}\right] = -\nabla^2\,(\mathrm{f}+\mathrm{h}) \tag{21}$$

The former result is obtained after replacing the first derivatives of the stream function by the corresponding components of the velocity vector. Cancelling terms, neglecting the harmonic function and multiplying by ν it yields

$$\nu\left(\nabla^2\Psi\right)^2 + v\frac{\partial v}{\partial t} + u\frac{\partial u}{\partial t} - v\frac{\partial f}{\partial y} - u\frac{\partial f}{\partial x} = -\nu\nabla^2\mathrm{f} \tag{22}$$

The time derivatives can be written in terms of f, since

$$v\frac{\partial v}{\partial t} + u\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\frac{u^2+v^2}{2}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\nabla\Psi.\nabla\Psi\right) = -\frac{\partial f}{\partial t} \tag{23}$$

The remaining first order terms in (22) cancel each other. Once

$$v\frac{\partial f}{\partial y} + u\frac{\partial f}{\partial x} = \vec{V}.\nabla f = \vec{V}.\nabla\left(\frac{1}{2}\nabla\Psi.\nabla\Psi\right) = \vec{V}.\nabla\Psi\nabla^2\Psi \tag{24}$$

and

$$\vec{V}.\nabla\Psi = v\frac{\partial \Psi}{\partial y} + u\frac{\partial \Psi}{\partial x} = -\frac{\partial \Psi}{\partial x}\frac{\partial \Psi}{\partial y} + \frac{\partial \Psi}{\partial y}\frac{\partial \Psi}{\partial x} = 0 \tag{25}$$

there are no advection terms in the auxiliary model, a result which was yet expected. Hence, equation (22) becomes

$$\nu\left(\nabla^2\Psi\right)^2 - \frac{\partial f}{\partial t} = -\nu\nabla^2\mathrm{f} \tag{26}$$

Here the nonlinear term is the square of the vorticity function. Thus, equation (26) can be regarded as an inhomogeneous diffusion model with a "source" of vorticity:

$$\frac{\partial f}{\partial t} = \nu\nabla^2\mathrm{f} + \nu\omega^2. \tag{27}$$

This equation can be converted into a linear model because the vorticity function may be expressed in terms of the new dependent variable. In order to carry out this mapping, it is important to observe that

$$\nabla \mathrm{f} = \nabla\left(-\frac{1}{2}\nabla\Psi.\nabla\Psi\right) = -\nabla\Psi\nabla^2\Psi = \omega\nabla\Psi \tag{28}$$

Therefore, it becomes possible to define the vorticity as a nonlinear operator applied over f. By performing a dot product by the gradient of the stream function, it yields

$$\nabla \mathrm{f}.\nabla\Psi = \omega\nabla\Psi.\nabla\Psi = 2\omega f$$

Solving for the vorticity, it results

$$\omega = \frac{1}{2\mathrm{f}}\nabla \mathrm{f}.\nabla\Psi \tag{29}$$

Therefore, the square of the vorticity function is defined as

$$\omega^2 = \omega.\omega = \frac{1}{4\mathrm{f}^2}\left(\nabla \mathrm{f}\,.\nabla \mathrm{f}\right)\left(\nabla\Psi.\nabla\Psi\right) = \frac{1}{2\mathrm{f}}\nabla \mathrm{f}.\nabla \mathrm{f}\,. \tag{30}$$

Hence, equation (27) becomes

$$\frac{\partial f}{\partial t} = \nu\left(\nabla^2 \mathrm{f} + \frac{1}{2\mathrm{f}}\nabla \mathrm{f}\,.\nabla \mathrm{f}\right) \tag{31}$$

The nonlinear term in (31) is obtained when the laplacian operator is applied over a function of the dependent variable:

$$\nabla^2 g(\mathrm{f}) = \nabla.\left[g'(f)\nabla \mathrm{f}\right] = g''(\mathrm{f})\nabla \mathrm{f}\cdot\nabla \mathrm{f} + g'(\mathrm{f})\nabla^2 \mathrm{f} \tag{32}$$

Thus, equation (31) can be written as a purely diffusion model in the form

$$\frac{\partial \mathrm{g}(\mathrm{f})}{\partial t} = \nu\nabla^2 \mathrm{g}(\mathrm{f}) \tag{33}$$

