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Coupled ABC and Spline Collocation Approach for a Class of Nonlinear Boundary Value Problems over Semi-Infinite Domains

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Abstract: In this article, we introduce a numerical scheme to solve a class of nonlinear two-point BVPs on a semi-infinite domain that arise in engineering applications and the physical sciences. The strategy is based on replacing the boundary condition at infinity by an asymptotic boundary condition (ABC) specified over a finite interval that approaches the given value at infinity. Then, the problem complimented with the resulting ABC is solved using a fourth order spline collocation approach constructed over uniform meshes on the truncated domain. A number of test examples are considered to confirm the accuracy, efficient treatment of the boundary condition at infinity, and applicability of the approach. The computational results show that the scheme is reliable and converges fast with a fourth order rate of convergence.

Keywords: Asymptotic boundary conditions, spline collocation, nonlinear two-point BVPs.

1 Introduction

In this paper, we present an approach that is based on asymptotic boundary conditions (ABCs) and a fourth-order cubic B-spline collocation to acquire numerical solutions for the following class of non-linear boundary-value problems (BVPs) on a semi-infinite interval:

$$y''(x)+p(x)y'(x)+q(x)y(x)=f(x,y(x)), \tag{1}$$

complimented with the following boundary conditions:

$$y(0)=\alpha, \quad y(\infty)=\beta, \tag{2}$$

where $p(x)$, $q(x)$, $q(x)$ and $f(x,y)$ are continuous.

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BVPs over the positive half-line arise frequently in many physical situations and engineering applications. For example, such problems include the study of stellar structures, isothermal gaseous sphere, thermal behavior of a spherical cloud of gas, and radially symmetric solutions of semilinear elliptic equations. Moreover, this class of BVPs usually arise when studying plasma physics and electrical potential theory, non-linear mechanics and non-Newtonian fluid flows, the study of unsteady flow of a gas through a semi-infinite porous medium as well as the study of theory of thermionic currents and the eigenvalue problem for the Schrödinger equation. Other examples include the Von Karman flows, combined forced and free convection over a horizontal plate, heat transfer in the radial flow between parallel circular disks, draining flows, circularly symmetric deformations of shallow membrane caps, heat transfer in the radial flow between parallel circular disks, modelling of vertex solitons. For more details on the various applications see [Agarwal and O'Regan (2004); Agarwal and O'Regan (2003); Fazio (2002); Tsynkov (1998)] and the references therein.

Boundary value problems that are formulated on an infinite or semi-infinite domains have attracted lots of attention in past years due to the numerical challenge they exhibit in handling the condition at infinity. Consequently, this necessitates the need for efficient and applicable numerical strategies to treat the difficulties encountered, due to dealing with an unbounded domain. Tangible advancement have been made recently in tackling such problems in which the condition at infinity was handled relatively successfully. The simplest common approach is to reduce the unbounded interval into a finite one so that the condition at infinity is specified at a finite N point instead. This simple strategy is referred to as domain truncation, and is applicable if N is chosen to be large enough. However, the setback is that the accuracy of finite difference methods or finite element methods worsens when imposing artificial boundary conditions on the truncated interval. For details on setting the artificial boundary conditions see the paper by Tsynkov (1998), which includes an extensive survey and provides a comparative assessment of different existing methods for constructing the artificial boundary conditions and proposes a new technique as well. A number of notable approaches are available in the literature, for instance, De Hoog and Weiss (1980), Lentini and Keller (1980) and Markowich (1982) performed an asymptotic analysis to find the relevant boundary conditions to be imposed at a truncated boundary and since such conditions are related to the asymptotic behavior of the solution, this method often yields relatively better accurate solutions. De Hoog and Weiss (1980) proposed an analytical transformation of the independent variable that simplifies the original problem to a boundary value problem over a finite interval, which resulted in a singularity of the second kind at the origin that was handled by difference methods. Spectral methods have been implemented by using polynomials that are orthogonal over infinite domains, such as the Hermite spectral and the Laguerre spectral methods (see Shen (2000)). The disadvantage of such an approach is that it introduces relatively large errors due to the use of quadratures. Abbasbandy and Shivanian (2011) considered a model of mixed convection in a porous medium with boundary conditions on semi-infinite interval using pseudo-spectral collocation method. Gavrilyuk, Hermann, Kutnivc, and Makarovd (2009) used adaptive algorithms based on exact difference schemes for nonlinear BVPs on the half-axis. Some other direct approaches for solving such problems were used such as: methods based on rational approximations, spectral methods by using mutually orthogonal systems of rational functions, pseudospectral methods, Padé approximants, and the use of a suitable mapping to transfer infinite domains to the finite domains and then applying the standard spectral methods for the transformed problems in finite domains. For more details on such the various methods see [Agarwal and O'Regan (2004); Agarwal and O'Regan (2003); Boyd (1997); Fazio (2002); Maleki, Hashim, and Abbasbandy (2012); Sarler (2005); Tsynkov (1998)] and the references therein.

