On the Axisymmetric Time-harmonic Lamb's Problem for a System Comprising a Half-space and a Covering Layer with Finite Initial Strains

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Abstract: By employing the Three-dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TLTEWISB) the time-harmonic Lamb's problem for a system comprising a finite pre-strained half-space and finite pre-strained covering layer made of incompressible materials is examined for the case where the material of the covering layer is stiffer than that of the half-space material. It is assumed that on the upper free face plane of the covering layer the point-located time-harmonic force acts. The elasticity relations of the materials are described through Treloar's potential. The corresponding boundary-value problem is solved by employing the Hankel integral transformation. The corresponding inverse transformations are found (numerically) by utilizing the Sommerfeld contour. Numerical results regarding the stresses acting on the interface plane are presented and discussed. The main focus is on the frequency response of these stresses and the influence of the initial strains on them. In particular, it is established that the mechanical behavior of the forced vibration of the system under consideration is similar to that of the system comprising a mass, a parallel connected spring and a dashpot. Moreover, it is established that by increasing the stiffness of the covering layer material as well as with initial stretching of the covering layer, the "resonance" values of the stresses decrease.

Keywords: time-harmonic Lamb's problem, frequency response, resonance, forced vibration, Sommerfeld contour.

1 Introduction

It is known that covering layers are widely used in various branches of modern industry such as civil engineering, mechanical engineering, aeronautics, rocketry,

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etc. Covering layers can be used as armor to prevent external static and especially dynamic power and thermal effects, as well as to isolate the basic material of the structural element from direct contact with the external environment. In addition, the covering layers may be used to redistribute the concentrated external influences (forces) to some part of the structural element. These and many other examples not listed here, make the fundamental research related to the mechanical behavior of a system composed of a coating layer and base material under the influence of external dynamic loads, important. Therefore investigations related to the corresponding problems have not only theoretical but also great practical significance. Note that among these problems there are also many which cannot be solved within the framework of the classical linear theory of elastodynamics, such as that of elastodynamic problems related to elastic systems with initial strains. The initial strains (or stresses) in the elements of construction may arise as a result of the action of various factors. For example, the strains and stresses which arise as a result of the action of the operating forces can be taken as the initial ones caused by additional static or dynamic forces. Moreover, the initial stresses and strains in elements of construction can arise as a result of the change in the environmental conditions etc.

Up to now a large number of investigations have been made in this field. Note that almost all these investigations were made by utilizing the so-called Three-dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TL-TEWISB). The relations and equations of the TLTEWISB are obtained from the exact relations and equations of the non-linear theory of elastodynamics by linearization with respect to small dynamical perturbations. The general questions of the TLTEWISB have been elaborated in many investigations such as in works by Biot (1965), Truestel (1961), Eringen and Suhubi (1975), Guz (2004) and others. It should be noted that there are some versions of the TLTEWISB which were detailed in the monograph by Guz (2004). These versions of the TLTEWISB are distinguished from each other with respect to the magnitude of the initial strains. The version of the TLTEWISB developed for high-elastic materials, according to which the initial strains in the bodies are determined within the scope of the non-linear theory of elasticity without any restrictions on the magnitude of the initial strains, is called the large (or finite) initial deformation version. The version of the TLTEWISB, according to which, an initial stress-strain state in bodies is determined within the scope of the geometrical non linear theory of elasticity and under which changes to the elementary areas and volumes as a result of the initial deformation are not taken into account, is called the first version of the small initial deformation theory of the TLTEWISB. The second version of the small initial deformation theory of the TLTEWISB is the version, according to which, an initial stress-strain state in bodies is determined within the scope of the classical linear

theory of elasticity.

The subject of the present article, namely, axisymmetrically forced vibrations of a system composed of a finite pre-strained covering layer and a finite pre-strained half-space, also applies to the basic research specified above. Consider now a brief review of these works beginning with a paper by Akbarov (2006a) dealing with the study of the axisymmetric forced vibration of a system comprising a pre-stressed covering layer and a pre-stressed half-space which was studied within the scope of the above-noted first version of the small initial deformation theory of the TL-TEWISB. It was assumed that the materials of the constituents are compressible and their elasticity relations are described through the Murnaghan potential.

The same problem for a system comprising a finite pre-strained covering layer and a finite pre-strained half-space made of high elastic incompressible materials was also studied by Akbarov (2006b) by utilizing the large (or finite) initial deformation version of the TLTEWISB. Numerical results were presented for the case where the material of the half-space is stiffer than that of the covering layer and the mechanical relations of the constituents are described through the Treloar potential. Below, we will return to discuss this paper in more detail because the object of investigation of the present paper is also a finite pre-strained covering layer and a finite pre-strained half-space made of high-elastic incompressible materials. In another paper by Akbarov (2006c), the axisymmetric forced vibration of a finite pre-strained two-layered slab made from high-elastic incompressible Treloar materials resting on an absolute rigid foundation was studied. The investigations were also made by utilizing the large initial deformation version of the TLTEWISB. Moreover, within the assumptions of the last two papers, an axisymmetric problem on the frequency response of the pre-strained slab made of incompressible functionally graded material and resting on an absolute rigid foundation was studied in a paper by Akbarov (2006d).

Within the scope of the second version of the TLTEWISB in a paper by Akbarov and Guler (2007), the problem related to the plane-strain state in a half-plane covered with a pre-stretched layer under the action of arbitrary linearly located time-harmonic forces was solved. Specific numerical results on the interface stress distribution were presented for cases where steel and aluminum are taken as the materials of the constituents.

In the above-mentioned investigations it was assumed that the materials of the constituents of the system under consideration are isotropic. In a paper by Akbarov and Ilhan (2010) these investigations were developed for the case where the materials of the constituents are orthotropic. Moreover, within the framework of the foregoing assumptions and theories, in papers by Akbarov and Ilhan (2008, 2009), problems related to the dynamics of moving and oscillating moving loads acting on

the aforementioned system were investigated. Note that in these papers by Akbarov and Ilhan the investigations were carried out within the scope of the second version of the TLTEWISB.

The dynamics of the oscillating moving load acting on the finite pre-strained two-layered slab resting on an absolute rigid foundation were also considered in a paper by Akbarov and Salmanova (2009) by utilizing the large initial deformation version of the TLTEWISB. The materials of the layers of the slab are assumed to be compressed, the elasticity relations of which are described through the harmonic potential.

In all the foregoing works, two-dimensional problems related to the axisymmetric and plane-strain states were studied. Development of these studies for the corresponding 3D problems was first made in papers by Akbarov et al. (2005) and Emiroglu et al. (2009). In these papers the time-harmonic Lamb's problem was considered for a system comprising a bi-axially pre-stressed covering layer and a bi-axially pre-stressed half-space. Note that the investigations in these papers were made by employing the second version of the TLTEWISB.

This concludes our review of related works and note that despite the fact that in these works it has been assumed that there are initial stresses or strains in the constituents of the system and that these investigations have been made by utilizing the TLTEWISB, they are also relevant in the cases where the initial stresses in these constituents are absent. Consequently, in the works reviewed above, the theoretical and numerical results related to the classical linear theory of elasticity are also obtained as particular cases of the general theoretical and numerical results, respectively. Moreover, in the previous works the corresponding problems were solved by employing various types of integral transformation techniques. The integrals related to the inverse of these transformations are wavenumber integrals. Note that, in the foregoing works of the author and his students these integrals were calculated according to Cauchy's principal value sense. As has been noted in works by Lamb (1904), Tsang (1978), Jensen et al. (2011) and many others listed therein, more accurate and physically correct results can be obtained if these wave-number integrals are calculated along the Sommerfeld contour. Therefore in the paper by Akbarov and Ilhan (2013) which relates to the study of the Lamb's problem for a system consisting of the piezoelectric covering layer and piezoelectric half-plane the corresponding wavenumber integrals are calculated by the use of the Sommerfeld contour.

In the present paper we study again the problem which was considered in the paper by Akbarov (2006 b) within the assumption that the stiffness of the covering layer material is greater than that of the material of the half-space material. We recall that in the paper by Akbarov (2006 b) it was assumed that the material of

the half-space is stiffer than the material of the covering layer and the wavenumber integrals mentioned above are calculated by the use of Cauchy's principal value sense algorithm. Note that in the present case, i.e. in the case where the stiffness of the covering layer material is greater than that of the material of the half-space, the calculation of the wavenumber integrals by the use of the Cauchy principal value sense algorithm is not stable and consequently is not applicable. Therefore in the present paper for calculation of such wavenumber integrals the algorithm based on the Sommerfeld contour noted above is developed. Numerical results on the frequency response of stresses acting on the interface plane are presented and discussed. Some attempts are also made for discussion of the difference between the two foregoing algorithms which are employed for calculation of the wavenumber integrals.

2 Formulation of the problem
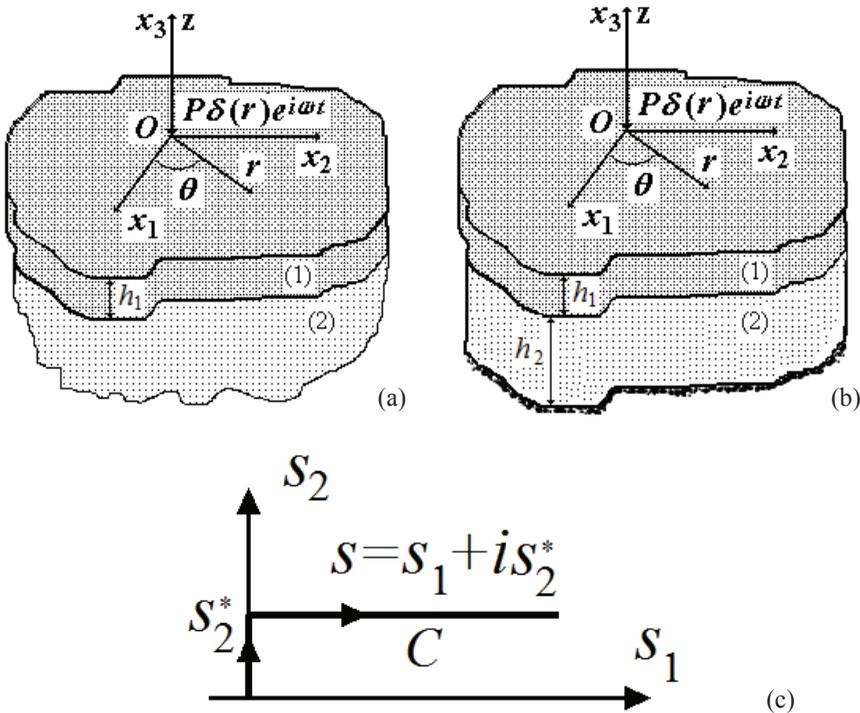
Figure 1: The geometries of the system consisting of the covering layer and half space (a), the bi-layered slab resting on a rigid foundation (b) and the Sommerfeld contour (c).

As in the paper by Akbarov (2006 b), we consider the half-space covered with the layer (Fig. 1a), the thickness of which in the natural state is h_1. We determine the position of the points of the layer and half-space in the natural state by the Lagrange coordinates in the cylindrical system of coordinates $Or\theta z$. We assume that the layer and half-space, before being compounded with each other, are pre-strained separately along the radial direction and in each of them the homogeneous axisymmetric initial finite strain state has appeared. With the initial state of the layer and half-space, we associate the Lagrangian cylindrical system of coordinates $O'r'\theta'z'$. Assume that the material of the constituents is incompressible and the values related to the layer and half-space are denoted by upper indices (1) and (2), respectively. Furthermore, we denote the values related to the initial state by upper index 0. Thus, according to the above, the initial state in the layer and half-space can be written as follows:

$$u_r^{(k),0} = (\lambda_1^{(k)} - 1)r, \; u_z^{(k),0} = (\lambda_3^{(k)} - 1)z, \; u_\theta^{(k),0} = 0, \; k = 1,2, \; (\lambda_1^{(k)})^2\lambda_3^{(k)} = 1, \; (1)$$

where $u_r^{(k),0}$, $u_\theta^{(k),0}$ and $u_z^{(k),0}$ are the radial, circumferential and axial displacements respectively, and $\lambda_1^{(k)}$ and $\lambda_3^{(k)}$ are constants.

