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Fuzzy Analysis of Structures with Imprecisely Defined Properties

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Abstract: This paper targets to analyse the static response of structures with fuzzy parameters using fuzzy finite element method. Here the material, geometrical properties and external load applied to the structures are taken as uncertain. Uncertainties presents in the parameters are modelled through convex normalised fuzzy sets. Fuzzy finite element method converts the problem into fuzzy or fully fuzzy system of linear equations for static analysis. As such here, two new methods are proposed to solve the fuzzy and fully fuzzy system of linear equations. Numerical examples for structures with uncertain system parameters that are in term of triangular fuzzy number are presented to illustrate the computational aspects of the proposed methods. The results obtained are depicted in term of plots.

Keywords: Stepped bar**,** beam, triangular fuzzy number, fuzzy system of linear equations, fuzzy finite element method.

1 Introduction

In the last few decades for various scientific and engineering problems finite element method has become a more powerful tool for solving the complex systems. In this method the complicated structures/domains are discretized into small finite elements, giving the element wise behavior. Assembling together for all the elements and applying the respective conditions, it gives the output. The system parameters involved in the traditional finite element method such as mass, geometry, material properties, external loads, or boundary conditions are considered as crisp or defined exactly. But, rather than the particular value we may have only the vague, imprecise and incomplete information about the variables and parameters being a result of errors in measurement, observations, experiment, applying different operating

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conditions or it may be maintenance induced error, etc. which are uncertain in nature. Basically these uncertainties can be modeled through probabilistic approach, interval analysis and fuzzy theory.

In probabilistic practice, the variables of uncertain nature are assumed as random variables with joint probability density functions. If the structural parameters and the external load are modeled as random variables with known probability density functions, the response of the structure can be predicted using the theory of probability and stochastic processes in Elishakoff (1983). Also the probabilistic concept is already well established for the extension of the deterministic finite element method towards uncertain assessment. This has led to a number of probabilistic and stochastic finite element procedures [Holder and Mohadevan (2000); Antonio and Hoff Bauer (2010)]. Unfortunately, probabilistic methods are not able to deliver reliable results at the required precision without sufficient experimental data. It may be due to the probability density functions involved in it. As such in the recent decades, interval analysis and fuzzy theory are becoming powerful tools for many real life applications. In these approaches, the uncertain variables and parameters are represented by interval and fuzzy numbers, vectors or matrices.

Various aspects of interval analysis along with applications are explained by Moore (1979). If only incomplete information is available, it is possible to establish the minimum and maximum favorable response of the structures using interval analysis or convex models [Ben-Haim and Elishakoff (1990); Ganzerli and Pantelides (2000)]. Moreover structural analysis with interval parameters using interval based approach has been studied by various authors [Rao and Berke (1997); Muhanna and Mullen (2001); Qui et al. (2006)].

Fuzzy set theoretical concept was developed by Zadeh (1965) which is further used in the uncertain analysis of structures in a wide range. As discussed above, if the structural parameters and the external loads are described in imprecise terms, then fuzzy theory can be applied. As such Behera and Chakraverty (2013a) proposed a solution method to study the uncertain behavior of an electric circuit. Fuzzy diferential equations are solved by [Fatullayev and Köroglu (2012); Tapaswini and Chakraverty (2012); Tapaswini and Chakraverty (2013a); Tapaswini and Chakraverty (2013b)]. Valliappan and Pham (1995) used fuzzy logic for the numerical modeling of engineering problems. An optimization algorithm is developed by Munck et al. (2008) for fuzzy properties based on response surface for the calculation of fuzzy envelope and fuzzy response functions of models. Fuzzy structural analysis using $\alpha-$level optimization is excellently studied by [Moller et al. (2000)]. The transformation method has been applied for the simulation and analysis of systems with uncertain parameters by Hanss (2002). Also an important book is written by Hanss (2005) in which applications of fuzzy arithmetic into engineering problems are described. Fuzzy behavior of mechanical systems with uncertain boundary conditions is investigated by Chekri et al. (2000). Nonlinear membership function for fuzzy optimization of mechanical and structural systems is discussed in Dhingra et al. (1992). When the Finite Element Method (FEM) is described with fuzzy theory it is then known as Fuzzy Finite Element Method (FFEM).

Recently various generalized model of uncertainty have been applied to finite element method to solve the structural problems with fuzzy parameters. Although FEM for structural problems [Zienkiewicz (1979)] is well known and there exits large number of papers related to this. As such few papers that are related to fuzzy FEM are discussed here. Fuzzy finite element approach is applied to describe structural systems with imprecisely defined parameters in an excellent way by Rao and Sawyer (1995). Verhaeghe et al. (2010) discussed the fuzzy finite element analysis technique to describe the static analysis of structures which is based on interval computation. Both fuzzy static and dynamic analysis of structures is explained by Akpan et al. (2001a) using fuzzy finite element approach. Vertex method and VAST software is used in it for the fuzzy finite element analysis. Also Akpan, et al. (2001b) derived fuzzy finite element method for smart structures. Fuzzy finite element method is formulated by Muhanna and Mullen (1999) for mechanics problems. Hanss and Willner (2000) used fuzzy arithmetical approach for the solution of finite element problems with fuzzy parameters. Very recently [Balu and Rao (2011a); Balu and Rao (2011b)] investigated the structural problems with fuzzy parameters. They have used an interesting approach viz. High Dimensional Model Representation (HDMR) along with FEM is for the analysis. Also Balu and Rao (2012) explained both static and dynamic responses of structures using FFEM with HDMR.