In recent years, much attention has been to the development, analysis and implementation of stable methods, including the cubic B-spline finite element collocation approach, for the numerical solution of a wide spectrum of IVPs and BVPs (see [Christara and Sun (2006); Khuri and Sayfy (2011); Khuri and Sayfy (2012); Mai-Duy and Tran-Cong (2003); Sarler (2005); Shokri and Dehghan (2012); Zhang, Dong, Alotaibi, and Alturi (2013)]). B-spline functions possess nice properties since that are piecewise polynomials with compact support that can be integrated and differentiated easily. Numerical methods in which B-spline functions are used as basis functions lead to simple matrix systems including band matrices. Such systems can be handled with low computational cost. Furthermore, the fact that the cubic B-spline method has a fourth order rate of convergence, makes it very attractive approach and is suggested in many studies for obtaining numerical solutions. However one challenge in implementing this method occurs when tackling BVPs over infinite domains. To surmount this difficulty, we will incorporate the ABCs in order to replace the condition at infinity with an asymptotic boundary condition that approaches the given value at infinity over a large finite interval. An analogous approach was used by Kanth, Ravi and Reddy (2003) for the solution of a two-point boundary value problem posed on an infinite interval involving a second order linear differential equation. They reduced the infinite interval to a finite interval that is large enough and then imposed approximate asymptotic boundary condition at the far end, and the resulting boundary value problem was treated by a finite difference method.

In this article, we propose a strategy aimed at obtaining numerical solution for a class of nonlinear two point boundary value problems over the positive half-line. The method is based on implementation of asymptotic boundary conditions on artificial boundaries as well as a fourth order cubic B-spline collocation technique, which first introduced by Christara and NG (see Christara and Sun (2006) and Christara and Sun (2006)). We discuss the incorporation of the ABCs into the collocation scheme and present the results of the numerical experiments on the solution of a number of examples including one which models a mixed convection flow past a plane of arbitrary shape embedded in a porous medium. The method yielded a convergence rate of order four and much more accurate results than what was obtained by other papers that exist in the literature.

A method for tackling nonlinear, two point boundary value problems over a semi-infinite domain is illustrated in this paper. The method is based on reducing the semi-infinite interval to a large finite one while replacing the condition at infinity with an asymptotic boundary condition. The resulting boundary value problem is handled by the 4th order adaptive cubic B-spline collocation approach. The application of the theory is illustrated to a couple of problems and the numerical results were compared to analytic solutions in order to demonstrate the applicability of the method.

The paper is organized as follows. In Section 2, we derive the asymptotic boundary conditions for the class of BVPs under consideration. In section 3, we describe and provide details of the spline collocation approach aimed at acquiring numerical solution for the class of BVPs on a semi-infinite domain. In section 4, a number of examples are presented to test the applicability and efficiency of the method. The last section 5 includes a conclusion that briefly summarizes the numerical outcomes.

2 Asymptotic Boundary Conditions (ABCs)

In this section, we show how the condition at infinity is to be replaced by an asymptotic boundary condition. This method was developed by Lentini and Keller (1980) as well as De Hoog and Weiss (1980). After obtaining the ABCs, we truncate the solution interval and then apply the fourth order cubic B-Spline collocation approach together with Newton's method in order to tackle the nonlinearity.