Within the framework above, let us investigate the stress state in the considered system in the case where the time-harmonic point-located normal force acts on the free face plane of the covering layer. We make this investigation by the use of the large initial deformation version of the TLTEWISB.

We introduce the notation

$$r' = \lambda_1^{(k)}r, \quad z' = \lambda_3^{(k)}z, \quad h_1' = \lambda_3^{(k)}h_1, \tag{2}$$

Below, the values related to the system of coordinates associated with the initial state, i.e. with $O'r'\theta'z'$ are denoted by an upper prime.

Thus, according to the monograph by Guz (2004), we write the basic relations of the TLTEWISB for the incompressible body under the axisymmetric state. These relations are satisfied within the layer and half-space because we use the piecewise homogeneous body model.

The equations of motion are

$$\frac{\partial}{\partial r'}Q_{r'r'}^{'(k)} + \frac{\partial}{\partial z'}Q_{r'z'}^{'(k)} + \frac{1}{r'}(Q_{r'r'}^{'(k)} - Q_{\theta'\theta'}^{'(k)}) = \rho^{'(k)}\frac{\partial}{\partial t^2}u_{r'}^{'(k)},$$

$$\frac{\partial}{\partial r'}Q_{r'z'}^{'(k)} + \frac{\partial}{\partial z'}Q_{z'z'}^{'(k)} + \frac{1}{r'}Q_{r'z'}^{'(k)} = \rho^{'(k)}\frac{\partial}{\partial t^2}u_{z'}^{'(k)}. \tag{3}$$

The mechanical relations are

$$Q_{r'r'}^{'(k)} = \chi_{1111}^{'(k)}\frac{\partial u_{r'}^{'(k)}}{\partial r'} + \chi_{1122}^{'(k)}\frac{u_{r'}^{'(k)}}{r'} + \chi_{1133}^{'(k)}\frac{\partial u_{z'}^{'(k)}}{\partial z'} + p^{'(k)},$$

$$Q_{\theta'\theta'}^{'(k)} = \chi_{2211}^{'(k)} \frac{\partial u_{r'}^{'(k)}}{\partial r'} + \chi_{2222}^{'(k)} \frac{u_{r'}^{'(k)}}{r'} + \chi_{2233}^{'(k)} \frac{\partial u_{z'}^{'(k)}}{\partial z'} + p^{'(k)},$$

$$Q_{z'z'}^{'(k)} = \chi_{3311}^{'(k)} \frac{\partial u_{r'}^{'(k)}}{\partial r'} + \chi_{3322}^{'(k)} \frac{u_{r'}^{'(k)}}{r'} + \chi_{3333}^{'(k)} \frac{\partial u_{z'}^{'(k)}}{\partial z'} + p^{'(k)},$$

$$Q_{r'z'}^{'(k)} = \chi_{1313}^{'(k)} \frac{\partial u_{r'}^{'(k)}}{\partial z'} + \chi_{1331}^{'(k)} \frac{\partial u_{z'}^{'(k)}}{\partial r'}, \quad Q_{z'r'}^{'(k)} = \chi_{3113}^{'(k)} \frac{\partial u_{r'}^{'(k)}}{\partial z'} + \chi_{3131}^{'(k)} \frac{\partial u_{z'}^{'(k)}}{\partial r'}. \tag{4}$$

In (3) and (4) through $Q_{r'r'}^{'(k)},\ldots, Q_{z'r'}^{'(k)}$ the perturbations of the components of the Kirchhoff stress tensor are determined. The notation $u_{r'}^{'(k)}$, $u_{z'}^{'(k)}$ shows the perturbations of the components of the displacement vector, while $p^{'(k)} = p^{'(k)}(r',z',t)$ is an unknown function (a Lagrange multiplier). The constants $\chi_{1111}^{'(k)},\ldots, \chi_{3131}^{'(k)}$ in (4) are determined through the mechanical constants of the layer and half-space materials and through the initial strain state. $\rho^{'(k)}$ is the density of the $k-th$ material. Note that the constants $\chi_{1111}^{'(k)},\ldots, \chi_{3131}^{'(k)}$, $\rho^{('k)}$ are given through their expression in the system of coordinates $Or\theta z$ (we denote them by $\chi_{1111}^{(k)},\ldots, \chi_{3131}^{(k)}$, $\rho^{(k)}$, respectively) by the following formulae:

$$\chi_{1111}^{'(k)} = (\lambda_1^{(k)})^2 \chi_{1111}^{(k)}, \quad \chi_{1122}^{(k)} = (\lambda_1^{(k)})^2 \chi_{1122}^{(k)}, \quad \chi_{1133}^{(k)} = (\lambda_1^{(k)})^{-1} \chi_{1133}^{(k)},$$

$$\chi_{2222}^{'(k)} = (\lambda_1^{(k)})^2 \chi_{2222}^{(k)}, \quad \chi_{1221}^{'(k)} = (\lambda_1^{(k)})^2 \chi_{1221}^{(k)}, \quad \chi_{1313}^{'(k)} = (\lambda_1^{(k)})^{-1} \chi_{1313}^{(k)},$$

$$\chi_{1331}^{'(k)} = (\lambda_1^{(k)})^2 \chi_{1331}^{(k)}, \quad \chi_{3131}^{'(k)} = \chi_{1313}^{'(k)}, \quad \chi_{2211}^{'(k)} = \chi_{1122}^{'(k)}, \tag{5}$$

$$\chi_{2233}^{'(k)} = \chi_{1133}^{'(k)}, \quad \chi_{3311}^{'(k)} = \chi_{1133}^{'(k)} = \chi_{3322}^{'(k)} = \chi_{2233}^{('k)}, \quad \chi_{3113}^{'(k)} = (\lambda_1^{(k)})^2 \chi_{3113}^{(k)},$$

$$\chi_{3333}^{'(k)} = (\lambda_1^{(k)})^{-4} \chi_{333}^{(k)}, \quad \rho^{'(k)} = \rho^{(k)}.$$

The explicit expressions of the constants $\chi_{1111}^{(k)},\ldots, \chi_{3131}^{(k)}$ are determined through the elastic energy function (potential). In the present investigation, as in the paper by Akbarov (2006 b), we assume that the elasticity relations of the layer and half-space materials are given by the Neo-Hooken type (Treloar's) potential. This potential is given as follows:

$$\Phi = C_{10}(I_1 - 3), \quad I_1 = 3 + 2A_1 \quad , \quad A_1 = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}, \tag{6}$$

where C_{10} is the elastic constant, A_1 is the first algebraic invariant of Green's strain tensor and ε_{rr}, $\varepsilon_{\theta\theta}$ and ε_{zz} are the components of this tensor. For the considered axisymmetric case, the components of Green's strain tensor are determined through the components of the displacement vector by the following expressions:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial u_r}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_z}{\partial z}\right)^2 \quad , \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{2}\left(\frac{u_r}{r}\right)^2,$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2}\left(\frac{\partial u_r}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u_z}{\partial z}\right)^2, \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} + \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial z}\right).$$

$$(7)$$

In this case, the components of the Lagrange stress tensor S are determined as follows:

$$S_{rr} = \frac{\partial \Phi}{\partial \varepsilon_{rr}} + pg^*_{rr}, \quad S_{\theta\theta} = \frac{\partial \Phi}{\partial \varepsilon_{\theta\theta}} + pg^*_{\theta\theta}, \quad S_{zz} = \frac{\partial \Phi}{\partial \varepsilon_{zz}} + pg^*_{zz},$$

$$S_{rz} = \frac{1}{2}\left(\frac{\partial}{\partial \varepsilon_{rz}} + \frac{\partial}{\partial \varepsilon_{zr}}\right)\Phi, \quad S_{rz} = S_{zr}, \quad g^*_{rr} = 1 + 2\frac{\partial u_r}{\partial r} + \left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2,$$

$$g^*_{zz} = 1 + 2\frac{\partial u_z}{\partial z} + \left(\frac{\partial u_r}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2, \quad g^*_{\theta\theta} = 1 + 2\frac{u_r}{r} + \frac{1}{r^2}\left(\frac{u_r}{r}\right)^2. \quad (8)$$

Note that the expressions (6), (7) and (8) are written in the arbitrary cylindrical coordinate system without any restrictions related to the association of this system to the natural or initial state of the considered layer and half-space.

For the considered case the relations between the perturbation of the Kirchhoff stress tensor and perturbation of the components of the Lagrange stress tensor can be written as follows:

$$Q'^{(k)}_{r'r'} = \lambda_1^{(k)} S'^{(k)}_{r'r'} + S'^{(k),0}_{r'r'}\frac{\partial u'^{(k)}_{r'}}{\partial r'} \quad , \quad Q'^{(k)}_{\theta'\theta'} = \lambda_1^{(k)} S'^{(k)}_{\theta'\theta'} + S'^{(k),0}_{r'r'}\frac{u'^{(k)}_{r'}}{r'},$$

$$Q'^{(k)}_{z'z'} = \left(\lambda_1^{(k)}\right)^{-2} S'^{(k)}_{z'z'}, \quad Q'^{(k)}_{r'z'} = \lambda_1^{(k)} S'^{(k)}_{r'z'} + S'^{(k),0}_{r'r'}\frac{\partial u'^{(k)}_{z'}}{\partial r'}, \quad Q'^{(k)}_{z'r'} = \left(\lambda_1^{(k)}\right)^{-2} S'^{(k)}_{z'r'}.$$

$$(9)$$

By linearization of the equation (8) and taking (9) and (1) into account we obtain the following expressions for the constants $\chi_{1111}^{(k)}, \dots, \chi_{3131}^{(k)}$ in (5).

$$\chi_{1111}^{(k)} = \chi_{2222}^{(k)} = 2C_{10}^{(k)}\left(\lambda_1^{(k)}\right)^2\left(\left(\lambda_1^{(k)}\right)^2 + \left(\lambda_1^{(k)}\right)^{-4}\right), \quad \chi_{1122}^{(k)} = \chi_{1133}^{(k)} = \chi_{2233}^{(k)} =$$

$$\chi_{3311}^{(k)} = \chi_{2211}^{(k)} = \chi_{3322}^{(k)} = 0, \quad \chi_{1331}^{(k)} = 2C_{10}^{(k)}, \quad \chi_{1221}^{(k)} = 2C_{10}^{(k)}, \quad \chi_{3333}^{(k)} = 4C_{10}^{(k)},$$

$$\chi_{1313}^{(k)} = \chi_{3131}^{(k)} = 2C_{10}^{(k)}\left(\lambda_1^{(k)}\right)^{-3}, \quad \chi_{3113}^{(k)} = 2C_{10}^{(k)}. \quad (10)$$

It should be noted that to the above equations, the incompressibility condition

$$\frac{\partial u'^{(k)}_{r'}}{\partial r'} + \frac{u'^{(k)}_{r'}}{r'} + \frac{\partial u'^{(k)}_{z'}}{\partial z'} = 0 \quad (11)$$

of the layer and half-space materials must be added.

