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Time-Domain BEM Analysis for Three-dimensional Elastodynamic Problems with Initial Conditions

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Abstract: In this paper, a time-domain boundary element method formulation for the analysis of three-dimensional elastodynamic problems with arbitrary, non-null initial conditions is presented. The formulation is based on the convolution quadrature method, by which the numerical stability is improved significantly. In order to take into account the non-null initial conditions in this formulation, a general method is developed to replace the initial conditions by equivalent pseudo-forces based on the pseudo-force method. The original governing equation is transformed into a new one subjected to null initial conditions. In the numerical examples, longitudinal vibrations of a free beam and a cantilevered beam are considered as the illustrative problems to evaluate the new formulation. Results are shown to be in good agreement with the analytical solutions or the finite element method solutions.

Keywords: time-domain BEM, convolution quadrature method, non-null initial conditions, pseudo-force method.

1 Introduction

The boundary element method (BEM) is an efficient numerical technique for solving engineering problems [Brebbia (1978); Cheng and Cheng (2005)], such as Laplace equation, Navier equation, Helmholtz equation and linear diffusion-reaction equation [Burton and Miller (1971); Jaswon and Symm (1977); Rizzo (1967); Rizzo and Shippy (1970)]. For transient elastodynamic problems, according to the different approximation solution strategies in time space, the BEM generally follows two approaches namely, the frequency-domain approaches [Ahmad and Manolis (1987); Xiao et al. (2012)] and the time-domain approaches [Manolis and Beskos (1988); Yao (2009)]. A review about these methods is published by Beskos

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(1987; 1997).

In this paper, we concern the time-domain method. The implementation of the time-domain approach is straightforward. The time-domain formulation is more general and the problem can be solved not only for harmonic or periodic load, but also for a much wider range of loads. At the earliest Mansur (1983) established the general formulation for the scalar wave equation. Subsequently, Banerjee and Manolis et al. (1986) presented the direct BEM formulation for three-dimensional (3D) transient elastodynamic analysis, which is the most widely used and classical formulation at present. However, numerical results have shown that the standard BEM formulation may be unstable in some applications. This phenomenon has been first mentioned in a paper of Cole and Kosloff (1978). An in-depth study on this

unstable behavior was carried out by Dominguez and Gallego (1991), and conclusions are summarized as: (I) a too small time-step may cause instability in the numerical scheme, (II) while a too large time-step may give rise to strong numerical damping in the results. In recent years, several approaches have been proposed to improve the stability. Among them, three ways should be mentioned as follows: (I) the first one is called 'linear θ method' which employs modified numerical time marching procedures proposed by Araujo et al. (1999). They achieved more stable results by introducing weighting integrals. Other similar and effective schemes include: the ε method, the half-step method [Birgisson et al. (1999)], the constant velocity prediction method [Marrero et al. (2003)], and the $\alpha - \delta$ method [Soares and Mansur (2007)]; (II) the second one is called 'Heaviside fundamental solution method' which employs a modified time dependent fundamental solution proposed by Coda and Venturini (1995). Another similar scheme is using a new boundary integral equation (BIE) based on the velocity reciprocal theorem to reduce instability in BEM formulations, proposed by Panayiotopoulos and Manolis (2011); (III) the third one is called 'convolution quadrature method' (CQM) based on the theorem proposed by Lubich (1988). This was then developed and applied to time domain BEM formulations by Schanz and Antes (1997; Schanz, 2010). In this formulation, the convolution integral is numerically approximated by a quadrature formula whose weights are determined by the Laplace transformed fundamental solutions and a linear multistep method. Through a comparative numerical study of these three groups of methods, the CQM outperforms the other methods in terms of stability and accuracy [Li and Zhang (2014)]. Besides, a much older paper of Gaul and schanz (1999) compared the performances of three approaches, including the calculation in Laplace domain, the calculation of convolution analytically in time domain and the calculation based on convolution quadrature method (CQM), to calculate the transient response of viscoelastic solids. The numerical results also demonstrated that the formulation based on the convolution quadrature is not as sensitive as others in terms of time step size.

It is worth noting that the CQM approach provides the solution directly in the time domain without the need of an inverse transformation in spite of employing the frequency domain fundamental solution. This BEM formulation was initially developed for elastodynamic problems with null initial conditions [Schanz and Antes (1997)]. Nonzero initial conditions give rise to domain integrals in time-domain integral equation approaches. These terms involve domain integrals without time convolution, so can not be integrated numerically approximated by CQM. If we compute these terms directly using domain discretization methods and the time-dependent fundamental solutions just like the conventional time-domain BEM, judgment of wave fronts and spherical surface integrals are required because that Heaviside functions and Dirac delta functions are included in the time-dependent fundamental solutions. This will lead to complicate geometry operations, especially for complicate structures. In order to overcome this limitation, a general procedure is proposed in this paper to replace the initial conditions by equivalent pseudo-forces based on the pseudo-force method. The pseudo-force method was already successfully employed in the frequency-domain analysis and the CQM-based time-domain analysis of 2D scalar wave propagation problems [Abreu et al. (2006); Mansur et al. (2004)]. It was also employed in a fast Fourier transform-based spectral element method for the linear continuum dynamic systems subjected to non-null initial conditions by Lee and Cho (2008). To the authors' best knowledge, the CQM-based time-domain BEM has not yet been developed for the analysis of 3D elastodynamic problems with non-null initial conditions.

