 CMES, vol.100, no.4, pp.323-345, 2014

Algebraic Multigrid Methods Based on Generic Approximate Banded Inverse Matrix Techniques

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Abstract: Since the introduction of the Algebraic MultiGrid algorithm (AMG) over twenty years ago, significant progress has been made in improving the coarsening and the convergence behavior of the method. In this paper, an AMG method is introduced that utilizes a new generic approximate inverse algorithm as a smoother in conjunction with common coarsening techniques, such as classical Ruge-Stüben coarsening, CLJP and PMIS coarsening. The proposed approximate inverse scheme, namely Generic Approximate Banded Inverse (GenAbI), is a banded approximate inverse based on Incomplete LU factorization with zero fill–in (ILU(0)). The new class of Generic Approximate Banded Inverse can be computed for any sparsity pattern of the coefficient matrix, in an analogous way as the explicit approximate inverse, yielding a suitable smoother to be used in conjunction with an Algebraic Multigrid method. The proposed smoother is parameterized and thus by increasing the "retention" parameter the smoothing scheme becomes more effective in terms of required number of cycles for convergence. Finally, the applicability and effectiveness of the proposed AMG method along with implementation issues, based on the Generic Approximate Banded Inverse matrix, is demonstrated by solving two and three dimensional problems and numerical results on the convergence behavior and convergence factor are given.

Keywords: Sparse linear systems, algebraic multigrid methods, coarsening techniques, incomplete LU factorization, generic approximate banded inverse smoothing, DOUR algorithm.

1 Introduction

Multilevel methods have seen a popularity surge for efficiently solving partial differential equations (PDE) in recent years. The more popular multilevel methods in-

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clude the various multigrid methods. Multigrid methods are based on the idea that from a series of coarser and coarser grids, sufficient corrections can be acquired and then applied to a sequence of finer grids in order to speed up convergence. This is possible due to the behavior of the components of the error, which consists of low and high frequencies. Moreover, multigrid methods exhibit h-independent convergence behavior for various classes of problems and near-optimal complexity. More on the h-ellipticity property of multigrid methods, as well as the Local Fourier Analysis (LFA) associated with it, can be found in [Trottenberg, Osterlee and Schuller (2000)]. The convergence behavior of different variations of the multigrid method has been studied extensively by researchers and theoretical estimates and bounds have been presented, [Bank and Douglas (1985); Brandt (1977); Hackbusch (1985b); Tang and Wan (2000)].

In [Brandt, McCormick and Ruge (1984); Ruge and Stüben (1987)] the Algebraic MultiGrid (AMG) method was first presented. A plethora of developments have happened since then, leading to new coarsening algorithms, [Cleary, Falgout, Henson and Jones (1998); De Sterck, Yang and Heys (2006);Henson and Yang (2002); Livne 2004], and interpolation schemes, [Brezina, Falgout, MacLachlan, Manteuffel, McCormick and Ruge (2005); De Sterck, Falgout, Nolting and Yang (2008)]. Considering a linear system Au=s, AMG only requires the coefficient matrix A in order to create the multilevel hierarchy through coarsening algorithms. Thus, AMG methods are suitable for solving problems arising from unstructured grids and complicated domains. It should be noted that Algebraic Multigrid methods have been used extensively for solving various complex engineering problems such as problems arising in Computational Fluid Dynamics or Computational Electromagnetism, [Trottenberg, Osterlee and Schuller (2000)].

The use of approximate inverses in the context of multilevel methods has been described in [Bröker and Grote (2002); Meurant (2002); Tang and Wan (2000)]. Classes of approximate inverse matrices based on the minimization of the Frobenius norm (SPAI, FSAI, etc) or the explicit approximate inverses have been recently proposed, [Grote and Huckle (1995); Kolotolina and Yeremin (1993); Lipitakis and Evans (1987)]. Extensive discussions on the methodology of explicit approximate inverse preconditioning as well as on the rate of convergence of explicit approximate inverse preconditioning have been presented in [Gravvanis(2009,1996); Gravvanis and Lipitakis (1996b); Lipitakis and Evans (1987); Lipitakis and Gravvanis (1995)]. The new proposed scheme GENeric Approximate Banded Inverse (GenAbI), used as a smoother for the AMG method, is based on a generalized storage format, namely the Compressed Sparse Row (CSR) and is generic in contradiction to its "dedicated" predecessors, [Gravvanis and Lipitakis (1996b); Lipitakis and Evans (1987); Lipitakis and Gravvanis (1995)].

In this article, an algebraic multigrid method is presented, based on a new class of generic approximate banded inverse matrices, derived from the incomplete factorization, as smoothers. In Section 2, the proposed AMG method is presented divided into the setup and solution phase. In Section 3, a new algorithmic technique for computing the elements of the generic approximate banded inverse matrix, based on incomplete factorization procedure, for any sparsity is presented and used as a smoother for the solution phase. Finally in Section 4, the applicability and effectiveness of the proposed algebraic multigrid method on two and three dimensional boundary value problems is presented along with numerical results on the convergence behavior and convergence factors.

2 The AMG Method

In this section, the proposed AMG method is discussed in detail. AMG methods can be split into two distinct phases: the setup phase, where the basic components are constructed, and the solution phase, where through the recursive use of the coarse-grid correction process, [Brandt (1977); Briggs, Henson and McCormick (2000); Hackbusch (1985a); Trottenberg, Osterlee, and Schuller (2000)], the solution is obtained.

AMG is comprised of four essential components, [Yang (2006)], (superscripts indicate grid level): grids $\Omega^1 \supset \Omega^2 \supset ... \supset \Omega^{\mathrm{N}}$ containing two disjoint subsets $C^k, k = 1,...,N-1$ (Coarse points set) and $F^k, k = 1,...,N-1$ (Fine points set), grid operators $A^1, A^2, ..., A^N$, interpolation and restriction operators $I_{k+1}^k, k = 1,...,N-1$ and $I_k^{k+1}, k = 1,...,N-1$ and finally a smoother (relaxation scheme), used at each level. Considering the linear system Au=s, where $A = (a_{i,j})$, $i, j \in [1,n]$ is an (n×n) sparse coefficient matrix, a "grid" is defined as a set of indices of the variables, the first grid being $\Omega = \{1,2,...,n\}$. In AMG, the coarsening process is responsible for constructing the coarser grid levels, along with the transfer operators, where the corrections to the solution will later be obtained through the cycle scheme. The setup phase of the AMG method presented uses the RS, CLJP and PMIS coarsening algorithms in conjunction with direct interpolation.

A key component of the setup phase is the coarsening algorithm. Its goal is to determine the sets of coarse and fine grid points respectively, as well as a small set $C_i \subset C$ of interpolating points for each fine grid point, [Ruge and Stüben (1987); Stüben (2001)]. Interpolation is then defined as:

$$\left(I_{k+1}^k u^{k+1}\right)_i = \begin{cases} u_i^{k+1}, i \in C \\ \sum\limits_{j \in C_i} w_{ij} u_j^{k+1}, i \in F \end{cases} \tag{1}$$

It is essential that strong influence and dependence are defined within the scope of

coarse-grid correction methods. A point i depends on point j if matrix element $a_{ij}$is "sufficiently large". The set of dependencies for variable i is then denoted as:

$$S_i = \left\{ j \neq i, -a_{ij} \geq \theta \max_{k \neq i}(-a_{ik}) \right\} \tag{2}$$

where θ is the strength threshold which determines the stencil size and affects convergence, [Yang (2006)]. A typical value for θ is 0.25.