Thus, the stress-strain in the considered system will be investigated by the use of Eqs. (3) – (11). In this case, we will assume that the following boundary conditions are satisfied on the free face plane of the covering layer.

$$Q'^{(1)}_{z'z'}\Big|_{z'=0} = -P_0\delta(r')e^{i\omega t}\frac{1}{(\lambda_1^{(1)})^2} \quad , \quad Q'^{(1)}_{z'r'}\Big|_{z'=0} = 0, \tag{12}$$

where $\delta(r')$ is the Dirac function. Also, we assume that

$$\left\{\left|Q'^{(2)}_{r'r'}\right|,\left|Q'^{(2)}_{\theta'\theta'}\right|,\left|Q'^{(2)}_{z'z'}\right|,\left|Q'^{(2)}_{r'z'}\right|,\left|Q'^{(2)}_{z'r'}\right|,\left|u'^{(2)}_{z'}\right|,\left|u'^{(2)}_{r'}\right|\right\} < M = const \text{ for } z' \to -\infty \tag{13}$$

Moreover, we assume that on the interface between the covering layer and half-space the complete contact conditions

$$Q'^{(1)}_{z'z'}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2} = Q'^{(2)}_{z'z'}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2}, \quad Q'^{(1)}_{z'r'}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2} = Q'^{(2)}_{z'r'}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2},$$

$$u'^{(1)}_{r'}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2} = u'^{(2)}_{r'}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2}, \quad u'^{(1)}_{z'}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2} = u'^{(2)}_{z'}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2}. \tag{14}$$

are satisfied.

This completes the formulation of the problem under consideration. It should be noted that in the case where $\lambda_1^{(1)} = \lambda_1^{(2)} = 1.0$, Eqs. (3) – (5) and (9) – (11), and conditions (12) – (14) transform to the corresponding ones of the classical linear theory of elastodynamics. Note that the similar relations for the compressible materials even in more complicated cases were analyzed in a paper by Akbarov (2013).

The foregoing mathematical formulation can be easily transformed to the formulation of the corresponding problem for the bi-layered slab resting on a rigid foundation (Fig. 1b). In the latter case the boundary condition (13) must be replaced with the following one:

$$u'^{(2)}_{r'}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2-h_2/(\lambda_1^{(1)})^2} = 0, \quad u'^{(2)}_{z'}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2-h_2/(\lambda_1^{(1)})^2} = 0, \tag{15}$$

where h_2 is the thickness of the lower layer of the slab. Note that the problem related to the forced vibration of the slab with a stiff upper layer and soft lower layer, which are shown in Fig. 1b, was investigated in the paper by Akbarov (2006c).

At the same time, the foregoing formulation can be changed with respect to the contact conditions (13). Namely, shear-spring type imperfect contact conditions can be assumed instead of the perfect conditions (13). In this case the third condition in (13) must be replaced with the following one:

$$u_{r'}^{'(1)}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2} - u_{r'}^{'(2)}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2} = \frac{Fh_1}{C_{10}^{(1)}} Q_{z'r'}^{'(1)}\Big|_{z'=-h_1/(\lambda_1^{(1)})^2}. \tag{16}$$

Here the parameter F characterizes the degree of the imperfectness of the contact conditions and changes in the interval $[0, \infty]$. The case where $F = 0$ corresponds to perfect contact conditions, but the case where $F = \infty$, corresponds to full slipping contact conditions between the constituents. Note that investigations related to the axisymmetric wave propagation in the pre-strained and with the foregoing type imperfectly bonded bi-layered cylinders were made by Akbarov and Ipek (2010, 2012).

3   Solution method

For the solution of the problem formulated above, we use the Hankel integral transformation with respect to the space coordinate r'. First, we substitute expression (4) into the equation (3) and obtain the following equation of motion in displacement terms:

$$\chi_{1111}^{'(k)} \frac{\partial^2 u_{r'}^{'(k)}}{\partial r'^2} + \chi_{1122}^{'(k)} \frac{\partial}{\partial r'}\left(\frac{u_{r'}^{'(k)}}{r'}\right) + \left(\chi_{1133}^{'(k)} + \chi_{1331}^{'(k)}\right)\frac{\partial^2 u_{z'}^{'(k)}}{\partial z' \partial r'} + \chi_{1313}^{'(k)} \frac{\partial^2 u_{r'}^{'(k)}}{\partial z'^2} +$$

$$\frac{1}{r'}\left(\chi_{1111}^{'(k)} - \chi_{2211}^{'(k)}\right)\frac{\partial u_{r'}^{'(k)}}{\partial r'} + \left(\chi_{1122}^{'(k)} - \chi_{2222}^{'(k)}\right)\frac{u_{r'}^{'(k)}}{r'^2} + \left(\chi_{1133}^{'(k)} - \chi_{2233}^{'(k)}\right)\frac{1}{r'}\frac{\partial u_{z'}^{'(k)}}{\partial z'} =$$

$$\rho^{(k)}\frac{\partial^2 u_{r'}^{'(k)}}{\partial t^2} - \frac{\partial p^{'(k)}}{\partial r'},$$

$$\chi_{3322}^{'(k)} \frac{\partial^2 u_{r'}^{'(k)}}{\partial z' \partial r'} + \chi_{3131}^{'(k)} \frac{\partial^2 u_{z'}^{'(k)}}{\partial r'^2} + \frac{1}{r'}\chi_{3113}^{'(k)} \frac{\partial u_{r'}^{'(k)}}{\partial z'} + \frac{1}{r'}\chi_{3131}^{'(k)} \frac{\partial u_{z'}^{'(k)}}{\partial r'} +$$

$$\frac{1}{r'}\chi_{3322}^{'(k)} \frac{\partial u_{r'}^{'(k)}}{\partial z'} + \chi_{3333}^{'(k)} \frac{\partial^2 u_{z'}^{('k)}}{\partial z'^2} = \rho^{(k)}\frac{\partial^2 u_{z'}^{'(k)}}{\partial t^2} - \frac{\partial p^{'(k)}}{\partial z'}. \tag{17}$$

Eqs. (11) and (17) compose the complete system with respect to the unknown functions $u_{r'}^{'(k)}$, $u_{z'}^{'(k)}$ and $p^{'(k)}$. According to the monograph by Guz (2004), we use the following representation for the displacement and unknown function $p^{'(k)}$:

$$u_{r'}^{'(k)} = -\frac{\partial^2 X^{'(k)}}{\partial r' \partial z'}, \quad u_{z'}^{'(k)} = \Delta_1' X^{'(k)},$$

$$p'^{(k)} = \left[\left(\chi_{1111}'^{(k)} - \chi_{1133}'^{(k)} - \chi_{1313}'^{(k)} \right) \Delta_1' + \chi_{3113}'^{(k)} \frac{\partial^2}{\partial z'^2} - \rho^{(k)} \frac{\partial^2}{\partial t^2} \right] \frac{\partial}{\partial z'} X'^{(k)}, \tag{18}$$

where

$$\Delta_1' = \frac{d^2}{dr'^2} + \frac{1}{r'} \frac{d}{dr'} \tag{19}$$

and the function $X'^{(k)}$ satisfies the equation

$$\left[\left(\Delta_1' + \left(\xi_2'^{(k)} \right)^2 \frac{\partial^2}{\partial z'^2} \right) \left(\Delta_1' + \left(\xi_3'^{(k)} \right)^2 \frac{\partial^2}{\partial z'^2} \right) - \frac{\rho^{(k)}}{\chi_{1331}'^{(k)}} \left(\Delta_1' + \frac{\partial^2}{\partial z'^2} \right) \frac{\partial^2}{\partial t^2} \right] X'^{(k)} = 0, \tag{20}$$

The constants $\xi_2'^{(k)}$ and $\xi_3'^{(k)}$ in (20) are determined for the case under consideration as

$$\xi_2'^{(k)} = 1, \quad \xi_3'^{(k)} = \left(\lambda_1^{(k)} \right)^{-6}. \tag{21}$$

According to the problem statement, all dependent variables become harmonic with respect to time and can be presented as $g(r', z', t) = \bar{g}(r', z')e^{i\omega t}$ where a superimposed dash denotes the amplitude of the relevant quantity. Below we will omit this superimposed dash.

If the presentation $g(r', z', t) = \bar{g}(r', z')e^{i\omega t}$ is employed in the foregoing equations, by replacing the operator $\partial^2/\partial t^2$ with $-\omega^2$, we obtain the same equations and conditions for the amplitude of the sought quantities. Consequently, introducing the dimensionless coordinates $r' \rightarrow r'/h_1'$, $z' \rightarrow z'/h_1'$ and the dimensionless frequency

$$\Omega^2 = \frac{(\omega h_1')^2 \rho^{(2)}}{2C_{10}^{(2)}} \tag{22}$$

we obtain the following equation for the potential $X'^{(k)}$

$$\left[\left(\Delta_1' + \left(\xi_2'^{(k)} \right)^2 \frac{\partial^2}{\partial z'^2} \right) \left(\Delta_1' + \left(\xi_3'^{(k)} \right)^2 \frac{\partial^2}{\partial z'^2} \right) - \frac{\Omega^2}{\left(\lambda_1^{(k)} \right)^2} \left(\Delta_1' + \frac{\partial^2}{\partial z'^2} \right) \frac{C_{10}^{(2)} \rho^{(k)}}{C_{10}^{(k)} \rho^{(2)}} \right] X'^{(k)} = 0. \tag{23}$$

For the solution to Eq. (23) we use the Hankel integral representation for the function $X'^{(k)}$:

$$X'^{(k)} = \int_0^\infty Y_1^{(k)} e^{\gamma^{(k)}} J_0(sr')s \, ds, \tag{24}$$

where $J_0(sr')$ is the Bessel function with zeroth order.

Substituting (24) into (23) we obtain the following algebraic equation for $\gamma^{(k)}$.

$$A^{(k)} \left(\gamma^{(k)} \right)^4 + B^{(k)} \left(\gamma^{(k)} \right)^2 + C^{(k)} = 0, \tag{25}$$

where

$$A^{(k)} = \left(\lambda_1^{(k)} \right)^{-6} \quad , \quad B^{(k)} = \frac{1}{\left(\lambda_1^{(2)} \right)^2} \frac{C_{10}^{(2)}}{C_{10}^{(k)}} \frac{\rho^{(k)}}{\rho^{(2)}} \Omega^2 - \left(1 + \left(\lambda_1^{(2)} \right)^{-6} \right) s^2 \quad ,$$

$$C^{(k)} = s^4 - s^2 \frac{1}{\left(\lambda_1^{(2)} \right)^2} \frac{C_{10}^{(2)}}{C_{10}^{(k)}} \frac{\rho^{(k)}}{\rho^{(2)}} \Omega^2 \quad . \tag{26}$$

We obtain from Eq. (25)

$$\left(\gamma^{(k)} \right)^2 = \frac{-B^{(k)} \pm \sqrt{\left(B^{(k)} \right)^2 - 4A^{(k)} C^{(k)}}}{2A^{(k)}} \quad . \tag{27}$$

By using the expression (26), by direct verification and transformation, it is proven that

$$\left(\gamma_1^{(k)} \right)^2 = \frac{-B^{(k)} + \sqrt{\left(B^{(k)} \right)^2 - 4A^{(k)} C^{(k)}}}{2A^{(k)}} = s^2,$$

$$\left(\gamma_2^{(k)} \right)^2 = \frac{-B^{(k)} - \sqrt{\left(B^{(k)} \right)^2 - 4A^{(k)} C^{(k)}}}{2A^{(k)}} = s^2 \left(\lambda_1^{(k)} \right)^6 \left(1 - \frac{\Omega_k^2}{s^2} \right), \tag{28}$$

where

$$\Omega_k^2 = \frac{1}{\left(\lambda_1^{(2)} \right)^2} \frac{C_{10}^{(2)}}{C_{10}^{(k)}} \frac{\rho^{(k)}}{\rho^{(2)}} \Omega^2. \tag{29}$$