The design and analysis of many engineering problems require the solution of linear systems of equations. For example, the finite element formulation of equilibrium and steady state problems lead to a set of simultaneous algebraic linear equations. Accordingly FFEM converts the problem to a Fuzzy System of Linear Equations (FSLE) [Friedman et al. (1998); Guo and Gong (2010); Behera and Chakraverty (2012); Chakraverty and Behera (2013); Amrahov and Askerzade (2011); Behera and Chakraverty (2013d)] or Fully Fuzzy System of Linear Equations (FFSLE) [Senthilkumar and Rajendran (2011); Dehghan and Hashemi (2006); Das and Chakraverty (2012)] for the static analysis of structural problems. There is a difference between fuzzy linear system and fully fuzzy linear system. The coefficient matrix is treated as crisp in the fuzzy linear system, but in the fully fuzzy linear system all the parameters and variables are considered to be fuzzy numbers. Various solution methods have been proposed for the solution of FFSLE by Skalna et al. (2008) and applied in structural mechanics problems. Behera et al. (2011) developed a method to find finite element solution of a stepped rectangular bar in presence of fuzziness in material properties. Recently [Behera and Chakraverty (2013b), Behera and Chakraverty (2013c)] studied the uncertain static behavior of structures using fuzzy finite element method when applied forces are considered as fuzzy.

As such it is an important issue to develop mathematical models and numerical techniques that would appropriately treat the general fuzzy or fully fuzzy linear systems because subtraction and division of fuzzy numbers are not the inverse operations of addition and multiplication respectively. So, this is an important area of research in the recent years. This paper targets to propose new methods for fuzzy and fully fuzzy system of linear equations and applied those methods to the analysis of structural problems using FFEM. In the following sections first preliminaries is discussed. Then, solution methods are explained for fuzzy and fully fuzzy system of linear equations. Next, numerical examples of bar and beam with various effects of uncertain parameters are discussed using fuzzy finite element method to find fuzzy static responses. Lastly conclusions are drawn.

2 Preliminaries

In the following paragraph some definitions related to the present work are given [Ross (2004); Zimmermann (2001)].

Definition 2.1 *Fuzzy number*

A fuzzy number $\tilde{U}$ is convex normalised fuzzy set $\tilde{U}$ of the real line R such that

$$\{\mu_{\tilde{U}}(x) : R \to [0,\ 1],\ \forall\, x \in\ R\}$$

where, $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set and it is piecewise continuous.

Definition 2.2 *Triangular fuzzy number (TFN)*

A triangular fuzzy number $\tilde{U}$ is a convex normalized fuzzy set $\tilde{U}$ of the real line R such that

- There exists exactly one $x_0 \in R$ with $\mu_{\tilde{U}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{U}$), where $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set.
- $\mu_{\tilde{U}}(x)$ is piecewise continuous.

Let us consider an arbitrary triangular fuzzy number $\tilde{U} = (a, b, c)$ as depicted in

Fig. 1. The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$may be define as follows

$$\mu_{\tilde{U}}(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ \dfrac{c-x}{c-b}, & b \le x \le c \\ 0, & x \ge c. \end{cases}$$

The triangular fuzzy number $\tilde{U} = (a,b,c)$ can be represented with an ordered pair of functions through $\alpha-$cut approach as $[\underline{u}(\alpha), \bar{u}(\alpha)] = [(b-a)\alpha + a,\ -(c-b)\alpha + c]$ where, $\alpha \in [0,1]$. This satisfies the following requirements

- $\underline{u}(\alpha)$ is a bounded left continuous non-decreasing function over $[0,\ 1]$.
- $\bar{u}(\alpha)$ is a bounded right continuous non-increasing function over $[0,\ 1]$.
- $\underline{u}(\alpha) \le \bar{u}(\alpha)$, $0 \le \alpha \le 1$.

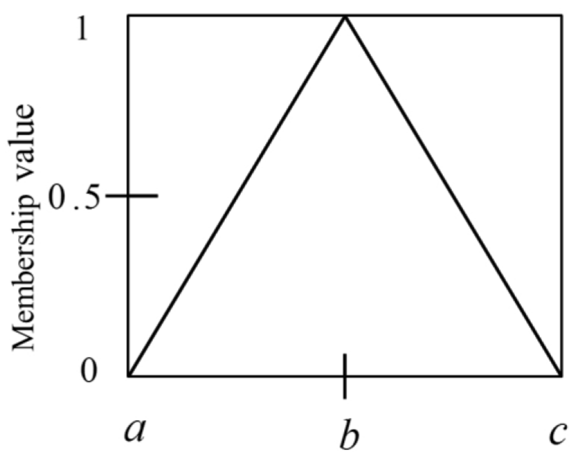

Figure 1: Triangular fuzzy number

Definition 2.3 *Fuzzy arithmetic*

As discussed above, fuzzy numbers may be transformed into an interval through $\alpha-$cut approach. So, for any arbitrary fuzzy number $\tilde{x} = [\underline{x}(\alpha), \bar{x}(\alpha)]$, $\tilde{y} = [\underline{y}(\alpha), \bar{y}(\alpha)]$ and scalar k, we have the interval based fuzzy arithmetic as

- $\tilde{x} = \tilde{y}$ if and only if $\underline{x}(\alpha) = \underline{y}(\alpha)$ and $\bar{x}(\alpha) = \bar{y}(\alpha)$

- $\tilde{x}+\tilde{y}=[\underline{x}(\alpha)+\underline{y}(\alpha),\ \bar{x}(\alpha)+\bar{y}(\alpha)]$
- $\tilde{x}-\tilde{y}=[\underline{x}(\alpha)-\bar{y}(\alpha),\ \bar{x}(\alpha)-\underline{y}(\alpha)]$
- $\tilde{x}\times\tilde{y}=[\min\,(S),\ \max\,(S)]$,
  where $S=\{\underline{x}(\alpha)\times\underline{y}(\alpha),\ \underline{x}(\alpha)\times\bar{y}(\alpha),\ \bar{x}(\alpha)\times\underline{y}(\alpha),\ \bar{x}(\alpha)\times\bar{y}(\alpha)\}$
- $k\tilde{x}=\begin{cases}[k\bar{x}(\alpha),k(\alpha)]\,,k<0,\\ [k(\alpha),k\bar{x}(\alpha)]\,,k\geq 0.\end{cases}$

As discussed above, for the static analysis of structures the finite element equations reduced to system of linear equations, hence for fuzzy parameters of the structures the equation of motion obtained by FFEM reduces to fuzzy or fully system of linear equations. As such here, solution method for fuzzy and fully fuzzy system of linear equations is presented as follows to find the fuzzy static response of the structures.