Consider the following two-point BVP on a semi-infinite interval

$$y''(x) = f\left(x, y(x), y'(x)\right), \tag{3}$$

complimented with the boundary conditions

$$y(0) = \alpha, \quad y(\infty) = 0, \tag{4}$$

To get the equivalent ABC for this problem, the following procedure is applied: Let $u_1 = y$ and $u_2 = y'$, then we can rewrite the BVP in matrix form as

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} u_2 \\ f(x, u_1, u_2) \end{bmatrix} = F \tag{5}$$

The first step is to calculate the Jacobian J of F with respect to u_1 and u_2:

$$J(x) = \begin{bmatrix} 0 & 1 \\ \frac{\partial f}{\partial u_1} & \frac{\partial f}{\partial u_2} \end{bmatrix}. \tag{6}$$

Now define A_∞ as

$$A_\infty = \lim_{x\to\infty} J(x), \tag{7}$$

and consider the eigenvalues λ_1 and λ_2 of such a matrix. If $\mathrm{Re}(\lambda_1) > 0$ then define $P_m = [1,0]$ and if $\mathrm{Re}(\lambda_2) > 0$ then define $P_m = [0,1]$. Let E be the matrix whose columns represent the eigenvectors of A_∞ with the first column corresponding to λ_1 and the second column corresponding to λ_2. Then we can define the asymptotic boundary condition as:

$$g(x_N) \equiv \lim_{x\to\infty} P_m E^{-1} G\Big|_{x_N} = 0. \tag{8}$$

In other words, we can replace ∞ with x_N after taking the limit as $x \to \infty$ for the terms in which x appears explicitly, where N the number of mesh points for the numerical solution as is described in the next section and x_N is the endpoint of the interval on which the numerical solution is to be found. There are, of course, certain conditions that should be satisfied and are outlined in Lentini and Keller (1980).

3 Fourth Order Spline Collocation Method

In this section, we describe a strategy, based on the fourth order cubic spline collocation method, for the numerical solution of the nonlinear two-point boundary-value problem (1)-(2) over semi-infinite interval. Originally, this collocation method is developed by Christara and Sun (2006) for the numerical solution of linear boundary-value problem over a closed interval $[a,b]$, using uniform and non-uniform mesh points. Consider such the second order linear differential equation:

$$y'' + p(x)y' + q(x)y = f(x), \tag{9}$$

subject to the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta. \tag{10}$$

To construct a numerical solution on $[a,b]$, we discretize the interval by defining the mesh points

$x_j = jh, \quad j = 0,1,2,..,N; \quad h = (b-a)/N,$

for a given number of subdivisions N. Let $\Psi(x)$ be a shape function that satisfies the boundary condition (10) and is expressed as a linear combination of $N+3$ spline functions given by

$$\Psi(x) = \sum_{j=-3}^{N-1} a_j \phi_j(x). \tag{11}$$

The set of coefficients $\{a_j\}$ are the unknowns to be found, while $\phi_j(x)$ is the cubic B-spline function defined on uniform mesh points given by:

$$\phi_j(x) = \frac{1}{h^3}\begin{cases} (x-x_j)^3, & [x_j, x_{j+1}] \\ h^3 + 3h^2(x-x_{j+1}) + 3h(x-x_{j+1})^2 - 3(x-x_{j+1})^3, & [x_{j+1}, x_{j+2}] \\ h^3 + 3h^2(x_{j+3}-x) + 3h(x_{j+3}-x)^2 - 3(x_{j+3}-x)^3, & [x_{j+2}, x_{j+3}] \\ (x_{j+4}-x)^3, & [x_{j+3}, x_{j+4}] \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

where $h = x_{j+1} - x_j$. Using (11) and (12), we can express the values of the shape function $\Psi(x)$ along with the first and second derivatives at the mesh points in terms of the a_j's.

$$\begin{aligned} \Psi(x_j) &= a_{j-3} + 4a_{j-2} + a_{j-1}, \\ \Psi'(x_j) &= \frac{3}{h}(a_{j-3} - a_{j-1}), \\ \Psi''(x_j) &= \frac{6}{h^2}(a_{j-3} - 2a_{j-2} + a_{j-1}), \end{aligned} \tag{13}$$

for each $j = 0,1,2,...,N$.

