

# Investigation of Squeezing Unsteady Nanofluid Flow Using the Modified Decomposition Method

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**Abstract:** In this paper, we use the modified decomposition method (MDM) to solve the unsteady flow of a nanofluid squeezing between two parallel equations. Copper as nanoparticle with water as its base fluid has considered. The effective thermal conductivity and viscosity of nanofluid are calculated by the Maxwell-Garnetts (MG) and Brinkman models, respectively. The effects of the squeeze number, the nanofluid volume fraction, Eckert number,  $\delta$  on Nusselt number and the Prandtl number are investigated. The figures and tables clearly show high accuracy of the method to solve the unsteady flow.

**Keywords:** nonlinear differential equation, unsteady nanofluid flow, modified decomposition method.

## 1 Introduction

Most phenomena in our world are essentially nonlinear and are described by nonlinear equations. Since the appearance of high-performance digit computers, it becomes easier and easier to solve a linear problem. However, generally speaking, it is still difficult to obtain accurate solutions of nonlinear problems. In particular, it is often more difficult to get an analytic approximation than a numerical one of a given nonlinear problem, although we now have high performance supercomputers and some high-quality symbolic computation software such as Mathematica, Maple, and so on. The numerical techniques generally can be applied to nonlinear problems in complicated computation domain; this is an obvious advantage of numerical methods over analytic ones that often handle nonlinear problems in simple domains. However, numerical methods give discontinuous points of a curve and thus it is often costly and time consuming to get a complete curve of results. Besides, from numerical results, it is hard to have a whole and essential understanding

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of a nonlinear problem. Numerical difficulties additionally appear if a nonlinear problem contains singularities or has multiple solutions. The numerical and analytic methods of nonlinear problems have their own advantages and limitations, and thus it is unnecessary for us to do one thing and neglect another.

Therefore, many different methods have been introduced to obtain analytical approximate solutions for these nonlinear problems, such as the perturbation method [Holmes (2013); He (2000)], orthogonal polynomial and wavelet methods [Lakestani, Razzaghi, and Dehghan (2006)], methods of travelling wave solutions [Jafari, Borhanifar, and Karimi (2009)], the Adomian decomposition method (ADM) and the Variational iteration method. The method, which requires neither linearization nor perturbation, works efficiently for a large class of initial value or boundary value problems including linear or nonlinear equations. For parameter analysis, approximate analytical solutions are more practical than numerical solutions.

One of the most applicable analytical techniques is the ADM [Lu and Duan (2014); Duan, Rach, and Wazwaz (2013); Fu, Wang, and Duan (2013); Lai, Chen, and Hsu (2008); Adomian (1983, 1986, 1989, 1994); Wazwaz (2009, 2011); Serrano (2011); Adomian and Rach (1983); Duan, Rach, Baleanu, and Wazwaz (2012); Rach (2012)]. It is a practical technique for solving nonlinear functional equations, including ordinary differential equations, partial differential equations, integral equations, integro-differential equations, etc. The ADM provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering without unphysical restrictive assumptions such as required by linearization and perturbation. The accuracy of the analytic approximate solutions obtained can be verified by direct substitution.

In the ADM, the solution  $u(x)$  is represented by a decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad (1)$$

and the nonlinearity comprises the Adomian polynomials

$$Nu(x) = \sum_{n=0}^{\infty} A_n(x), \quad (2)$$

where the Adomian polynomials  $A_n(x)$  is defined for the nonlinearity  $Nu = f(u)$  as [Adomian and Rach (1983)]

$$A_n(x) = A_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} f\left(\sum_{k=0}^{\infty} \lambda^k u_k(x)\right) \Bigg|_{\lambda=0}. \quad (3)$$

Different algorithms for the Adomian polynomials have been developed by Rach [Rach (2008, 1984)], Wazwaz [Wazwaz (2000)], Abdelwahid [Abdelwahid (2003)]

and several others [Abbaoui, Cherruault, and Seng (1995); Zhu, Chang, and Wu (2005); Biazar, Ilie, and Khoshkenar (2006)]. Recently new algorithms and sub-routines in MATHEMATICA for fast generation of the Adomian polynomials to high orders have been developed by Duan [Duan (2010b,a, 2011)]. The solution components are determined by recursion scheme. The  $n$ th-stage approximation is given as  $\phi_n(x) = \sum_{k=0}^{n-1} u_k(x)$ .

