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A Wavelet Method for the Solution of Nonlinear Integral Equations with Singular Kernels

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Abstract: In this paper, we propose an efficient wavelet method for numerical solution of nonlinear integral equations with singular kernels. The proposed method is established based on a function approximation algorithm in terms of Coiflet scaling expansion and a special treatment of boundary extension. The adopted Coiflet bases in this algorithm allow each expansion coefficient being explicitly expressed by a single-point sampling of the function, which is crucially important for dealing with nonlinear terms in the equations. In addition, we use the technique of integration by parts to transform the original integral equations with non-smooth or singular kernels into regular ones with smooth kernels. Then, we incorporate the proposed function approximation algorithm into the Galerkin method for the solution of the transformed nonlinear integral equations. Numerical examples show that the proposed wavelet method is much more accurate and efficient than several others.

Keywords: Coilfet, wavelet Galerkin method, nonlinear integral equations, singular kernel.

1 Introduction

An integral equation is a functional equation with unknown function under the integration sign [Baker (1977)]. Integral equations arise in a great many branches of science, including potential theory, acoustics, elasticity, fluid mechanics, radiative transfer and even theory of population.

Recently, using multiresolution techniques and wavelets to develop numerical schemes for the solution of differential and integral equations has become increasingly popular [Beylkin, Coifman, and Rokhlin (1991); Alpert, Beylkin, Coifman, and

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Rokhlin (1993); Amaratunga, Williams, Qian, and Weiss (1994); Sweldens (1995); Chen, Micchelli, and Xu (1997); Kaneko, Noren, and Novaprateep (2003); Wang (2001); Zhou, Wang, and Zheng (1999); Zhou, Wang, Wang, and Liu (2011); Liu, Wang, and Zhou (2013); Liu, Wang, Zhou and Wang (2013)]. Although in applications, many mathematical tools have been demonstrated valid, yet wavelet applications to the solution of nonlinear integral equations with singular kernels arising in different areas of mechanics, physics and engineering have been very limited. It was Belykin et al. [Beylkin, Coifman, and Rokhlin (1991)] who first proposed a wavelet method to solve the integral equations. Their work focused on the sparse discretization of linear integral operators, which leading to a fast numerical algorithm. The reason of the sparse and accurate discretization is mainly due to the compact support and vanishing moment properties of wavelets. Later on, works by Alpert et al. [Alpert, Beylkin, Coifman, and Rokhlin (1993)] and Chen et al. [Chen, Micchelli, and Xu (1997)] continuously attempted to develop a fast wavelet algorithm for the linear second kind integral equations. For the nonlinear integral equations with continuous kernel, several wavelet methods have been proved available, which includes the Galerkin methods based on the Legendre wavelets [Mahmoudi (2005)] and the B-spline wavelets [Sahu and Ray (2013)], and collocation method based on the Haar wavelet [Babolian and Shahsavaran (2009)]. However, only a class of simple nonlinear integral equations with common nonlinear terms in the form of polynomials was studied. For the nonlinear integral equations with singular kernels, efficient methods with high accuracy are still very limited [Liang, Liu and Che (2001); Xiao, Wen and Zhang (2006); Gao and Jiang (2007); Galperin, Kansa, Makroglou and Nelson (2000); Panigrahi and Nelakanti (2012)].

Coiflets are the most efficient wavelets in constructing one-point quadrature formula with very high precision [Sweldons and Piessens (1994); Wang (2001); Maleknejad, Lotfi and Rostami (2007)]. This interesting property is very convenient in dealing with nonlinear differential and integral equations [Liang, Liu and Che (2001); Maleknejad, Lotfi and Rostami (2007); Liu, Wang, and Zhou (2013); Liu, Wang, Zhou and Wang (2013)]. For most wavelet Galerkin methods without using this property, solution procedures will inevitably involve complicated integral computations associated with the scaling functions even in solving certain simple nonlinear integral equations [Avudainayagam and Vani (2000); Liu, Qin, Liu and Cen (2010)]. Based on this fact, a Coiflet-based algorithm for numerical solution of the second kind integral equations with continuous and weakly singular kernels is proposed in the present study. Numerical examples are considered to verify the efficiency and accuracy of the proposed method.

2 Function approximation based on series expansion of scaling functions

For a function $f(x) \in L^2(\mathbf{R})$, if $\varphi(x)$ represents a Coiflet scaling function [Daubechies (1993)], then we can have [Daubechies (1993)]

$$f(x) = \lim_{j\to\infty} \sum_{k=-\infty}^{+\infty} c_{j,k}\varphi_{j,k}(x) \approx \sum_{k=-\infty}^{+\infty} c_{n,k}\varphi_{n,k}(x) \tag{1}$$

in which $c_{n,k} = \int_{-\infty}^{+\infty} f(x)\varphi_{n,k}(x)dx$, $\varphi_{n,k}(x) = 2^{n/2}\varphi(2^n x - k)$, and integer n is the so-called resolution level.

Table 1: Coiflet filter coefficients for N = 2, 4, and 6.

k	$N = 2$ ($M_1 = 4$)	$N = 4$ ($M_1 = 4$)	$N = 6$ ($M_1 = 7$)
0	5.456145913796356e-02	1.689380907695821e-03	−2.392638657280051e-03
1	-1.795614591379636e-01	-1.816639282073453e-02	−4.932601854180402e-03
2	-1.091229182759271e-01	3.507862062605389e-02	2.714039971139949e-02
3	83591229182759271e-01	7.074394036809258e-02	3.064755594619984e-02
4	1.054561459137964e+00	-2.197082915811749e-01	−1.393102370707997e-01
5	3.204385408620364e-01	-1.013118304071172e-01	−8.060653071779983e-02
6		8.067593419102440e-01	6.459945432939942e-01
7		1.061135780078056e+00	1.116266213257999e+00
8		3.968448038803485e-01	5.381890557079980e-01
9		-1.047986487449172e-02	−9.961543386239989e-02
10		-2.066385574316280e-02	−7.992313943479994e-02
11		-1.921632058008399e-03	5.149146293240031e-02
12			1.238869565706006e-02
13			−1.583178039255944e-02
14			−2.717178600539990e-03
15			2.886948664020020e-03
16			6.304993947079994e-04
17			−3.058339735960013e-04