The set of influences is the transpose of the dependencies set:

$$S_i^T = \left\{ j, i \in S_j \right\} \tag{3}$$

The above concepts, coupled with the following two heuristics, are essential within classical AMG coarsening algorithms, [Cleary, Falgout, Henson and Jones (1998); Ruge and Stüben (1987); Wesseling (1980)]:

- **H1**: For each point j that strongly influences a fine grid point i, j is either a coarse grid point or strongly depends on a coarse grid point that also strongly influences i.
- **H2**: The set C of coarse grid points is a maximal subset of all the points, i.e. no coarse grid point influences another coarse grid point.

The first coarsening technique to be introduced for AMG was the Ruge-Stüben (RS) algorithm. It is based on enforcing condition H1, while using condition H2 as a guideline, since it is impossible to satisfy both heuristics simultaneously. Further details on the RS coarsening scheme can be found in [Ruge and Stüben (1987)].

The Cleary-Luby-Jones-Plassmann (CLJP) coarsening algorithm, [Cleary, Falgout, Henson and Jones (1998)], was the first inherently parallel coarsening algorithm to be presented. Based on Luby's parallel maximal independent set algorithm, [Luby (1986)], CLJP selects more than one C-points in each loop, unlike RS coarsening which selects a single C-point in each loop. More details about the heuristics used in the CLJP algorithm can be found in [Cleary, Falgout, Henson and Jones (1998)].

Another parallel coarsening scheme that has been introduced is the Parallel Modified Independent Set (PMIS) coarsening algorithm. It is known that in three dimensions (3D), imposing heuristic H1 may result in unwanted complexity growth, [De Sterck, Yang and Heys (2006)]. In order to create sparser coarse grids while maintaining good convergence factors, the PMIS algorithm enforces heuristic H2, while H1 is replaced by a significantly less restrictive heuristic, [Krechel and Stüben (2001)]:

- **H1'**: Each fine-grid point must strongly depend on at least one coarse-grid point.

The PMIS algorithm is also based on Luby's parallel maximal independent set algorithm, [Luby (1986)], resulting in a simplified version of the CLJP algorithm. Detailed information about the algorithm can be found in [De Sterck, Yang, and Heys (2006)].

The major advantage of the CLJP and PMIS coarsening algorithms is their inherent parallelism, compared to RS coarsening's sequential nature. The PMIS coarsening is significantly faster and creates grids with lower operator complexities than CLJP, while sacrificing convergence behavior. Detailed comparisons of popular coarsening schemes can be found in [De Sterck, Yang, and Heys (2006); Yang (2006)].

The interpolation formula used for the purposes of this article is the direct interpolation, in which the weights w_{ij} are defined as follows, [Yang (2006)]:

$$w_{ij} = -\left(\frac{\sum_{k\in N_i} a_{ik}}{\sum_{l\in C_i} a_{il}}\right)\frac{a_{ij}}{a_{ii}}. \tag{4}$$

The drawback of direct interpolation is that it can potentially have a negative effect on the convergence rate compared to interpolation formulas that use extended neighborhoods, [Yang (2006)]. The interpolation operator can then be computed according to (1). Based on the Galerkin condition, [McCormick (1985)], the restriction operator is annotated as the transpose of the interpolation operator. The next level grid operator is defined as the triple matrix product of the restriction operator, the finer grid operator and the interpolation operator. When the current grid is considered "coarse enough", the setup phase terminates leading to the solution phase. The solution phase of the Algebraic Multigrid method proposed is based on the use of Generic Approximate Banded Inverses, based on ILU(0) factorization, as smoothers.

The cycle strategy is vital to any multigrid algorithm and refers to the sequence in which the various grids are visited and the respective coarse grid corrections are obtained. The most common cycle strategy is the V–cycle algorithm. Successive applications of the cycle lead to the solution according to arbitrary termination criterion. The solution phase of the presented AMG scheme can be defined by the below compact recursive algorithmic scheme, [Matskanidis and Gravvanis (2012)]:

Algorithm : $MGV(A^k, \left(M^{\delta l}\right)^k, I_k^{k+1}, I_{k+1}^k, u^k, f^k)$

If $k = N$, $Relax\left(A^N, \left(M^{\delta l}\right)^N, u^N, f^N\right)$ ν_3 times

Else

Relax $\left(A^k, \left(M^{\delta l}\right)^k, u^k, f^k\right) \nu_1$ times

Perform coarse grid correction:

Set $u^{k+1} = 0$.

Set $f^{k+1} = I_k^{k+1}(f^k - A^k u^k)$

Solve level k+1: $MGV(A^{k+1}, \left(M^{\delta l}\right)^{k+1}, I_{k+1}^{k+2}, I_{k+2}^{k+1}, u^{k+1}, f^{k+1})$

Correct the solution by $u_k = u_k + I_{k+1}^k u_{k+1}$

Relax $\left(A^k, \left(M^{\delta l}\right)^k, u^k, f^k\right) \nu_2$ times

One of the essential multigrid components is the smoother to be used at each level which can be described as:

$$x_{(i+1)}^k = x_{(i)}^k + M^k r^k, r^k = f^k - A^k x_{(i)}^k \tag{5}$$

where f^k, A^k are the right-hand side and coefficient matrix (at the k-th coarse level) and $x_{(i)}^k$ is the solution vector at the i-th iterative step. Equation (5) describes a family of stationary iterative methods, based on the M^k matrix. A commonly used smoother for AMG schemes is the Gauss – Seidel method with M^k matrix of the form $M^k = (D+L)^{-1}$, where D is the diagonal part of the coefficient matrix and L is the strictly lower part of the coefficient matrix. Further discussions and proofs about classical smoothers can be found in [Hackbusch (1985a,1985b)]. Approximate inverses based on the minimization of the Frobenius norm of the error have also been used as smoothers in the multigrid method, [Bröker and Grote (2002); Bröker, Grote, Mayer and Reusken (2001); Frederickson (1996)].

3 Generic Approximate Banded Inverse Smoothing

When choosing $M^k = (M^k)^{\delta l}$, where $(M^k)^{\delta l}$ is a class of approximate inverses, generic approximate banded inverses can be utilized as smoothers. The new class of smoothing methods can be described as follows, [Matskanidis and Gravvanis (2012)]:

$$x_{(i+1)}^k = x_{(i)}^k + \omega \left(M^k\right)_r^{\delta l} \left(f^k - A^k x_{(i)}^k\right) \tag{6}$$

with ω being the damping parameter, $0 < \omega \leq 1$.

The computation of the new class of generic approximate banded inverse requires the Incomplete LU factorization of the coefficient matrix A. The Incomplete LU Factorization is based on a modified Gaussian Elimination procedure, [Axelsson (1996); Saad (1996)], which retains certain fill – in terms based on a set of elements in specific places in the LU decomposition procedure. The most commonly used

variant of LU factorization is the zero fill – in variant, namely ILU(0) factorization. The ILU(0) factorization retains the profile and sparsity pattern of the coefficient matrix, without allowing fill – in terms. The algorithm for ILU(0) factorization can be found in [Axelsson (1996)].