We remark that the convergence of the Adomian series has already been proven by several investigators [Rach (2008); Abbaoui and Cherruault (1994, 1995); Abdelrazec and Pelinovsky (2011)]. For example, Abdelrazec and Pelinovsky [Abdelrazec and Pelinovsky (2011)] have published a rigorous proof of convergence for the ADM under the aegis of the Cauchy-Kovalevskaya theorem. In point of fact the Adomian decomposition series is found to be a computationally advantageous rearrangement of the Banach-space analog of the Taylor expansion series about the initial solution component function.

In this paper, we use the modified decomposition method(MDM) [Duan and Rach (2011)] to solve the unsteady flow of a nanofluid squeezing between two parallel equations.

## **2 Governing equations**

The study of heat transfer for unsteady squeezing viscous flow between two parallel plates has been regarded as one of the most important research topics due to its wide range of scientific and engineering applications such as hydrodynamical machines, polymer processing, lubrication system, chemical processing equipment, formation and dispersion of fog, damage of crops due to freezing, food processing and cooling towers.

We consider the heat transfer analysis in the unsteady two-dimensional squeezing nanofluid flow between the infinite parallel plates. The two plates are placed at  $z = \pm l(1 - \alpha t)^{1/2} = \pm h(t)$ . For  $\alpha > 0$ , the two plates are squeezed until they touch  $t = 1/\alpha$  and for  $\alpha < 0$  the two plates are separated. The viscous dissipation effect, the generation of heat due to friction caused by shear in the flow, is retained. This effect is quite important in the case when the fluid is largely viscous or flowing at a high speed. This behavior occurs at high Eckert number ( $\gg 1$ ). Further the symmetric nature of the flow is adopted. The fluid is a water based nanofluid containing Cu (copper).

The nanofluid is a two component mixture with the following assumptions: incompressible; no-chemical reaction; negligible viscous dissipation; negligible radiative heat transfer; and nano-solid-particles and the base fluid are in thermal equilibrium and no slip occurs between them. The thermo-physical properties of the nanofluid

Table 1: Thermo physical properties of water and nanoparticles

	$\rho(kg/m^3)$	$C_p(j/kgk)$	$k(w/m.k)$
Pure water	997.1	4179	0.613
Copper (Cu)	8933	385	401

are given in Table 1.

The governing equations for momentum and energy in unsteady two dimensional flow of a nanofluid are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (4)$$

$$\rho_{nf} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \mu_{nf} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (5)$$

$$\rho_{nf} \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = - \frac{\partial p}{\partial y} + \mu_{nf} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \quad (6)$$

$$\begin{aligned} \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} &= \frac{k_{nf}}{(\rho C_p)_{nf}} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \\ &+ \frac{\mu_{nf}}{(\rho C_p)_{nf}} \left( 4 \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right), \end{aligned} \quad (7)$$

where  $u$  and  $v$  are the velocities in the  $x$  and  $y$  directions respectively,  $T$  is the temperature,  $p$  is the pressure, the effective density ( $\rho_{nf}$ ), the effective dynamic viscosity ( $\mu_{nf}$ ), the effective heat capacity  $(\rho C_p)_{nf}$  and the effective thermal conductivity  $k_{nf}$  of the nanofluid are defined as

$$\begin{aligned} \rho_{nf} &= (1 - \phi) \rho_f + \phi \rho_s, \\ \mu_{nf} &= \frac{\mu_f}{(1 - \phi)^{2.5}}, \\ (\rho C_p)_{nf} &= (1 - \phi) (\rho C_p)_f + \phi (\rho C_p)_s, \\ \frac{k_{nf}}{k_f} &= \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + 2\phi(k_f - k_s)}, \end{aligned} \quad (8)$$

where relevant boundary conditions as

$$\begin{aligned} v &= v_w = \frac{dh}{dt}, \quad T = T_H \text{ at } y = h(t), \\ v &= \frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = 0 \text{ at } y = 0, \end{aligned} \quad (9)$$

where these parameters as

$$\eta = \frac{y}{[l(1-\alpha t)^{1/2}]}, \quad u = \frac{\alpha x}{[2(1-\alpha t)]} f'(\eta), \quad (10)$$

$$v = -\frac{\alpha l}{[2(l-\alpha t)^{1/2}]} f(\eta), \quad \theta = \frac{T}{T_H}, \quad A_1 = (1-\phi) + \phi \frac{\rho_s}{\rho_f}.$$

Substituting the above variables into (5) and (6) and then eliminating the pressure gradient from the resulting equations give:

$$f^{(4)} - SA_1(1-\phi)^{2.5}(\eta f''' + 3f'' + f' f'' - f f''') = 0, \quad (11)$$