In the following Section 2, we will develop a general method to transform the original governing equations of linear elastic dynamic system subjected to arbitrary, non-null initial conditions into a new set of governing equations with completely null initial conditions based on the pseudo-force method. In our method, the whole transformation process is based on the governing equations. Thus, this method can also be applied to the conventional time-domain BEM formulations, which employ the time-dependent fundamental solutions, to avoid spherical surface integrals of the terms involved initial conditions. After that, a brief introduction to the CQM-based time-domain BEM formulation and its numerical implementation are presented in Section 3. To verify the applicability of the proposed method, longitudinal vibration analyses of a slender free beam and a cantilevered beam are considered as the illustrative examples in Section 4. And finally, the paper ends with conclusions in Section 5.

2 The transformation of initial conditions by pseudo-force method

The pseudo-force method was initially proposed by Mansur et al. (2004) and developed by Abreu et al. (2006) both for 2D scalar wave propagation problems. Now we develop it to deal with the non-zero initial conditions for 3D elastodynamic problems but not directly. The basic idea of the pseudo-force method is transforming the contribution of initial conditions to equivalent body forces. Although the methodology is quite general, the transformation procedure is somewhat different from that of 2D case because the governing equation of 3D elastodynamics is more complicate. Furthermore, a new governing equation with zero initial conditions will be deduced to help the transformation procedure easier to understand in this section, which is not presented in the original paper of 2D case.

2.1 The transformation of initial conditions for the general case

The governing equation of elastodynamic problems presented here is corresponding to small displacement theory for homogeneous, isotropic, linearly elastic materials. The displacement equations of motion can be written as

$$(\lambda+G)u_{j,ij}(q,t)+Gu_{i,jj}(q,t)+\rho b_i(q,t)=\rho\ddot{u}_i(q,t) \tag{1}$$

where $i,j=1,2,3$, u_i is the displacement component, b_i represents the body force per mass, ρ is the medium density and $\ddot{u}_i$ is the second derivative of displacement with respect to time t, λ and G are the Lame constants given by

$$\lambda=\frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)} \quad \text{and} \quad G=\frac{E}{2(1+\upsilon)} \tag{2}$$

in which E and ν represent Young's modulus and Poisson's ratio respectively.
Eq. (1) satisfies the boundary conditions

$$\begin{aligned} &u_i(q,t)=\bar{u}_i(q,t), && \forall q\in S^{ui}\\ &p_i(q,t)=\sigma_{ij}(q,t)n_j(q)=\bar{p}_i(q,t), && \forall q\in S^{pi} \end{aligned} \tag{3}$$

in which σ_{ij} and p_i are the stress components and boundary traction respectively, n_j is the normal vector to boundary S at point q, $\bar{u}_i$ and $\bar{p}_i$ are prescribed boundary displacement and traction on S^{ui} and S^{pi}, respectively, and the non-zero initial conditions

$$\begin{aligned} &u_i(q,0)=u_i^0(q), && \forall q\in V\\ &\dot{u}_i(q,0)=v_i^0(q), && \forall q\in V \end{aligned} \tag{4}$$

in which $\dot{u}_i$ is the first derivative of displacement with respect to time.

Firstly, on the basis of the superposition principle valid for linear time-invariant systems, we can assume the complete solution of Eq. (1) can be computed by adding the initial displacement to the displacement increment, just as

$$u_i(q,t) = u_i(q,0) + \Delta u_i(q,t) \tag{5}$$

Substituting Eq. (5) into Eq. (1) gives

$$\begin{aligned}&(\lambda + G)\Delta u_{j,ij}(q,t) + G\Delta u_{i,jj}(q,t) + [(\lambda + G)u_{j,ij}(q,0) + Gu_{i,jj}(q,0)]\\&+\rho b_i(q,t) = \rho \Delta \ddot{u}_i(q,t)\end{aligned} \tag{6}$$

We can define the pseudo-force per mass for the initial displacement as

$$f_i^{u0}(q,t) = \frac{1}{\rho}[(\lambda + G)u_{j,ij}(q,0) + Gu_{i,jj}(q,0)]H(t-0) \tag{7}$$

where $H(t-0)$ represents the Heaviside function. It is different from that in 2D wave propagation problems. It should be mentioned that this pseudo-force is divided by the density in order to preserve uniformity with the definition of body force in the governing equation.

Then, based on the momentum principle "impulse=momentum change", the contribution of the change in momentum from $t = 0^-$ to $t = 0$ is equivalent to that of the impulse force $f_i^{v0}(q,t)$ per mass, i.e.

$$\int_V \rho \int_{0^-}^{0} f_i^{v0}(q,t)dtdV(q) = \int_V \rho[\dot{u}_i(q,0) - \dot{u}_i(q,0^-)]dV(q) \tag{8}$$

in which, $\dot{u}_i(q,0^-) = 0$. It means the system is initially at rest until an impulse force is applied. There is no instantaneous change in the displacement because the time duration is too short for displacement to develop.