Thus, we obtain the following expression for the function $X'^{(k)}$

$$X'^{(k)} = \int_0^\infty \left[Y_1^{(k)} e^{sz'} + Y_2^{(k)} e^{-sz'} + Y_3^{(k)} e^{\gamma_2^{(k)} z'} + Y_4^{(k)} e^{-\gamma_2^{(k)} z'} \right] J_0(sr') s \, ds. \tag{30}$$

Substituting (30) into expressions (18) and (4), the displacements and stresses which enter the boundary and contact conditions (12) – (16) are determined as follows:

$$u_{r'}^{'(1)} = \int\limits_0^\infty \left[Y_1^{(1)} s e^{sz'} + Y_2^{(1)} s e^{-sz'} + Y_3^{(1)} \gamma_2^{(1)} e^{\gamma_2^{(1)} z'} - Y_4^{(1)} \gamma_2^{(1)} e^{-\gamma_2^{(1)} z'} \right] J_1(sr') s^2 ds,$$

$$u_{z'}^{'(1)} = -\int\limits_0^\infty s^2 \left[Y_1^{(1)} e^{sz'} + Y_2^{(1)} e^{-sz'} + Y_3^{(1)} e^{\gamma_2^{(1)} z'} - Y_4^{(1)} e^{-\gamma_2^{(1)} z'} \right] J_0(sr') s ds,$$

$$Q_{z'r'}^{'(1)} = C_{10}^{(1)} \int\limits_0^\infty \left[\frac{2s^2}{\left(\lambda_1^{(1)}\right)^4} \left(Y_1^{(1)} e^{sz'} + Y_1^{(1)} e^{-sz'} \right) + \left(s^2 + s^2 \left(\lambda_1^{(1)}\right)^2 \left(1 - \frac{\Omega_1^2}{s^2} \right) \right) \times \right.$$
$$\left. \frac{1}{\left(\lambda_1^{(1)}\right)^4} \left(Y_3^{(1)} e^{\gamma_2^{(1)} z'} + Y_4^{(1)} e^{-\gamma_2^{(1)} z'} \right) \right] s^2 J_1(sr') ds,$$

$$Q_{z'z'}^{'(1)} = C_{10}^{(1)} \int\limits_0^\infty \left[\left(\Omega_1^2 - s^2 \left(\lambda_1^{(1)}\right)^2 - \frac{s^2}{\left(\lambda_1^{(1)}\right)^4} \right) s \left(Y_1^{(1)} e^{sz'} - Y_2^{(1)} e^{-sz'} \right) + \right.$$
$$\left. \gamma_2^{(1)} \left(\Omega_1^2 - s^2 \left(\lambda_1^{(1)}\right)^2 - \frac{s^2}{\left(\lambda_1^{(1)}\right)^4} \right) \left(Y_3^{(1)} e^{\gamma_2^{(1)} z'} - Y_4^{(1)} e^{-\gamma_2^{(1)} z'} \right) \right] J_0(sr') s ds,$$

$$u_{r'}^{'(2)} = \int\limits_0^\infty \left[Y_1^{(2)} s e^{sz'} + Y_3^{(2)} \gamma_2^{(2)} e^{\gamma_2^{(2)} z'} \right] J_1(sr') s^2 ds,$$

$$u_{z'}^{'(2)} = -\int\limits_0^\infty s^2 \left[Y_1^{(2)} e^{sz'} + Y_3^{(2)} e^{\gamma_2^{(2)} z'} \right] J_0(sr') s ds,$$

$$Q_{z'r'}^{'(2)} = C_{10}^{(2)} \int\limits_0^\infty \left[\frac{2s^2}{\left(\lambda_1^{(2)}\right)^4} Y_1^{(2)} e^{sz'} + \left(s^2 + s^2 \left(\lambda_1^{(2)}\right)^2 \left(1 - \frac{\Omega_1^2}{s^2} \right) \right) \times \right.$$
$$\left. \frac{1}{\left(\lambda_1^{(2)}\right)^4} Y_3^{(2)} e^{\gamma_2^{(2)} z'} \right] s^2 J_1(sr') ds,$$

$$Q_{z'z'}^{'(2)} = C_{10}^{(2)} \int\limits_{0}^{\infty} \left[\left(\Omega_1^2 - s^2 \left(\lambda_1^{(2)} \right)^2 - \frac{s^2}{\left(\lambda_1^{(2)} \right)^4} \right) sY_1^{(2)} e^{sz'} + \right.$$

$$\left. \gamma_2^{(2)} \left(\Omega_1^2 - s^2 \left(\lambda_1^{(2)} \right)^2 - \frac{s^2}{\left(\lambda_1^{(2)} \right)^4} \right) Y_3^{(2)} e^{\gamma_2^{(2)} z'} \right] J_0(sr') s ds. \tag{31}$$

Note that under selection of the solution to the function $X'^{(2)}$, the boundary condition (13) is taken into consideration and this condition can also be estimated as "there are no reflected waves" at infinity. Moreover, note that, according to (28), the foregoing expressions are written for the case where $\Omega_k < s$. But in the cases where $\Omega_k > s$, in the foregoing expressions, $i \left| \gamma_2^{(k)} \right|$ must be written instead of $\gamma_2^{(k)}$ and the real part of these expressions must be taken for determination of the stresses and displacements. To find the unknowns $Y_1^{(1)}$, $Y_2^{(1)}$, $Y_3^{(1)}$, $Y_4^{(1)}$, $Y_1^{(2)}$ and $Y_3^{(2)}$ in (31) we use the boundary (12) and contact (14) conditions. For this purpose we determine the Hankel transformation of the right-hand side of the first condition in (12). Using the equality $P_0 \delta(r') = \lim\limits_{r' \to 0} \left(P_0/(\pi r'^2) \right)$ we obtain $P_0/(2\pi)$ for the Hankel transformation of $P_0 \delta(r')$ from $\lim\limits_{\varepsilon \to 0} \left(\int\limits_0^\varepsilon \frac{P_0}{\pi\varepsilon^2} r' J_0(sr') dr' \right)$.

Thus, we derive the equations

$$\sum_{j=1}^{4} Y_j^{(1)} \left. \alpha_{ij}^{(1)}(z',s) \right|_{z'=0} = -P_0 \left/ \left(2\pi(\lambda_1^{(1)})^2 \right) \delta_i^1 \right., \quad \delta_1^1 = 1, \quad \delta_2^1 = 0, \quad i = 1,2, \tag{32}$$

from the boundary condition (12), and the following equations

$$\sum_{j=1}^{4} Y_j^{(1)} \left. \alpha_{ij}^{(1)}(z',s) \right|_{z'=-h'/(\lambda_1^{(1)})^2} - Y_1^{(2)} \left. \alpha_{i1}^{(2)}(z',s) \right|_{z'=-h'/(\lambda_1^{(1)})^2} -$$

$$Y_3^{(2)} \left. \alpha_{i3}^{(2)}(z',s) \right|_{z'=-h'/(\lambda_1^{(1)})^2} = 0, \quad i = 3,4,5,6, \tag{33}$$

from the contact condition (13).

The coefficients of the unknowns in Eqs. (32) and (33) are determined through the expression (31). Thus, we determine the unknowns $Y_1^{(1)}$, $Y_2^{(1)}$, $Y_3^{(1)}$, $Y_4^{(1)}$, $Y_1^{(2)}$ and $Y_3^{(2)}$ from Eqs. (32) and (33), and the stresses and displacements are determined from the expressions in (31).

Note that the foregoing solution procedure can also be developed easily for the corresponding problem related to the finite pre-strained bi-layered slab resting on a rigid foundation which has an additional boundary condition (15). Moreover, the foregoing solution procedure can easily be repeated for the case where the shear-spring type imperfect contact condition (16) takes place between the constituents.

4   Some remarks on the calculation of the integrals in (31)

Thus, taking the foregoing discussions into account, we can represent the unknowns $Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}, Y_4^{(1)}, Y_1^{(2)}$ and $Y_3^{(2)}$ as follows:

$$\left\{ Y_1^{(1)}, ..., Y_3^{(2)} \right\} = \frac{1}{\det \left\| \alpha_{ij}(s) \right\|} \left(\det \left\| \beta_{ij}^{Y_1^{(1)}}(s) \right\|, ..., \left\| \beta_{ij}^{Y_3^{(2)}}(s) \right\| \right). \tag{34}$$

It should be noted that the equation

$$\det \left\| \alpha_{ij}(s) \right\| = 0 \tag{35}$$

coincides with the dispersion equation of the axisymmetric near-surface waves guided in the radial direction in the system under consideration with respect to the dimensionless velocity (or to the frequency) $\Omega(22)$, if we take the parameter s as the wavenumber of the waves. Therefore the integrals in expressions (31) are called the wavenumber integrals and the equation (35) has an infinite number of real roots through which dispersion diagrams, i.e. graphs of the dependencies between Ω and sh_1, are constructed. These diagrams give a lot of information about the possible dynamical behavior of the related system. For instance, if the dispersion diagram has a point at which $d\Omega/d(sh_1) = 0$, the frequency which corresponds to this point is the resonance frequency. A similar issue was also noted in a paper by Dieterman and Metrikine (1997).

Thus, we turn to calculation of the integrals in (31). We note that similar integrals in papers by Akbarov (2006 b, c) have been calculated by employing Cauchy's principal value sense approach, according to which, first, the singular points of the integrated expressions, i.e. the roots of the equation (35) are determined and after the corresponding isolation of these singularities, the traditional algorithm is applied for calculation of the integrals.

However, the algorithm based on Cauchy's principal value sense approach has the following disadvantages: a) this approach does not take into account the waves which are generated by the external forces and guided from the force source to infinity and namely in this sense is estimated as physical incorrect, and b) this approach is more sensitive to the accuracy of the determination of the roots of the dispersion equation (35) and to the location character of the integration points in the

near vicinity of the singular points. In the high frequencies of the external forces, for instance in the cases where $\Omega > 1$ the disadvantage of (b) causes more serious difficulties under calculation of these integrals and avoiding these difficulties is almost impossible in the cases where the stiffness of the covering layer material is greater than that of the half-space material, i.e. in the cases which are investigated in the present paper. Moreover, in general, successive use of the algorithm based on Cauchy's principal value sense approach requires symmetry of the location of Gauss's integration points with respect to the singular points. However, the disadvantage of (a) can become more significant under calculation of the stresses or displacements at the points which are very far from the external force source.

In a paper by Akbarov and Ilhan (2013), according to Tsang (1978), Jensen et al. (2011) and many others listed in these references, the foregoing disadvantages are avoided by the use of the Sommerfeld contour, i.e. the foregoing type of wavenumber integrals are evaluated along the Sommerfeld contour. In the present investigation, for calculation of the integrals (31) we will also use the algorithm based on employing the Sommerfeld contour, according to which, using Cauchy's theorem, the contour $[0, \infty]$ is deformed into the contour C (Fig. 1c) in the complex plane $s = s_1 + is_2$ and in this way the real roots of the equation (35) are avoided.