Let us assume the incomplete LU factorization, such that

$$A \approx LU + R \tag{7}$$

where L and U are upper and lower matrices of the same nonzero structure as the lower and upper parts of coefficient matrix A respectively and R is an error matrix. Let $M^{\delta l} = (\mu_{i,j}), i \in [1,n], j \in [i-\delta l+1, i+\delta l-1]$ be the GENeric Approximate Banded Inverse (GenAbI) matrix of the coefficient matrix A, where δl is the so called "retention" parameter which denotes the number of outer diagonals in the upper and lower parts of the approximate inverse to be computed. Its elements can then be computed by the recursive solution of the following systems, [Gravvanis (2002); Lipitakis and Evans (1987)]:

$$UM^{\delta l} = I \quad \text{and} \quad M^{\delta l} L = 0 \tag{8}$$

The form of the Generic Approximate Banded Inverse matrix for n=8 and δl=4 is as follows:

$$\begin{array}{c} \leftarrow \quad \delta l \quad \rightarrow \\ \left[\begin{array}{cccccccc} \mu_{1,1} & \mu_{1,2} & \mu_{1,3} & \mu_{1,4} & & & & \\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3} & \mu_{2,4} & \mu_{2,5} & & 0 & \\ \mu_{3,1} & \mu_{3,2} & \mu_{3,3} & \mu_{3,4} & \mu_{3,5} & \mu_{3,6} & & \\ \mu_{4,1} & \mu_{4,2} & \mu_{4,3} & \mu_{4,4} & \mu_{4,5} & \mu_{4,6} & \mu_{4,7} & \\ & \mu_{5,2} & \mu_{5,3} & \mu_{5,4} & \mu_{5,5} & \mu_{5,6} & \mu_{5,7} & \mu_{5,8} \\ & & \mu_{6,3} & \mu_{6,4} & \mu_{6,5} & \mu_{6,6} & \mu_{6,7} & \mu_{6,8} \\ & 0 & & \mu_{7,4} & \mu_{7,5} & \mu_{7,6} & \mu_{7,7} & \mu_{7,8} \\ & & & & \mu_{8,5} & \mu_{8,6} & \mu_{8,7} & \mu_{8,8} \end{array}\right] \end{array} \tag{9}$$

The approximate inverse is stored in CSR (Compressed Sparse Row) storage format and the number of nonzero elements can be computed by the formula $nnz = n + (2\delta l - 2)\, n - \delta l\, (\delta l - 1)$. It has been studied and proven, that the value of the "retention" parameter δl can be chosen as multiples of the semi – bandwidth parameters of the coefficient matrix, [Gravvanis (2002,1996); Gravvanis and Lipitakis (1996b); Lipitakis and Evans (1987); Lipitakis and Gravvanis (1995)].

The elements of the new class of Generic Approximate Banded Inverse matrix (the GenAbI algorithm) can be expressed by the following compact algorithmic scheme:

For $i = n, \ldots, 1$

If $i = n$ **then**

$$\mu_{n,n} = 1/u_{n,n} \tag{10}$$

Else

$$\mu_{i,i} = \frac{1 - \sum_{j=1}^{min(n-i,\delta l-1)} \mu_{i+j,i} u_{i,i+j}}{u_{i,i}} \tag{11}$$

EndIf

For $j = 1, \ldots, min(i-1, \delta l - 1)$

$$\mu_{i,i-j} = - \sum_{k=i-j+1}^{min(i+\delta l-1,n)} \mu_{i,k} l_{k,i-j} \tag{12}$$

$$\mu_{i-j,i} = - \frac{\sum_{k=i-j+1}^{min(i+\delta l-1,n)} \mu_{k,i} u_{i-j,k}}{u_{i-j,i-j}} \tag{13}$$

EndFor

EndFor

The computation of the approximate inverse is commencing through "inverse L" shaped entities for each respectable diagonal element as shown in Figure 1, [Gravvanis, Filelis-Papadopoulos and Matskanidis (2013)]. Each i – loop denotes the respective "inverse L" shaped entity. However, the j – loop denotes the element within the "inverse L" shaped entity. The complexity of the Generic Approximate Banded Inverse Matrix is $O\left(n\delta l^2\right)$ multiplications. It should be noted that for δl=1 the computation of the approximate inverse is limited to the inversion of the diagonal elements of the upper factor of the incomplete factorization resulting in a fast approximate inverse.

The Generic Approximate Banded Inverse Matrix is fast to compute in CSR format, since the positions of its elements are a priori known. Moreover, the Generic Approximate Banded Inverse Matrix and the Explicit Optimized Banded Approximate Inverse matrices have been designed using the same "fishbone" computational approach, which allows a high level of inherent parallelism, [Gravvanis (2009)]. The parallel performance, as well as the parallelization techniques, of the Explicit Optimized Banded Approximate Inverse Matrices has been presented in [Gravvanis (2009)].

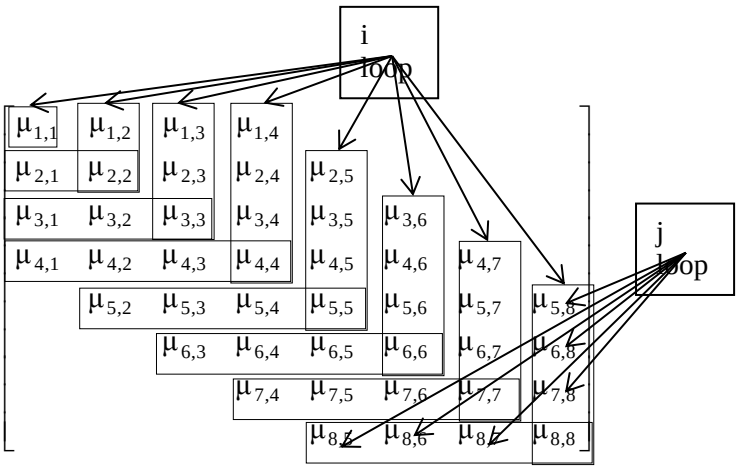

Figure 1: Schematic representation of the computation scope of the elements of the Generic Approximate Banded Inverse Matrix.

In multigrid convergence theory two properties must be satisfied in order for the coarse-grid correction process to converge: the smoothing and the approximation property, [Briggs, Henson and McCormick (2000); Bröker, Grote, Mayer and Reusken (2001); Hackbusch (1985a, 1985b)].

For symmetric positive definite (SPD) coefficient matrices, convergence for the V–Cycle independent of the levels k is also implied, [Hackbusch (1985b)]. The approximation property is independent of the smoother and depends only on the discretization and the transfer operators. It has been shown to be satisfied for various elliptic boundary value problems, [Hackbusch (1985b)].

The smoothing property for the finite difference (FD) Optimized Banded Generalized Approximate Inverse Matrix (OBGAIM) was proven in [Filelis-Papadopoulos and Gravvanis (2013b)]. In an analogous way, it can be proven for the Generic Approximate Banded Inverse (GenAbI). Additionally, sharp generalized estimates have been presented for the convergence of various smoothing schemes in [Bank and Douglas (1985)]. These estimates also apply for using Generic Approximate Banded Inverse (GenAbI) as a smoother.