By substituting the initial velocity in Eq. (4) into the Eq. (8), we can obtain the pseudo-force for initial velocity just as

$$f_i^{v0}(q,t) = v_i^0(q)\delta(t) \tag{9}$$

in which, $\delta(t)$ is the Dirac delta function centered at time $t = 0$ and its value is zero at all times other than zero. This is same as the pseudo-force for initial velocity in 2D case [Abreu et al. (2006)] although the formulations are different. A discrete time sampling of the pseudo-force will be presented in the next sub-section.

Finally, using the pseudo-forces in Eq. (7) and Eq. (9), the Eq. (1) with non-zero initial conditions can be transformed into a new set of governing equation with respect to the displacement increment $\Delta u_i(q,t)$ as

$$(\lambda+G)\Delta u_{j,ij}(q,t)+G\Delta u_{i,jj}(q,t)+\rho[b_i(q,t)+f_i^{u0}(q,t)+f_i^{v0}(q,t)]=\rho\Delta\ddot{u}_i(q,t) \tag{10}$$

with completely null initial conditions

$$\begin{aligned} &\Delta u_i(q,0) = 0, \quad \forall q \in V \\ &\Delta \dot{u}_i(q,0) = 0, \quad \forall q \in V \end{aligned} \tag{11}$$

and new boundary conditions

$$\begin{aligned} \Delta\bar{u}_i(q,t) &= \bar{u}_i(q,t) - u_i(q,0), \quad \forall q \in S^{ui} \\ \Delta\bar{p}_i(q,t) &= \bar{p}_i(q,t) - p_i(q,0), \quad \forall q \in S^{pi} \end{aligned} \tag{12}$$

The complete solution of the original governing Eq. (1) can be obtained by first solving the transformed Eq. (10) with completely null initial conditions by considering the pseudo-forces $f_i^{u0}(q,t)$ and $f_i^{v0}(q,t)$ as the effective external forces, and then adding the result $\Delta u_i(q,t)$ to the initial displacement $u_i(q,0)$.

2.2 Calculation of the pseudo-forces

Before solving the Eq. (10), one must find the nodal values of the pseudo-force $f_i^{u0}(q,t)$ and $f_i^{v0}(q,t)$ at all domain points. When t>0, we can simplify the $f_i^{u0}(q,t)$ in Eq. (7) into

$$f_i^{u0}(q) = \frac{1}{\rho}[(\lambda + G)u_{j,ij}(q,0) + Gu_{i,jj}(q,0)] \tag{13}$$

As mentioned in the paper of Mansur et al. (2004), the pseudo-force $f_i^{u0}(q)$ can be computed through two ways:

I. When the analytical expression for $u_i(q,0)$ is available, it is sufficient to compute the Eq. (13) analytically to obtain $f_i^{u0}(q)$.

II. When the values of $u_i(q,0)$ can not be expressed analytically, we can compute $f_i^{u0}(q)$ numerically by solving the following well-known Navier equation

$$(\lambda + G)u_{j,ij}(q,0) + Gu_{i,jj}(q,0) + [-\rho f_i^{u0}(q)] = 0 \tag{14}$$

which is the governing equation of elastostatics problem. In Eq. (14), the displacement $u_i(q,0)$ is introduced as the input data and the body force $f_i^{u0}(q)$ represents the unknown variable. Numerical implementation in detail can refer to the reference [Brebbia (1978)]. However, for 2D scalar wave propagation problems, it will become a Poisson equation.

If we impose the effect of the impulse pseudo-force $f_i^{v0}(q,t) = v_i^0(q)\delta(t)$ to a short time duration $[0,\Delta t]$ averagely, the equivalent pseudo-force for initial velocity will become

$$f_i^{v0}(q) = v_i^0(q)/\Delta t \tag{15}$$

which can be computed easily no matter the values of $v_i^0(q)$ can be expressed analytically or not. After computing all nodal values of the pseudo-force $f_i^{u0}(q,t)$ and $f_i^{v0}(q,t)$, we can employ the CQM-based BEM to solve the new governing equation (10).

3 The CQM-based BEM formulation and its numerical implementation

The time-domain boundary integral formulation corresponding to the governing Eq. (10) can be expressed as (Manolis and Beskos, 1988):

$$
\begin{aligned}
C_{ij}(p)\Delta u_j(p,t) = &\int_s \int_0^t u_{ij}^s(p,q;t-\tau)\Delta p_j(q,\tau)d\tau dS(q) \\
&- \int_s \int_0^t p_{ij}^s(p,q;t-\tau)\Delta u_j(q,\tau)d\tau dS(q) \\
&+ \rho \int_V \int_0^t u_{ij}^s(p,q;t-\tau)b_j(q,\tau)d\tau dV(q) \\
&+ \rho \int_V \int_0^t u_{ij}^s(p,q;t-\tau)f_j^{v0}(q,\tau)d\tau dV(q) \\
&+ \rho \int_V \int_0^t u_{ij}^s(p,q;t-\tau)f_j^{u0}(q,\tau)d\tau dV(q)
\end{aligned}
\tag{16}
$$

where $C_{ij} = \delta_{ij}/2$ when the source point p belongs to smooth surfaces, $C_{ij} = \delta_{ij}$ when p in domain ($\delta_{ij} = 1$ when $i = j$, otherwise $\delta_{ij} = 0$), $u_{ij}^s(p,q;t-\tau)$ and $p_{ij}^s(p,q;t-\tau)$ are time-dependent fundamental solutions, represent the displacement and traction in direction j at field point q and at time t due to a unit concentrated load at source point p in direction i and at time τ , respectively.