Using (10), Eqs.(6) and (7) reduce to the following differential equations as

$$\theta'' + PrS\left(\frac{A_2}{A_3}\right)(f\theta' - \eta\theta') + \frac{PrEc}{A_3(1-\phi)^{2.5}}((f'')^2 + 4\delta^2(f')^2) = 0, \quad (12)$$

where  $A_2$  and  $A_3$  are constants given as

$$A_2 = (1-\phi) + \phi \frac{(\rho C_p)_s}{(\rho C_p)_f}, \quad A_3 = \frac{k_{nf}}{k_f} = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + 2\phi(k_f - k_s)}, \quad (13)$$

where these boundary conditions as

$$f(0) = 0, \quad f''(0) = 0, \quad f(1) = 1, \quad f'(1) = 0, \quad \theta'(0) = 0, \quad \theta(1) = 1, \quad (14)$$

In Eq.(12),  $S$  is the squeeze number,  $Pr$  is the Prandtl number and  $Ec$  is the Eckert number, which are defined as

$$S = \frac{\alpha l^2}{2v_f}, \quad Pr = \frac{\mu_f(\rho C_p)_f}{\rho_f k_f}, \quad Ec = \frac{\rho_f}{(\rho C_p)_f} \left(\frac{\alpha x}{2(1-\alpha t)}\right)^2, \quad \delta = \frac{1}{x}. \quad (15)$$

### 3 Solution of the heat transfer of Cu-water nanofluid

We consider the nonlinear BVP for heat transfer of a nanofluid flow as

$$f^{(4)} - SA_1(1-\phi)^{2.5}(\eta f''' + 3f'' + f' f'' - f f''') = 0, \quad (16)$$

$$\theta'' + PrS\left(\frac{A_2}{A_3}\right)(f\theta' - \eta\theta') + \frac{PrEc}{A_3(1-\phi)^{2.5}}((f'')^2 + 4\delta^2(f')^2) = 0, \quad (17)$$

where these boundary conditions as

$$f(0) = 0, \quad f''(0) = 0, \quad f(1) = 1, \quad f'(1) = 0, \quad \theta'(0) = 0, \quad \theta(1) = 1. \quad (18)$$

In Adomian operator-theoretic notation we have

$$L_1 f(\eta) = Nf(\eta), L_2 \theta(\eta) = N\theta(\eta), \quad (19)$$

where

$$L_1(\cdot) = \frac{d^4}{d\eta^4}(\cdot), Nf(\eta) = SA_1(1-\phi)^{2.5}(\eta f''' + 3f'' + f' f'' - f f'''), \quad (20)$$

$$L_2(\cdot) = \frac{d^2}{d\eta^2}(\cdot), N\theta(\eta) = -PrS\left(\frac{A_2}{A_3}\right)(f\theta' - \eta\theta') \\ - \frac{PrEc}{A_3(1-\phi)^{2.5}}((f'')^2 + 4\delta^2(f')^2). \quad (21)$$

According to the Duan-Rach modified decomposition method for BVPs, we take the inverse linear operator as

$$L_1^{-1}(\cdot) = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta (\cdot) d\eta d\eta d\eta d\eta, L_2^{-1}(\cdot) = \int_1^\eta \int_0^\eta (\cdot) d\eta d\eta, \quad (22)$$

Then, we have

$$L_1^{-1}L_1 f(\eta) = \int_0^\eta \int_0^\eta \int_0^\eta \int_0^\eta f^{(4)}(\eta) d\eta d\eta d\eta d\eta = f(\eta) - \Phi_1(\eta), \quad (23)$$

$$L_2^{-1}L_2 \theta(\eta) = \int_1^\eta \int_0^\eta \theta''(\eta) d\eta d\eta = \theta(\eta) - \Phi_2(\eta), \quad (24)$$

where

$$\Phi_1(\eta) = f(0) + \eta f'(0) + \frac{\eta^2}{2} f''(0) + \frac{\eta^3}{6} f'''(0), \quad (25)$$

$$\Phi_2(\eta) = \theta(1) + (\eta - 1)\theta'(0). \quad (26)$$

Applying the operator  $L_1^{-1}(\cdot)$  and  $L_2^{-1}(\cdot)$  to both sides of Eq.(19) yield

$$f(\eta) = \Phi_1(\eta) + L_1^{-1}Nf(\eta), \quad (27)$$

$$\theta(\eta) = \Phi_2(\eta) + L_2^{-1}N\theta(\eta), \quad (28)$$

Using the boundary conditions (18), we have from Eq.(25) and (26) as

$$\Phi_1(\eta) = \eta f'(0) + \frac{\eta^3}{6} f'''(0), \quad (29)$$

$$\Phi_2(\eta) = 1. \quad (30)$$

Upon substitution of the formula Eq.(29) and (30) into Eq.(27) and (28), we obtain