3 Fuzzy System of linear Equations

The $n\times n$ fuzzy system of linear equations may be written as

$$\begin{aligned}
&a_{11}\tilde{x}_1+a_{12}\tilde{x}_2+\cdots+a_{1n}\tilde{x}_n=\tilde{b}_1\\
&a_{21}\tilde{x}_1+a_{22}\tilde{x}_2+\cdots+a_{2n}\tilde{x}_n=\tilde{b}_2\\
&\qquad\qquad\vdots\\
&a_{n1}\tilde{x}_1+a_{n2}\tilde{x}_2+\cdots+a_{nn}\tilde{x}_n=\tilde{b}_n
\end{aligned}\tag{1}$$

In matrix notation the above system may be written as $[A]\{\tilde{X}\}=\{\tilde{b}\}$, where the coefficient matrix $[A]=(a_{kj}),\ 1\leq k\leq n,\ j\leq n$ is a crisp real $n\times n$ matrix, $\{\tilde{b}\}=\{\tilde{b}_k\},\ 1\leq k$ is a column vector of fuzzy number and $\{\tilde{X}\}=\{\tilde{x}_j\}$ is the vector of fuzzy unknown.

The above system, $[A]\{\tilde{X}\}=\{\tilde{b}\}$, can be written as

$$\sum_{j=1}^{n}a_{kj}\tilde{x}_j=\tilde{b}_k,\quad\text{for}\quad k=1,2,\cdots,n.\tag{2}$$

As per the parametric form we may write the real fuzzy unknown and the right hand real fuzzy number vector as $\tilde{x}_j=[\underline{x}_j(\alpha),\bar{x}_j(\alpha)]$ and $\tilde{b}_k=[\underline{b}_k(\alpha),\ \bar{b}_k(\alpha)]$. Substituting these expressions in Eq. (2), we have

$$\sum_{j=1}^{n}a_{kj}[\underline{x}_j(\alpha),\ \bar{x}_j(\alpha)]=[\underline{b}_k(\alpha),\ \bar{b}_k(\alpha)],\quad\text{for}\quad k=1,2,\cdots,n.\tag{3}$$

Now Eq. (3) can equivalently be written as the following two equations Eqs. (4) and (5)

$$\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha) + \sum_{a_{kj}<0} a_{kj}\bar{x}_j(\alpha) = \underline{b}_k(\alpha) \tag{4}$$

and

$$\sum_{a_{kj}\geq 0} a_{kj}\bar{x}_j(\alpha) + \sum_{a_{kj}<0} a_{kj}\underline{x}_j(\alpha) = \bar{b}_k(\alpha). \tag{5}$$

In the following section a new method is proposed for solving the fuzzy system of linear equations as defined in Eq. (1).

3.1 Proposed method for solving fuzzy real system of linear equations

In this section we proposed a new method to solve fuzzy real system of linear equations. Before discussing the method, a related theorem is first stated and proved in the following paragraphs.

Theorem 1 If $[A]\{\tilde{X}\} = \{\tilde{b}\}$ then $\{\underline{X}+\bar{X}\}$ is the solution vector of the system $[A]\{\underline{X}+\bar{X}\} = \{\underline{b}+\bar{b}\}$.

Proof. Now one may write $[A]\{\underline{X}+\bar{X}\}$ as

$$\sum_{j=1}^{n} a_{kj}\{\underline{x}_j(\alpha)+\bar{x}_j(\alpha)\}, \quad \text{for } k = 1,2,\cdots,n\,.$$

This can be written as

$$\sum_{a_{kj}\geq 0} a_{kj}\{\underline{x}_j(\alpha)+\bar{x}_j(\alpha)\} + \sum_{a_{kj}<0} a_{kj}\{\underline{x}_j(\alpha)+\bar{x}_j(\alpha)\}.$$

It is equivalent to

$$\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha) + \sum_{a_{kj}\geq 0} a_{kj}\bar{x}_j(\alpha) + \sum_{a_{kj}<0} a_{kj}\underline{x}_j(\alpha) + \sum_{a_{kj}<0} a_{kj}\bar{x}_j(\alpha).$$

Using Eqs. (4) and (5), the above expression can be written as $\{\underline{b}_k(\alpha)+\bar{b}_k(\alpha)\} = \{\underline{b}+\bar{b}\}$. Accordingly, one may conclude $[A]\{\underline{X}+\bar{X}\} = \{\underline{b}+\bar{b}\}$. This proves that $\{\underline{X}+\bar{X}\}$ is the solution vector of the system $[A]\{\underline{X}+\bar{X}\} = \{\underline{b}+\bar{b}\}$.

Now using Theorem 1 one can find the solution vector $\{\underline{X}+\bar{X}\}$ of $[A]\{\underline{X}+\bar{X}\} = \{\underline{b}+\bar{b}\}$. Let us consider the solution vector $\{\underline{X}+\bar{X}\}$ as $\{P\} = \{P_j(\alpha)\}$. So, this can be written as $\{\underline{x}_j(\alpha)+\bar{x}_j(\alpha)\} = \{P_j(\alpha)\}$. The lower and upper bounds of the solution vector may be obtained as $\{\underline{x}_j(\alpha)\} = \{P_j(\alpha)\} - \{\bar{x}_j(\alpha)\}$ and $\{\bar{x}_j(\alpha)\} = \{P_j(\alpha)\} - \{\underline{x}_j(\alpha)\}$ respectively.

Theorem 2 The monotonic increasing solution vector $\{\underline{x}_j(\alpha)\}$ can be obtained by replacing $\{\bar{x}_j(\alpha)\}$ in terms of $\{\underline{x}_j(\alpha)\}$ and $\{P_j(\alpha)\}$ in Eq. (3).

Proof. As Eq. (3) converts into two crisp systems (4) and (5), so substituting $\{\bar{x}_j(\alpha)\} = \{P_j(\alpha)\} - \{\underline{x}_j(\alpha)\}$ in any one equation and solving we may find $\{\underline{x}_j(\alpha)\}$. So this proves the theorem.