Thus, in the expressions in (31) the integrals in the form $\int_0^\infty f(s)ds$ are replaced with the corresponding ones in the form $\int_C f(s)ds$ and the stresses and displacements in (31) are determined as a real part of this integral, i.e. as $Re \int_C f(s)ds$. According to Fig. 1c, we can write the following relation:

$$\int_C f(s)\mathrm{d}s = i\int_0^{s_2^*} f(is_2)\mathrm{d}s_2 + \int_0^\infty f(s_1 + is_2^*)\mathrm{d}s_1, \tag{36}$$

Taking into account the fact that the values of the integral $\int_C f(s)\mathrm{d}s$ are independent of the values of the parameter $s_2^* > 0$, then as usual (see, for example Jensen et al. (2011) and Tsang (1978)), in order to simplify the calculation procedure of the integral $\int_C f(s)e^{isx}\mathrm{d}s$, the parameter s_2^* is assumed to be small. According to this assumption, we can write

$$\left| \int_0^{s_2^*} f(is_2)\mathrm{d}s_2 \right| = O(s_2^*). \tag{37}$$

Taking the estimation (37) into account, for calculation of the integral $\int_C f(s)\mathrm{d}s$ we

can use the following approximate expression:

$$\int\limits_C f(s)\mathrm{d}s \approx \int\limits_0^\infty f(s_1 + is_2^*)\mathrm{d}s_1.\tag{38}$$

The accuracy of the expression (38) with respect to values of the parameter s_2^* will be discussed below. Moreover, under the calculation procedure, the improved integral $\int\limits_0^{+\infty} (\bullet)\mathrm{d}s_1$ in (38) is replaced by the corresponding definite integral $\int\limits_0^{+S_1^*} (\bullet)\mathrm{d}s_1$. The values of S_1^* are determined from the convergence requirement of the numerical results. Note that under calculation of the latter integral, the interval $[0, +S_1^*]$ is further divided into a certain number of shorter intervals, which are used in the Gauss integration algorithm. In this integration procedure the values of the integrated expressions, i.e. the values of the unknowns $Y_1^{(1)}$, $Y_2^{(1)}$, $Y_3^{(1)}$, $Y_4^{(1)}$, $Y_1^{(2)}$ and $Y_3^{(2)}$ in Gauss's integration points are determined through (34). In the aforementioned integration procedure it is assumed that in each of the shorter intervals the sampling interval of the numerical integration s_1 must satisfy the relation $|s_1| \ll \min\{s_2^*, 1/r'\}$. All these procedures are performed automatically with the PC by use of the corresponding programs constructed by the author in MATLAB.

5 Numerical results and discussions

Here we focus our attention on the numerical results related to the frequency response of the stresses $Q'_{z'z'}$ and $Q'_{z'r'}$, where

$$Q'_{z'z'}(r') = Q'^{(1)}_{z'z'}(r',z')\Big|_{z'=-h_1\big/(\lambda_1^{(1)})^2} = Q'^{(2)}_{z'z'}(r',z')\Big|_{z'=-h_1\big/(\lambda_1^{(1)})^2},$$

$$Q'_{z'r'}(r') = Q'^{(1)}_{z'r'}(r',z')\Big|_{z'=-h_1\big/(\lambda_1^{(1)})^2} = Q'^{(2)}_{z'r'}(r',z')\Big|_{z'=-h_1\big/(\lambda_1^{(1)})^2}.\tag{39}$$

In all further numerical investigations we will assume that $\rho^{(1)}\big/\rho^{(2)} = C_{10}^{(1)}\big/C_{10}^{(2)}$ and the influence of the initial strains on the numerical results we will estimate through the parameters $\lambda_1^{(1)}$ $(\lambda_3^{(1)} = (\lambda_1^{(1)})^{-2})$ and $\lambda_1^{(2)}$ $(\lambda_3^{(2)} = (\lambda_1^{(2)})^{-2})$ which enter into the expression (1). Consequently, according to expression (1), in the case where $\lambda_1^{(k)} = 1.0(k = 1,2)$ there are no initial strains in the $k-th$ constituent of the system under consideration.

First, we consider a convergence of the results obtained for various values of the parameter s_2^*, calculated with the use of the expression (37). According to the well-known mathematical, mechanical and physical considerations, the results obtained
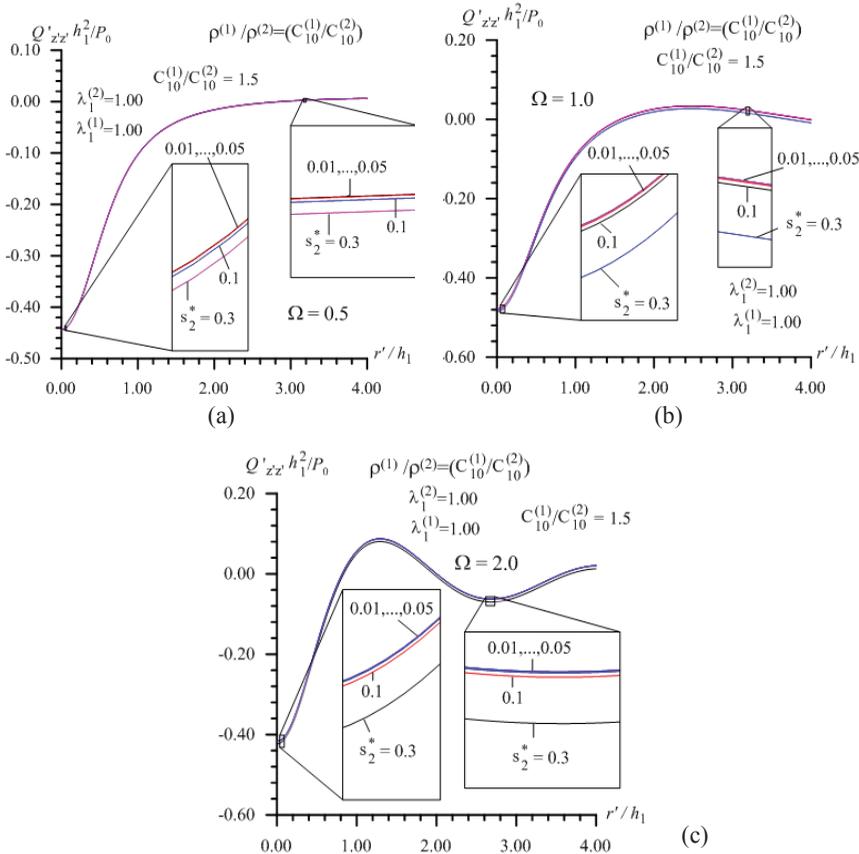
Figure 2: The convergence of the Sommerfield contour integration results with respect to the parameter s_2^* in the cases where $\Omega = 0.5$ (a), 1.0 (b) and 2.0 (c)

with the use of the expression (38) must approach a certain asymptote by decreasing the parameter s_2^*. We check these considerations with respect to the dependence between $Q'_{z'z'}(r')$ (39) and r'/h_1 calculated for various values of the dimensionless frequency Ω under $C_{10}^{(1)}/C_{10}^{(2)} = 1.5$ and $\lambda_1^{(1)} = \lambda_1^{(2)} = 1.0$. The graphs of this dependence are given in Figs. 2a, 2b and 2c for Ω=0.5, 1.0 and 2.0, respectively. It follows from the analyses of these graphs and others (which are not given here) that the obtained numerical results approach the corresponding limit values by decreasing the parameter s_2^* and in the cases where $0.01 \le s_2^* \le 0.05$ these results coincide with very high accuracy, i.e. with an accuracy in the order not greater than 10^{-5}. Therefore, it is enough to take $s_2^* = 0.01$ to obtain guaranteed numeri-
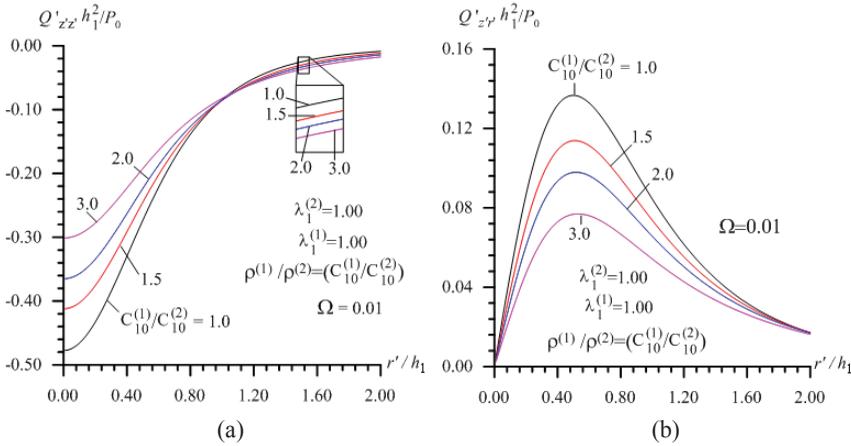
Figure 3: The testing of the calculation algorithm with respect to the Boussinesq problem (see Timoshenko and Goodier (1975)) which corresponds to the case where $C_{10}^{(1)}/C_{10}^{(2)} = 1.0$ under calculation of the normal stress $Q'_{z'z'}$ (a) and shear stress $Q'_{z'r'}$ (b).

cal results by the use of the expression (38). Taking this conclusion into account, all the numerical results, which will be discussed below, are calculated in the case where $S_1^* = 50$ in $\int_0^{+S_1^*} (\bullet) \mathrm{d}s_1$. Note that the value $S_1^* = 50$ is determined from the convergence requirement of the numerical results with respect to the parameter S_1^*. Now we test the algorithm and programs used in the present investigation. For this purpose we consider the graphs given in Fig. 3 which show the distribution of $Q'_{z'z'}$ (Fig. 3a) and $Q'_{z'r'}$ (Fig. 3b) with respect to r'/h_1. Note that these graphs are constructed for various values of $C_{10}^{(1)} \big/ C_{10}^{(2)}$ in the case where $\Omega = 0.01$ and $\lambda_1^{(1)} = \lambda_1^{(2)} = 1.0$. According to the results by Akbarov (2006b, 2006c), the case where $\Omega = 0.01$ can be considered as a static loading case and the results obtained under $C_{10}^{(1)} \big/ C_{10}^{(2)} = 1.0$ must be very near to the corresponding ones obtained for the Boussinesq problem (see Timoshenko and Goodier (1975)). Moreover, according to the well-known mechanical considerations, an increase in the values of the $C_{10}^{(1)} \big/ C_{10}^{(2)}$ must cause a decrease in the absolute maximum values of the stresses $Q'_{z'z'}$ and $Q'_{z'r'}$. By direct verification it is established that the foregoing predictions are confirmed with the graphs given in Fig. 3 and this confirmation can be taken as the successful testing of the algorithm and programs used in the present investi-
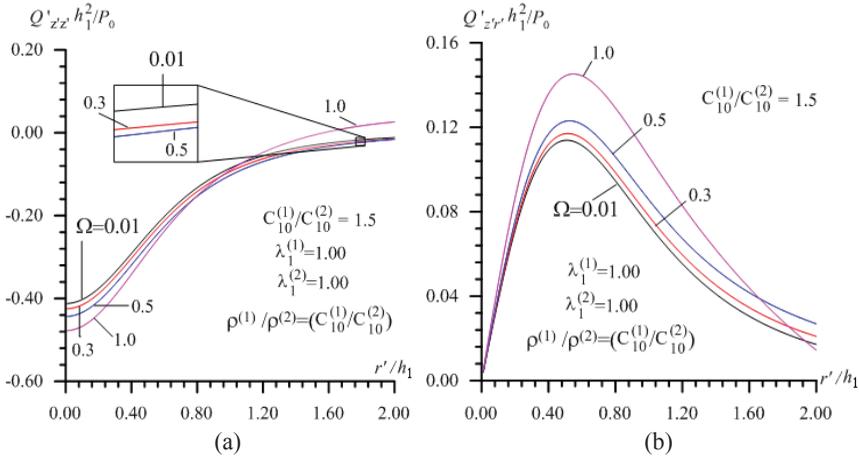
Figure 4: The testing of the calculation algorithm with respect to the influence of the inertial terms, i.e. the frequency Ω on the distribution of the normal $Q'_{z'z'}$ (a) and shear $Q'_{z'r'}$(b) stresses.

gation. Moreover, the graphs given in Figs. 4a and 4b which also show the same distributions given in Figs. 3a and 3b can be taken as confirmation of the testing of the algorithm and programs used in the present investigation with respect to the mechanical consideration related to the influence of the inertial terms on the values of the foregoing stresses. Note that the graphs given in Figs. 4a and 4b are constructed for various values of the dimensionless frequency Ω under $C_{10}^{(1)}\big/C_{10}^{(2)} = 1.5$ and show that for the considered change range, i.e. for $0.01 \leq \Omega \leq 1.0$ the absolute maximum values of the stresses increase with Ω. Moreover, Figs. 3 and 4 show that the absolute maximum values of the stress $Q'_{z'z'}$ (of the stress $Q'_{z'r'}$) take place at $r'\big/h_1 = 0$ (at $r'\big/h_1 \approx 0.5$). Therefore under construction, below, of the frequency response of these stresses, the values of $Q'_{z'z'}$ (of $Q'_{z'r'}$) are calculated at $r'\big/h_1 = 0$ (at $r'\big/h_1 = 0.5$).