$$f(\eta) = \eta f'(0) + \frac{\eta^3}{6} f'''(0) + L_1^{-1} Nf(\eta), \quad (31)$$

$$\theta(\eta) = 1 + L_2^{-1} N\theta(\eta). \quad (32)$$

Before we design a modified recursion scheme, we determine the two undetermined  $f'(0)$  and  $f'''(0)$  in advance. Evaluating  $f(\eta)$  at  $\eta = 1$  and using the boundary condition  $f(1) = 1$ , we have

$$f'(0) + \frac{1}{6} f'''(0) + [L_1^{-1} Nf(\eta)]_{\eta=1} = 1, \quad (33)$$

where this nonlinear Fredholm integral is

$$[L_1^{-1} Nf(\eta)]_{\eta=1} = \int_0^1 \int_0^\eta \int_0^\eta Nf(\eta) d\eta d\eta d\eta, \quad (34)$$

Differentiating Eq.(31) then evaluating  $f'(\eta)$  at  $\eta = 1$  and using the boundary condition  $f'(1) = 0$ , we have

$$f'(0) + \frac{1}{2} f'''(0) + \left[ \frac{dL_1^{-1} Nf(\eta)}{d\eta} \right]_{\eta=1} = 0, \quad (35)$$

where this nonlinear Fredholm integrate is

$$\left[ \frac{dL_1^{-1} Nf(\eta)}{d\eta} \right]_{\eta=1} = \int_0^1 \int_0^\eta \int_0^\eta Nf(\eta) d\eta d\eta d\eta. \quad (36)$$

From the system of Eq.(33) and (35), which constitutes two linearly independent equations in two unknowns, we readily obtain

$$f'(0) = -\frac{3}{2} [L_1^{-1} Nf(\eta)]_{\eta=1} + \frac{1}{2} \left[ \frac{dL_1^{-1} Nf(\eta)}{d\eta} \right]_{\eta=1} + \frac{3}{2}, \quad (37)$$

$$f'''(0) = 3 [L_1^{-1} Nf(\eta)]_{\eta=1} - 3 \left[ \frac{dL_1^{-1} Nf(\eta)}{d\eta} \right]_{\eta=1} - 3. \quad (38)$$

Substituting Eq.(37) and (38) into Eq.(31), we obtain the integral equation for the solution

$$\begin{aligned} f(\eta) = & \frac{3\eta}{2} - \frac{\eta^3}{2} - \left( \frac{3\eta}{2} - \frac{\eta^3}{2} \right) [L_1^{-1} Nf(\eta)]_{\eta=1} \\ & + \left( \frac{\eta}{2} - \frac{\eta^3}{2} \right) \left[ \frac{dL_1^{-1} Nf(\eta)}{d\eta} \right]_{\eta=1} + L_1^{-1} Nf(\eta), \end{aligned} \quad (39)$$

Thus, we have converted the nonlinear BVP into an equivalent nonlinear integral equation without any undetermined coefficients.

Next, we substitute the adomian decomposition series for the solution  $f(\eta)$  and  $\theta(\eta)$ , the series of the Adomian polynomials for the nonlinearity  $Nf(\eta)$  and  $N\theta(\eta)$  as

$$f(\eta) = \sum_{m=0}^{\infty} f_m(\eta) \text{ and } Nf(\eta) = \sum_{m=0}^{\infty} A_m(\eta), \quad (40)$$

$$\theta(\eta) = \sum_{m=0}^{\infty} \theta_m(\eta) \text{ and } N\theta(\eta) = \sum_{m=0}^{\infty} B_m(\eta). \quad (41)$$

Substitution (40) into Eq.(39) we have

$$\begin{aligned} \sum_{m=0}^{\infty} f_m(\eta) &= \frac{3\eta}{2} - \frac{\eta^3}{2} - \left(\frac{3\eta}{2} - \frac{\eta^3}{2}\right) [L_1^{-1}(\sum_{m=0}^{\infty} A_m(\eta))]_{\eta=1} + \left(\frac{\eta}{2} \right. \\ &\quad \left. - \frac{\eta^3}{2}\right) \left[ \frac{dL_1^{-1}(\sum_{m=0}^{\infty} A_m(\eta))}{d\eta} \right]_{\eta=1} + L_1^{-1}(\sum_{m=0}^{\infty} A_m(\eta)), \end{aligned} \quad (42)$$