Theorem 3 Crisp linear systems (viz. Eqs. 4 and 5) give exactly same $\{\underline{x}_j(\alpha)\}$ when the upper bound of the fuzzy variable $\{\bar{x}_j(\alpha)\}$is replaced by $\{P_j(\alpha)\} - \{\underline{x}_j(\alpha)\}$in Eq. (3).

Proof. Let us first consider Eq. (4),

$$\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha) + \sum_{a_{kj}<0} a_{kj}\bar{x}_j(\alpha) = \underline{b}_k(\alpha).$$

Now substituting $\{\bar{x}_j(\alpha)\} = \{P_j(\alpha)\} - \{\underline{x}_j(\alpha)\}$ in the above equation we have

$$\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha) + \sum_{a_{kj}<0} a_{kj}\left[\{P_j(\alpha)\} - \{\underline{x}_j(\alpha)\}\right] = \underline{b}_k(\alpha).$$

The above expression may equivalently be written as

$$\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha) + \sum_{a_{kj}<0} a_{kj}\{P_j(\alpha)\} - \sum_{a_{kj<0}} a_{kj}\{\underline{x}_j(\alpha)\} = \underline{b}_k(\alpha). \tag{6}$$

Eq. (6) now be represented as

$$\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha) + \sum_{a_{kj}<0} a_{kj}\{P_j(\alpha)\} = \underline{b}_k(\alpha). \tag{7}$$

But, the fuzzy system$[A]\{\underline{X}+\bar{X}\} = \{\underline{b}+\bar{b}\}$ is written as

$$\sum_{j=1}^{n} a_{kj}\{\underline{x}_j(\alpha) + \bar{x}_j(\alpha)\} = \{\underline{b}_k(\alpha) + \bar{b}_k(\alpha)\}.$$

Above is now equivalently expressed as

$$\sum_{a_{kj}\geq 0} a_{kj}\{\underline{x}_j(\alpha) + \bar{x}_j(\alpha)\} + \sum_{a_{kj}<0} a_{kj}\{\underline{x}_j(\alpha) + \bar{x}_j(\alpha)\} = \{\underline{b}_k(\alpha) + \bar{b}_k(\alpha)\}$$

and it may reduce the following as

$$\sum_{a_{kj}\geq 0} a_{kj}\{P_j(\alpha)\} + \sum_{a_{kj}<0} a_{kj}\{P_j(\alpha)\} = \{\underline{b}_k(\alpha) + \bar{b}_k(\alpha)\}.$$

This is similar to

$$\sum_{a_{kj}<0} a_{kj}\{P_j(\alpha)\} = \{\underline{b}_k(\alpha) + \bar{b}_k(\alpha)\} - \sum_{a_{kj}\geq 0} a_{kj}\{P_j(\alpha)\}.$$

Substituting this in Eq. (7) we have

$$\sum_{a_{kj}\geq 0} a_{kj}\underline{x}_j(\alpha) = \sum_{a_{kj}\geq 0} a_{kj}\{P_j(\alpha)\} - \bar{b}_k(\alpha). \tag{8}$$

From this one may conclude that Eq. (4) is equivalent to Eq. (8). Similarly one may prove that Eq. (5) is equivalent to Eq. (8). Hence, it may be concluded that Eqs. (4) and (5) are exactly same. Thus Theorem 3 is proved.

Theorem 4 The monotonic decreasing solution vector $\{\bar{x}_j(\alpha)\}$ can be obtained by replacing $\{\underline{x}_j(\alpha)\}$ in terms of $\{\bar{x}_j(\alpha)\}$ and $\{P_j(\alpha)\}$ in Eq. (3).

Proof . The proof is straight forward as Theorem 2.

Theorem 5 Crisp linear systems (viz. Eqs. 4 and 5) give exactly same $\{\bar{x}_j(\alpha)\}$ when the lower bound of the fuzzy variable $\{\underline{x}_j(\alpha)\}$is replaced by $\{P_j(\alpha)\} - \{\bar{x}_j(\alpha)\}$.

Proof. The proof is straight forward as Theorem 3.

4 Fully Fuzzy System of Linear Equations

The $n \times n$ fully fuzzy system of linear equations may be written as

$$\begin{aligned}
\tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + \cdots + \tilde{a}_{1n}\tilde{x}_n &= \tilde{b}_1 \\
\tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 + \cdots + \tilde{a}_{2n}\tilde{x}_n &= \tilde{b}_2 \\
&\vdots \\
\tilde{a}_{n1}\tilde{x}_1 + \tilde{a}_{n2}\tilde{x}_2 + \cdots + \tilde{a}_{nn}\tilde{x}_n &= \tilde{b}_n
\end{aligned} \tag{9}$$

In matrix notation the above system may be written as $[\tilde{A}]\{\tilde{X}\} = \{\tilde{b}\}$,where the coefficient matrix $[\tilde{A}] = (\tilde{a}_{kj})$, $1 \leq k \leq n$, $j \leq n$ is a fuzzy $n \times n$ matrix, $\{\tilde{b}\} = \{\tilde{b}_k\}$, $1 \leq k$ is a column vector of fuzzy number and $\{\tilde{X}\} = \{\tilde{x}_j\}$ is the vector of fuzzy unknown.

The above system (9), $[\tilde{A}]\{\tilde{X}\} = \{\tilde{b}\}$ can be written as

$$\sum_{j=1}^{n} \tilde{a}_{kj}\tilde{x}_j = \tilde{b}_k, \quad \text{for} \quad k = 1, 2, \cdots, n.. \tag{10}$$

As per the parametric form we may write the fuzzy coefficient matrix, real fuzzy unknown and the right hand real fuzzy number vector as $\tilde{a}_{kj} = [\underline{a}_{kj}(\alpha), \bar{a}_{kj}(\alpha)]$,$\tilde{x}_j = [\underline{x}_j(\alpha), \bar{x}_j(\alpha)]$ and $\tilde{b}_k = [\underline{b}_k(\alpha), \bar{b}_k(\alpha)]$.