The choice of the relaxation parameter governs the smoothing properties of the generic approximate inverse. In addition, the value of the relaxation parameter should satisfy the smoothing condition, and hence the Dynamic Over / Under Relaxation (DOUR) algorithm is used, [Haelterman, Viederndeels and Van Heule (2009)].

As seen in equation (6), the proposed generic approximate inverse smoothing scheme requires a relaxation parameter ω in order to be more effective. The choice of the relaxation parameter is non-trivial for a wide variety of problems and various choices of the "retention" parameter δl, [Matskanidis and Gravvanis (2012)]. A "predictor–corrector"-like scheme, based on the DOUR (Dynamic Over/Under Relaxation) scheme, [Haelterman, Viederndeels and Van Heule (2009)], was used in conjunction with the GenAbI smoother in order to compute the relaxation parameter dynamically, [Matskanidis and Gravvanis (2012)]:

$$x_{(i+1)}^{k} = x_{(i)}^{k} + \omega_e \left(S\left(x_{(i)}^{k}\right) - x_{(i)}^{k} \right), \omega_e = \omega\left(1+\kappa\right) \tag{14}$$

where ω_e is the effective relaxation parameter. This equation denotes a two-stage non-stationary generic approximate inverse smoother. The complexity of the DOUR PGenAbI smoother is (5n+(2δl-2)n-δl(δl-1)+3nnz(A)+2)ν multiplications and (7n +(2δl-2)n-δl(δl-1)+3nnz(A)+2)ν additions. Thus the complexity is $\approx \mathrm{O}\left(n\delta l\right)$.

4 Numerical Results

In this section numerical results are presented for the proposed AMG scheme. The results were obtained using the Visual Studio 2010 C++ environment. The convergence factor depends on the required number of iterations for convergence, [Briggs, Henson and McCormick (2000); Bröker, O., Grote, Mayer and Reusken (2001); Trottenberg, Osterlee and Schuller (2000)], and is defined as follows with respect to the 2-norm:

$$q = \sqrt[m]{\left\|r_m\right\|_2 / \left\|r_0\right\|_2} \tag{15}$$

where r_m is the residual vector at the m-th iteration. The termination criterion for all model problems is $\left\|r_m\right\|_2 < 10^{-10}\left\|r_0\right\|_2$ and the numbering of the grid is lexicographical. The maximum number of iterations was set to 300 iterations.

The strength threshold θ was set to 0.25. The values for the pre-smoothing and post-smoothing steps were set to v_1, v_2=2. The coarsest level solver used v_3=6 steps.

Model Problem I: Let us consider the following convection–diffusion equation in two space variables, [Briggs, Henson and McCormick (2000)], modeling transport phenomena, such as fluid flow in Computational Fluid Dynamics, [Trottenberg, Osterlee and Schuller (2000)]:

$$-\varepsilon\left(\Delta u\right) + \alpha\frac{\partial u}{\partial x} = A sin\left(\ell\pi y\right)\left(c_2 x^2 + c_1 x + c_0\right), \quad (x,y) \in \Omega \tag{16}$$

$$u(x,y) = 0, \quad (x,y) \in \partial\Omega \tag{16a}$$

where $c_2 = -\varepsilon\ell^2\pi^2, c_1 = \varepsilon\ell^2\pi^2 - 2\alpha, c_2 = \alpha + 2\varepsilon$, Δ is the Laplace operator, Ω is the unit square and $\partial\Omega$ denotes the boundary of Ω. The region Ω is covered by a triangular or quadrilateral grid, respectively, with mesh size $h = 1/(\sqrt{n}+1)$ where n denotes the order of the linear system. The values of the parameters for the convection–diffusion P.D.E are set arbitrarily to $\varepsilon = 0.1, \alpha = 2.5, \mathrm{A} = 1, \ell = 3$.

In Table 1, the convergence factors and convergence behavior for various coarsening techniques, values of the order of the linear system n and "retention" parameter δl of the generic approximate banded inverse matrix are presented for the convection-diffusion problem discretized with quadrilateral elements. Additionally, the convergence factors and convergence behavior for the Gauss-Seidel method as a smoother are also given. In Figures 2 and 3, the behavior of the error measures $\|r_i\|_2$ are presented for the CLJP and PMIS coarsening schemes respectively, for various smoothers with the order of the linear system n=65025, for the convection-diffusion problem discretized with quadrilateral elements.

In Table 2, the convergence factors and convergence behavior for various coarsening techniques, values of the order of the linear system n and "retention" parameter δl of the generic approximate banded inverse matrix are presented for the convection-diffusion problem discretized with triangular elements. Additionally, the convergence factors and convergence behavior for the Gauss-Seidel method as a smoother are also given.

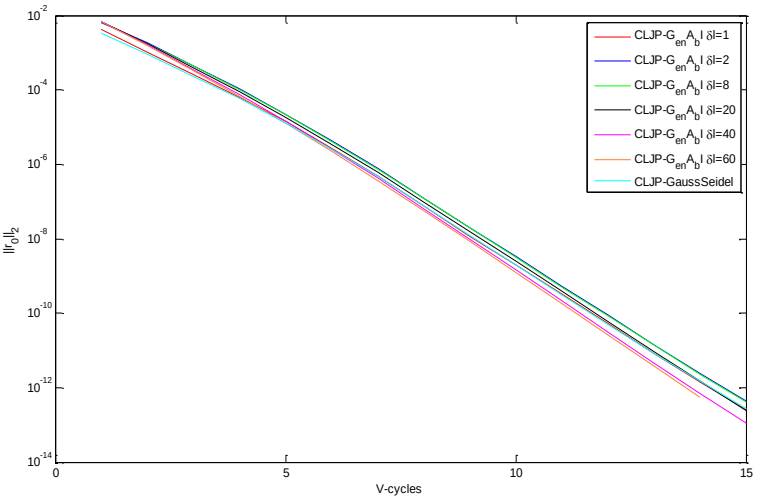

Figure 2: Behavior of the error measures $\|r_i\|_2$ for various smoothers with CLJP coarsening and n=65025, for model problem I (quadrilaterals).

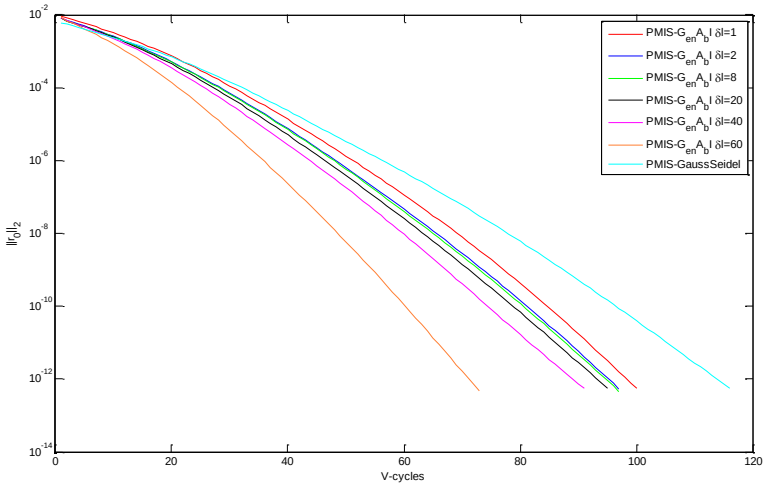

Figure 3: Behavior of the error measures $\|r_i\|_2$ for various smoothers with PMIS coarsening and n=65025, for model problem I (quadrilaterals).