Substitution (41) into Eq.(32) we have

$$\sum_{m=0}^{\infty} \theta_m(\eta) = 1 + L_2^{-1}(\sum_{m=0}^{\infty} B_m(\eta)). \quad (43)$$

Using the modified recursion scheme, we have

$$\begin{aligned} f_0(\eta) &= \frac{3\eta}{2} - \frac{\eta^3}{2}, \\ \theta_0(\eta) &= 1, \end{aligned}$$

$$\begin{aligned} f_{m+1}(\eta) &= -\left(\frac{3\eta}{2} - \frac{\eta^3}{2}\right) [L_1^{-1}(A_m(\eta))]_{\eta=1} \\ &\quad + \left(\frac{\eta}{2} - \frac{\eta^3}{2}\right) \left[ \frac{dL_1^{-1}(A_m(\eta))}{d\eta} \right]_{\eta=1} + L_1^{-1}(A_m(\eta)), \end{aligned} \quad (44)$$

$$\theta_{m+1}(\eta) = L_2^{-1}(B_m(\eta)), \quad (45)$$

We can compute the solution components  $f_m(\eta)$  and  $\theta_m(\eta)$ ,  $m \geq 1$ , where we can use any one of several efficient MATHEMATICA subroutine for generation of the



Adomian polynomials,

$$\begin{aligned}
 f_1(\eta) &= -\frac{1}{10}S\eta^5(1-\phi)^{2.5}A_1 + \frac{1}{280}S\eta^7(1-\phi)^{2.5}A_1 \\
 &\quad -\frac{19}{40}S\left(\frac{\eta}{2} - \frac{\eta^3}{2}\right)(1-\phi)^{2.5}A_1 \\
 &\quad -\frac{27}{280}S\left(-\frac{3\eta}{2} + \frac{\eta^3}{2}\right)(1-\phi)^{2.5}A_1, \\
 \theta_1(\eta) &= -\frac{3\text{Ec Pr } \eta (5(-4 + \eta^3) + 2\delta^2(-16 + 15\eta - 5\eta^3 + \eta^5))}{20(1-\phi)^{2.5}A_3}, \\
 &\quad \dots
 \end{aligned}$$

The  $n$ th-stage solution approximate is

$$\Psi_n(\eta) = \sum_{k=0}^{n-1} f_k(\eta), \quad (46)$$

$$\Phi_n(\eta) = \sum_{k=0}^{n-1} \theta_k(\eta). \quad (47)$$

Since the exact solution cannot be obtain in general for the case of most nonlinear operator equations, we instead consider the error remainder function in our context of the particular nonlinear differential equation  $L_1f(\eta) - Nf(\eta) = 0$  and  $L_2\theta(\eta) - N\theta(\eta) = 0$

$$\begin{aligned}
 \text{ER1}_n(\eta) &= L_1\Psi_n(\eta) - N_1\Psi_n(\eta) \\
 &= \Psi_n^{(4)}(\eta) - SA_1(1-\phi)^{2.5}(\eta\Psi_n'''(\eta) + 3\Psi_n''(\eta)) \\
 &\quad + \Psi_n'(\eta)\Psi_n''(\eta) - \Psi_n(\eta)\Psi_n'''(\eta),
 \end{aligned}$$

$$\begin{aligned}
 \text{ER2}_n(\eta) &= L_2\Phi_n(\eta) - N_2\Phi_n(\eta) \\
 &= \Phi_n''(\eta) + \text{Pr}S\frac{A_2}{A_3}(\Psi_n(\eta)\Phi_n'(\eta) - \eta\Phi_n'(\eta)) \\
 &\quad + \frac{\text{PrEc}}{A_3(1-\phi)^{2.5}}\Phi_n''(\eta)\Phi_n''(\eta) + 4\delta^2\Phi_n'(\eta)\Phi_n'(\eta).
 \end{aligned}$$

to verify the convergence of our solution and the maximal error remainder parameter

$$\text{MER1}_n = \max_{0 \leq \eta \leq 0.5} | \text{ER1}_n(\eta) |, \quad \text{MER2}_n = \max_{0 \leq \eta \leq 0.5} | \text{ER2}_n(\eta) |. \quad (48)$$

which can be conveniently computed by the MATHEMATICA native command 'NMaximize' for the  $n$ th-stage approximate  $\Psi_n(\eta)$  and  $\Phi_n(\eta)$ .

#### 4 Simulation results

The obtained analytical approximations include many parameters. Here we present simulation results of the proposed scheme for heat transfer of a nanofluid flow which is squeezed between parallel plates.