Noting that the Coiflet scaling function $\varphi(x)$ has a compact support = [0, $3N-1$], then $\varphi(x)$ can be constructed by using the filter coefficients a_k, k=0,1,2,3…$3N$-1, in Tab 1 [Wang (2001)] in terms of the relation below,

$$\varphi(x) = \sum_{k=0}^{3N-1} a_k \varphi(2x - k) \tag{2}$$

Such a scaling function has the unique property of shifted moments

$$\int_{-\infty}^{+\infty} (t - M_1)^k \varphi(t)dx = 0 \quad 1 \le k < N - 1 \tag{3}$$

Considering Eq. (3) one can have

$$c_{n,k}=\int_{-\infty}^{+\infty} f(x)\varphi_{n,k}(x)dx \approx 2^{n/2} f(\frac{k+M_1}{2^n}) \tag{4}$$

with a degree of accuracy of $N-1$. Combining Eqs. (1) and (4) yields

$$f(x)\approx \sum_{k=-\infty}^{+\infty} f(\frac{k+M_1}{2^n})\varphi(2^n x-k) \tag{5}$$

Such an approach to approximation is attractive because of its simplicity and a degree of accuracy of $N-1$. Moreover, if it has $f(x)\in \mathbf{C}^{\gamma}$, $\gamma \le N-1$, the precision of Eq. (5) immediately becomes [Sweldons and Piessens (1994); Wang (2001)]

$$||f(x)-\bar{\mathbf{P}}_n f(x)||_2 \le O(2^{-n\gamma}). \tag{6}$$

And by using Eq. (5), we can derive the following rules [Wang (2001)]:
Rule 1: for the composite function $\mathbf{\Pi}[f(x)]$ of $f(x)$, we have

$$\mathbf{\Pi}[f(x)] \approx \sum_k \mathbf{\Pi}[f(\frac{k+M_1}{2^n})]\varphi(2^n x-k) \tag{7}$$

Rule 2: for a derivative or integration operator $\mathbf{D}$, we have

$$\mathbf{D}f(x) \approx \sum_k f(\frac{k+M_1}{2^n})\mathbf{D}\varphi(2^n x-k) \tag{8}$$

As we know, the wavelet theory is established on the whole real line. For many applications, the functions involved are usually defined on a bounded interval. In order to apply wavelets in these applications, some modifications will have to be made. Several constructions of wavelets on a bounded interval have become available [Cohen, Daubechies and Vial (1993); Sweldens (1995)]. However, all these constructions are mathematically difficult, and the resulting wavelets are complicated to be applied to numerical analysis. We thus need to find a simple alternative solution. To be specific, let us consider the case of a unit interval [0, 1]. Given a function $f(x)$ on [0, 1], the most obvious approach is to set $f(x) = 0$ outside [0, 1], and then using wavelet theory on the whole real line. However, for a general function $f(x)$ this padding with 0 usually introduces discontinuities at the endpoints 0 and 1, for instance the simple function $f(x) = 1, x\in[0,1]$. Just as Cohen et al. [Cohen, Daubechies and Vial (1993)] and Sweldens [Sweldens (1995)] have pointed out that, because wavelets are effective for detecting singularities, the presence of artificial discontinuities is likely to introduce significant errors. Another approach is to consider the function to be periodic with period one, $f(x+1) = f(x)$. However, unless the behavior of the function $f(x)$ at 0 matches that at 1 the periodic version of $f(x)$ has singularities there. A simple function like $f(x) = x, x \in [0,1]$, gives a good illustration of this. A third method, which works if the bases functions are symmetric, is to use reflection across the edges. This preserves continuity, but introduces discontinuities in the first derivative.

What is really needed is that functions do not introduce discontinuities in up to a certain order of derivative at the endpoints 0 and 1.

Consider a function$g(x) \in L^2[0,1]$, we define

$$d_{0,i} = \frac{d^i g(0)}{dx^i}, d_{1,i} = \frac{d^i g(1)}{dx^i}, i = 0,1,2,\cdots. \tag{9}$$

By applying Taylor expansion, the function can continue at endpoints 0, 1 as

$$g(x) = \begin{cases} \sum\limits_{i=0}^{M} \frac{d_{0,i}}{i!} x^i \; x \in (-\delta,\ 0) \\ g(x) \; x \in [0,\ 1] \\ \sum\limits_{i=0}^{M} \frac{d_{1,i}}{i!} (x-1)^i \; x \in (1,\ 1+\delta) \end{cases} \tag{10}$$

where $\delta > 0$, i= 0, 1, 2,..., M. It can be seen that such a boundary extension treatment does not introduce discontinuities at endpoints 0 and 1. However, in practical applications, the derivatives at endpoints sometimes are unknown or even do not exist at all. And therefore we have to apply equidistant numerical difference to approximate or replace (when they do not exist.) them via the discrete points in [0, 1] as

$$d_{0,i} = \sum_{j=0}^{m} p_{0,i,j} g_j, d_{1,i} = \sum_{j=0}^{m} p_{1,i,j} g_{2^n-j} \tag{11}$$

where $p_{0,i,j}$, $p_{1,i,j}$are parameters associated with numerical difference and $g_k = g(k/2^n)$, i= 1,2,..., m. Hence Eq. (10) becomes

$$g(x) = \begin{cases} \sum\limits_{i=0}^{M} \frac{1}{i!} x^i \sum\limits_{j=0}^{m} p_{0,i,j} g_j & x \in (-\delta,\ 0) \\ g(x) & x \in [0,\ 1] \\ \sum\limits_{i=0}^{M} \frac{1}{i!} (x-1)^i \sum\limits_{j=0}^{m} p_{1,i,j} g_{2^n-j} & x \in (1,\ 1+\delta) \end{cases}$$

$$= \begin{cases} \sum\limits_{j=0}^{m} g_j \sum\limits_{i=0}^{M} p_{0,i,j} \frac{1}{i!} x^i & x \in (-\delta,\ 0) \\ g(x) & x \in [0,\ 1] \\ \sum\limits_{j=0}^{m} g_{2^n-j} \sum\limits_{i=0}^{M} p_{1,i,j} \frac{1}{i!} (x-1)^i & x \in (1,\ 1+\delta) \end{cases}$$