Thus, we consider the frequency response graphs given in Fig. 5a (Fig. 5b) for the stress $Q'_{z'z'}$ (for the stress $Q'_{z'r'}$) and constructed for various values of $C_{10}^{(1)}\big/C_{10}^{(2)}$ under $\lambda_1^{(1)} = \lambda_1^{(2)} = 1.0$. It follows from Fig. 5 that the dependence between the normal stress $Q'_{z'z'}$ and Ω, as well as between the shear stress $Q'_{z'r'}$ and Ω has a non monotonic character. In other words, the mechanical behavior of the forced vibration of the system consisting of the stiff covering layer and of the soft half-space is similar to that of the system which comprises a mass, a parallel connected spring and a dashpot. At the same time, Fig. 5 shows that the "resonance" frequency (de-

note it by $\Omega*$) and the "resonance" values of the stresses decrease with $C_{10}^{(1)}\big/C_{10}^{(2)}$. Note that, according to Lamb (1904), Gladwell (1968) and others, the behavior of the half-space or half-plane under forced vibrations is also similar to that of the system which comprises a mass, a parallel connected spring and a dashpot. Moreover, similar behavior is also observed under dynamical (vibrating) contact problems (see Johnson (1985)).
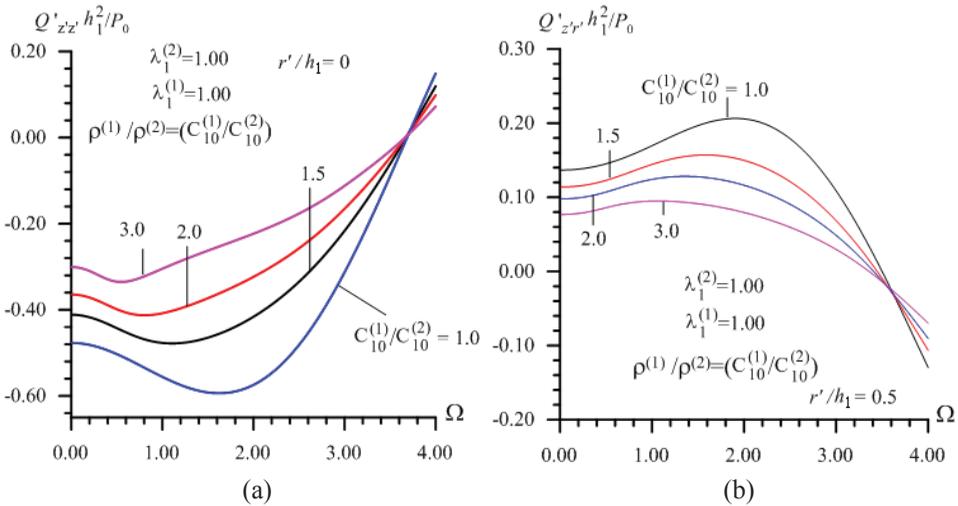
Figure 5: The influence of the parameter $C_{10}^{(1)}/C_{10}^{(2)}$ on the frequency response of normal $Q'_{z'z'}$ (a) and shear $Q'_{z'r'}$ (b) stresses.

In general, the character of the frequency response of elastic systems has an important significance for understanding of the dynamic behavior of these systems. For instance, according to the frequency response graphs, the range of the forced vibration frequency under which the resonance type behavior takes place can be determined. Note that the character of the frequency response graphs depends not only on the ratio of the mechanical constants of the constituents, but also on their characteristic geometrical size. However in the system under consideration, i.e. in the system consisting of the covering layer and half-space, such geometric size is h_1 only and this size is taken under consideration when the distribution of the stresses with respect to the space coordinates is taken into account. But, for the systems such as those shown in Fig. 1b, the ratio of the geometric sizes, i.e. the ratio of the thicknesses of the layers $h_2\big/h_1$ can influence significantly the character of the foregoing frequency response. For illustration of this prediction we consider the graphs given in Figs. 6a and 6b which show the dependence between $Q'_{z'z'}$ and Ω
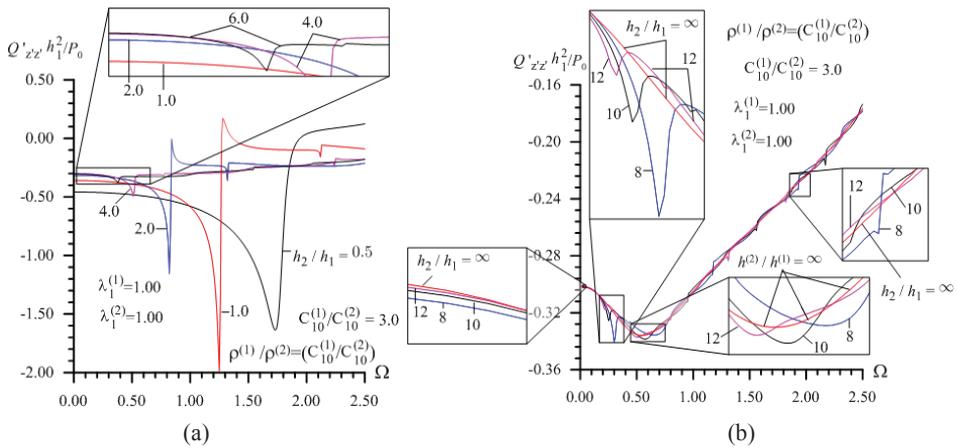
Figure 6: The comparison of the frequency response of the normal stress $Q'_{z'z'}$ obtained for the system shown in Fig. 1a with the corresponding one obtained for the system shown in Fig. 1b in the cases where $h_2/h_1 = 0.5, 1.0, 2.0, 4.0, 6.0$ (a) and $h_2/h_1 = 8.0, 10.0, 12.0, \infty$ (b)

for the system depicted in Fig. 1b. These graphs are constructed for various values of h_2/h_1 under $C_{10}^{(1)}/C_{10}^{(2)} = 3.0$ and $\lambda_1^{(1)} = \lambda_1^{(2)} = 1.0$. For clarity of the illustration in Fig. 6a, graphs are given related to the cases where $h_2/h_1 = 0.5, 1.0, 2.0, 4.0$ and 6.0, but in Fig. 6b graphs are given related to the cases where $h_2/h_1 = 8.0, 10.0, 12.0$ and ∞. It is evident that the case where $h_2/h_1 = \infty$ corresponds to the system consisting of the covering layer and half-space. Note that under construction of these graphs the values of $Q'_{z'z'}$ are calculated at $z' = -h_1/(\lambda_1^{(1)})^2$ by the use of the corresponding integral expression which is determined by utilizing the method discussed in the previous section and these integrals are also evaluated along the Sommerfeld contour. Moreover, we note that the forced vibration problem related to the system shown in Fig. 1b was already investigated in a paper by Akbarov (2006 c) in which the numerical results were given in the cases where $h_2/h_1 = 0.05, 0.10, 0.5,$ 1.0 and 2.0. We recall that the numerical results discussed in the paper by Akbarov (2006 c) were obtained by the use of the algorithm based on Cauchy's principal value sense approach. Consequently, we can compare the results obtained in the cases where $h_2/h_1 = 0.5, 1.0$ and 2.0 and which are shown in Fig. 6a with the corresponding ones given in the paper by Akbarov (2006 c). This comparison shows that for the problem related to the system shown in Fig. 1b the results obtained by the use of the algorithm based on the Sommerfeld contour integration coincide with very high accuracy with the corresponding results obtained by utilizing Cauchy's

principal value sense approach. This comparison also confirms the reliability of the algorithm and programs used in the present investigation. At the same time, the results given in Fig. 6 show that in the quantitative sense the results obtained for the system shown in Fig. 1b approach the corresponding ones obtained for the system shown in Fig. 1a with h_2/h_1. However, the results obtained for the system depicted in Fig. 1b do not coincide "exactly" (i.e. such as in the corresponding static problems) with the corresponding ones obtained for the system depicted in Fig. 1a because the system in Fig. 1b takes the waves reflected from the lower boundary $z' = -h_1 / (\lambda_1^{(1)})^2 - h_2 / (\lambda_1^{(2)})^2$ into account, but the system in Fig. 1a does not.

It follows from the analyses of the results given in Fig. 6 that indeed, the frequency response of the system shown in Fig. 1b depends significantly on the values of the ratio of the geometrical sizes, i.e. on the ratio h_2/h_1. For instance, in the cases where h_2/h_1=1.0 and 2.0 for certain values of the dimensionless frequency Ω, a sudden jump occurs in the values of the stress. Note that this jump is similar to that which arises in the near vicinity of the ordinary resonance frequency. Besides all of this, the results given in Fig 6 and their comparison with the corresponding ones obtained in the case where $h_2/h_1 = \infty$ give a certain orientation on the application fields of the results obtained for the system consisting of the covering layer and the half-space in real cases.

Thus, we turn again to consideration of the results related to the system consisting of the covering layer and half space (Fig.1a), and analyze the influence of the initial strains, i.e. the influence of the parameters $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$ on the frequency response of the stresses $Q'_{z'z'}$ and $Q'_{z'r'}$. The graphs of these responses for the normal stress $Q'_{z'z'}$ are given in Figs. 7 and 8 for the cases where $C_{10}^{(1)}/C_{10}^{(2)} = 1.5$ and 3.0, respectively, but for the shear stress $Q'_{z'r'}$ in Fig. 9 for the case where $C_{10}^{(1)}/C_{10}^{(2)} = 1.5$. The graphs given in the figures which are indicated by the letter $a(b)$ are obtained in the case where the initial strains occur in the covering layer (in the half-space) only, but the graphs given in the figures indicated by the letter c are obtained for the case where the initial strains occur in both constituents simultaneously and are equal to each other.