Unlike dealing with the time integration analytically in the conventional time-domain BEM, the convolution between the fundamental solutions and the corresponding nodal values in Eq. (16) is performed numerically in the CQM-based BEM. If we discretize the time interval $[0,t]$ into M time steps of duration Δt, the discretized time nodal $t_m = m\Delta t$, where m=*1,2,...,M*. Then, apply the CQM procedure and the pseudo-forces $f_i^{u0}(q)$ in Eq. (13) and $f_i^{v0}(q)$ in Eq. (15) to Eq. (16), the following representation arises

$$
\begin{aligned}
C_{ij}(p)\Delta u_j(p,m\Delta t) &= \sum_{a=0}^{m}\int_s \omega_{m-a}(\hat{u}_{ij}^s,p,\Delta t)\Delta p_j(q,a\Delta t)ds(q) \\
&-\sum_{a=0}^{m}\int_s \omega_{m-a}(\hat{p}_{ij}^s,p,\Delta t)\Delta u_j(q,a\Delta t)ds(q) \\
&+\rho\sum_{a=0}^{m}\int_V \omega_{m-a}(\hat{u}_{ij}^s,p,\Delta t)b_j(q,a\Delta t)dV(q) \\
&+\rho\int_V \omega_m(\hat{u}_{ij}^s,p,\Delta t)f_j^{v0}(q)dV(q)+\rho\sum_{a=0}^{m}\int_V \omega_{m-a}(\hat{u}_{ij}^s,p,\Delta t)f_j^{u0}(q)dV(q)
\end{aligned}
\tag{17}
$$

In this formulation, the effect of the pseudo-force $f_i^{v0}(q,t)=v_i^0(q)\delta(t)$ is imposed to the first time step $[0,\Delta t]$ averagely, being null at the subsequent ones.

For more details of the derivation process of CQM, please refer to Reference [Schanz and Antes (1997)]. The quadrature weights in the above formulation are

$$\omega_m^(\hat{u}_{ij}^s,p,\Delta t)=\frac{\Re^{-m}}{M+1}\sum_{l=0}^{M}[\hat{u}_{ij}^s(p,q,\frac{\gamma(\Re e^{il\frac{2\pi}{M+1}})}{\Delta t})e^{-ilm\frac{2\pi}{M+1}}] \tag{18}$$

$$\omega_m^(\hat{p}_{ij}^s,p,\Delta t)=\frac{\Re^{-m}}{M+1}\sum_{l=0}^{M}[\hat{p}_{ij}^s(p,q,\frac{\gamma(\Re e^{il\frac{2\pi}{M+1}})}{\Delta t})e^{-ilm\frac{2\pi}{M+1}}] \tag{19}$$

Here, $\gamma(z)$ (where $z=\Re e^{il\frac{2\pi}{M+1}}$) is the quotient of the characteristic polynomials of a linear multistep method according to Lubich (1988), for example, the backward differentiation formula of second order $\gamma(z)=1.5-2z+0.5z^2$. If an error ε ($\varepsilon \geq 10^{-10}$ according to Abreu et al. (2006)) is assumed in the computation of $\hat{u}_{ij}^s(p,q,\frac{\gamma(z)}{\Delta t})$, the choice of $\Re^{M+1}=\sqrt{\varepsilon}$ yields an error of order $O(\sqrt{\varepsilon})$. Note that the calculation of the quadrature weights (19) and (20) is based on the Laplace transformed fundamental solutions $\hat{u}_{ij}^s(p,q,s)$ and $\hat{p}_{ij}^s(p,q,s)$, see Reference [Manolis and Beskos (1988)].

In order to treat the corners of a body surfaces in a simple manner, we adopted discontinuous elements to discretize the boundary and domain of the model. After discretizing all faces with *BN* discontinuous elements and domain with *DN* discontinuous cells, integrals in Eq. (17) will be carried out over all elements and cells.

The discretized version of Eq. (17) for each source point q_k can be written as:

$$\begin{aligned} C_{ij}^{(}q_k)\Delta u_j^m(q_k) &= \sum_{a=0}^{m}\sum_{n=0}^{BN-1}\Delta p_j^a(q_n)G_{ij}^a(q_k,q_n) - \sum_{a=0}^{m}\sum_{n=0}^{BN-1}\Delta u_j^a(q_n)H_{ij}^a(q_k,q_n) \\ &+\rho\sum_{a=0}^{m}\sum_{n=0}^{DN-1}(b_j^a(q_n)+f_j^{u0}(q_n))F_{ij}^a(q_k,q_n)+\rho\sum_{n=0}^{DN-1}f_j^{v0}(q_n)F_{ij}^m(q_k,q_n) \end{aligned} \tag{20}$$

It should be mentioned that in order to reduce the geometric error, the boundary face method (BFM) which is proposed by Zhang et al. (2009), is employed in the numerical implementation in this paper. The BFM is implemented directly based on the boundary representation data structure (B-rep) that is used in most CAD packages for geometry modeling. Each bounding surface of geometry model is represented as parametric form by the geometric map between the parametric space and the physical space. Both boundary integration and variable approximation are performed in the parametric space. The integrand quantities are calculated directly from the faces rather than from elements, and thus geometric error will be reduced significantly. This method has been successfully used in the calculations of 3D potential problems [Qin, et al. (2010)], elasticity problems [Huang et al. (2012); Zhou et al. (2012)], transient heat conduction problems [Guo et al. (2013); Zhou et al. (2013)], acoustic problems [Wang et al. (2013)] and buckling problems [Li et al. (2014)].