or

$$g(x)=\begin{cases} \sum\limits_{j=0}^{m} g_j T_{0,j}(x) & x\in(-\delta,\ 0) \\ g(x) & x\in[0,\ 1] \\ \sum\limits_{j=0}^{m} g_{2^n-j} T_{1,j}(x) & x\in(1,\ 1+\delta) \end{cases} \tag{12}$$

where $T_{0,j}(x)=\sum\limits_{i=0}^{M} p_{0,i,j}\frac{1}{i!}x^i T_{1,j}(x)=\sum\limits_{i=0}^{M} p_{1,i,j}\frac{1}{i!}(x-1)^i$. Taking n, which satisfy $2^n > m+1$, as the scale of scaling series, and then applying the approximation formula (5) to function (12), it yields

$$\begin{aligned} g(x) &\approx \sum_{k=2-3N}^{2^n-1} g\left(\frac{M_1+k}{2^n}\right)\varphi(2^n x-k) \\ &\approx \sum_{k=2-3N+M_1}^{-1}\sum_{j=0}^{m} g_j T_{0,j}\left(\frac{k}{2^n}\right)\varphi(2^n x-k+M_1) \\ &+\sum_{k=0}^{2^n} g_k\varphi(2^n x-k+M_1)+\sum_{k=2^n+1}^{2^n+M_1}\sum_{j=0}^{m} g_{2^n-j}T_{1,j}\left(\frac{k}{2^n}\right)\varphi(2^n x-k+M_1) \\ &=\sum_{j=0}^{m} g_j\sum_{k=2-3N+M_1}^{-1} T_{0,j}\left(\frac{k}{2^n}\right)\varphi(2^n x-k+M_1) \\ &+\sum_{k=0}^{2^n} g_k\varphi(2^n x-k+M_1)+\sum_{j=0}^{m} g_{2^n-j}\sum_{k=2^n+1}^{2^n+M_1} T_{1,j}\left(\frac{k}{2^n}\right)\varphi(2^n x-k+M_1) \\ &=\sum_{k=0}^{m} g_k[b_{0,k,n}(x)+\varphi(2^n x-k+M_1)] \\ &+\sum_{k=m+1}^{2^n-m-1} g_k\varphi(2^n x-k+M_1)+\sum_{k=2^n-m}^{2^n} g_k[b_{1,k,n}(x)+\varphi(2^n x-k+M_1)] \end{aligned}$$

or

$$g(x)\approx\sum_{k=0}^{2^n} g_k\Phi_{n,k}(x) \tag{13}$$

where

$$\Phi_{n,k}(x)=\begin{cases} b_{0,k,n}(x)+\varphi(2^n x-k+M_1), & if\quad 0\le k\le m \\ \varphi(2^n x-k+M_1), & if\quad m+1\le k\le 2^n-m-1 \\ b_{1,k,n}(x)+\varphi(2^n x-k+M_1), & if\quad 2^n-m\le k\le 2^n \end{cases}, \tag{14}$$

and

$$
\begin{aligned}
b_{0,k,n}(x) &= \sum_{j=2-3N+M_1}^{-1} T_{0,k}(\frac{j}{2^n})\varphi(2^n x - j + M_1), \\
b_{1,k,n}(x) &= \sum_{j=2^n+1}^{2^n+M_1} T_{1,2^n-k}(\frac{j}{2^n})\varphi(2^n x - j + M_1).
\end{aligned}
\tag{15}
$$

Thus, after applying the rules of scaling transform and Taylor expansion, we obtain the modified approximation series (13). It can be seen that such approximation manner does not introduce discontinuity up to Mth order derivatives at the end-points 0 and 1.

When we use the Coiflets-like bases to the numerical example at the end of the paper, here, for M = 3, m = 3, we take the 4-point *Malkoff* formula of numerical difference:

$$
\{2^{-in} p_{0,i,j}\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{pmatrix} \quad \{2^{-in} p_{1,i,j}\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{11}{6} & -3 & \frac{3}{2} & -\frac{1}{3} \\ 2 & -5 & 4 & -1 \\ 1 & -3 & 3 & -1 \end{pmatrix}
\tag{16}
$$

where i=0,1,2,3, j=0,1,2,3 and from here onward, such a numerical difference formula will be used in all the relative numerical examples of this paper.

3 Nonlinear integral equations with continuous kernel

We consider the nonlinear Fredholm integral equations of the second kind with continuous kernel as below

$$
f(x) + \int_0^1 K(x,y,f(y))dy = q(x), \quad 0 \le x \le 1 \tag{17}
$$

where f, K are given smooth functions. Two approaches have been considered here.

3.1 Approach 1: Galerkin method with $L^2(R)$ Coiflet bases

If f, K, and q all belong to $L^2(\mathbf{R})$, the functions in Eq. (17) can be approximated by the Coiflet scaling bases on $L^2(\mathbf{R})$ as

$$\begin{aligned} f(x) &\approx \sum_{k=2-3N}^{2^n-1} f(x_k)\varphi(2^n x-k) \\ q(x) &\approx \sum_{k=2-3N}^{2^n-1} q(x_k)\varphi(2^n x-k) \\ K(x,y,f(y)) &\approx \sum_{j=2-3N}^{2^n-1}\sum_{k=2-3N}^{2^n-1} K(x_j,y_k,f_k)\varphi(2^n y-j)\varphi(2^n x-k) \end{aligned} \tag{18}$$

where $x_k=\frac{k+M_1}{2^n}$, $x_j=\frac{j+M_1}{2^n}$, $y_k=\frac{k+M_1}{2^n}$ and $f_k=f(x_k)$. Substituting Eq. (18) into Eq. (17) gives

$$\begin{aligned} &\sum_{k=2-3N}^{2^n-1} f(x_k)\varphi(2^n x-k)+\sum_{k=2-3N}^{2^n-1}\sum_{j=2-3N}^{2^n-1} K(x_k,y_j,f_j)\varphi(2^n x-k)\int_0^1 \varphi(2^n y-j)dy \\ &\approx \sum_{k=2-3N}^{2^n-1} q(x_k)\varphi(2^n x-k) \end{aligned} \tag{19}$$