Now, if we substitute (13) in the boundary conditions and the differential equation given in (10) and (9) for $j = 0,1,...,N$ and solve the obtained $(N+3)\times(N+3)$ linear system, the resulting numerical solutions will be of a second order of rate of convergence. To achieve a fourth order rate of convergence, a correction term should be added to the second derivative approximation $\Psi''(x_j)$. The correction terms are defined in terms of 4th derivative of the shape function as follows:

$$\begin{aligned} \sigma_0 &= \frac{h^2}{12}\left[2\Psi^{(4)}(x_1) - \Psi^{(4)}(x_2)\right], \\ \sigma_j &= \frac{h^2}{12}\Psi^{(4)}(x_j), \quad j = 0,1,...,N-1, \\ \sigma_N &= \frac{h^2}{12}\left[2\Psi^{(4)}(x_{N-1}) - \Psi^{(4)}(x_{N-2})\right], \end{aligned} \tag{14}$$

where

$$\begin{aligned}\Psi^{(4)}(x_j) &= \frac{\Psi''(x_{j-1})-2\Psi''(x_j)+\Psi''(x_{j+1})}{h^2},\\ &= \frac{6}{h^2}\left(a_{j-4}-4a_{j-3}+6a_{j-2}-4a_{j-1}+a_j\right),\end{aligned} \tag{15}$$

where $j=0,1,2,...,N$. As a consequence, the shape functions are modified as follows:

$$\begin{aligned}\Psi(x_j) &= a_{j-3}+4a_{j-2}+a_{j-1},\\ \Psi'(x_j) &= \frac{3}{h}\left(a_{j-3}-a_{j-1}\right),\\ \Psi''(x_j) &= \frac{6}{h^2}\left(a_{j-3}-2a_{j-2}+a_{j-1}\right)+\sigma_j,\end{aligned} \tag{16}$$

where $j=0,1,2,...,N$. To approximate the solution of the boundary value problem (9)-(10) by the shape function (11), we substitute (16) in (9) for each mesh point to end up with $N+1$ equations in $N+3$ unknowns:

$$\begin{aligned}&\frac{6}{h^2}\left(a_{j-3}-2a_{j-2}+a_{j-1}\right)+\sigma_j+\frac{3p(x_j)}{h}\left(a_{j-3}-a_{j-1}\right)\\ &+q(x_j)\left(a_{j-3}+4a_{j-2}+a_{j-1}\right)=f(x_j),\end{aligned} \tag{17}$$

where $j=0,1,2,...,N$. The boundary conditions (10) give the two equations:

$$a_{-3}+4a_{-2}+a_{-1}=\alpha, \quad a_{N-3}+4a_{N-2}+a_{N-1}=\beta. \tag{18}$$

Solving the $(N+3)\times(N+3)$ linear system (17)-(18) for the a_j's yields a numerical solution $\{y_j\}$, where

$$y_j=a_{j-3}+4a_{j-2}+a_{j-1}\approx y(x_j), \quad j=0,1,2,...,N. \tag{19}$$

Now, implementing the above collocation scheme on our problem (1)-(2) yields to solve the nonlinear system

$$\begin{aligned}&\Psi''(x_j)+p(x_j)\Psi'(x_j)+q(x_j)\Psi(x_j)=f\left(x_j,\Psi(x_j)\right), \quad j=0,1,2,...,N,\\ &\Psi(x_0)=\alpha,\\ &g\left(\Psi(x_N)\right)=0,\end{aligned} \tag{20}$$

on the interval $[0,x_N]$. An iterative scheme arise from Newton's method can be used to solve the nonlinear system (20). In other word, starting with initial values $\Psi_0(x_j)$, $j=0,1,2,...,N$, the following linear system solved iteratively for n=0, 1, ... , M-1 for the a_j's, using (16).