4 Numerical examples

Two numerical applications, e.g. longitudinal vibrations of a free beam and a cantilevered beam, are analyzed next in order to verify the accuracy of the numerical results provided by the formulation presented in this work. Both beams have the same geometric and material properties: the length $L = 8.0m$ and the width and height $W = H = 2.0m$ as shown in Fig. 1, the Young's modulus $E = 1.1\times10^5 N/m^2$ and the mass density $\rho = 2.0kg/m^3$. By setting the Poisson's ratio $\upsilon = 0.0$, the 3D problem is reduced to a 1D problem, of which the analytical solution is available [Graff (1975)].

The dimensionless parameter $\beta = c_1\Delta t/d$, where d is the characteristic element length, was used to control time-step length. In the traditional time-domain BEM, it is suggested that $\beta > 0.5$ because a too small time-step may cause instability in the numerical scheme. But in the CQM-based BEM, much smaller value of β can be adopted and will not induce the instability. In this two examples, we adopt $\beta = 0.2$ to achieve relatively more accurate results. The beam is discretized into 72 linear quadrilateral elements and 32 linear hexahedral elements as shown in Fig. 2(a). To treat the corners of a body surfaces in a simple manner, we adopted discontinuous

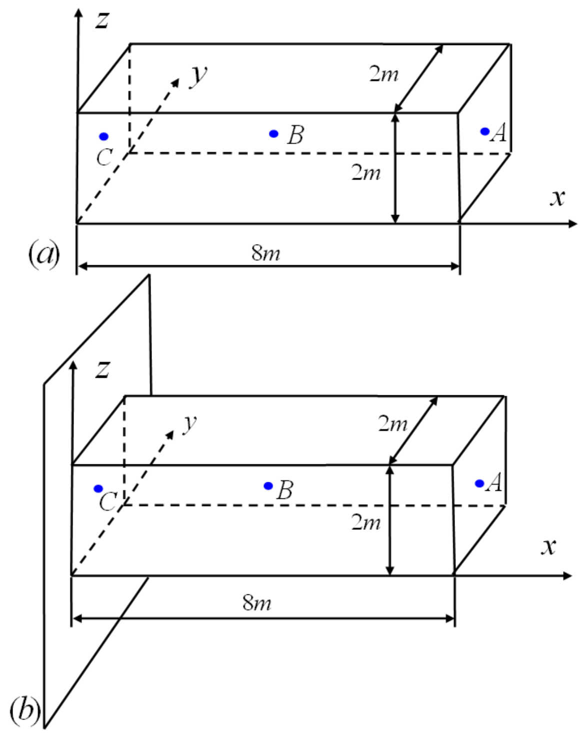

Figure 1: Geometry of the beam and the boundary conditions: (a) free beam; (b) cantilevered beam.

elements as shown in Fig. 2(b) and 2(c). The nodes on the face edge are shrunk to inside the face or the domain by a parameter λ, which is defined by $\lambda = a/b$, where b stands for the length of the element, a the distance of the node to the edge. It was observed that the best value for λ is 0.25 for linear quadrilateral element, and 0.167 for quadratic quadrilateral element.

4.1 longitudinal vibrations of a free beam under initial conditions

This example considers a slender beam free at two ends. Two analyses are performed: in the first analysis initial displacement is applied as

$$u_0(q) = -0.005x + 0.02, \quad (0 \le x \le 8) \tag{21}$$

and in the second analysis initial velocity is applied as

$$v_0(q) = 2.5x - 10, \quad (0 \le x \le 8) \tag{22}$$

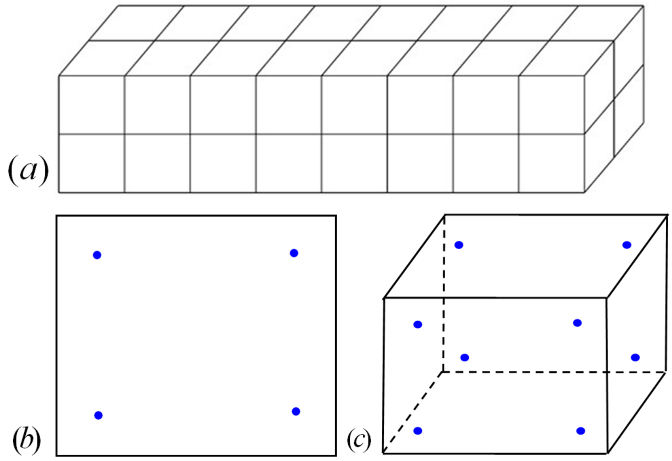

Figure 2: Discretized model and the discontinuous elements: (a) discretized model; (b) linear discontinuous quadrilateral element; (c) linear discontinuous hexahedral element.

For the case of $u_0(x) = ax + b$ and $v_0(x) = c_{v0}x + d$, the analytical solution of one-dimensional rod is given by

$$u(x,t) = \sum_{n=1,3,5...}^{\infty} \left(\frac{-4al}{n^2\pi^2}\cos\frac{cn\pi}{l}t + \frac{-4c_{v0}l^2}{cn^3\pi^3}\sin\frac{cn\pi}{l}t\right)\cdot\cos\frac{n\pi x}{l} \tag{23}$$

in which l represents the length of rod. For detail derivation, please refer to [Graff (1975)].