Define

$$\varphi^{\underbrace{\int\cdots\int}_{m}}(x)=\int_{-\infty}^{x}\cdots\int_{-\infty}^{t_{m-1}}\varphi(t_m)dt_1\cdots dt_m. \tag{20}$$

Then we have

$$\begin{aligned} &\sum_{k=2-3N}^{2^n-1}\{f(x_k)+1/2^n\sum_{j=2-3N}^{2^n-1} K(x_k,y_j,f_j)[\varphi^{\int}(2^n-j)-\varphi^{\int}(-j)]\}\varphi(2^n-x) \\ &\approx \sum_{k=2-3N}^{2^n-1} q(x_k)\varphi(2^n x-k). \end{aligned} \tag{21}$$

The Galerkin discretization scheme is applied to Eq. (21), giving the system of nonlinear algebraic equations

$$\sum_{k=2-3N}^{2^n-1}\{f(x_k)+1/2^n\sum_{j=2-3N}^{2^n-1} K(x_k,y_j,f_j)[\varphi^{\int}(2^n-j)-\varphi^{\int}(-j)]\}\Gamma_{i,k}^{0,0}\approx\sum_{k=2-3N}^{2^n-1} q(x_k)\Gamma_{i,k}^{0,0}$$

$$(22)$$

where $\Gamma_{i,k}^{0,0} = \int_0^1 \varphi(2^n x - i)\varphi(2^n x - k)dx = 2^{-n}\delta_{i,k}$. Then Eq. (22) can be changed into

$$f_k + 1/2^n \sum_{j=2-3N}^{2^n-1} K(x_k, y_j, f_j)[\varphi^{\int}(2^n - j) - \varphi^{\int}(-j)] \approx q_k \tag{23}$$

in which $k = 2-3N, 3-3N, \cdots, 2^n - 1$ and $\varphi^{\overbrace{\int\cdots\int}^{k}}(x)$ can be obtained from references [Wang (2001); Chen, Wang, and Shih (1996)]. By solving Eq. (23) using Newton-Rapson method, then the solution can be obtained.

3.2 Approach 2: Galerkin method with modified Coiflets bases

Considering Eq. (17) if f, K,and q all belong to $L^2[0,1]$, then we have

$$\begin{aligned} f(x) &\approx \sum_{k=0}^{2^n} f(x_k)\Phi_{n,k}(x) \\ q(x) &\approx \sum_{k=0}^{2^n} q(x_k)\Phi_{n,k}(x) \\ K(x,y,f(y)) &\approx \sum_{k=0}^{2^n}\sum_{j=0}^{2^n} K(x_j, y_k, f_k)\Phi_{n,k}(y)\Phi_{n,j}(x) \end{aligned} \tag{24}$$

where $x_k = \frac{k}{2^n}$,$x_j = \frac{j}{2^n}$, $y_k = \frac{k}{2^n}$ and $f_k = f(x_k)$. Substituting Eq. (24) into Eq. (17) gives

$$\sum_{k=0}^{2^n} f(x_k)\Phi_{n,k}(x) + \sum_{k=0}^{2^n}\sum_{j=0}^{2^n} K(x_k, y_j, f_j)\Phi_{n,k}(x)\int_0^1 \Phi_{n,j}(y)dy \approx \sum_{k=0}^{2^n} q(x_k)\Phi_{n,k}(x). \tag{25}$$

Define

$$\Phi_{n,k}^{\overbrace{\int\cdots\int}^{m}}(x) = \int_{-\infty}^{x}\cdots\int_{-\infty}^{t_{m-1}} \Phi_{n,k}(t_m)dt_1\cdots dt_m \tag{26}$$

which can be obtained easily by the definition (19), thus Eq. (26) becomes

$$\sum_{k=0}^{2^n}\{f(x_k) + \sum_{j=0}^{2^n} K(x_k, y_j, f_j)\Phi_{n,j}^{\int}(1)\}\Phi_{n,k}(x) \approx \sum_{k=0}^{2^n} q(x_k)\Phi_{n,k}(x). \tag{27}$$

Let $\Phi_{n,i}(x)$, $i=0,1,\cdots,2^n$ be the weighted functions, multiplying both sides of Eq. (27) by them and then taking integration from 0 and 1, we can obtain

$$\sum_{k=0}^{2^n}\{f(x_k)+\sum_{j=0}^{2^n}K(x_k,y_j,f_j)\Phi_{n,j}^{\int}(1)\}\bar{\Gamma}_{i,k}^{0,0}\approx\sum_{k=0}^{2^n}q(x_k)\bar{\Gamma}_{i,k}^{0,0} \tag{28}$$

where $\bar{\Gamma}_{i,k}^{0,0}=\int_0^1\Phi_{n,i}(x)\Phi_{n,k}(x)dx$. Because the matrix $\{\bar{\Gamma}_{i,k}^{0,0}\}$ is nonsingular, then Eq. (28) can be changed into

$$f_k+\sum_{j=2-3N}^{2^m-1}K(x_k,y_j,f_j)\Phi_{n,j}^{\int}(1)\approx q_k. \tag{29}$$

By solving Eq. (29) using Newton-Raphson method, the solution $\{f_k, k{=}0,1,\cdots,2^n\}$ can be obtained readily.

4 Integral equations with weakly singular kernel

We consider the Fredhlom integral equations of the second kind with weakly singular kernel given by

$$f(x)+\int_0^1 s(x,y)f(y)dy=q(x) \tag{30}$$

where $s(x,y)$ satisfies the conditions

$$\left|\frac{\partial^r s(x,y)}{\partial x^r}\right|\leq M|x-y|^{\alpha-r},\left|\frac{\partial^r s(x,y)}{\partial y^r}\right|\leq M|x-y|^{\alpha-r},x\neq y \tag{31}$$

in which M is a positive constant, $-1<\alpha\leq 0$ and r is nonnegative integer. First, we will use integration by parts to change the form of the original Eq. (30) to make the singular kernel become very smooth, then solve the resulting equation, eventually convert the solution back to that of the original Eq. (30). The details are as follows.