$$\begin{aligned}&\Psi''_{n+1}(x_j)+p(x_j)\Psi'_{n+1}(x_j)+\left[q(x_j)-\frac{\partial f}{\partial \Psi}\left(x_j,\Psi_n(x_j)\right)\right]\Psi_{n+1}(x_j),\\ &=f\left(x_j,\Psi_n(x_j)\right)-\Psi_n(x_j)\frac{\partial f}{\partial \Psi}\left(x_j,\Psi_n(x_j)\right),\end{aligned} \tag{21}$$

for $j = 0, 1, 2, \ldots, N$, and the boundary conditions including the ABC are given by

$$\begin{aligned} &\Psi_{n+1}(x_0) = \alpha, \\ &\Psi_{n+1}(x_N)\frac{\partial g}{\partial \Psi}(x_j, \Psi_n(x_j)) = g(x_j, \Psi_n(x_j)) - \Psi_n(x_N)\frac{\partial g}{\partial \Psi}(x_j, \Psi_n(x_N)), \end{aligned} \tag{22}$$

where M is the number of iterations in Newton's method, which usually does not have to be relatively large ($M \leq 5$). Then the numerical approximation for the boundary value problem (1)-(2) is given by

$$y_j = \Psi_M(x_j) \equiv y(x_j), \quad j = 0, 1, 2, \ldots, N. \tag{23}$$

4 Numerical Examples

In this section, the ABC-collocation approach is applied to a number of boundary value problems defined on semi-infinite intervals. The numerical results are compared with exact and/or numerical solutions that are available in the literature. The rate of convergence for the numerical solutions is verified numerically using the logarithmic ratio. The test examples conform the high accuracy and applicability of the current proposed strategy.

Example 1. We will consider a BVP that models steady mixed convection flow in a porous medium past a plane of arbitrary shape, with boundary conditions specified on semi-infinite interval (see Abbasbandy and Shivanian (2011)), namely,

$$2f'''(x) + f'(x) - \left(f'(x)\right)^2 = 0, \tag{24}$$

subject to

$$f(0) = 0, \qquad f'(0) = b + 1, \qquad f'(\infty) = 1. \tag{25}$$

Problem (24)-(25) admits exact solutions for $f'(x)$ given by

$$f'(x) = -\frac{1}{2} + \frac{3}{2}\tanh^2\left[\frac{x}{2\sqrt{2}} \pm \frac{1}{2}\ln\left(\frac{\sqrt{3}+\sqrt{3+2b}}{\sqrt{3}-\sqrt{3+2b}}\right)\right].$$

This problem will be solved for $f'(x)$. In other words, let $u(x) = f'(x)$ and hence it reduces to

$$2u'' + u - u^2 = 0, \tag{26}$$

subject to

$$u(0) = b + 1, \qquad u(\infty) = 1. \tag{27}$$

In order to effectively apply the ABC, we transform the problem as follows: Let $v(x) = u(x) - 1$ and therefore we have

$$2v'' = v^2 + v, \tag{28}$$

complimented with the boundary conditions

$$v(0) = b, \qquad v(\infty) = 0, \tag{29}$$

where $b \in [-3/2, 0)$. We solve the above problem for the choice of $b = -1$ and then find $u(x) = v(x) + 1$. This is done analogous to the procedure for finding the ABC that requires the condition at infinity to be zero.

To demonstrate the strategy, we rewrite the above differential equation in matrix form as follows: let $z_1 = v$, $z_2 = v'$ and hence we get

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} z_2 \\ \frac{z_1^2}{2} + \frac{z_1}{2} \end{bmatrix} = F. \tag{30}$$

Calculating the jacobian (J) of F gives

$$J(x) = \begin{bmatrix} 0 & 1 \\ \frac{z_1}{2} + \frac{1}{2} & 0 \end{bmatrix} \tag{31}$$

and so A_∞ is equal to

$$A_\infty = \lim_{x \to \infty} J(x) = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}. \tag{32}$$

Calculating the eigenvalues of A_∞ we obtain

$$\lambda_1 = \sqrt{2}/2, \quad \lambda_1 = -\sqrt{2}/2.$$

Consequently, we set $P_m = [1\ 0]$ and we obtain $E^{-1} = \begin{bmatrix} \sqrt{2}/4 & 1/2 \\ -\sqrt{2}/4 & 1/2 \end{bmatrix}$. We conclude by finding the equivalent ABC as

$$\begin{aligned} 0 &= \lim_{x \to \infty} P_m E^{-1} F = \lim_{x \to \infty} [1 \quad 0] \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} z_2 \\ \frac{z_1^2}{2} + \frac{z_1}{2} \end{bmatrix} \\ &= \lim_{x \to \infty} \left(\frac{\sqrt{2}}{4} z_2 + \frac{1}{4}(z_1^2 + z_1) \right), \end{aligned} \tag{33}$$