To demonstrate the accuracy of the CQM-based formulation, in Fig. 3 and Fig. 4 the displacement responses at point A(8,1,1) due to the initial displacement filed and due to the initial velocity filed, respectively, are compared with analytical solutions. One can observe that the numerical responses present a good agreement with the analytical ones initially, but the accuracy of results get worse with the increase in time. This is because the information from all earlier time steps must be used in time-domain BEM, the errors of numerical integration will accumulate at later time steps. To reduce the numerical damping, we can improve the precision of numerical integration, for examples, use an inadequate number of sub-elements or Gauss points and use high order interpolation functions. Recently, a Runge-Kutta convolution quadrature method is presented in Reference [Banjai et al. (2012)], which is preferable with regard to less numerical oscillations in the solution and better representation of wave fronts. It will be very helpful to improve the accuracy of results.

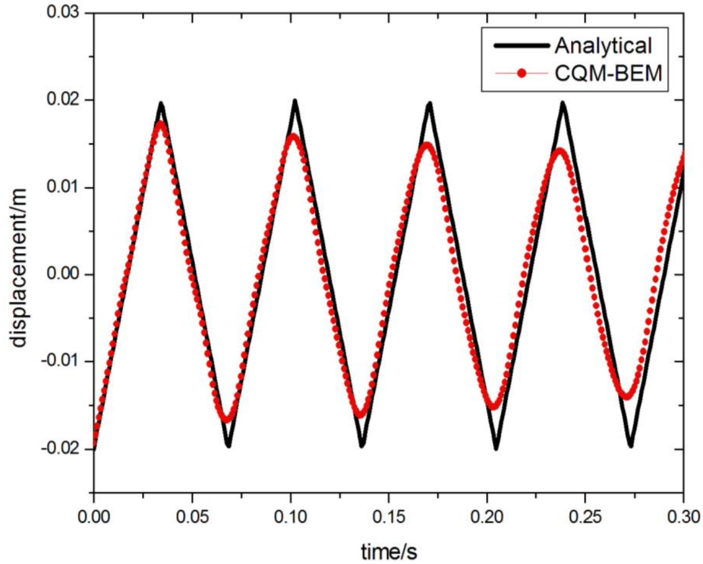

Figure 3: Displacement responses at point A of the free beam under initial displacement.

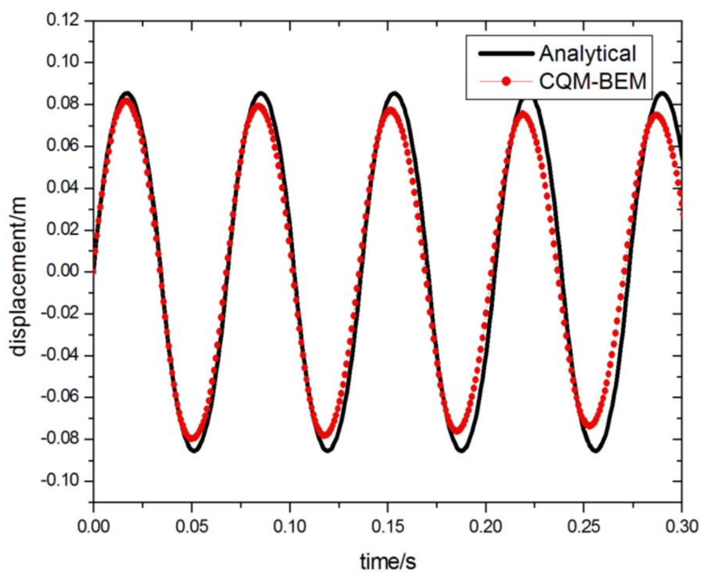

Figure 4: Displacement responses at point A of the free beam under initial velocity.

4.2 longitudinal vibrations of a cantilevered beam under initial conditions

In this case, a cantilevered beam fixed at one end (x=0) and free at the other end (x=8) is considered. Two analyses were performed. The first analyses is carried out by assuming initial conditions are

$$\begin{aligned} u_0(q) &= 0.005x, \quad (0 \le x \le 8) \\ v_0(q) &= 2.5, \qquad (0 \le x \le 8) \end{aligned} \tag{24}$$

For the case of $u_0(x) = ax + b$ and $v_0(x) = c_{v0}x + d$, the analytical solution of one-dimensional rod is given by

$$u(x,t) = \sum_{n=1,3,5\ldots}^{\infty} \left\{ \frac{4}{n\pi} \left[(-1)^{\frac{n-1}{2}} \frac{2al}{n\pi} + b \right] \cos \frac{cn\pi}{2l} t + \frac{8l}{cn^2\pi^2} \left[(-1)^{\frac{n-1}{2}} \frac{2c_{v0}l}{n\pi} + d \right] \sin \frac{cn\pi}{2l} t \right\} \cdot \sin \frac{n\pi x}{2l} \tag{25}$$

in which l represents the length of rod, $c = \sqrt{E/\rho}$ is the wave propagation velocity. The displacement responses at point A(8,1,1) and point B(4,1,1) obtained by the present CQM-based BEM are compared with analytical solutions in Fig. 5 and Fig. 6 respectively. The traction response at the fixed end of beam is compared with analytical solutions in Fig. 7. The numerical results are in a good agreement with the analytical ones, demonstrating the accuracy of the proposed formulation.