Define

$$s_k(x,y)=\underbrace{\int\cdots\int}_{k}s(x,y)dy,\ k=1,2,3,\cdots, \tag{32}$$

and $I_k(x,y)=h_k(x,y)-h_k(x,1)$, $k=1,2,3,\cdots$, in which

$$h_k(x,y)=s_k(x,y)-\sum_{i=1}^{k-1}h_i(x,1)y^{k-i}/(k-i)!.$$

Considering Eq. (31), it can be easily known that $s_k(x,y)$, $I_k(x,y) \in \mathbf{C}^{k-1}$. Using integration by parts to $\int_0^1 s(x,y)f(y)dy$, Eq. (30) can be reduced to

$$f(x)+[f(y)I_1(x,y)]|_{y=0}^{y=1}-\int_0^1 I_1(x,y)f'(y)dy=q(x). \tag{33}$$

Because we have $I_1(x,1)=0$, then

$$f(x)-f(0)I_1(x,0)-\int_0^1 I_1(x,y)f'(y)dy=q(x). \tag{34}$$

Repeat the integration by parts to $\int_0^1 I_1(x,y)f'(y)dy$ and consider the fact of $I_k(x,1)=0$, we have

$$f(x)+\sum_{\xi=1}^{J}(-1)^{\xi}I_{\xi}(x,0)f^{(\xi-1)}(0)+(-1)^J\int_0^1 I_J(x,y)f^{(J)}(y)dy=q(x). \tag{35}$$

Let $g(x)=f^{(J)}(x)$, and

$$g^{\overbrace{\int\cdots\int}^{n}}(x)=\int_0^x\cdots\int_0^{t_{\eta-1}}g(t_1)dt_1\cdots dt_{\eta}, \tag{36}$$

then we have

$$f(x)=g^{\overbrace{\int\cdots\int}^{J}}(x)+\sum_{k=0}^{J-1}\frac{1}{k!}x^k f^{(k)}(0). \tag{37}$$

Thus Eq. (35) can be changed into

$$\begin{aligned}&g^{\overbrace{\int\cdots\int}^{J}}(x)+\sum_{\xi=1}^{J}[(-1)^{\xi}I_{\xi}(x,0)+\frac{1}{(\xi-1)!}x^{(\xi-1)}]f^{(\xi-1)}(0)\\&+(-1)^J\int_0^1 I_J(x,y)g(y)dy=q(x).\end{aligned} \tag{38}$$

Substituting $x=t_1,t_2,t_3,\cdots,t_J\in[0,1]$ into Eq. (38), $t_i\neq\rho/2^n$, where ρ is an arbitrary integer, we have

$$\begin{aligned}&g^{\overbrace{\int\cdots\int}^{J}}(t_i)+\sum_{\xi=1}^{J}[(-1)^{\xi}I_{\xi}(t_i,0)+\frac{1}{(\xi-1)!}t_i^{(\xi-1)}]f^{(\xi-1)}(0)\\&+(-1)^J\int_0^1 I_J(t_i,y)g(y)dy=q(t_i)\end{aligned} \tag{39}$$

in which $i = 1, 2, \cdots, J$. In order to get the expression of $f^{(k)}(0)$, $k = 1, 2, \cdots$, we change the formation of Eq. (39) to

$$\begin{aligned} &\sum_{\xi=1}^{J} [(-1)^{\xi} I_{\xi}(t_i, 0) + \frac{1}{(\xi-1)!} t_i^{(\xi-1)}] f^{(\xi-1)}(0) \\ &= q(t_i) - g^{\overbrace{\int\cdots\int}^{J}}(t_i) - (-1)^J \int_0^1 I_J(t_i, y) g(y) dy. \end{aligned} \tag{40}$$

Rewriting Eq. (40) to a matrix form, we can obtain

$$\mathbf{AF} = \mathbf{B} \tag{41}$$

in which $\mathbf{A} = \{a_{i,j}\}$, $\mathbf{F} = \{f_{0j}\}$, $\mathbf{B} = \{b_i\}$, and

$$\begin{aligned} &a_{i,j} = (-1)^j I_j(t_i, 0) + \frac{t_i^{(j-1)}}{(j-1)!}, \qquad f_{0j} = f^{(j-1)}(0), \\ &b_i = q(t_i) - g^{\overbrace{\int\cdots\int}^{J}}(t_i) - (-1)^J \int_0^1 I_J(t_i, y) g(y) dy. \end{aligned} \tag{42}$$

In order to solve Eq. (41) with unknown $\mathbf{F}$, we denote the adjoint matrix of the matrix $\mathbf{A}$ as $adj\mathbf{A} = \{c_{i,j}, i, j = 1, 2, \cdots, J\}$. Then

$$\mathbf{F} = \mathbf{A}^{-1}\mathbf{B} = (adj\mathbf{A})\mathbf{B}/det\mathbf{A}. \tag{43}$$

Substituting Eq. (43) into Eq. (38), it yields

$$g^{\overbrace{\int\cdots\int}^{J}}(x) - \sum_{i=1}^{J} \alpha_i(x) g^{\overbrace{\int\cdots\int}^{J}}(t_i) + \int_0^1 \Theta(x, y) g(y) dy = Q(x) \tag{44}$$

where

$$\begin{aligned} &Q(x) = q(x) - \sum_{i=1}^{J} \alpha_i(x) q(t_i), \\ &\Theta(x, y) = (-1)^J I_J(x, y) - \sum_{i=1}^{J} \alpha_i(x) (-1)^J I_J(t_i, y), \\ &\alpha_i(x) = \sum_{\xi=1}^{J} [(-1)^{\xi} I_{\xi}(x, 0) + \frac{x^{(\xi-1)}}{(\xi-1)!}] c_{\xi,i}/det\mathbf{A}. \end{aligned} \tag{45}$$

Thus we obtain the transformed Eq. (44) after smoothing treatment.