Substituting the above expressions in Eq. (10), one may have

$$\sum_{j=1}^{n}[\underline{a}_{kj}(\alpha),\bar{a}_{kj}(\alpha)][\underline{x}_j(\alpha),\bar{x}_j(\alpha)] = [\underline{b}_k(\alpha),\bar{b}_k(\alpha)]. \tag{11}$$

Here we will obtain a non-negative solution of the fully fuzzy linear system (11) where, $\{\tilde{X}\} \geq 0$ as follows.

4.1 Proposed method for solving fully fuzzy system of linear equations

Eq. (11) can equivalently be written as the following two Eqs. (12) and (13) by applying fuzzy arithmetic as

$$\sum_{\underline{a}_{kj}(\alpha)\geq 0}\underline{a}_{kj}(\alpha)\underline{x}_j(\alpha)+\sum_{\underline{a}_{kj}(\alpha)<0}\underline{a}_{kj}(\alpha)\bar{x}_j(\alpha)=\underline{b}_k(\alpha) \tag{12}$$

and

$$\sum_{\bar{a}_{kj}(\alpha)\geq 0}\bar{a}_{kj}(\alpha)\bar{x}_j(\alpha)+\sum_{\bar{a}_{kj}(\alpha)<0}\bar{a}_{kj}(\alpha)\underline{x}_j(\alpha)=\bar{b}_k(\alpha). \tag{13}$$

One may write explicitly the combined form of Eqs. (12) and (13) as follows

$$\begin{pmatrix}
\underline{a}_{11}(\alpha) & \underline{a}_{12}(\alpha) & \cdots & \underline{a}_{1n}(\alpha) & -(\underline{a}_{11}(\alpha)) & -(\underline{a}_{12}(\alpha)) & \cdots & -(\underline{a}_{1n}(\alpha)) \\
\underline{a}_{21}(\alpha) & \underline{a}_{22}(\alpha) & \cdots & \underline{a}_{2n}(\alpha) & -(\underline{a}_{21}(\alpha)) & -(\underline{a}_{22}(\alpha)) & \cdots & -(\underline{a}_{2n}(\alpha)) \\
\vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\
\underline{a}_{n1}(\alpha) & \underline{a}_{n2}(\alpha) & \cdots & \underline{a}_{nn}(\alpha) & -(\underline{a}_{n1}(\alpha)) & -(\underline{a}_{n2}(\alpha)) & \cdots & -(\underline{a}_{nn}(\alpha)) \\
-(\bar{a}_{11}(\alpha)) & -(\bar{a}_{12}(\alpha)) & \cdots & -(\bar{a}_{1n}(\alpha)) & \bar{a}_{11}(\alpha) & \bar{a}_{12}(\alpha) & \cdots & \bar{a}_{1n}(\alpha) \\
-(\bar{a}_{21}(\alpha)) & -(\bar{a}_{22}(\alpha)) & \cdots & -(\bar{a}_{2n}(\alpha)) & \bar{a}_{21}(\alpha) & \bar{a}_{22}(\alpha) & \cdots & \bar{a}_{2n}(\alpha) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\
-(\bar{a}_{n1}(\alpha)) & -(\bar{a}_{n2}(\alpha)) & \cdots & -(\bar{a}_{nn}(\alpha)) & \bar{a}_{n1}(\alpha) & \bar{a}_{n2}(\alpha) & \cdots & \bar{a}_{nn}(\alpha)
\end{pmatrix}
\begin{pmatrix}
\underline{x}_1(\alpha) \\ \underline{x}_2(\alpha) \\ \vdots \\ \underline{x}_n(\alpha) \\ \bar{x}_1(\alpha) \\ \bar{x}_2(\alpha) \\ \vdots \\ \bar{x}_n(\alpha)
\end{pmatrix}
=\begin{pmatrix}
\underline{b}_1(\alpha) \\ \underline{b}_2(\alpha) \\ \vdots \\ \underline{b}_n(\alpha) \\ \bar{b}_1(\alpha) \\ \bar{b}_2(\alpha) \\ \vdots \\ \bar{b}_n(\alpha)
\end{pmatrix}. \tag{14}$$

Now solving the above crisp system of linear equations one may get the lower and upper bound of the fuzzy solution vector. Also the negative solution of the fully fuzzy system of linear equations may be obtained in the similar manner.

5 Numerical Examples and Results

Example 1 Let us consider a 2×2 real fuzzy system as

$$\tilde{x}_1 - \tilde{x}_2 = [\alpha,\ 2-\alpha]$$

$$\tilde{x}_1 + 3\tilde{x}_2 = [4+\alpha,\ 7-2\alpha]. \tag{15}$$

According to Theorem 1 the solution vector is $\begin{bmatrix} \underline{x}_1(\alpha)+\bar{x}_1(\alpha) \\ \underline{x}_2(\alpha)+\bar{x}_2(\alpha) \end{bmatrix} = \begin{bmatrix} \frac{17}{4}-\frac{1}{4}\alpha \\ \frac{9}{4}-\frac{1}{4}\alpha \end{bmatrix}$.

Now using Theorems 2 and 4 this we have $\underline{x}_1(\alpha) = \frac{11}{8}+\frac{5}{8}\alpha$, $\bar{x}_1(\alpha) = \frac{23}{8} - \frac{7}{8}\alpha$, $\underline{x}_2(\alpha) = \frac{7}{8}+\frac{1}{8}\alpha$ and $\bar{x}_2(\alpha) = \frac{11}{8}-\frac{3}{8}\alpha$.

Finally we may write in standard form $\tilde{x}_1 = \left[\frac{11}{8}+\frac{5}{8}\alpha, \frac{23}{8}-\frac{7}{8}\alpha\right]$ and $\tilde{x}_2 = \left[\frac{7}{8}+\frac{1}{8}\alpha, \frac{11}{8}-\frac{3}{8}\alpha\right]$. Plots for $\tilde{x}_1$ and $\tilde{x}_2$ are given in Figs. 2 (a) and 2(b) respectively.