Figs. 7a, 8a and 9a show that the initial stretching of the covering layer causes a decrease in the "resonance' values of the stresses and an increase in the "resonance' frequency $\Omega*$. Moreover Figs. 7b, 8b and 9b show that after a certain frequency (before which the influence of the initial strains of the half-space on the stresses is insignificant), as a result of the initial compressing (stretching) of the half-space, the "resonance" values of the stresses increase (decrease). It follows from the fore-
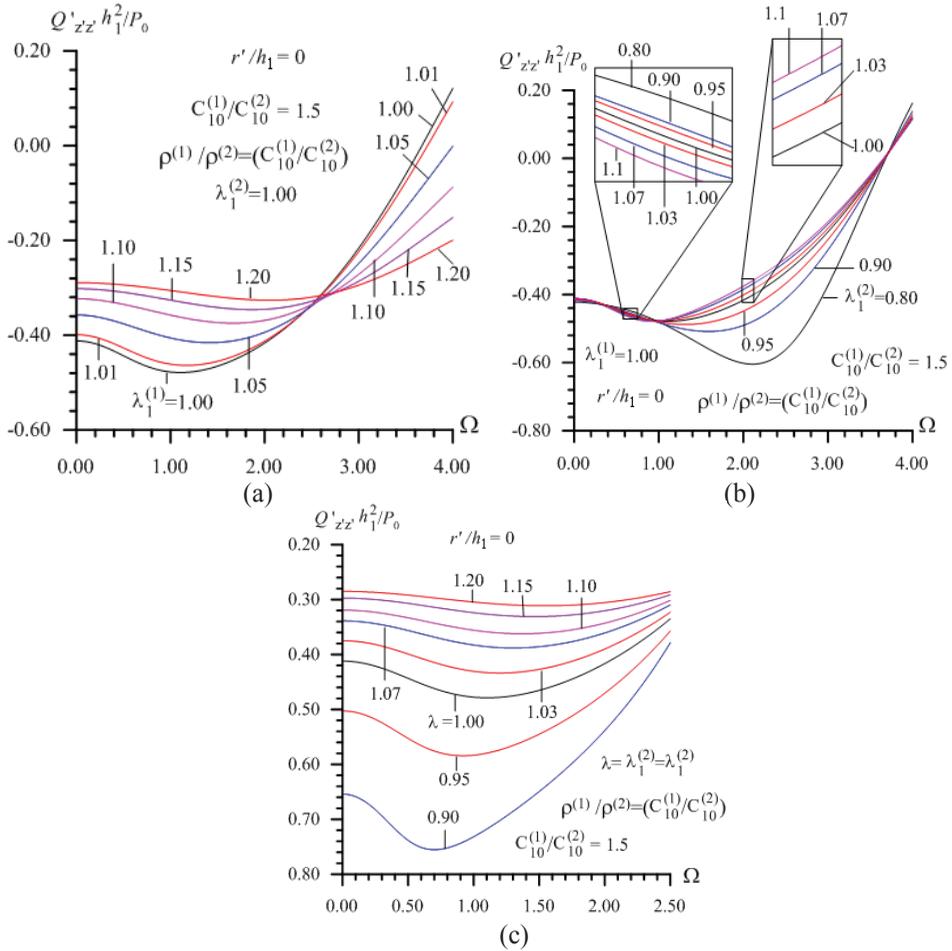
Figure 7: The influence of the initial strains on the frequency response of the normal stress $Q'_{z'z'}$ in the cases where the initial strains exist in the covering layer only (a), in the half-space only (b) and in both the constituents simultaneously and which are equal to each other (c) under $C_{10}^{(1)}/C_{10}^{(2)} = 1.5$.

going results that the influence of the initial compression of the half-space on the stresses is more significant than the influence of the initial stretching.

The analyses of the Figs. 7c, 8c and 9c show that the simultaneous initial compression (stretching) of the constituents of the system under consideration causes an increase (a decrease) in the resonance values of the stresses. In this case the resonance values of the frequency Ω_* increase with $\lambda \left(= \lambda_1^{(1)} = \lambda_1^{(2)}\right)$. At the same
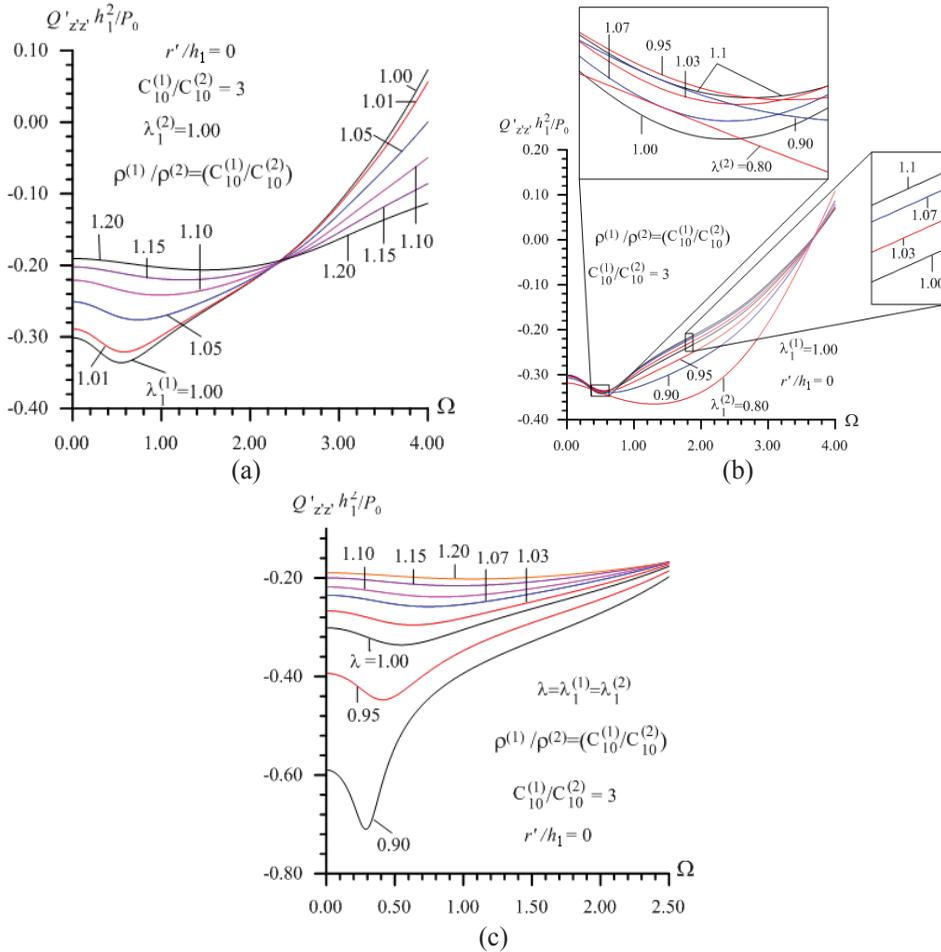
Figure 8: The influence of the initial strains on the frequency response of the normal stress $Q'_{z'z'}$ in the cases where the initial strains exist in the covering layer only (a), in the half-space only (b) and in both the constituents simultaneously and which are equal to each other (c) under $C_{10}^{(1)}/C_{10}^{(2)} = 3.0$.

time, Figs. 7c, 8c and 9c show that under simultaneous initial compression of the constituents of the system, after a certain λ the "resonance" values of the stresses increase sharply with decreasing λ. For a clear illustration of this sharp increase, in Fig. 10 the graphs of the dependence between $Q'_{z'z'}$ and Ω are given for smaller values of the parameter λ in the cases where $C_{10}^{(1)} \Big/ C_{10}^{(2)} = 1.5$ (Fig. 10a) and 3.0 (Fig. 10b). Note that the "resonance" values of the stresses approach infinity as
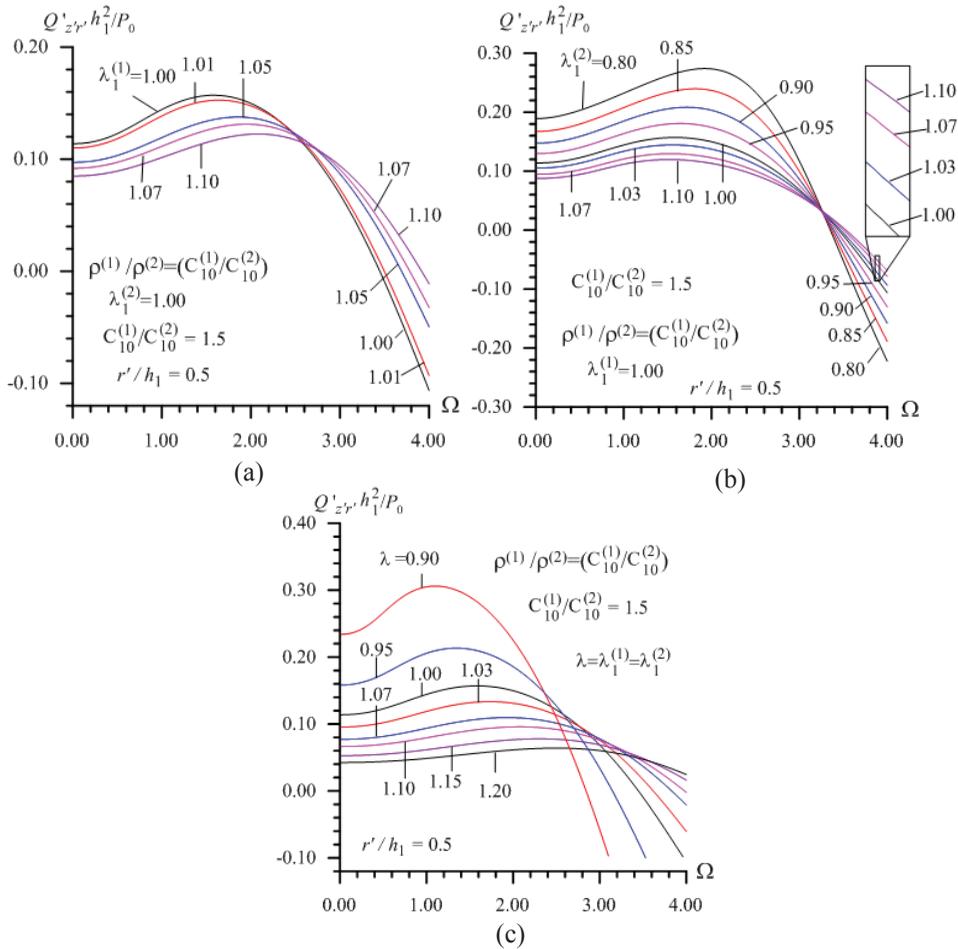
Figure 9: The influence of the initial strains on the frequency response of the normal stress $Q'_{z'r'}$ in the cases where the initial strains exist in the covering layer only (a), in the half-space only (b) and in both the constituents simultaneously and which are equal to each other (c) under $C_{10}^{(1)}/C_{10}^{(2)} = 1.5$.