and by truncating the infinite interval we get

$$\frac{\sqrt{2}}{4}v'(x_N)+\frac{1}{4}\left[v^2(x_N)+v(x_N)\right]=0.$$

In other words, we now solve the following BVP over the interval $[0,x_N]$:

$$2v''=v^2+v, \tag{34}$$

subject to

$$v(0)=b, \qquad \frac{\sqrt{2}}{4}v'(x_N)+\frac{1}{4}\left[v^2(x_N)+v(x_N)\right]=0. \tag{35}$$

It is worth mentioning that for $b\neq 0$ this BVP has two solutions. For $b=-1$ and $x_N=10$, the B-spline collocation scheme is used to obtain numerical approximations for the corresponding two solutions. The numerical solutions, for the two branches, are obtained by applying Newton's method on the resulting nonlinear system, which includes the ABC, for different starting values. The graphs of numerical approximations for the two branch solutions are presented together with true solutions in Fig. 1. The errors of the numerical approximations of the two branch solutions are presented in Table (1) and Table (2), respectively, at specific values and for different mesh sizes. The order of the rate of convergence for the two branch solutions are computed and presented in Tables (1) and (2) and is verified to be of order four. The numerical results are compared with those approximations obtained in Abbasbandy and Shivanian (2011), which uses Chebyshev pseudo-spectral method after reducing the problem to a singular BVP via a change of variables (we refer to their results in Tables 1 and 2 as [AS]). These results are depicted in Tables (1) and (2) and clearly show the superiority of our proposed scheme.

Table 1: Numerical results of the first branch solution of Example 1.

h	x=2		x=6		x=10	
	Error	Order	Error	Order	Error	Order
1/5	2.73095(-7)	-	3.10270(-8)	-	6.86707(-11)	-
1/10	1.42313(-8)	4.3	1.72003(-9)	4.2	1.72441(-11)	4.1
1/15	2.69289(-9)	4.1	3.31124(-10)	4.1	3.94541(-12)	4.0
1/20	8.38433(-10)	4.1	1.03731(-10)	4.0	1.31065(-12)	4.0
1/25	3.40791(-10)	4.0	4.22873(-11)	4.0	5.49052(-13)	4.0
Error in [AS]	7.59453(-8)		8.54071(-7)		9.37897(-7)	

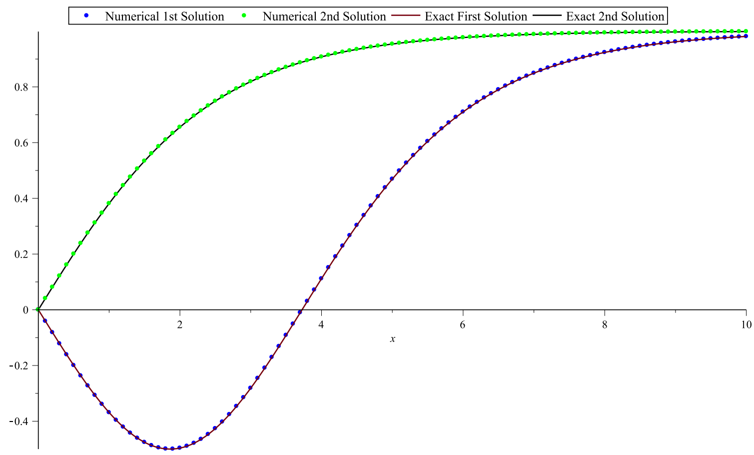

Figure 1: Exact and Numerical solutions of Example 1 on $[0,10]$ using $N = 200$.

