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A Wavelet Method for Solving Nonlinear Time-Dependent Partial Differential Equations

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Abstract: A wavelet method is proposed for solving a class of nonlinear time-dependent partial differential equations. Following this method, the nonlinear equations are first transformed into a system of ordinary differential equations by using the modified wavelet Galerkin method recently developed by the authors. Then, the classical fourth-order explicit Runge-Kutta method is employed to solve the resulting system of ordinary differential equations. To justify the present method, the coupled viscous Burgers' equations are solved as examples, results demonstrate that the proposed wavelet algorithm have a much better accuracy and efficiency than many existing numerical methods, and the order of convergence of such a wavelet method can even reach about 5.

Keywords: modified wavelet Galerkin method, Runge-Kutta method, nonlinear time-dependent partial differential equations, Burgers' equation

1 Introduction

The nonlinear time-dependent partial differential equations (PDEs) provide a quantitative description for many nonlinear phenomena which play crucial roles in almost every scientific and engineering field [Caffarelli, Golse, Guo, Kenig and Vasseur (2012); Yi and Chen (2012); Zhou, Wang, Wang and Liu (2011); Chen, Sun, Li and Fu (2013); Kuo, Gu, Young and Lin (2013); Wei, Chen, Li, and Yi (2012)]. Solving these nonlinear PDEs are critically important for the resolutions of many practical problems in science and engineering. However, until now, it is still very difficult to obtain high accurate solutions of nonlinear PDEs, either theoretically or numerically [Baines, Hubbard and Jimack (2011); Tadmor (2012); Eslami and Mirzazadeh (2013); Feng, Glowinski and Neilan (2013); Su and Chen (2013)].

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Most nonlinear time-dependent PDEs, including for instance the nonlinear vibration equation of beams, nonlinear reaction-diffusion equations and Navier–Stokes equation of fluids etc. can be converted into the general form as

$$\begin{cases} \partial u_i(x,t)/\partial t = \sum\limits_{l=1}^{M} \mathbf{L}_{i,l}^0 u_l(x,t) + \sum\limits_{l=1}^{M_2} \mathbf{L}_{i,l}^1 \mathbf{N}_{i,l}[u_1(x,t),\ldots,u_M(x,t),x,t] + f_i(x,t),\ 0<x<1,\ t>0 \\ \partial^{j_{i,s}^0} u_i(x,t)/\partial^{j_{i,s}^0} x|_{x=0} = \partial^{j_{i,s}^1} u_i(x,t)/\partial^{j_{i,s}^1} x|_{x=1} = 0,\ u_i(x,0) = g_i(x),\ i=1,2,\ldots,M \end{cases} \tag{1}$$

in which $\mathbf{L}_{i,l}^0$ and $\mathbf{L}_{i,l}^1$ denote differential operators in the spatial variable x, $\mathbf{N}_{i,l}$ is a nonlinear functional of the unknown functions $u_i(x,t)$, $j_{i,s}^0$, $j_{i,s}^1$ and s are nonnegative integers.

A typical example of Eq. (1) is the so-called coupled viscous Burgers' equations, as the fundamental partial differential equations from fluid mechanics, which can be expressed as

$$\begin{cases} \frac{\partial u}{\partial t} = \varsigma \frac{\partial^2 u}{\partial x^2} - \frac{\eta}{2}\frac{\partial u^2}{\partial x} - \alpha \frac{\partial (uv)}{\partial x} + p(x,t) \\ \frac{\partial v}{\partial t} = \xi \frac{\partial^2 v}{\partial x^2} - \frac{\varepsilon}{2}\frac{\partial v^2}{\partial x} - \beta \frac{\partial (uv)}{\partial x} + q(x,t) \end{cases},\ x \in (0,\ 1),\ t>0 \tag{2}$$

with the boundary conditions $u(0,t) = u_0(t)$, $u(1,t) = u_1(t)$, $v(0,t) = v_0(t)$ and $v(1,t) = v_1(t)$, and the initial conditions $u(x,0) = f(x)$ and $v(x,0) = g(x)$, where ζ, η, α, ξ, ε and β are constants.