Applying the chain rule, it results

$$g'(\mathrm{f})\frac{\partial f}{\partial t} = \nu\left[g''(\mathrm{f})\nabla \mathrm{f}.\nabla \mathrm{f} + g'(\mathrm{f})\nabla^2 \mathrm{f}\right] \tag{34}$$

Dividing by g'(f) it yields

$$\frac{\partial f}{\partial t} = \nu\left(\nabla^2 \mathrm{f} + \frac{\mathrm{g}''(\mathrm{f})}{\mathrm{g}'}\right)\nabla \mathrm{f}\,.\nabla \mathrm{f} \tag{35}$$

A direct comparison with (31) furnishes an ordinary differential equation from which g(f) is defined:

$$\frac{g''(\mathrm{f})}{g'(\mathrm{f})} = \frac{1}{2\mathrm{f}} \tag{36}$$

Solving for g(f), it results

$$g = c_0 + c_1 \mathrm{f}^{\frac{3}{2}} \tag{37}$$

The inverse function is given by

$$f = \left(\frac{g - c_0}{c_1}\right)^{\frac{2}{3}} \tag{38}$$

Once f=0 at the boundaries (Γ), due to the classical no slip and no penetration conditions (u=v=0 at Γ equation (33) is accompanied by a first kind boundary condition in g, which prescribes the value g=c0/c1 at Γ.

The function defined by equation (38) determines the change of variable to be employed to transform any solution of the unsteady diffusion equation (33) into a solution of (31). Thus, once obtained an exact solution g of the diffusion equation, f can be immediately obtained from (38), and the first order linear equation defined by (28) still must be solved in order to find the stream function:

$$\nabla\Psi = \frac{\nabla f}{\omega} \tag{39}$$

In this equation the vorticity is obtained from (30):

$$\omega = \pm\sqrt{\frac{1}{2\mathrm{f}}\nabla \mathrm{f}\,.\nabla \mathrm{f}} \tag{40}$$

Therefore, the stream function can be obtained by direct integration respect to the spatial variables. It is also possible to avoid the integration by finding the velocity field from equation (39). Once $\nabla\psi$ = (-v,u), the components of the velocity vector are explicitly defined after finding f:

$$u = \frac{\frac{\partial f}{\partial y}}{\sqrt{\frac{1}{2\mathrm{f}}\nabla \mathrm{f}.\nabla \mathrm{f}}} \tag{41}$$

Hence, it becomes possible to plot the velocity field instead of integrating (41) and (42) to obtain the stream function.

$$v = \frac{-\frac{\partial f}{\partial x}}{\sqrt{\frac{1}{2\mathrm{f}}\nabla \mathrm{f}.\nabla \mathrm{f}}} \tag{42}$$

4 Solving the diffusion model

The auxiliary model can be readily solved by standard techniques. A suitable solution for problems in infinite media, such as flow around obstacles, is easily obtained via integral transforms. For instance, applying the Fourier transform in x over equation (33) it yields

$$\frac{\partial h}{\partial t} = -\nu \mathrm{s}^2 \mathrm{h} + \nu \frac{\partial^2 \mathrm{h}}{\partial y^2} \tag{43}$$

In this equation, h denotes the Fourier transform of g respect to x, and s the corresponding independent variable in frequency domain. Applying now the Fourier transform in y, we obtain

$$\frac{\partial m}{\partial t} = -\nu \left(\mathrm{s}^2 + \mathrm{r}^2\right) \tag{44}$$

In this equation m is the double Fourier transform of g. Equation (44) can be solved by direct integration. Indeed, dividing both sides by m and multiplying by dt, it results

$$\frac{\partial m}{m} = -\nu \left(\mathrm{s}^2 + \mathrm{r}^2\right) \partial t \tag{45}$$

Once each infinitesimal contribution in both sides of equation (45) are equal, the corresponding sums over m and t are also equivalent. Hence, equation (44) can be treated as an ordinary separable one. Therefore, integrating the left hand side in m and the right hand side in t, the following implicit solution in frequency domain is obtained:

$$\ln(m) = \ln \mathrm{c}(\mathrm{r},\mathrm{s}) - \nu \left(\mathrm{s}^2 + \mathrm{r}^2\right) t \tag{46}$$