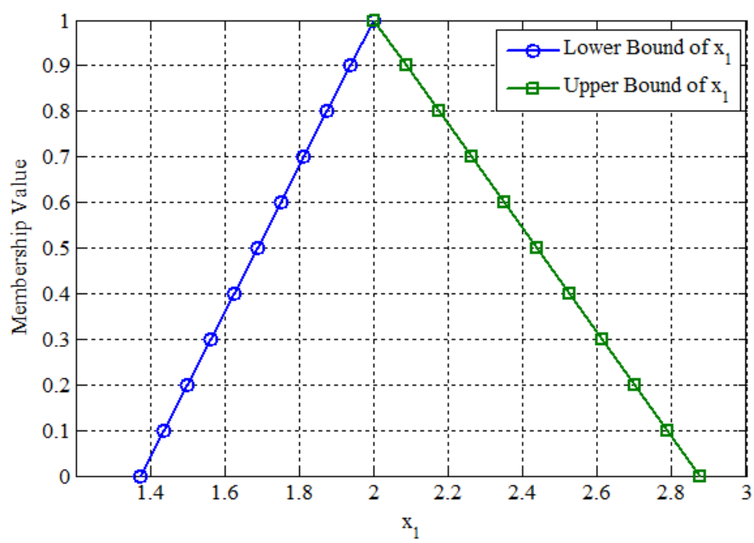

Figure 2(a): Plot of $\tilde{x}_1$ (Example 1)

Example 2 (Three stepped bar) Here we have considered a three stepped bar as shown in Fig. 3. This was previously considered by (Balu and Rao 2012). Similar type of study has been reported in (Akpan et al. 2001a, Rao and Sawyer 1995). For the uncertain static response, three different cases have been considered here. The input variables for all the cases are shown in Table 1. In Case A only the load $P_3$is fuzzy. In Case B the load (P_3) as well as the Young's modulus (E_i) is

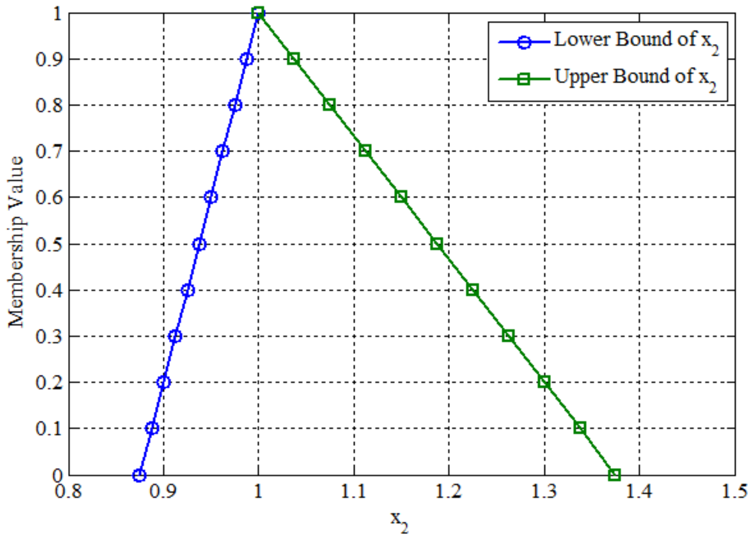

Figure 2(b): Plot of $\tilde{x}_2$ (Example 1)

having fuzziness and in Case C all the properties viz., cross sectional areas (A_i), lengths (L_i) ,Young's modulus for the bar elements and the load applied at free end are taken as fuzzy variables. Here i varies from 1 to 3. All the fuzzy variables are assumed as triangular fuzzy number viz. (a,b,c). Through $\alpha-$ cut approach this can be represented as $[(b-a)\alpha+a,\ -(c-b)\alpha+c]$ where $\alpha \in [0,\ 1]$. This defines a triangular membership function, where a and c are the lower and upper bounds of the fuzzy number at $\alpha = 0$ and b is the exact or crisp value at $\alpha = 1$.

Using the proposed methodologies obtained fuzzy translational displacement at nodes 2, 3 and 4 are depicted in Figs. 4, 5 and 6 respectively for all the cases. From Figs. 4 and 6 one can observe that the larger width is obtained for both the figures when fuzziness appears only in the applied external load viz. for Case A. The spread in the fuzzy displacements are gradually decreases when we have introduced fuzziness in the stiffness matrix viz. for Cases B and C respectively. But in Fig. 5 the translational displacement at node 3 represents weak fuzzy responses as obtained for all the cases. Fig. 5 demonstrates the opposite behavior from Figs. 4 and 6 that is the spread of the fuzzy responses are gradually increasing when we introduce fuzziness in the stiffness matrix. It also gives the smaller width for Case A that is when only fuzziness appears in the external load. Fuzzy displacements obtained at the free end gives similar behavior as the observations reported in [Rao and Sawyer (1995)]. Case A demonstrates the proposed method for fuzzy system of linear equations and Cases B and C use proposed method for fully fuzzy system of linear equations. Moreover the proposed solution methods estimate narrow bounds for the structural responses.

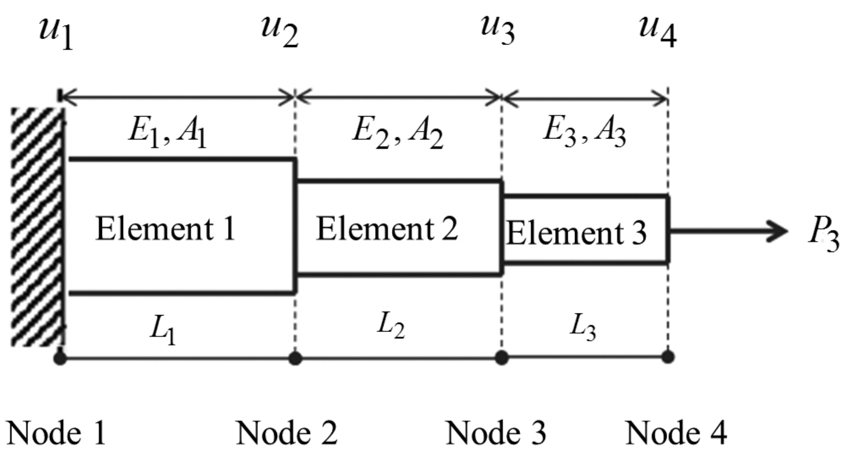

Figure 3: Discretization of a stepped bar into three elements with force applied at the free end