Model Problem II: Let us consider the Poisson equation,:

$$\Delta u(x,y) = f(x,y), \quad (x,y) \in \Omega \tag{17}$$

$$u(x,y) = 0, \quad (x,y) \in \partial\Omega \tag{17a}$$

usually used to model gravitational phenomena in various methods used for N-Body simulation such as the Particle Mesh methods, [Kyziropoulos, Filelis-Papadopoulos and Gravvanis (2013)]. The following 9-point stencil $\frac{1}{3}\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$ has been used.

The right hand side vector was chosen as the product of the coefficient matrix by the solution vector with all its components set to unity. The initial guess of the solution vector was the zero vector.

In Table 3, the convergence factors and convergence behavior are presented for various coarsening techniques, smoothers, values of the order of the linear system n and "retention" parameter δl for the model problem II.

It should be noted that the use of generic approximate banded inverses as smoothers significantly improves the convergence behavior, compared to the Gauss-Seidel method, which is considered as a classical smoother. It should also be mentioned that increasing the value of the "retention" parameter δl leads to better convergence

Table 1: Convergence factors and convergence behavior of the V-Cycle for various coarsening techniques, smoothers and values of n and δl for model problem I (quadrilateral grid).

			n=3969	5 LEV-S	**n=16129**	5 LEVS	**n=65025**	7 LEVS
Coarsening	**Smoother**	**δl**	**q**	**its**	**q**	**its**	**q**	**its**
RS0	**GenAbI**	**1**	0.1591	13	0.1062	11	0.1305	12
CJLP			0.1007	11	0.1455	13	0.2025	15
PMIS			0.3706	24	0.5757	43	0.7932	100
RS0	**GenAbI**	**2**	0.1150	11	0.1243	12	0.1323	12
CLJP			0.1057	11	0.1531	13	0.2097	15
PMIS			0.3722	24	0.5852	43	0.7873	97
RS0	**GenAbI**	**8**	0.1119	11	0.1209	12	0.1306	12
CLJP			0.108	11	0.1543	13	0.209	12
PMIS			0.3742	24	0.5883	44	0.7859	97
RS0	**GenAbI**	**20**	0.1089	11	0.1208	12	0.1322	12
CLJP			0.1013	11	0.1456	13	0.202	15
PMIS			0.3345	22	0.5797	43	0.7836	95
RS0	**GenAbI**	**40**	0.0868	10	0.1260	12	0.1360	12
CLJP			0.0992	10	0.1419	12	0.1917	15
PMIS			0.2998	20	0.5339	37	0.7752	91
RS0	**GenAbI**	**60**	0.0866	10	0.1248	12	0.1361	12
CLJP			0.0953	10	0.1405	12	0.1912	14
PMIS			0.2949	19	0.5068	34	0.7262	73
RS0	**G-S**	**-**	0.1176	11	0.0916	10	0.0948	10
CLJP			0.0867	10	0.1436	12	0.2034	15
PMIS			0.3523	23	0.6054	47	0.8187	116

behavior, especially when approximate inverse smoothing is used in conjunction with CLJP and PMIS coarsening, as shown in Figures 2 and 3.

Model Problem III: Let us consider the Poisson equation in three space variables:

$$\Delta u(x,y,z) = f(x,y,z), \quad (x,y,z) \in \Omega \tag{18}$$

$$u(x,y,z) = 0, \quad (x,y,z) \in \partial\Omega \tag{18a}$$

discretized with the seven point finite difference stencil, where Δ is the Laplace operator, Ω is the unit cube and $\partial\Omega$ denotes the boundary of Ω. Such equations are often used in modeling gravitational phenomena or electrostatics, etc.

The right hand side vector was chosen as the product of the coefficient matrix by the solution vector with all its components set to unity. The initial guess of the solution vector was the zero vector.

Table 2: Convergence factors and convergence behavior of the V-Cycle for various coarsening techniques, smoothers and values of n and δl for model problem I (triangular grid).

			n=3969	5 LEV-S	**n=16129**	6 LEVS	**n=65025**	7 LEVS
Coarsening	**Smoother**	**δl**	**q**	**its**	**q**	**its**	**q**	**its**
RS0	**GenAbI**	**1**	0.0788	10	0.0766	9	0.0870	10
CJLP			0.0868	10	0.1571	13	0.2507	17
PMIS			0.5681	41	0.8409	134	*	>300
RS0	**GenAbI**	**2**	0.0965	10	0.0971	10	0.1036	11
CLJP			0.0998	10	0.1449	12	0.2421	17
PMIS			0.5068	34	0.8011	104	*	>300
RS0	**GenAbI**	**8**	0.0914	10	0.0937	10	0.0983	11
CLJP			0.0934	10	0.1376	12	0.2086	15
PMIS			0.5111	35	0.8139	113	*	>300
RS0	**GenAbI**	**20**	0.0837	10	0.09	10	0.0973	10
CLJP			0.0896	10	0.1284	12	0.1909	14
PMIS			0.4964	34	0.8114	111	*	>300
RS0	**GenAbI**	**40**	0.1214	11	0.0879	10	0.0948	10
CLJP			0.1033	11	0.1164	11	0.1787	14
PMIS			0.4315	28	0.8077	108	*	>300
RS0	**GenAbI**	**60**	0.1185	11	0.0879	10	0.0947	10
CLJP			0.1050	11	0.1148	11	0.1712	14
PMIS			0.3481	22	0.7292	74	*	>300
RS0	**G-S**	**-**	0.0753	10	0.0839	10	0.1089	11
CLJP			0.0662	9	0.1297	12	0.2196	16
PMIS			0.5163	35	0.8191	116	*	>300
* The convergence factor cannot be obtained since maximum number of iterations was exceeded.								

In Table 4, the convergence factors and convergence behavior are presented for various coarsening techniques, smoothers, values of the order of the linear system n and "retention" parameter δl for the model problem III.

It can be observed that the use of PMIS coarsening leads to increased number of iterations compared to the classical RS and CLJP coarsening, as has already been shown in [De Sterck, Yang and Heys (2006)]. However, as has been discussed in [De Sterck, Yang and Heys (2006)], PMIS also provides advantages compared to the above since it is highly parallel, while RS is sequential, and produces significantly lower operator complexities, especially in three dimensional problems, compared to CLJP.