In this equation c(r,s) denotes an arbitrary function of its arguments, which belongs to the null space of the time derivative. Isolating m in (46) it results

$$m = \mathrm{c}(\mathrm{r},\mathrm{s})\, \mathrm{e}^{-\nu\left(s^2+r^2\right)t} \tag{47}$$

Applying now the inverse Fourier transform in r, we recover h(s,y):

$$\mathrm{h} = \mathrm{b}(\mathrm{s},\mathrm{y}) *_y \mathrm{F}_r^{-1} \mathrm{e}^{-\nu\left(s^2+r^2\right)t} \tag{48}$$

Here b(s,y) is an arbitrary function, *y denotes convolution respect to y and F_r^{-1} stands for the inverse Fourier transform in r. The former result can be recast as

$$\mathrm{h} = \mathrm{e}^{-\nu s^2 t} \mathrm{b}(\mathrm{s},\mathrm{y}) *_y \mathrm{F}_r^{-1} \mathrm{e}^{-\nu r^2 t} \tag{49}$$

Finally, applying the inverse Fourier transform in s, an explicit solution for g is obtained

$$\mathrm{g} = \mathrm{F}_s^{-1}\mathrm{e}^{-\nu s^2 t} *_x \mathrm{a}(\mathrm{x},\mathrm{y}) *_y \mathrm{F}_r^{-1}\mathrm{e}^{-\nu r^2 t} \tag{50}$$

Since the inverse transforms are known, namely

$$\mathrm{F}_s^{-1} e^{-\nu s^2 t} = \frac{e^{\frac{-x^2}{4\nu t}}}{\sqrt{4\pi\nu t}} \tag{51}$$

and

$$\mathrm{F}_r^{-1} e^{-\nu r^2 t} = \frac{e^{\frac{-y^2}{4\nu t}}}{\sqrt{4\pi\nu t}} \tag{52}$$

Since the inverse transforms are known, namely and Hence, function g can be written in the form

$$\mathrm{g} = \iint^{a} (\mathrm{x},\mathrm{y}) \frac{e^{\frac{-(x-X)^2-(y-Y)^2}{4\nu t}}}{4\pi\nu \mathrm{t}} \,\mathrm{dXdY}\ , \tag{53}$$

In this equation, X and Y denote dummy integration variables. The arbitrary function a(x,y) is specified by applying an initial condition whose meaning is now discussed. Suppose that there is a viscous flow around obstacles for $t < 0$. Then, at t=0 all solid bodies are suddenly removed from the field flow. Hence, the corresponding vector field would evolve in time and reach a steady state where obstacles no longer exist. This unperturbed flow is obviously uniform, so the field "forgets" the influence of the solid bodies. Hence, the arbitrary function a(x,y) describes the shape of the bodies which were removed at t=0, so equation (33) must be accompanied by the following initial condition:

$$\mathrm{g} = \mathrm{a}(\mathrm{x},\mathrm{y}) \quad (\mathrm{t}=0) \tag{54}$$

For instance, if the only solid body is a thin wire centered at the origin and whose orientation is perpendicular to the xy plane, the initial condition is approximated by

$$\mathrm{g} = \delta(\mathrm{x})\,\delta(\mathrm{y}) \quad (\mathrm{t}=0) \tag{55}$$

In this case, equation (54) produces the classical two-dimensional Gaussian solution:

$$g = \frac{e^{\frac{-\left(x^2+y^2\right)}{4\nu t}}}{4\pi\nu t} \tag{56}$$

This expression will be employed to generate some preliminary results whose importance is crucial to develop more sophisticated exact solutions, in order to describe realistic velocity fields.