$\lambda \to \lambda_{cr}$, whereas under $\lambda = \lambda_{cr}$ (the values of λ_{cr} are given in the field of Fig. 10), the near-surface stability loss of the system under consideration takes place. According to Biot (1965), Guz (2004), Akbarov (2013) and others, this near-surface stability loss occurs in the case where the stiffness of the covering layer material (i.e. the elastic constant $C_{10}^{(1)}$ in the expressions (6) and (10)) is greater than that of the half-space material (i.e. than the elastic constant $C_{10}^{(2)}$). Consequently, in the
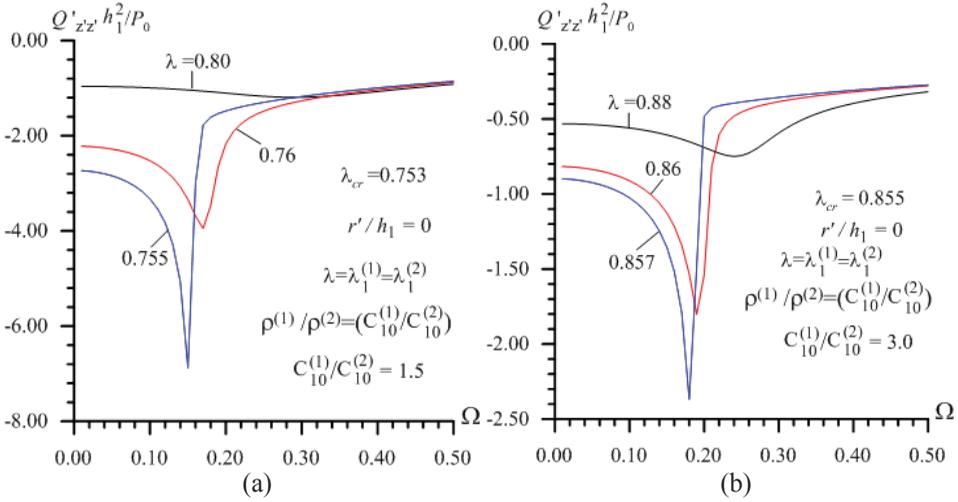
Figure 10: The illustration of the parametric resonance appearing with simultaneous initial compression of the constituents, i.e. with the approach of the parameter λ to λ_{cr}, which corresponds to the near surface stability loss of the system under consideration in the cases where $C_{10}^{(1)}/C_{10}^{(2)} = 1.5$ (a) and 3.0 (b).
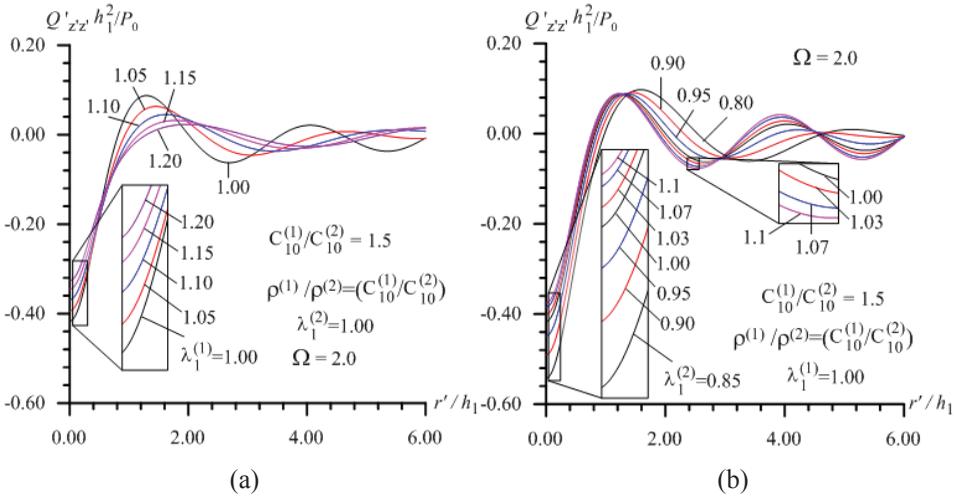
Figure 11: The influence of the initial strains on the distribution of the normal stress $Q'_{z'z'}$ with respect to the radial space coordinate in the cases where the initial strains occur in the covering layer only (a) and in the half-space only (b).
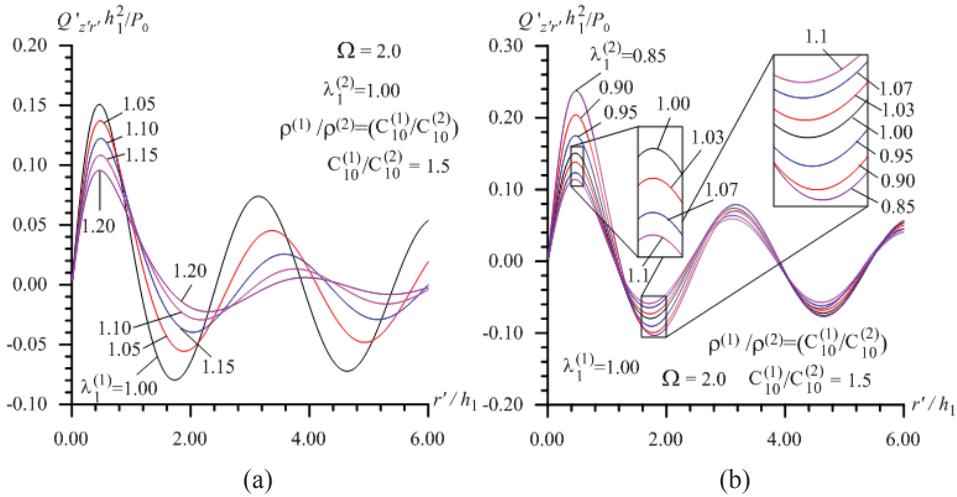
Figure 12: The influence of the initial strains on the distribution of the shear stress $Q'_{z'r'}$ with respect to the radial space coordinate in the cases where the initial strains occur in the covering layer only (a) and in the half-space only (b).
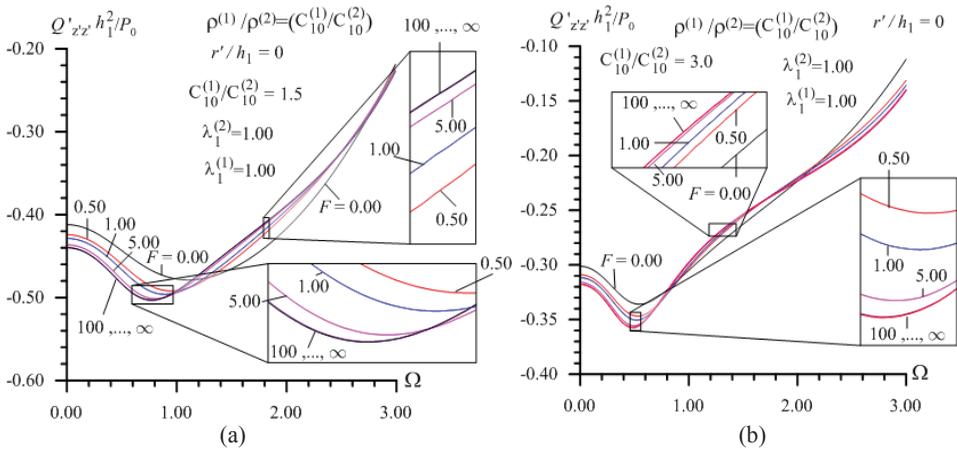
Figure 13: The influence of the shear-spring type imperfection parameter on the frequency response of the normal stress $Q'_{r'r'}$ in the cases where $C_{10}^{(1)}/C_{10}^{(2)} = 1.5$ (a) and 3.0 (b).

case where $\lambda = \lambda_{cr}$ the ordinary resonance of the system takes place and this resonance can be called a parametric resonance. This conclusion shows that the initial strains in the constituents of the system act on dynamical behavior of this system not only qualitatively, but also quantitatively.

Now we consider the results illustrated by the influence of the initial strains on the distribution of the stresses with respect to the radial coordinate r'/h_1. These results are given in Figs. 11 and 12 for the stresses $Q'_{z'z'}$ and $Q'_{z'r'}$ respectively in the case where $\Omega = 2.0$ and $C_{10}^{(1)} / C_{10}^{(2)} = 1.5$. Note that the graphs given in Figs. 11a and 12a (Figs. 11b and 12b) show the influence of the initial strains in the covering layer (in the half-space) on the distributions. It follows from these figures that the initial stretching in the covering layer causes the stresses to decay more rapidly with r'/h_1. But the initial compression of the half-space is the opposite, that is it causes the stresses to decay more slowly with r'/h_1.

Note that all the numerical results discussed above have been obtained in the case where, between the covering layer and half-space, complete contact conditions (14) are satisfied. Now we consider how the shear-spring type imperfectness (16) acts on the frequency responses. For this purpose we consider the graphs given in Fig. 13 which show the frequency response of the normal stress $Q'_{z'z'}$ in the cases where $C_{10}^{(1)} / C_{10}^{(2)} = 1.5$ (Fig.13a) and 3.0 (Fig.13b) under $\lambda_1^{(1)} = \lambda_1^{(2)} = 1.0$. Note that these graphs are constructed for various values of the shear-spring imperfection parameter F which enters into the condition (16). Thus, it follows from Fig. 13 that the shear spring type imperfectness of the contact conditions acts in the quantitative sense only on the frequency response. In this case an increase in the values of the parameter F causes an increase in the "resonance" values of the stress and a decrease of the resonance values of the frequency Ω_*. However, after a certain value of the frequency, it is the opposite, that is, an increase in the values of the parameter F causes a decrease in the absolute values of the stress $Q'_{z'z'}$. Also, it follows from Fig. 13 that the results obtained for the stress approach certain limit values with the parameter F and these limit values correspond to the results obtained under full slipping contact between the constituents, i.e. in the case where $F = \infty$.

6 Conclusions

Thus, in the present paper within the scope of the piecewise homogeneous body model, with the use of the large initial deformation version of the three-dimensional linearized theory of elastic waves in an initially stressed body, the axisymmetric time-harmonic Lamb's problem for a system consisting of a pre-strained half-space and a pre-strained covering layer made of incompressible materials has been stud-

ied for the case where the elastic constant $C_{10}^{(1)}$, which enters the expression of Treloar's potential (6), of the covering layer material is greater than the elastic constant $C_{10}^{(2)}$ of the half-space material. For the solution of the corresponding boundary value problems, the Hankel integral transformation method is employed. The algorithm based on the Sommerfeld contour integration method is developed for calculation of the corresponding inverse transformations. By utilizing this algorithm, numerical results related to the distribution and the frequency response of the stresses acting on the interface plane between the constituents are presented and discussed. In this discussion attention is also focused on the influence of the initial strains in the constituents on the foregoing frequency responses and distributions. Numerical results related to the same problem, which has been formulated for the pre-strained bi-layered slab resting on a rigid foundation are also presented and compared with the corresponding ones obtained for the system under consideration. At the same time, the numerical results which illustrate the influence of the shear-spring type imperfect contact conditions between the constituents on the foregoing frequency responses, are presented and analyzed.

According to the numerical results the following main conclusions can be made

- the frequency response for stresses has a non-monotonic character, i.e. the mechanical behavior of the axisymmetric forced vibration of the system consisting of the stiff covering layer and soft half-space is similar to that of the system comprising a mass, a parallel connected spring and a dashpot;

- the "resonance" values of the frequency and absolute maximum values of the stresses decrease with an increase in the ratio $C_{10}^{(1)}\big/C_{10}^{(2)}$;

- the initial stretching of the covering layer causes the "resonance" values of the stresses to decrease, but the initial compression of the half-space causes them to increase;

- in the case where the magnitude of the simultaneous initial compression of the constituents is near to that which corresponds to the near-surface stability loss of the system under consideration, ordinary resonance under forced vibration is caused. Note that this resonance can be called a parametric resonance. Consequently, the influence of the initial strains on the frequency response of the system under consideration is not only quantitatively, but also qualitatively important.

The results obtained in the case where the initial strains are absent correspond to those obtained within the framework of the classical linear theory of elastodynamics. Consequently, the numerical results presented in the paper are also significant

from this viewpoint. At the same time, the obtained numerical results can be taken as standard for a stratified ground vibration under which the stratified ground is modeled by a system composed of a mass, a parallel connected spring and a dash-pot.

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