Table 3: Convergence factors and convergence behavior of the V-Cycle for various coarsening techniques, smoothers and values of n and δl for model problem II.

			n=3969	5 LEV-S	**n=16129**	7 LEVS	**n=65025**	8 LEVS
Coarsening	**Smoother**	**δl**	**q**	**its**	**q**	**its**	**q**	**its**
RS0	**GenAbI**	**1**	0.1269	12	0.1492	13	0.1553	13
CJLP			0.2572	17	0.3058	20	0.3830	25
PMIS			0.7568	83	0.9208	280	*	>300
RS0	**GenAbI**	**2**	0.1338	12	0.1701	14	0.1625	13
CLJP			0.2561	17	0.3106	20	0.3902	25
PMIS			0.7513	81	0.9170	268	*	>300
RS0	**GenAbI**	**8**	0.1332	12	0.1685	14	0.1641	13
CLJP			0.2377	17	0.3002	20	0.3811	24
PMIS			0.738	76	0.9159	263	*	>300
RS0	**GenAbI**	**20**	0.1268	12	0.1617	13	0.1643	13
CLJP			0.2218	16	0.2866	19	0.3591	23
PMIS			0.6854	61	0.9104	246	*	>300
RS0	**GenAbI**	**40**	0.1196	11	0.1603	13	0.1605	13
CLJP			0.2043	15	0.2611	18	0.3462	22
PMIS			0.6320	51	0.8847	189	*	>300
RS0	**GenAbI**	**60**	0.1203	11	0.1603	13	0.1596	13
CLJP			0.1882	14	0.2564	17	0.3291	21
PMIS			0.6125	47	0.8818	184	*	>300
RS0	**G-S**	**-**	0.1044	11	0.1314	12	0.1373	12
CLJP			0.2545	17	0.3041	20	0.3943	25
PMIS			0.7696	88	0.9223	285	*	>300
* The convergence factor cannot be obtained since maximum number of iterations was exceeded.								

It should also be mentioned that using the proposed AMG method as a preconditioner for a Krylov type method, such as GMRES (m), [Yang (2006)], can partly remedy the above PMIS disadvantage. An alternative technique that can be used along with PMIS coarsening in order to reduce the required cycles for convergence is employing higher order interpolation schemes, [De Sterck, Falgout, Nolting and Yang (2008)].

In Table 5, the convergence behavior for the AMG-GMRES(m) algorithm is presented for PMIS coarsening and various values of the order of the linear system n and "retention" parameter δl with the restart parameter m=2.

Model Problem IV: Let us consider the following convection-diffusion equation

Table 4: Convergence factors and convergence behavior of the V-Cycle for various coarsening techniques, smoothers and values of n and δl for model problem III.

			n=8000	8 LEV-S	**n=64000**	10 LEVS	**n=125000**	11 LEVS
Coarsening	**Smoother**	**δl**	**q**	**its**	**q**	**its**	**q**	**its**
RS0	**GenAbI**	**1**	0.0372	7	0.0450	8	0.0498	8
CJLP			0.0641	9	0.0932	10	0.1082	11
PMIS			0.5241	36	0.6777	60	0.7741	90
RS0	**GenAbI**	**2**	0.0514	8	0.0546	8	0.0581	9
CLJP			0.0619	9	0.0914	10	0.1056	11
PMIS			0.4627	30	0.6603	56	0.7711	89
RS0	**GenAbI**	**8**	0.0504	8	0.0522	8	0.0560	8
CLJP			0.0606	9	0.0893	10	0.1034	11
PMIS			0.4552	30	0.6562	55	0.7673	87
RS0	**GenAbI**	**20**	0.0476	8	0.051	8	0.0548	8
CLJP			0.0554	8	0.0841	10	0.0976	10
PMIS			0.4326	28	0.6404	52	0.7584	84
RS0	**GenAbI**	**40**	0.0279	7	0.0492	8	0.0531	8
CLJP			0.0435	8	0.0812	10	0.0924	10
PMIS			0.4176	27	0.6282	50	0.7514	81
RS0	**GenAbI**	**80**	0.0274	7	0.0305	7	0.0379	8
CLJP			0.0419	8	0.0744	9	0.0902	10
PMIS			0.4150	27	0.6215	49	0.7306	74
RS0	**G-S**	**-**	0.0345	8	0.0594	9	0.0666	9
CLJP			0.0502	8	0.0883	10	0.0993	10
PMIS			0.4551	30	0.6623	56	0.7591	84

Table 5: Convergence behavior of AMG-GMRES(2) for PMIS coarsening, and various values of n and δl for model problem III.

Iterations	**δl=1**	**δl=2**	**δl=8**	**δl=20**	**δl=40**	**δl=80**
n=8000	10	10	10	9	8	8
n=64000	13	13	13	12	12	11
n=125000	20	20	18	17	17	16

in three space variables:

$$u_{xx} + u_{yy} + u_{zz} + \beta u_y = f(x,y,z), \quad (x,y,z) \in \Omega \tag{19}$$

$$u(x,y,z) = 0, \quad (x,y,z) \in \partial\Omega \tag{19a}$$

where Ω is the unit cube and $\partial\Omega$ denotes the boundary of Ω. Such equations are used to model energy transport phenomena.

The problem was discretized by applying central differences for the first and second order derivatives. It should be mentioned that the parameter β controls the degree of asymmetry of the resulting linear system. Additionally, for large values of β the system loses the diagonal dominance property, which is an important factor for the convergence of many iterative methods, [Gravvanis and Lipitakis (1996a)].

The right hand side vector was chosen as the product of the coefficient matrix by the solution vector with all its components set to unity. The initial guess of the solution vector was the zero vector.

In Table 6, the convergence factors and convergence behavior are presented for order of the linear system n=27000 and various coarsening techniques, smoothers, values of the parameter β and "retention" parameter δl for the model problem IV. In Table 7, the convergence factors and convergence behavior are presented for order of the linear system n=64000 and various coarsening techniques, smoothers, values of the parameter β and "retention" parameter δl for the model problem IV.

The numerical results confirm the fact that for large values of the parameter β the system loses the diagonal dominance property, as previously mentioned. It should be noted that for those values, while Gauss-Seidel does not converge within the iterations limit, the GenAbI smoother converges, although slowly, when high values of the "retention" parameter δl are used. Additionally, for this particular problem PMIS significantly outperforms the other coarsening schemes as its convergence behavior still improves as β increases up to a certain extent, while also retaining its lower operator complexity in 3D problems.

It should be also mentioned that in order to handle effectively very ill-conditioned or unstructured problems a new generic approximate inverse matrix technique, based on sparsity patterns, has been recently proposed, [Filelis-Papadopoulos and Gravvanis (2013a)]. Finally, we state that the new proposed scheme based on the algebraic multigrid method in conjunction with the GenAbI algorithm can be efficiently used for solving non–linear initial/boundary value problems, [Gravvanis (2002)].