After obtaining any solution in the form given by (53), it becomes necessary to perform an extra convolution in the time variable. This convolution, which is carried out in order to account for the presence of the solid bodies for t $>$ 0, produces

$$g = \int\int\int a(x,y)\frac{e^{\frac{-(x-X)^2-(y-Y)^2}{4\nu(t-T)}}}{4\pi\nu(t-T)}\mathrm{dXdYdT} \tag{57}$$

Once obtained the final solution to the auxiliary model, function f can be easily evaluated using equation (38):

$$f = \left(\frac{\int\int\int a(x,y)\,\frac{e^{\frac{-(x-X)^2-(y-Y)^2}{4\nu(t-T)}}}{4\pi\nu(t-T)}\mathrm{dXdYT} - c_0}{c_1}\right)^{\frac{2}{3}} \tag{58}$$

At this point, a simple question about the pressure field may arise. Once the pressure terms were apparently neglected, it seems that no wall effects could be considered in this formulation. However, function f can be interpreted as the Bernoulli field pressure, whose reference value at infinity is $(c_0/\mathrm{c}_1)^{2/3}$. Therefore, as the velocity field evolves in such a way that the momentum is transferred by advection and diffusion, the pressure is produced near the wall and propagates, only by diffusion, to an infinite "buffer" which represents the free stream. Moreover, notice that additional solid interfaces can be easily taken into account by adding to the stream function any harmonic one containing branches whose shapes describe any extra obstacle. These additional terms belong to the null space of the laplacian operator except at the singularities, which not lie in the considered domain. Thus, despite the nonlinear character of the Helmholtz equation, the extra terms represents only trivial solutions, and hence can be added to the stream function in order to produce new exact ones.

5 Results and discussion

The proposed formulation was employed to map exact solutions of the diffusion model defined by equation (33) into velocity fields describing some basic structures arising in turbulent wakes. Figure 1 shows the field plot corresponding to a Gaussian peak given by

$$g = c_0\frac{e^{\frac{-\left(x^2+y^2\right)}{4t}}}{4\pi\nu t} \tag{59}$$

which represents a single vortex around the origin. In this case, the linear combination reduces to a one term solution with $c_0 = 2$, $t = 1$ and $\nu = 0{,}01$.

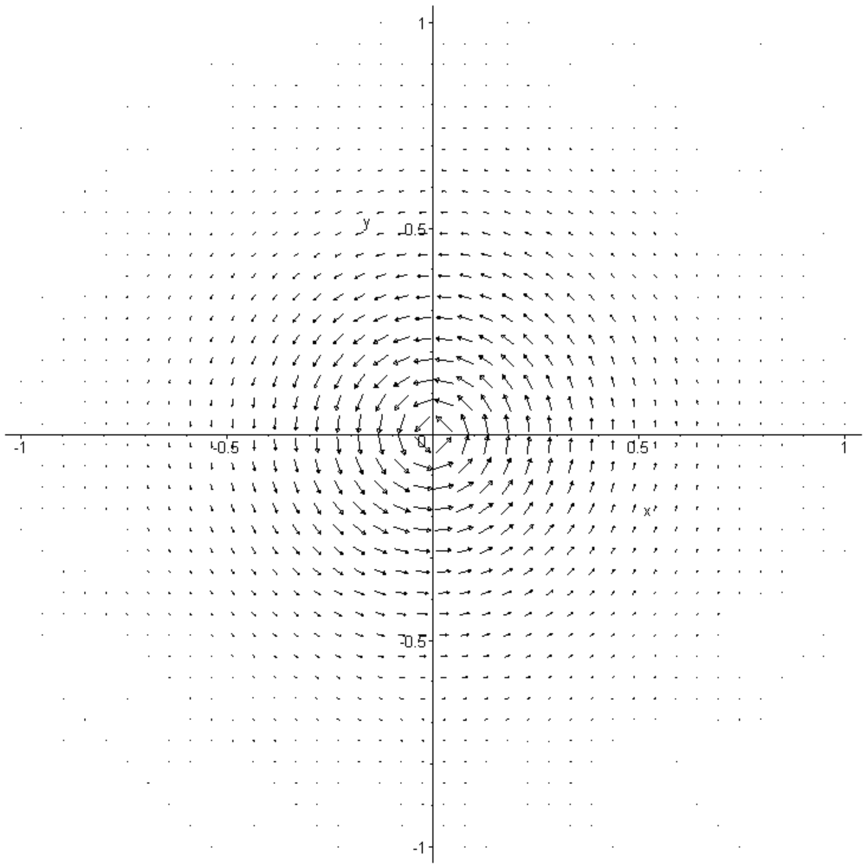

Figure 1: Single vortex generated by mapping from a Gaussian function.