In the previous analyses, the pseudo-forces which account for initial displacements contribution are computed analytically because the analytical expression of $u_0(q)$ is available. But in the second analysis of this sub-section, the cantilevered beam is subjected to a non-uniform loading

$$p(x = 8) = 500z, \quad (0 \le z \le 2) \tag{26}$$

at the free end as shown in Fig. 8. After the status of the beam is stable, release the external force and observe the transient dynamic response of the beam. That is to say, the initial displacement in this analysis is due to the elastic deformation. In this case, the pseudo-forces which account for initial displacements contribution are computed numerically according to the procedure described in the Section 2.

In Fig. 9, the displacement response of x direction at point A(8,1,1) obtained by the CQM-based BEM is compared with the results provided by finite element method (FEM) software. Traction response of x direction at point C(0,1,1) is depicted in Fig. 10. Again, comparison is made with the results provided by the FEM software. The results are in a good agreement, demonstrating the applicability of the proposed method again.

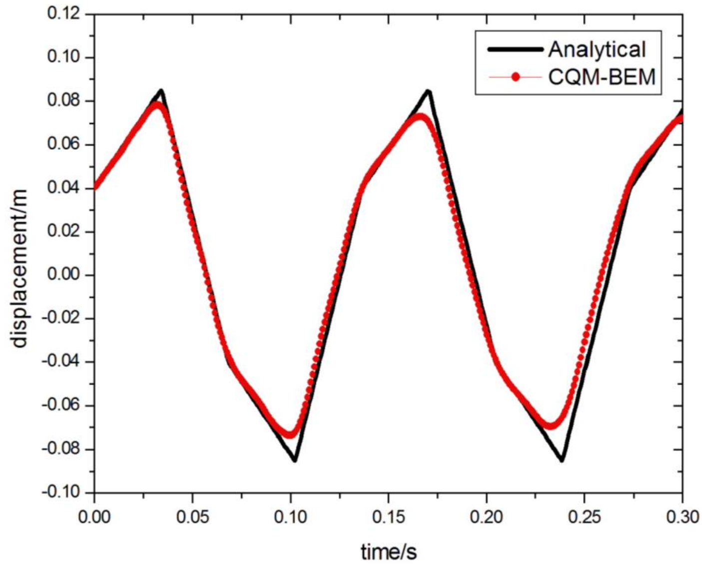

Figure 5: Displacement responses at point A of the cantilevered beam under initial conditions.

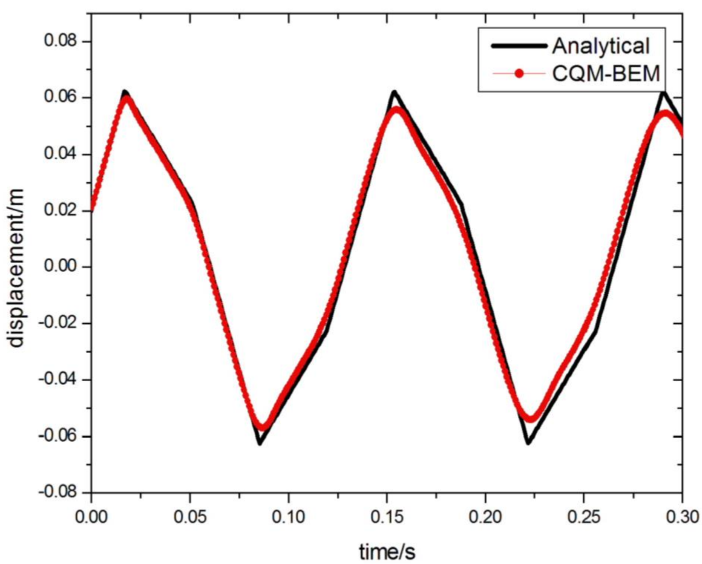

Figure 6: Displacement responses at point B of the cantilevered beam under initial conditions.

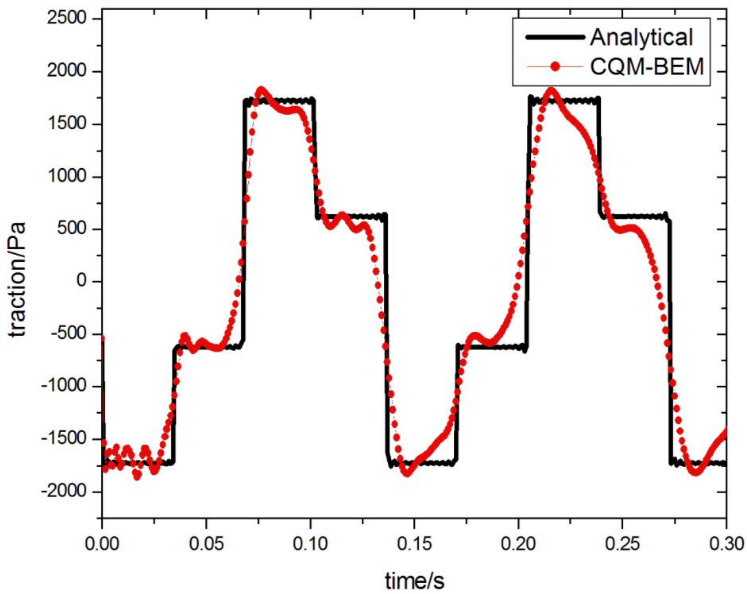

Figure 7: Traction responses at the fixed end of the cantilevered beam under initial conditions.