Table 2: Numerical results of the second branch solution of Example 1.

h	x=2		x=6		x=10	
	Error	Order	Error	Order	Error	Order
1/5	1.34791(-6)	-	1.55544(-6)	-	1.08591(-7)	-
1/10	8.56608(-8)	4.0	9.39732(-8)	4.0	6.55618(-9)	4.0
1/15	1.69714(-8)	4.0	1.84460(-8)	4.0	1.28681(-9)	4.0
1/20	5.37541(-9)	4.0	5.82372(-9)	4.0	4.06301(-10)	4.0
1/25	2.20281(-9)	4.0	2.38302(-9)	4.0	1.66297(-10)	4.0
Error in [AS]	1.32949(-6)		2.24473(-5)		1.67834(-6)	

Example 2**.** Consider the following BVP (see Gavrilyuk, Hermann, Kutnivc, and Makarovd (2009)):

$$u'' - 4u = 2u^3 + 6u^2, \tag{36}$$

subject to

$$u(0) = -1, \qquad u(\infty) = 0. \tag{37}$$

Its exact solution is $u(x) = \tanh(x) - 1$. The equivalent ABC is found to be

$$u^3(x_N) + 3u^2(x_N) + 2u(x_N) + u'(x_N) = 0.$$

Fig. 2 depicts the exact solutions and the numerical solutions obtained by using the B-spline collocations method for $x_N = 10$. As in the previous example the errors and the order of convergence at some points are shown in Table (3).

Table 3: Absolute errors and order for numerical results of Example 2.

h	Error at x=4	Order	Error at x=6	Order	Error at x=8	Order
1/2	4.75(-7)	-	6.23(-8)	-	2.45(-9)	-
1/4	9.35(-8)	2.34	6.03(-9)	3.37	1.90(-10)	3.69
1/6	2.85(-8)	2.93	1.37(-9)	3.66	4.06(-11)	3.80
1/8	1.01(-8)	3.62	4.51(-10)	3.86	1.32(-11)	3.91
1/10	4.29(-9)	3.82	1.88(-10)	3.93	5.44(-12)	3.96

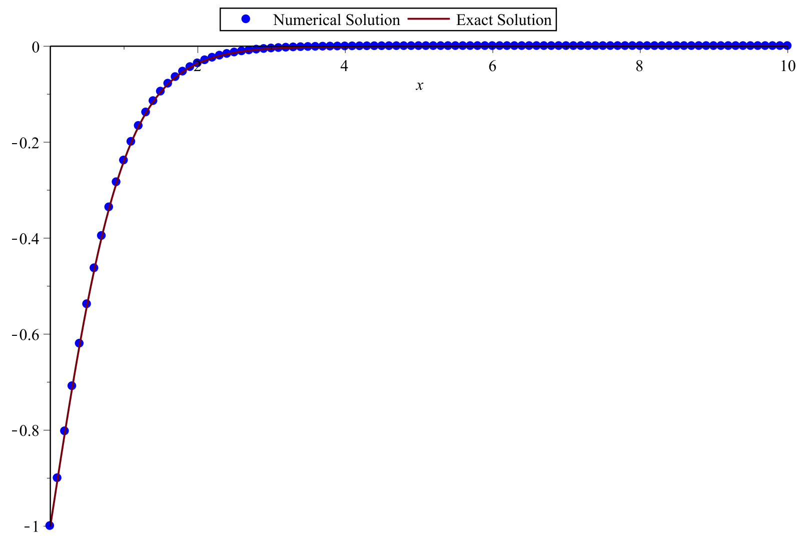

Figure 2: Exact and numerical solutions of Example 2 on $[0,10]$ using $N = 100$ and $x_N = 10$.

Comparing the results in Table (3) with the errors given in Agarwal and O'Regan (2004), we can say that our results require much less computations to achieve a certain accuracy.

Example 3. Consider the following BVP (see Gavrilyuk, Hermann, Kutnivc, and Makarovd (2009)):

$$u''(x) - 4u(x) = -\frac{4}{(1+x)^2} + \frac{7}{(1+x)^4} - u^2, \tag{38}$$

subject to

$$u(0) = 1, \qquad u(\infty) = 0. \tag{39}$$

It has the exact solution $u(x) = \dfrac{1}{(1+x)^2}$. The equivalent ABC is found to be

$6u(x_N) - u^2(x_N) = 0.$

The numerical results, for the choice $x_{50} = 10$, are presented in Fig. 3 and Table (4), which again confirm accurate approximations of the solution of the BVP with semi-infinite domain and a fourth order rate of convergence. We note from the

Table 4: Numerical results of Example 3 on $[0,10]$ using $N = 50$ and $x_N = 10$.