Once any linear combination of Gaussian functions are also exact solutions of the diffusion model, it becomes possible to generate structures analogous to the Kolmogorov cascade, by setting appropriate parameters defining the characteristic dimension of each component. For instance,

$$g = 2 + 3y + 1.994711402\{e^{-12.5\left[(x+1)^2+(y-0.5)^2\right]} + e^{-12.5\left[(x+0.5)^2+(y-1)^2\right]} + e^{-12.5\left[x^2+(y-0.5)^2\right]}\} \tag{60}$$

where the linear term in y represents an uniform flow, generates the wake depicted in figure 2.

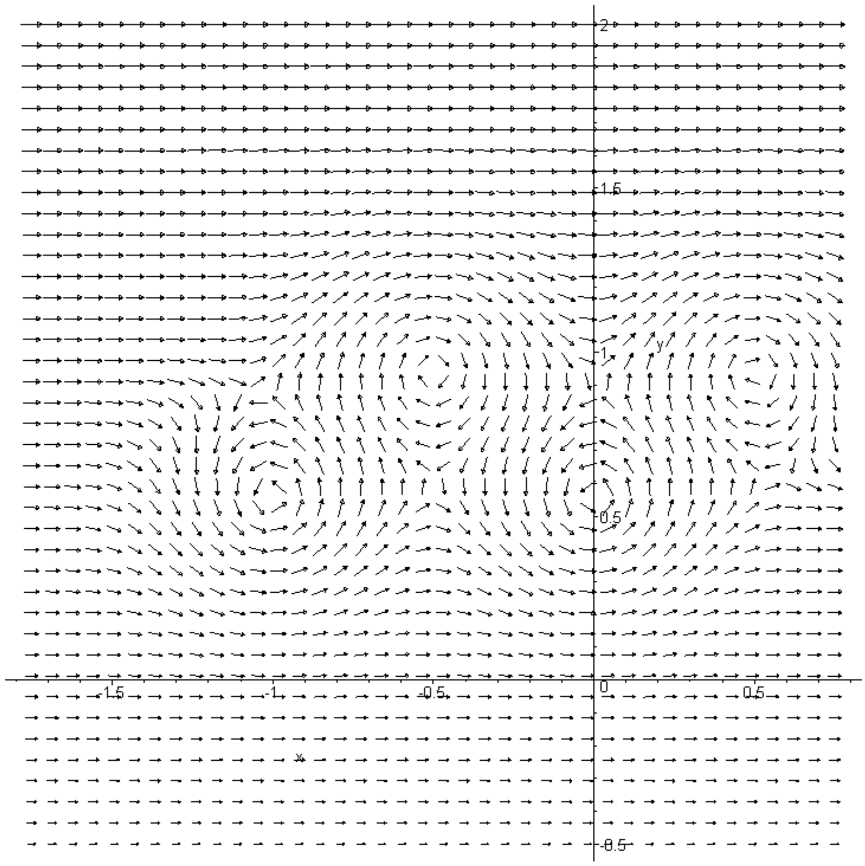

Figure 2: Wake obtained by mapping a linear combination of Gaussian peaks.

The parameters in equation (44) were chosen arbitrarily, in order to show the capabilities of the method, and stress an important feature of the proposed formulation. First, the time processing required to produce the maps is virtually negligible, even in low performance computers (about 10s using Maple V in an AMD Sempron 3100 processor). Moreover, this time processing is roughly proportional to the number of terms in the linear combination. It is also possible to include fluctuations in the velocity field by adding high frequency sinusoidal solutions of the diffusion model in the linear combination defining g. However, in order to determine the parameters in the linear combination which accounts for turbulence, it becomes necessary to estimate local values to the Reynolds number [Bodmann, Vilhena, Zabadal, Beck (2011)], which defines a set of wave numbers for the vorticity along the field.

In future works, our attention will be focused in formulating differential constraints to determine dispersion relations for the vorticity function, in order to obtain a realistic turbulence spectrum for a wide class of velocity fields.

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