We consider the error analytic function:  $\rho_s = 8933$ ,  $\rho_f = 997.1$ ,  $c_{ps} = 385$ ,  $c_{pf} = 4179$ ,  $k_s = 401$ ,  $k_f = 0.613$ ,  $Ec = 0.01$ ,  $Pr = 6.2$ ,  $\delta = 0.01$ ,  $\phi = 0.02$ ,  $S = 1$ , we plot the error remainder functions  $ER1_n(y)$  and  $ER2_n(y)$  for  $n = 4$  through 7 in Figs. 1. The maximal error remainder parameters  $MER1_n$  and  $MER2_n$  for  $n = 1$  through 8 are listed in Table 2 and Table 3. In Figs. 2, we display the logarithmic plots of the maximal error remainder parameters  $MER1_n$  and  $MER2_n$  versus  $n$  for  $\rho_s = 8933$ ,  $\rho_f = 997.1$ ,  $c_{ps} = 385$ ,  $c_{pf} = 4179$ ,  $k_s = 401$ ,  $k_f = 0.613$ ,  $Ec = 0.01$ ,  $Pr = 6.2$ ,  $\delta = 0.01$ ,  $\phi = 0.02$ ,  $S = 1$ , where the points lie almost in a straight line, which indicates that the maximal error remainder parameters decrease approximately at an exponential rate.

Table 2: The maximal error remainder parameters  $MER1_n$  for  $\rho_s = 8933$ ,  $\rho_f = 997.1$ ,  $c_{ps} = 385$ ,  $c_{pf} = 4179$ ,  $k_s = 401$ ,  $k_f = 0.613$ ,  $Ec = 0.01$ ,  $Pr = 6.2$ ,  $\delta = 0.01$ ,  $\phi = 0.02$ ,  $S = 1, 0 \leq \eta \leq 0.5$

$n$	1	2	3	4
$MER1_n$	6.19924	1.60181	0.392085	0.0959581
$n$	5	6	7	8
$MER1_n$	0.0240165	0.00617282	0.00162422	0.000435705

Table 3: The maximal error remainder parameters  $MER2_n$  for  $\rho_s = 8933$ ,  $\rho_f = 997.1$ ,  $c_{ps} = 385$ ,  $c_{pf} = 4179$ ,  $k_s = 401$ ,  $k_f = 0.613$ ,  $Ec = 0.5$ ,  $Pr = 5.0$ ,  $\delta = 0.1$ ,  $\phi = 0.06$ ,  $S = 1, 0 \leq \eta \leq 0.5$

$n$	1	2	3	4
$MER2_n$	0.13552	0.0901256	0.0550043	0.00591018
$n$	5	6	7	8
$MER2_n$	0.000832669	0.000154263	0.0000389321	$9.90308 * 10^{-6}$

In Figs. 3, we plot the curves of  $\Psi_{10}$  versus  $\eta$  for  $\rho_s = 8933$ ,  $\rho_f = 997.1$ ,  $c_{ps} = 385$ ,  $c_{pf} = 4179$ ,  $k_s = 401$ ,  $k_f = 0.613$ ,  $Ec = 0.5$ ,  $Pr = 6.2$ ,  $\delta = 0.1$ . For Figs. 3(a)

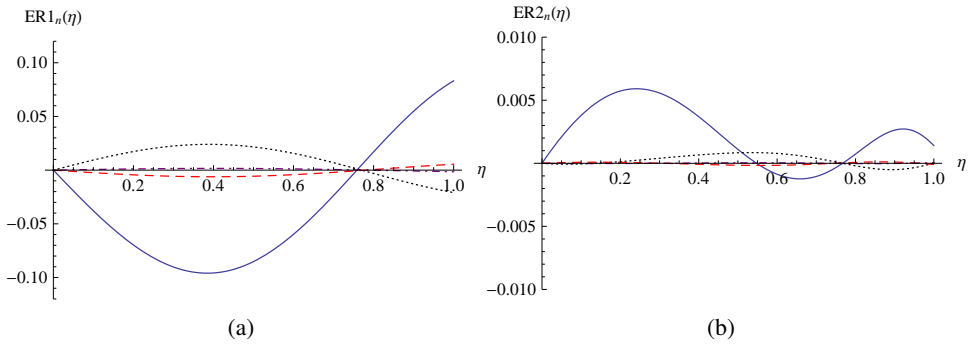


Figure 1: Curves of  $ER_n(\eta)$  versus  $\eta$  for  $n = 4$  (solid line),  $n = 5$  (dot line),  $n = 6$  (dash line),  $n = 7$  (dot-dash line), and for (a)  $ER1_n(\eta)$ , (b)  $ER2_n(\eta)$ .