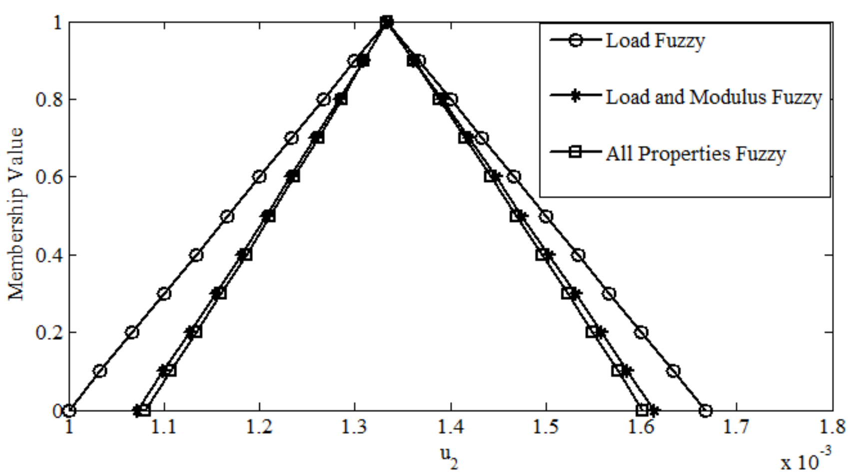

Figure 4: Fuzzy translational displacement at node 2 of three stepped bar (Example 2)

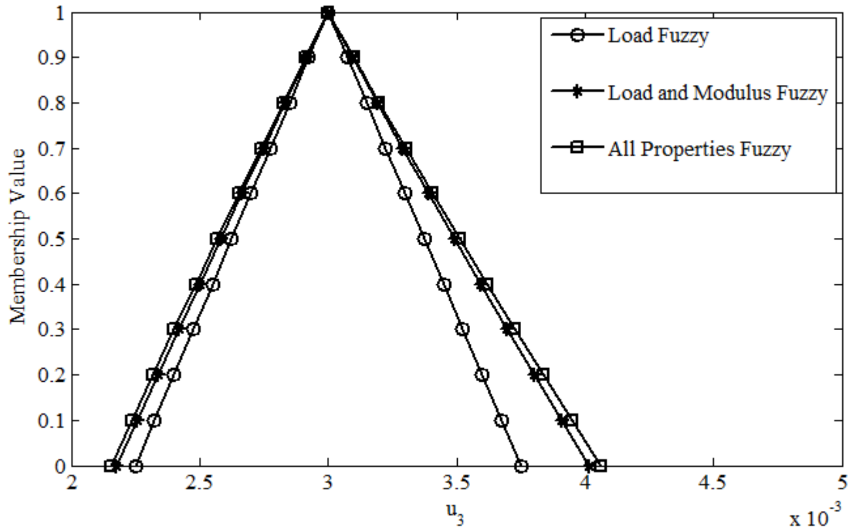

Figure 5: Fuzzy translational displacement at node 3 of three stepped bar (Example 2)

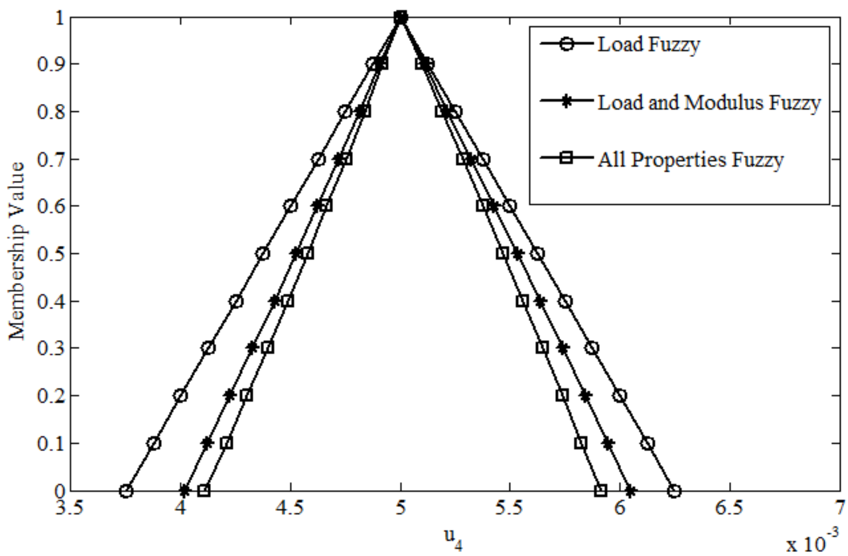

Figure 6: Fuzzy translational displacement at node 4 of three stepped bar (Example 2)

Table 1: Data of three-stepped bar with triangular fuzzy number (Example 2)

Parameters	Case A	Case B	Case C
A_1 (in.^{2})	3.00	3.00	(2.99,3.00,3.01)
A_2 (in.^{2})	2.00	2.00	(1.99,2.00,2.01)
A_3 (in.^{2})	1.00	1.00	(0.99,1.00,1.01)
L_1 (in.)	12.00	12.00	(11.95,12.00,12.05)
L_2 (in.)	10.00	10.00	(9.95,10.00,10.05)
L_3 (in.)	6.00	6.00	(5.95,6.00,6.05)
E_1, E_2, E_3 (psi)	3.0e7	(2.8e7,3.0e7,3.1e7)	(2.8e7,3.0e7,3.1e7)
P_1, P_2 (lb)	0.0	0.0	0.0
P_3 (lb)	(7500,10000,12500)	(7500,10000,12500)	(7500,10000,12500)

Example 3 (Fixed-Fixed beam)

In this example a fixed-fixed beam has been considered to compute fuzzy static response as shown in Fig. 7 using the proposed methodologies. It was studied earlier in Rao and Swayer (1995). Later on [Akpan et al. (2001a); Balu and Rao (2012)] also investigated the same problem. Three cases have been considered for the analysis. In Case A only the load is considered as fuzzy and the load is represented by the triplet (360,400,440). In Case B the modulus of elasticity represented by triplet (2.94e7, 3.0e7, 3.06e7) is considered as the only fuzzy variable. In Case C both the load and the modulus of elasticity were considered as in Cases A and B. The model parameters for each case are listed in the form of triangular fuzzy numbers in the Table 2. Two elements were used in each case.