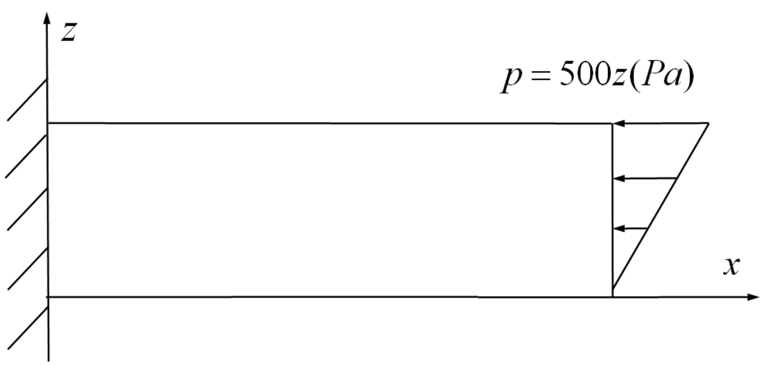

Figure 8: Cantilevered beam under a non-uniform loading.

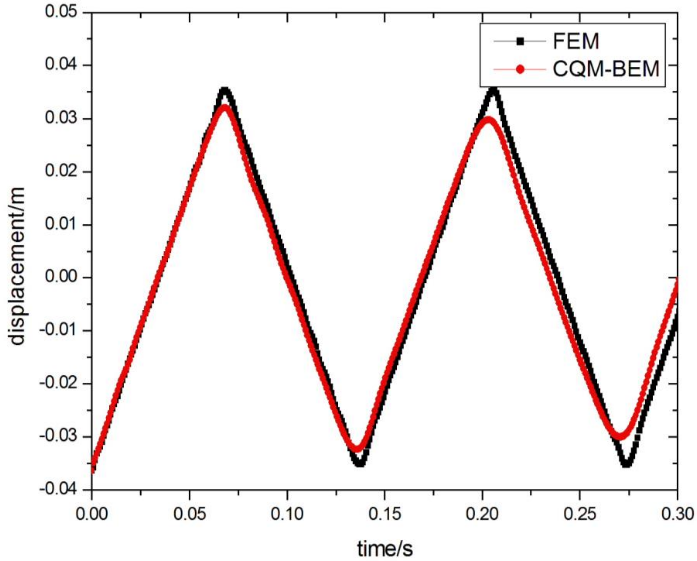

Figure 9: Displacement response of x direction at point A of the cantilevered beam under non-analytical initial displacement field.

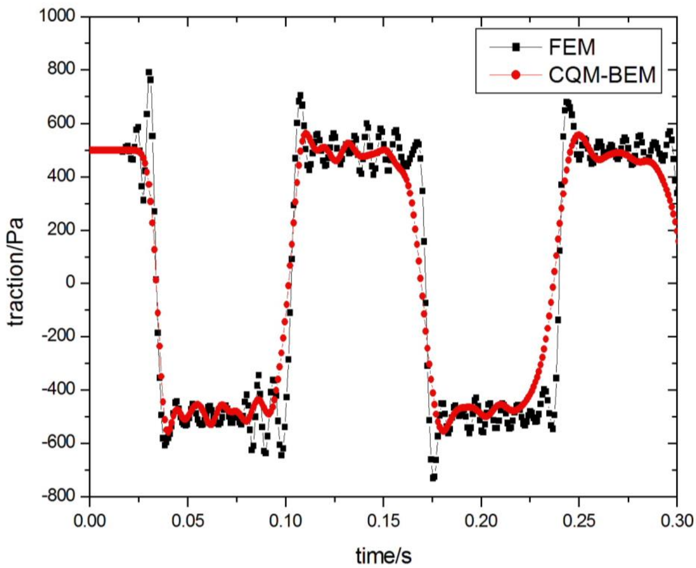

Figure 10: Traction response of x direction at point C of the cantilevered beam under non-analytical initial displacement field.

5 Conclusions

In this paper, a general method is presented to deal with non-null initial conditions in the governing equation for 3D elastodynamic problems. This method is developed from the pseudo-force method which is proposed by Mansur et al. (2004) for 2D scalar wave propagation problems but not directly. In this method, the initial conditions are transformed into equivalent pseudo-forces based on the linearity of the problem and the momentum principle. The basic steps for the transformation of the initial conditions, related to the displacement and its time derivative, are presented and discussed in detail. A new set of equations of motion subjected to completely null initial conditions is obtained finally. The CQM-based BEM formulation is applied to compute dynamic responses numerically. To evaluate the proposed method, longitudinal vibrations of a free beam and a cantilevered beam are analyzed. Numerical results are compared with the analytical solutions or the FEM solutions. They are in a good agreement, demonstrating the applicability of this method. Besides, the whole transformation process is based on the governing equations. Therefore, this method can also be applied to the traditional time-domain BEM formulation and other numerical method.

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