x	1	2	3	4	5
Numerical Solution	0.250016	0.111115	0.062501	0.040000	0.027778
Error	1.60(-5)	3.61(-6)	5.93(-7)	4.32(-8)	3.74(-7)
x	6	7	8	9	10
Numerical Solution	0.020405	0.015604	0.012193	0.008877	0.000000
Error	2.85(-6)	2.08(-5)	1.53(-4)	1.12(-3)	8.26(-3)

results presented in Table (4) that the accuracy of the numerical approximations on $[0,10]$, using $x_N = 10$ and $h = 1/5$, are deteriorating as x approaches $x_N = 10$. To improve the accuracy, we should choose x_N sufficiently large as mentioned earlier. Table (5) presents the results on $[0,10]$ for the same Example 3 but using $x_{100} = 20$. Obviously, better approximations are obtained using $x_N = 20$ and the same step size $h = 1/5$ as that for $x_N = 10$.

Table 5: Numerical results of Example 3 on $[0,10]$ using $N = 100$ and $x_N = 20$.

x	1	2	3	4	5
Numerical Solution	0.250016	0.111115	0.062501	0.040000	0.027778
Error	1.60(-5)	3.61(-6)	6.01(-7)	9.70(-8)	1.64(-8)
x	6	7	8	9	10
Numerical Solution	0.020408	0.015625	0.012346	0.010000	0.008264
Error	3.09(-9)	7.10(-10)	2.07(-10)	7.48(-11)	2.74(-11)

5 Conclusion

A numerical scheme is proposed and successfully implemented for the solution of a class of nonlinear second order boundary value problems over semi-infinite intervals. The approach is based on replacing the boundary condition at infinity by an asymptotic boundary condition ABC and solving numerically the resulting

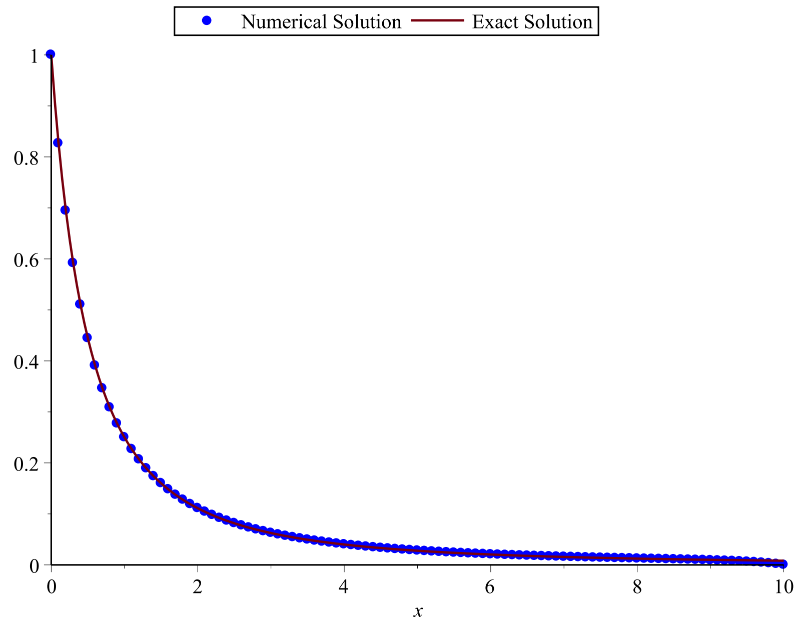

Figure 3: Exact and numerical solutions of Example 3 on $[0,10]$ using $N = 100$ and $x_N = 10$.

boundary value problem by a fourth order cubic spline collocation method over a sufficiently large bounded interval. The scheme requires a solution of a sequence of linear systems by using an iterative procedure that arises from Newton's method. The presented scheme is tested on three examples and the results showed very accurate approximations with fourth order convergence. Summarizing our observation, familiarity and involvement in proposing and implementing the ABC-collocation method, we can state and confirm that these asymptotic boundary conditions can be easily incorporated in the structure of the spline collocation method and can considerably reduce the computational time as compared with other methods while sustaining the high accuracy of the approximate solution, especially when the solution is required on a finite interval.

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