Using expansion formula (13) to unknown function $g(x)$, it yields

$$g(x) \approx \sum_{k=0}^{2^n} g(x_k)\Phi_{n,k}(x). \tag{46}$$

And consider the definition (36), it gives

$$g^{\overbrace{\int\cdots\int}^{\eta}}(x) = \int_0^x \cdots \int_0^{t_{\eta-1}} g(t_1)dt_1 \cdots dt_\eta = \sum_{k=0}^{2^n} g_k \Phi_{n,k}^{\overbrace{\int\cdots\int}^{\eta}}(x). \tag{47}$$

Use expansion formula (13) to Eq. (44) again, we have

$$\Theta(x,y)g(y) \approx \sum_{j=0}^{2^n} \Theta(x,y_j)g_j\Phi_{n,j}(y) \tag{48}$$

where $y_j = j/2^n, g_j = g(y_j)$. Then we have

$$\sum_{k=0}^{2^n} g_k \Phi_{n,k}^{\overbrace{\int\cdots\int}^{J}}(x) - \sum_{i=1}^{J} \alpha_i(x) \sum_{k=0}^{2^n} g_k \Phi_{n,k}^{\overbrace{\int\cdots\int}^{J}}(t_i) + \sum_{j=0}^{2^n} \Theta(x,y_j)g_j\Phi_{n,j}^{\int}(1) = Q(x). \tag{49}$$

Considering Eq. (49), and for the variable x, using the Eq. (13), we have

$$\sum_{l=0}^{2^n} \{\sum_{k=0}^{2^n} g_k \Phi_{n,k}^{\overbrace{\int\cdots\int}^{J}}(x_l) - \sum_{i=1}^{J} \alpha_i(x_l) \sum_{k=0}^{2^n} g_k \Phi_{n,k}^{\overbrace{\int\cdots\int}^{J}}(t_i) + \sum_{j=0}^{2^n} \Theta(x_l,y_j)g_j\Phi_{n,j}^{\int}(1)\}\Phi_{n,l}(x)$$
$$= \sum_{l=0}^{2^n} Q(x_l)\Phi_{n,l}(x). \tag{50}$$

Let $\Phi_{n,\varsigma}(x)$, $\varsigma = 0,1,\cdots,2^n$ be the weighted functions, multiplying both sides of Eq. (50) by them and then taking integration from 0 and 1, we can obtain

$$\sum_{l=0}^{2^n} \{\sum_{k=0}^{2^n} g_k \Phi_{n,k}^{\overbrace{\int\cdots\int}^{J}}(x_l) - \sum_{i=1}^{J} \alpha_i(x_l) \sum_{k=0}^{2^n} g_k \Phi_{n,k}^{\overbrace{\int\cdots\int}^{J}}(t_i) + \sum_{j=0}^{2^n} \Theta(x_l,y_j)g_j\Phi_{n,j}^{\int}(1)\}\bar{\Gamma}_{\varsigma,l}^{0,0}$$
$$= \sum_{l=0}^{2^n} Q(x_l)\bar{\Gamma}_{\varsigma,l}^{0,0}. \tag{51}$$

Because the matrix $\{\bar{\Gamma}^{0,0}_{\varsigma,l}\}$ is nonsingular, then Eq. (51) can be changed into

$$\sum_{k=0}^{2^n} g_k \Phi_{n,k}^{\overbrace{\int\cdots\int}^{J}}(x_l) - \sum_{i=1}^{J}\alpha_i(x_l)\sum_{k=0}^{2^n} g_k \Phi_{n,k}^{\overbrace{\int\cdots\int}^{J}}(t_i) + \sum_{j=0}^{2^n}\Theta(x_l,y_j)g_j\Phi_{n,j}^{\int}(1) = Q(x_l), \tag{52}$$

then

$$\sum_{k=0}^{2^n} g_k[\Phi_{n,k}^{\overbrace{\int\cdots\int}^{J}}(x_l) - \sum_{i=1}^{J}\alpha_i(x_l)\Phi_{n,k}^{\overbrace{\int\cdots\int}^{J}}(t_i) + \Theta(x_l,y_k)\Phi_{n,k}^{\int}(1)] = Q(x_l). \tag{53}$$

Also, it can be changed into matrix form as

$$\mathbf{Hg} = \mathbf{q} \tag{54}$$

where $\mathbf{H} = \{h_{l,k}\}$, $\mathbf{g} = \{g_k\}^T$, $\mathbf{q} = \{Q(x_l)\}^T$, $l,k = 0,1,\cdots,2^n$, and

$$h_{l,k} = \Phi_{n,k}^{\overbrace{\int\cdots\int}^{J}}(x_l) - \sum_{i=1}^{J}\alpha_i(x_l)\Phi_{n,k}^{\overbrace{\int\cdots\int}^{J}}(t_i) + \Theta(x_l,y_k)\Phi_{n,k}^{\int}(1). \tag{55}$$

Solving the Eq. (54) we can obtain the solution $g = \{g_k\}^T$.

5 Numerical examples

Six numerical examples are considered in this section, which include two linear integral equations with continuous kernels (Examples 1 and 2); two nonlinear integral equations with continuous kernel (Example 3 and 4); two nonlinear integral equations with weakly singular kernel (Example 5 and 6). All these examples were solved by using the scaling function with N=6 and $M_1 = 7$.

Example 1

As the first example, we consider [Liang, Liu and Che (2001); Xiao, Wen and Zhang (2006)]

$$f(x) + \int_0^1 \sin(4\pi x + 2\pi y)f(y)dy = \cos(2\pi x) + \frac{1}{2}\sin(4\pi x) \tag{56}$$

with exact solution: $f(x) = \cos(2\pi x)$.

Equation (56) has been solved by Liang et al. [Liang, Liu and Che (2001)] and Xiao et al. [Xiao, Wen and Zhang (2006)] by using different methods. Liang et al. [Liang, Liu and Che (2001)] have shown that using the Galerkin method of Daubechies wavelets in solving the integral equations has almost the same accuracy as that of noncontinuous multiwavelets. For the error defined by $\varepsilon = ||$*Exact solution – Approximation solution*$||_2$, Liang et al. [Liang, Liu and Che (2001)] shows that $\varepsilon = O(10^{-4})$ when the resolution level $n = 5$ and $O(10^{-7})$ when the resolution level $n = 8$. When Eq. (56) is solved by using the Galerkin method of periodic Daubechies wavelets [Xiao, Wen and Zhang (2006)], the maximum error becomes $O(10^{-7})$ when the resolution level $n = 5$.