Table 6: Convergence factors and convergence behavior of the V-Cycle for n=27000 and various coarsening techniques, smoothers and values of β and $\delta 1$ for model problem IV.

			n=27000				
			β=1	**β=25**	**β=50**	**β=100**	**β=200**
Coarsening	**Smoother**	**$\delta 1$**	**q (its)**	**q (its)**	**q (its)**	**q (its)**	**q (its)**
RS0	**GenAbI**	**1**	0.0411 (8)	0.0771 (8)	0.1983 (14)	0.5610 (45)	*
CJLP			0.0760 (8)	0.0611 (8)	0.0796 (9)	0.2876 (19)	*
PMIS			0.6097 (47)	0.3650 (23)	0.2124 (14)	0.4184 (27)	*
RS0	**GenAbI**	**2**	0.0523 (8)	0.0689 (8)	0.1081 (10)	0.5525 (42)	*
CLJP			0.0753 (8)	0.0519 (8)	0.0589 (8)	0.2867 (19)	*
PMIS			0.5955 (45)	0.3056 (20)	0.1560 (12)	0.4084 (26)	*
RS0	**GenAbI**	**8**	0.0514 (8)	0.0674 (8)	0.1053 (10)	0.5466 (40)	*
CLJP			0.0727 (8)	0.0524 (8)	0.0630 (8)	0.2816 (19)	*
PMIS			0.5878 (44)	0.2968 (19)	0.1624 (12)	0.4036 (26)	*
RS0	**GenAbI**	**20**	0.0499 (8)	0.0568 (8)	0.1023 (10)	0.3491 (22)	*
CLJP			0.0670 (8)	0.0507 (8)	0.0544 (8)	0.2629 (18)	*
PMIS			0.5670 (41)	0.2720 (18)	0.1565 (12)	0.3758 (24)	*
RS0	**GenAbI**	**40**	0.0294 (7)	0.0541 (8)	0.0705 (8)	0.1492 (12)	0.4074 (26)
CLJP			0.0607 (8)	0.0256 (7)	0.0332 (7)	0.0692 (8)	0.2798 (19)
PMIS			0.5537 (39)	0.2451 (17)	0.1035 (10)	0.1063 (10)	0.2943 (19)
RS0	**GenAbI**	**80**	0.0277 (7)	0.0492 (8)	0.0589 (8)	0.0635 (8)	0.2504 (18)
CLJP			0.0601 (8)	0.0265 (7)	0.0337 (7)	0.0404 (8)	0.1369 (12)
PMIS			0.5348 (37)	0.2439 (17)	0.0992 (10)	0.0706 (8)	0.1660 (13)
RS0	**GS**	**-**	0.0442 (8)	0.0545 (8)	0.0871 (9)	0.2516 (17)	*
CLJP			0.0678 (8)	0.0383 (8)	0.0559 (8)	0.2093 (14)	*
PMIS			0.5795 (43)	0.2974 (19)	0.1459 (12)	0.2994 (20)	*
* The convergence factor cannot be obtained since maximum number of iterations was exceeded.							

Table 7: Convergence factors and convergence behavior of the V-Cycle for n=64000 and various coarsening techniques, smoothers and values of β and δl for model problem IV.

			n=64000				
			β=1	**β=25**	**β=50**	**β=100**	**β=200**
Coarsening	**Smoother**	**δl**	**q (its)**	**q (its)**	**q (its)**	**q (its)**	**q (its)**
RS0	**GenAbI**	**1**	0.0460 (8)	0.0623 (8)	0.1467 (11)	0.7928(102)	*
CJLP			0.0913 (9)	0.0604 (8)	0.0750 (8)	0.1178 (10)	*
PMIS			0.6701 (58)	0.4248 (27)	0.2887 (19)	0.2169 (16)	*
RS0	**GenAbI**	**2**	0.0549 (8)	0.0690 (8)	0.1166 (10)	0.4634 (30)	*
CLJP			0.0902 (9)	0.0544 (8)	0.0613 (8)	0.1205 (10)	*
PMIS			0.6431 (53)	0.3807 (24)	0.2202 (16)	0.2135 (16)	*
RS0	**GenAbI**	**8**	0.0525 (8)	0.0665 (8)	0.1112 (10)	0.4505 (29)	*
CLJP			0.0882 (9)	0.0534 (8)	0.0628 (8)	0.0965 (9)	*
PMIS			0.6455 (53)	0.3754 (24)	0.2225 (16)	0.2074 (14)	*
RS0	**GenAbI**	**20**	0.0512 (8)	0.0607 (8)	0.0992 (9)	0.4160 (28)	*
CLJP			0.0833 (9)	0.0534 (8)	0.0597 (8)	0.0862 (9)	*
PMIS			0.6282 (50)	0.3592 (23)	0.2136 (14)	0.1902 (14)	*
RS0	**GenAbI**	**40**	0.0494 (8)	0.0551 (8)	0.0947 (9)	0.2351 (17)	*
CLJP			0.0806 (9)	0.0530 (8)	0.0552 (8)	0.0850 (9)	*
PMIS			0.6164 (48)	0.3367 (22)	0.2048 (14)	0.1530 (12)	*
RS0	**GenAbI**	**80**	0.0304 (7)	0.0498 (8)	0.0757 (8)	0.1244 (11)	0.1788 (14)
CLJP			0.0737 (8)	0.0300 (7)	0.0329 (7)	0.0370 (7)	0.1167 (11)
PMIS			0.6161 (48)	0.3269 (21)	0.1588 (12)	0.1030 (10)	0.1456 (12)
RS0	**GS**	**-**	0.0601 (8)	0.0634 (8)	0.1186 (10)	0.1749 (13)	*
CLJP			0.0848 (9)	0.0417 (8)	0.0511 (8)	0.1011 (10)	*
PMIS			0.6517 (54)	0.4011 (26)	0.1988 (14)	0.1899 (13)	*
* The convergence factor cannot be obtained since maximum number of iterations was exceeded.							

5 Conclusion

The new proposed class of generic approximate banded inverse matrices as a smoother for the AMG method results in an efficient solver for large sparse linear systems. The smoother was proven to be more effective than classical smoothers, such as Gauss-Seidel. It should be mentioned that the AMG method, based on the V-cycle in conjunction with the class of GenAbI matrices can be efficiently parallelized, [Gravvanis, Filelis-Papadopoulos and Matskanidis (2013)], and combined with parallel coarsening schemes, such as CLJP and PMIS. Further research is carried out on the parallelization issues of the algebraic multigrid method in conjunction with approximate inverses. Finally, the algebraic multigrid method in conjunction with of approximate inverse matrices can be efficiently used for solv-

ing complex engineering problems modelled by highly nonlinear partial differential equations such as Bratu problem and fluid mechanics problems, etc.

References

Axelsson, O. (1996): *Iterative Solution Methods*. Cambridge University Press.

Bank, R. E.; Douglas, C. C. (1985): Sharp estimates for multigrid rates of convergence with general smoothing and acceleration. *SIAM Journal on Numerical Analysis*, vol. 22, pp. 617-633.

Brandt, A. (1977): Multi-level adaptive solutions to boundary-value problems. *Math. Comp.*, vol. 31, pp. 333-390.

Brandt, A.; McCormick, S. F.; Ruge, J. W. (1984): Algebraic multigrid (AMG) for sparse matrix equations. In Evans, D.J. (Ed.): *Sparsity and Its Applications*, Cambridge University Press.

Brezina, M.; Falgout, R.; MacLachlan, S.; Manteuffel, T.; McCormick, S.; Ruge, J. (2005): Adaptive algebraic Multigrid. *SIAM Journal on Scientific Computing*, vol. 27, pp. 1261-1286.

Briggs, L. W.; Henson, V.; McCormick, S. F. (2000): *A multigrid tutorial*, SIAM.

Bröker, O.; Grote, M. J. (2002): Sparse approximate inverse smoothers for geometric and algebraic multigrid. *Applied Numerical Mathematics*, vol. 41, pp. 61-80.

Bröker, O.; Grote, M. J.; Mayer, C.; Reusken, A. (2001): Robust parallel smoothing for multigrid via sparse approximate inverses. *SIAM J. on Scientific Computing*, vol. 23, no 4, pp. 1396–1417.