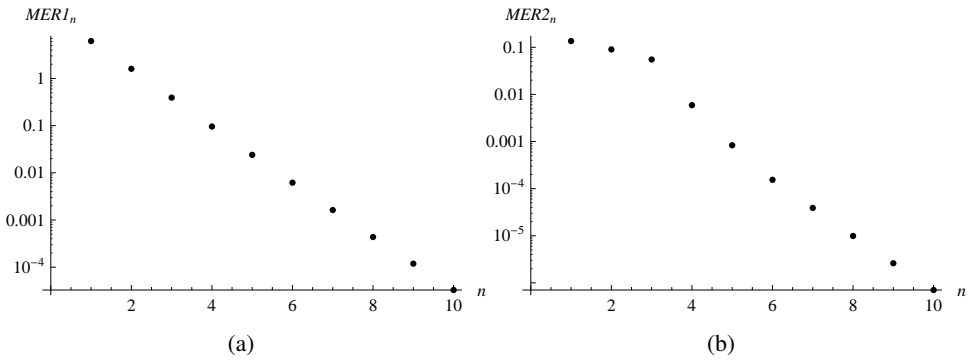


Figure 2: Logarithmic plots of the maximal errors remainder parameters  $MER_n$  and versus  $n$  for  $n = 1$  through 10, and for (a)  $MER1_n$ , (b)  $MER2_n$ .

and 3(b), we plot the curves of  $\Psi_{10}$  versus  $\eta$  for different values of  $S$  and  $\phi$ , respectively. For this case, when  $\phi = 0.06$ , increase in values of  $S$  is cause of decreasing in velocity. When  $S = 1$ , increase in values of  $\phi$  is cause of decreasing in velocity.

For Figs. 4(a) and 4(b), we plot the curves of  $\Phi_{10}$  versus  $\eta$  for different values of  $Pr$  and  $Ec$  when we fix  $S = 1$ ,  $\phi = 0.06$ ,  $\delta = 0.1$ , respectively. For this case, when  $Pr = 6.2$ , increase in values of  $Ec$  is cause of increasing in velocity. When  $Ec = 0.5$ , increase in values of  $Pr$  is cause of increasing in velocity. For Fig. 5, we plot the curves of  $\Phi_{10}$  versus  $\eta$  for different values of  $\delta$  when we fix  $S = 1$ ,  $\phi = 0.06$ ,  $Ec = 0.5$ ,  $Pr = 6.2$  respectively. For this case, increase in values of  $\delta$  is cause of increasing in velocity.

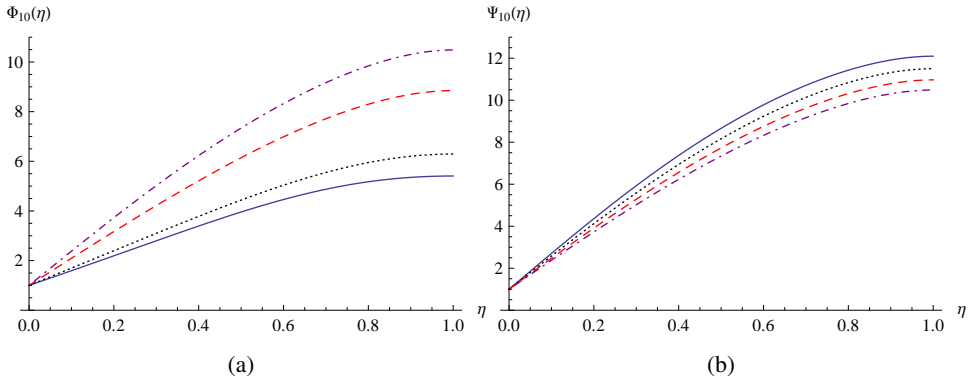


Figure 3: The curves of  $\Psi_{10}(\eta)$  versus  $\eta$  for (a)  $S = -1$  (solid line),  $S = -0.5$  (dot line),  $S = 0.5$  (dash line),  $S = 1$  (dot-dash line) and for  $\rho_s = 8933$ ,  $\rho_f = 997.1$ ,  $c_{ps} = 385$ ,  $c_{pf} = 4179$ ,  $k_s = 401$ ,  $k_f = 0.613$ ,  $Ec = 0.5$ ,  $Pr = 6.2$ ,  $\delta = 0.1$ ,  $\phi = 0.06$ , (b)  $\phi = 0$  (solid line),  $\phi = 0.02$  (dot line),  $\phi = 0.04$  (dash line),  $\phi = 0.06$  (dot-dash line) and for  $\rho_s = 8933$ ,  $\rho_f = 997.1$ ,  $c_{ps} = 385$ ,  $c_{pf} = 4179$ ,  $k_s = 401$ ,  $k_f = 0.613$ ,  $Ec = 0.5$ ,  $Pr = 6.2$ ,  $S = 1$ ,  $\delta = 0.1$ .