Table 2 shows the numerical results for example 1 based on our approach 1 when n = 3, 4, 5. It can be seen from Tab. 2 that numerical results obtained by the proposed method can reach much higher precision, with maximum absolute error on the order of $O(10^{-16})$, than the results obtained by other methods [Liang, Liu and Che (2001); Xiao, Wen and Zhang (2006)].

Table 2: Absolute Errors for Example 1.

x	n=3	n=4	n=5
0	2.220446049250313e-16	0.000000000000000E+00	0.000000000000000e+00
0.125	4.440892098500626e-16	0.000000000000000E+00	0.000000000000000e+00
0.25	6.901693502643938e-17	4.648517068615585e-17	7.80781917894856e-18
0.375	3.330669073875470e-16	0.000000000000000e+00	2.22044604925031e-16
0.5	4.440892098500626e-16	4.440892098500626e-16	2.22044604925031e-16
0.625	3.330669073875470e-16	1.110223024625157e-16	0.000000000000000e+00
0.75	2.431988140355359e-18	1.666961094280448e-17	5.12378556944903e-17
0.875	4.440892098500626e-16	1.110223024625157e-16	0.000000000000000e+00
1	0.000000000000000e+00	0.000000000000000e+00	1.11022302462516e-16

Example 2

$$f(x)+\int_0^1 \sin(\sqrt{y}+x)yf(y)dy = A\cos(x)+B\sin(x)+x \tag{57}$$

in which $A = 130\sin(1)-202\cos(1)$, $B = 130\cos(1)+202\sin(1)-240$. Exact solution: $f(x) = x$. As the kernel "$\sin(\sqrt{y}+x)y$" has no definition when $y < 0$, the approach 1 in this case is no longer valid. Thus we use the approach 2 based on the modified $L^2[0, 1]$ Coiflets-like bases to solve Eq. (57). Table 3 gives the results when n=3, 4, 5. It can be seen that, the proposed approach 2 can also have very high accuracy. The absolute error is on the order of $O(10^{-8})$ for n=5.

Table 3: Absolute Errors for Example 2.

x	n=3	n=4	n=5
0	1.252297980161853e-05	9.867446204977927e-07	8.28536162872373e-08
0.125	1.196574130396533e-05	9.479860868244483e-07	7.97777389982546e-08
0.25	1.122178134765894e-05	8.944344797034987e-07	7.54569567384955e-08
0.375	1.030270927804278e-05	8.269255532544761e-07	6.99586940045016e-08
0.5	9.222866867442114e-06	7.465127214167922e-07	6.33687492479496e-08
0.625	7.999104726819084e-06	6.544508311945307e-07	5.57899559883168e-08
0.75	6.650519214224104e-06	5.521764361304804e-07	4.73405796785897e-08
0.875	5.198154605534633e-06	4.412855452162745e-07	3.81524697390390e-08
1	3.664674494308073e-06	3.235085015429462e-07	2.83690009261761e-08

Example 3

$$y(x) = x^3 + \frac{1}{3}(\cos 1 - 1) + \int_0^1 s^2 \sin(y(s))ds. \tag{58}$$

Exact solution: $y(x) = x^3$. Table 4 shows the numerical results for example 3 based on approach 1 when n = 3, 4, 5. This nonlinear integral equation with continuous kernel also has been solved by the authors of reference [Galperin, Kansa, Makroglou and Nelson (2000)] based on the trapezoidal formula and other techniques associated with variable transformations. For the results with 40 grid points obtained by using the second order Korobov transformation [Galperin, Kansa, Makroglou and Nelson (2000)] and the third order sidi transformation [Galperin, Kansa, Makroglou and Nelson (2000)], the maximum absolute error is on the order of $O(10^{-6})$. However as shown in Tab. 4, our results have the maximum absolute error on the order of $O(10^{-7})$ when n = 4, corresponding to 16 grid points.

Table 4: Absolute Errors for Example 3.

x	n=3	n=4	n=5
0	3.531590655497861e-05	1.27207428490686e-07	6.13724326525990e-10
0.125	3.531590655497861e-05	1.27207428466487e-07	6.13724352371489e-10
0.25	3.531590655497796e-05	1.27207428490123e-07	6.13724333506371e-10
0.375	3.531590655497796e-05	1.27207428486653e-07	6.13724342179989e-10
0.5	3.531590655497796e-05	1.27207428479714e-07	6.13724335241095e-10
0.625	3.531590655497796e-05	1.27207428479714e-07	6.13724349118883e-10
0.75	3.531590655497796e-05	1.27207428424203e-07	6.13724293607731e-10
0.875	3.531590655503347e-05	1.27207428479714e-07	6.13724404630034e-10
1	3.531590655503347e-05	1.27207428479714e-07	6.13724404630034e-10

Example 4

$$f(x) - a\int_0^1 xyf^b(y)dy = q(x) \tag{59}$$

when a=1, b=3, and $q(x) = e^x$-$(x+2e^3x)/9$, the exact solution can be $f(x) = e^x$. This nonlinear integral equation has been solved in references [Mahmoudi (2005); Babolian and Shahsavaran (2009)] by numerical methods based on the Legendre and Haar wavelets, respectively. The maximum absolute error is about $O(10^{-2})$ and $O(10^{-3})$ when the resolution level is n = 5 in [Babolian and Shahsavaran (2009)] and n=3 in [Mahmoudi (2005)]. Table 5 lists the numerical results for example 4 based on our approach 1 when n = 3, 4, 5. However, as shown in Tab. 5, the results by using the proposed approach 1 have the maximum absolute error $O(10^{-6})$ for n=3, which is obviously much better.

When a=1/2, b=2, and $q(x)$ =7x/8. The exact solution is simply $f(x) = x$. This nonlinear integral equation has also been numerically solved by Sahu et al. [Sahu and Ray (2013)] by using the semi-orthogonal linear B-spline wavelets. They have shown that the maximum absolute error is about $O(10^{-5})$ when the resolution level is n=4 [Sahu and Ray (2013)]. Table 6 lists the absolute errors of the numerical solutions obtained by using the proposed approach 1 when n=3, 4, and 5, respectively. When the resolution level n=3, our results exhibit the absolute error nearly $O(10^{-13})$, being several orders of magnitude smaller.