Cleary, A.; Falgout, R.; Henson, V.; Jones, J. (1998): Coarse-grid selection for parallel algebraic multigrid. In *Proceedings of the Fifth International Symposium on Solving Irregularly Structured Problems in Parallel*, vol. 1457, Lecture Notes in Computer Science, pp. 104–115, Springer-Verlag.

De Sterck, H.; Falgout, R. D.; Nolting, J. W.; Yang, U. M. (2008): Distance-Two Interpolation for Parallel Algebraic Multigrid. *Numerical Linear Algebra with Applications*, vol. 15, pp. 115-139.

De Sterck, H.; Yang, U.; Heys, J. (2006): Reducing complexity in parallel algebraic multigrid preconditioners. *SIAM Journal on Matrix Analysis and Applications*, vol. 27, pp. 1019-1039.

Filelis-Papadopoulos, C. K.; Gravvanis, G. A. (2013a): Generic Approximate Sparse Inverse Matrix Techniques. *Int. J. Computational Methods*, in press.

Filelis-Papadopoulos, C. K.; Gravvanis, G. A. (2013b): On the multigrid method

based on Finite Difference Approximate Inverses. *Computer Modeling in Engineering and Sciences*, vol. 90, no 3, pp.233-353.

Frederickson, P. (1996): High performance parallel multigrid algorithms for unsctructured grids. In Melson, N.D.; Manteuffel, T.A.; McCormick, S.F.; Douglas, C.C. (Eds.): *Seventh Copper Mountain Conference on Multigrid Methods*, vol. CP 3339, pp. 317-326, NASA.

Gravvanis, G. A. (2009): High Performance Inverse Preconditioning. *Archives of Computational Methods in Engineering,* vol. 16, no 1, pp. 77-108.

Gravvanis, G. A. (2002): Explicit Approximate Inverse Preconditioning Techniques. *Archives of Computational Methods in Engineering*, vol. 9, no 4, pp. 371-402.

Gravvanis, G. A. (1996): The rate of convergence of explicit approximate inverse preconditioning. *Inter. J. Computer Mathematics*, vol. 60, pp. 77-89.

Gravvanis, G. A.; Lipitakis, E. A. (1996a): "A Three-Dimensional Explicit Preconditioned Solver. *Computers Math. Appl.*, vol. 32, no 2, pp. 111-131.

Gravvanis, G. A.; Lipitakis, E. A. (1996b): An explicit sparse unsymmetric finite element solver. *Communications in Numerical Methods in Engineering*, vol. 12, pp. 21-29.

Gravvanis, G. A.; Filelis-Papadopoulos, C. K.; Matskanidis, P. I. (2013): A survey on the parallelization issues of geometric and algebraic multigrid methods based on generic banded approximate inverses. In Topping, B.H.V.; Ivanyi, P. (Eds.): *Computational Technology Reviews*, vol. 7, pp. 65-98, Saxe-Coburg Publications.

Grote, M. J.; Huckle, T. (1995): Effective Parallel Preconditioning with Sparse Approximate Inverses. In *Proceedings SIAM Conference on Parallel Processing for Scientific Computing*, pp. 466-471, SIAM.

Hackbusch, W. (1985a): *Multi–grid Methods and Applications*, Springer-Verlag.

Hackbusch, W. (1985b): *Iterative Solution of Large Sparse Systems of Equations*, Springer-Verlag.

Haelterman, R.; Viederndeels, J.; Van Heule, D. (2009): Non–stationary two–stage relaxation based on the principle of aggregation multi-grid. *In Proceedings of Computational Fluid Dynamics 2006*, Part 3, pp. 243–248, Springer.

Henson, V.; Yang, U. (2002): BoomerAMG: a parallel algebraic multigrid solver and preconditioner. *Applied Numerical Mathematics*, vol. 41, pp. 155–177.

Kolotolina, Y. L.; Yeremin, Y. A. (1993): Factorized sparse approximate inverse preconditioning. I. Theory, *SIAM J. Matrix Analysis and Applications*, vol. 14, pp. 45-58.

Krechel, A.; Stüben, K. (2001): Parallel algebraic multigrid based on subdomain blocking. *Parallel Computing*, vol. 27, pp. 1009-1031.

Kyziropoulos, P. E.; Filelis-Papadopoulos, C. K.; Gravvanis, G. A. (2013): N-body simulation based on the Particle Mesh method using multigrid schemes. *In 2013 Federated Conference on Computer Science and Information Systems (FedCSIS),* pp. 471-478, IEEE Computer Society.

Lipitakis, E. A.; Evans, D. J. (1987): Explicit semi-direct methods based on approximate inverse matrix techniques for solving boundary-value problems on parallel processors. *Mathematics and Computers in Simulation*, vol. 29, pp. 1-17.

Lipitakis, E. A.; Gravvanis, G. A. (1995): Explicit preconditioned iterative methods for solving large unsymmetric finite element systems. *Computing*, vol. 54, no 2, pp. 167-183.

Livne, O. (2004): Coarsening by compatible relaxation. *Numerical Linear Algebra with Applications,* vol. 11, no 2, pp. 205–227.

Luby, M. (1986): A simple parallel algorithm for the maximal independent set problem. *SIAM Journal on Computing*, vol. 15, pp. 1036-1053.

Matskanidis, P. I.; Gravvanis, G. A. (2012): On the algebraic multigrid method based on generic approximate banded inverses. In *Proceedings 2012 Panhellenic Conference on Informatics (PCI 2012)*, pp. 211-216, IEEE Computer Society.

McCormick, S. F. (1985): Multigrid methods for variational problems: general theory for the V-cycle. *SIAM Journal on Numerical Analysis*, vol. 22, pp. 634-643.

Meurant, G. (2002): A multilevel AINV preconditioner. *Numerical Algorithms*, vol. 29, pp. 107-129.

Ruge, J. W.; Stüben, K. (1987): Algebraic multigrid (AMG). In McCormick, S.F. (Ed.): *Multigrid Methods*, Frontiers in Applied Mathematics, vol. 3, pp. 73–130, SIAM.

Saad, Y. (1996): *Iterative methods for sparse linear systems*, PWS Publishing.

Stüben, K. (2001): Algebraic multigrid (AMG): An introduction with applications. In Trottenberg, U.; Oosterlee, C.; Schuller, A. (Eds.): *Multigrid*, Academic Press.

Tang, P. W.; Wan, W. L. (2000): Sparse approximate inverse smoother for multigrid. *SIAM J. Matrix Analysis and Applications*, vol. 21, pp. 1236-1252.

Trottenberg, U.; Osterlee, C. W.; Schuller, A. (2000): *Multigrid*, Academic Press.

Walker, H. F. (1988): Implementation of the GMRES method using Householder transformations. *SIAM J. Scientific and Statistics Computing*, vol. 9, pp. 152-163.

Wesseling, P. (1980): The rate of convergence of a multiple grid method, Numeri-

cal analysis. In *Proceedings of the 8th bienn. Conference*, Lect. Notes Math., vol. 773, pp. 164-184.

Yang, U. M. (2006): Parallel Algebraic Multigrid Methods - High Performance Preconditioners. In Bruaset, A.M.; Tveito, A. (Eds.): *Numerical Solution of Partial Differential Equations on Parallel Computers*, vol. 51, pp. 209-236, Springer-Verlag.