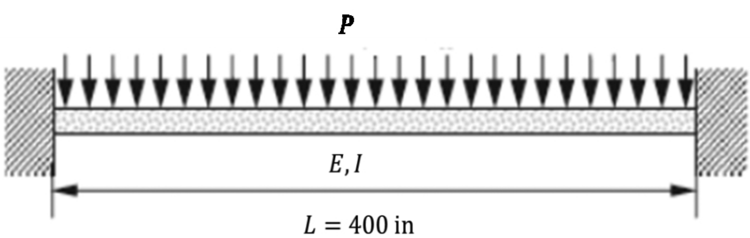

Figure 7: Configuration of fixed-fixed beam

Using the proposed methods obtained fuzzy vertical displacements and angle of rotations at the mid-span of the beam are shown in Figs. 8 and 9 respectively for all the cases. The results obtained by the proposed methods agree well with Balu and Rao (2012). Observing Fig. 8 it may be seen that spread of the fuzzy vertical

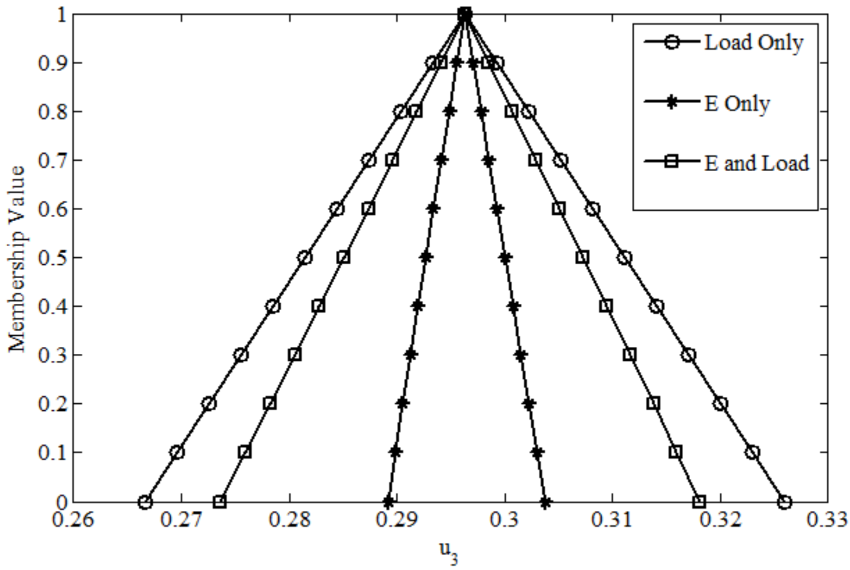

Figure 8: Fuzzy vertical displacement at the mid span of fixed-fixed beam (Example 3)

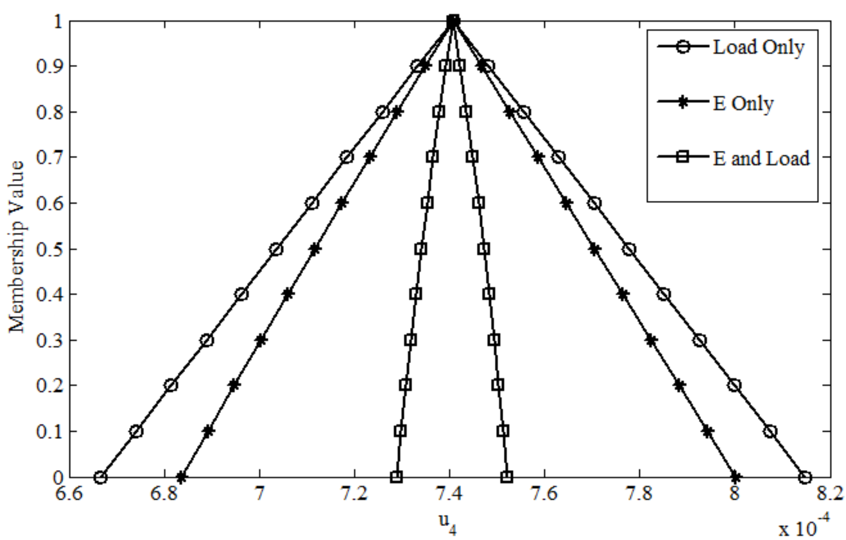

Figure 9: Fuzzy angle of rotation at the mid span of fixed-fixed beam (Example 3)

Table 2: Data for beam examples as triangular fuzzy numbers (Example 3)

Parameters	Case A	Case B	Case C
L (in.)	400	400	400
I (in.^{4})	3.0e3	3.0e3	3.0e3
E (psi)	3.0e7	(2.94e7,3.0e7,3.06e7)	(2.94e7,3.0e7,3.06e7)
P (lb/in.)	(360,400,440)	400	(360,400,440)

displacements for Case B is smaller where as for Case A it is larger. The spread for Case C is smaller than Case A but greater than Case B. Similar observations may be made for fuzzy angle of rotations obtained by the proposed methods which is depicted in Fig. (9). In this case smaller width is obtained for Case C and the larger width is seen for Case A. Here width of Case B is smaller than Case A but greater than Case C.

6 Conclusion

In this paper solution methods for fuzzy and fully fuzzy system of linear equations have been proposed to analyze the fuzzy structural response when fuzziness appears in the properties. The proposed methodologies are straight forward and easy to handle. It involves interval based computations in parametric form of fuzzy numbers. Stepped bar and fix ended beam have been considered for the present analysis. The uncertainties present in the geometry, material properties and external loads are represented by triangular fuzzy number. The results obtained by the proposed methods are compared with the existing results and are found to be in good agreement. The methods are based on analytic approach hence the errors arising in these procedures are minimum. Obtained results are depicted in term of plots to show the efficiency of the proposed methods.

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