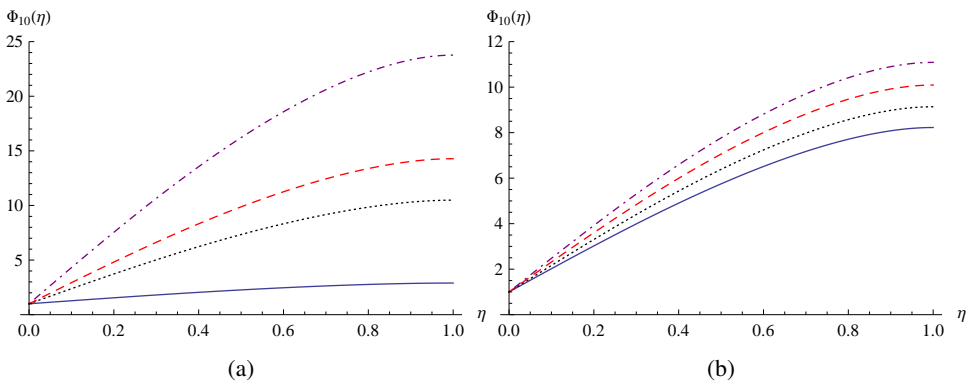


Figure 4: The curves of  $\Psi_{10}(\eta)$  versus  $\eta$  for (a)  $Ec = 0.1$  (solid line),  $Ec = 0.5$  (dot line),  $Ec = 0.7$  (dash line),  $Ec = 1.2$  (dot-dash line) and for  $\rho_s = 8933$ ,  $\rho_f = 997.1$ ,  $c_{ps} = 385$ ,  $c_{pf} = 4179$ ,  $k_s = 401$ ,  $k_f = 0.613$ ,  $Pr = 5.0$ ,  $\delta = 0.1$ ,  $\phi = 0.06$ ,  $S = 1$ , (b)  $Pr = 6.2$  (solid line),  $Pr = 5.5$  (dot line),  $Pr = 6.0$  (dash line),  $Pr = 6.5$  (dot-dash line) and for  $\rho_s = 8933$ ,  $\rho_f = 997.1$ ,  $c_{ps} = 385$ ,  $c_{pf} = 4179$ ,  $k_s = 401$ ,  $k_f = 0.613$ ,  $Ec = 0.5$ ,  $\delta = 0.1$ ,  $\phi = 0.06$ ,  $S = 1$ .

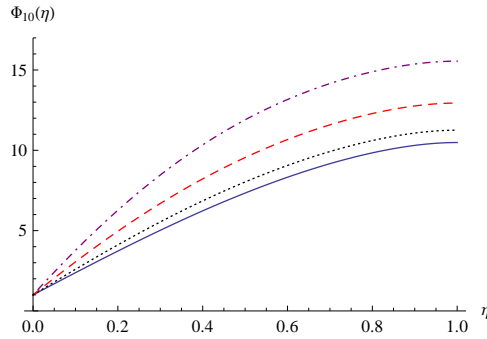


Figure 5: The curves of  $\Psi_{10}(\eta)$  versus  $\eta$  for (a)  $\delta = 0.1$  (solid line),  $\delta = 0.4$  (dot line),  $\delta = 0.7$  (dash line),  $\delta = 1.0$  (dot-dash line) and for  $\rho_s = 8933$ ,  $\rho_f = 997.1$ ,  $c_{ps} = 385$ ,  $c_{pf} = 4179$ ,  $k_s = 401$ ,  $k_f = 0.613$ ,  $Pr = 6.2$ ,  $Ec = 0.1$ ,  $\phi = 0.06$ ,  $S = 1$ .

## 5 Conclusions

In this research, the modified decomposition method was applied successfully to find the analytical solution of the unsteady flow of a nanofluid squeezing between two parallel. The figures and tables clearly show high accuracy of the method to solve the unsteady flow. Consequently, the present success of the modified decomposition method for the highly nonlinear problem verifies that the method is a useful tool nonlinear problems in science and engineering.

**Acknowledgement:** This work was supported by the Natural Science Foundation of Shanghai (No. 14ZR1440800) and the Innovation Program of the Shanghai Municipal Education Commission (No. 14ZZ161).

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