Table 5: Absolute Errors for Example 4 with a=1, b=3.

x	n=3	n=4	n=5
0	0.000000000000000e+00	0.000000000000000e+00	0.000000000000000e+00
0.125	1.323169851907835e-06	9.514252274911428e-09	7.15421055730303e-11
0.25	2.646339703815670e-06	1.902850477186746e-08	1.43084433190666e-10
0.375	3.969509555945550e-06	2.854275704677889e-08	2.14627204897511e-10
0.5	5.292679407631340e-06	3.805700954373492e-08	2.86168866381331e-10
0.625	6.615849259983264e-06	4.757126181864635e-08	3.57711193998966e-10
0.75	7.939019111891099e-06	5.708551409355778e-08	4.29254409795021e-10
0.875	9.262188964687113e-06	6.659976703460302e-08	5.00796737412657e-10
1	1.058535881526268e-05	7.611401908746984e-08	5.72337732762662e-10

Example 5

$$f(x) - \int_0^1 \ln|x-y|\,dy = x - 0.5[x^2\ln x + (1-x^2)\ln(1-x) - x - 0.5]. \tag{60}$$

Table 6: Absolute Errors for Example 4 with a=1/2, b=2.

x	n=3	n=4	n=5
0	0.000000000000000e+00	0.000000000000000e+00	0.000000000000000e+00
0.125	1.00114361245573e-13	5.01543251374414e-14	3.18634008067420e-14
0.25	2.00228722491147e-13	1.00308650274883e-13	6.37268016134840e-14
0.375	3.00370839312336e-13	1.50490730987940e-13	9.56457135714572e-14
0.5	4.00457444982294e-13	2.00617300549766e-13	1.27453603226968e-13
0.625	5.00599561803483e-13	2.50799381262823e-13	1.59428026336172e-13
0.75	6.00741678624672e-13	3.00981461975880e-13	1.91291427142914e-13
0.875	7.00772773143399e-13	3.51163542688937e-13	2.23154827949656e-13
1	8.00914889964588e-13	4.01234601099532e-13	2.54907206453936e-13

Exact solution: $f(x) = x$. This is an integral equation of second kind with weakly singular kernel. Equation (60) has been solved by the methods of reference [Liang, Liu and Che (2001); Panigrahi and Nelakanti (2012)]. When the error is defined as $\varepsilon = ||Exact\ solution - Approximation\ solution||_2$, Liang et al. [Liang, Liu and Che (2001)] have shown that this error associated with their results is on the order of $O(10^{-4})$ when the resolution level n = 5 and $O(10^{-7})$ when the resolution level n = 8. In addition, this error ε becomes $O(10^{-2})$ when 8 grid points are used in reference [Panigrahi and Nelakanti (2012)]. To solve this equation by our method, we define $J = 3$, $t_1 = 0.2$, $t_2 = 0.6$, $t_3 = 0.9$. Table 7 gives the numerical results when n = 3, 4, 5. Interestingly, for the present method, the maximum absolute error is $O(10^{-14})$ when only 8 grid points, corresponding to n=3, have been used.

Table 7: Absolute Errors for Example 5.

x	n=3	n=4	n=5
0	3.319566843629218e-14	8.881784197001252e-15	2.22044604925031e-16
0.125	3.574918139293004e-14	9.228728892196614e-15	1.24900090270330e-16
0.25	3.380629109983602e-14	8.215650382226158e-15	4.99600361081320e-16
0.375	3.169686735304822e-14	7.216449660063518e-15	9.43689570931383e-16
0.5	3.042011087472929e-14	6.994405055138486e-15	1.38777878078145e-15
0.625	3.042011087472929e14	7.438494264988549e-15	1.88737914186277e-15
0.75	3.252953462151709e-14	7.438494264988549e-15	2.44249065417534e-15
0.875	3.541611448554249e-14	7.438494264988549e-15	2.99760216648792e-15
1	3.375077994860476e-14	6.883382752675971e-15	3.55271367880050e-15

Example 6

$$y(x) = f(x) + \int_0^1 y(s)/\sqrt{|x-s|}ds \tag{61}$$

where $f(x) = x - 2x\sqrt{1-x} - \frac{2}{3}(1-x)^{3/2} - \frac{4}{3}x^{3/2}$. Exact solution: $y(x) = x$. This is also an integral equation of second kind with weakly singular kernel. Galperin et al. [Galperin, Kansa, Makroglou and Nelson (2000)] have solved Eq. (61) by using a method based on the trapezoidal formula and variable transformations. When the fourth order sidi transformation is used, for the results with 40 grid points, Galperin et al. [Galperin, Kansa, Makroglou and Nelson (2000)] have shown that the absolute maximum errors of their results is about $O(10^{-3})$. To solve this equation by using our method, we define J = 3, t_1 = 0.2, t_2 = 0.6, t_3 = 0.9. Table 8 gives the numerical results of Eq. (61) by using the proposed method when n = 3, 4, 5. It can be seen that the maximum absolute error can reach $O(10^{-12})$ with only 8 grid points has been used, which shows again much smaller error of the proposed method than the method given by Galperin et al. [Galperin, Kansa, Makroglou and Nelson (2000)].

Table 8: Absolute Errors for Example 6.

x	n=3	n=4	n=5
0	1.76285652742081e-11	7.24753590475302e-13	3.69482222595252e-13
0.125	5.77654590827592e-12	2.68646216383672e-13	1.64201985342061e-13
0.25	2.63541966027958e-12	6.27553564669370e-14	1.00475183728577e-14
0.375	7.62706564572113e-12	2.70117261891301e-13	9.29811783123569e-14
0.5	9.18681797301701e-12	3.53606033343112e-13	1.44995127016045e-13
0.625	7.30504545742861e-12	3.13304937549219e-13	1.45994327738208e-13
0.75	1.96931360108010e-12	1.49102952207159e-13	9.60342916300760e-14
0.875	6.80877576542116e-12	1.39110944985532e-13	4.99600361081320e-15
1	1.90141236089403e-11	5.51558798633778e-13	1.57207580286922e-13

6 Conclusions

We have proposed the Coiflet-based methods to deal with the nonlinear integral equations with weakly singular kernels. It can be seen from the numerical examples that the proposed approach is very efficient and accurate comparing with several existing methods.

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