In recent years, a number of numerical methods have been proposed to solve Eq. (2), because solving the coupled viscous Burgers' equations (2) has two-fold importance. On one hand, Eq. (2) has been widely adopted to describe various physical processes, including the polydispersive sedimentation, diffusion of two kinds of particles in fluid suspensions and fluid turbulence etc. [Nee and Duan (1998); Abdou and Soliman (2005); Mittal and Jiwari (2012)]. On the other hand, the coupled viscous Burgers' equations (2) have been used as a benchmark testing problem for many existing numerical methods on the solution of nonlinear time-dependent PDEs, such as the variational iteration method [Abdou and Soliman (2005)], the decomposition method [Kaya (2001)], the discrete Adomian decomposition method [Zhu, Shu and Ding (2010)], the implicit finite difference method [Srivastava, Singh and Awasthi (2013); Srivastava, Awsthi and Tamsir (2013)], the differential quadrature method [Mittal and Jiwari (2012)], the B-spline collocation [Mitttal and Arora (2011)], and the conventional Galerkin method [Zhang, Yu and Zhao (2011)]. These conventional methods are effective for the solution of nonlinear time-dependent PDEs under certain conditions. However, it is still very difficult for most of them to solve the general form (1) of time-dependent PDEs with arbitrary strong nonlinearities.

In our recent works [Liu, Zhou, Wang and Wang (2013); Liu, Wang and Zhou (2013)], we have developed a modified wavelet Galerkin method for solving nonlinear boundary value problems, which can simply and efficiently deal with arbitrary strong nonlinearities including the transcendental forms. And such a wavelet algorithm has a much better accuracy and a much faster convergence rate than many other numerical methods, including the finite difference method [Odejide and Aregbesola (2006)], the classical weighted residual method [Odejide and Aregbesola (2006)], the Non-polynomial spline method [Jalilian (2010)], the B-spline method [Caglar, Caglar, Özer, Valaristos and Anagnostopoulos (2010)], the Lie-group shooting method [Abbasbandy, Hashemi and Liu (2011)], the differential transformation method [Hassan and Ertürk (2007)], the Laplace transform decomposition method [Khuri (2004)] and the decomposition method [Deeba, Khuri and Xie (2000)]. More importantly, the computational accuracy of this wavelet method [Liu, Zhou, Wang and Wang (2013); Liu, Wang and Zhou (2013)] is insensitive to the nonlinear intensity of the equations. But on the contrary, the numerical accuracy of most other methods [Jalilian (2010); Caglar, Caglar, Özer, Valaristos and Anagnostopoulos (2010); Abbasbandy, Hashemi and Liu (2011); Hassan and Ertürk (2007); Khuri (2004); Deeba, Khuri and Xie (2000)] usually decays very fast along with the nonlinear intensity.

In this study, as a development of the wavelet method for solving nonlinear boundary value problems [Liu, Zhou, Wang and Wang (2013); Liu, Wang and Zhou (2013)], we have combined such a wavelet technique with the Runge-Kutta method to solve the class of nonlinear time-dependent PDEs in the form of Eq. (1). By using the modified wavelet Galerkin method [Liu, Zhou, Wang and Wang (2013); Liu, Wang and Zhou (2013)], the nonlinear PDEs (1) are reduced into a system of nonlinear ordinary differential equations (ODEs). Then the solution to PDEs (1) can be obtained by using the Runge-Kutta method to solve these ODEs. Further by using the coupled viscous Burgers' equations (2) as examples, we conduct a systematic investigation on the efficiency and accuracy of the proposed wavelet algorithm for solving nonlinear time-dependent PDEs.

2 Solution procedure for time-dependent PDEs

Following our previous work [Liu, Zhou, Wang and Wang (2013); Liu, Wang and Zhou (2013)], a function $f(x) \in L^2[0,\ 1]$ can be approximated by

$$f(x) \approx P^j f(x) = \sum_{k=0}^{2^j} f(k/2^j)\varphi_{j,k}(x),\ x \in [0,\ 1] \tag{3}$$

where the modified wavelet basis

$$\varphi_{j,k}(x)=\begin{cases}\sum\limits_{i=-9}^{-1}T_{0,k}(\frac{i}{2^j})\phi(2^jx-i+7)+\phi(2^jx-k+7) & k\in[0,\ 3]\\ \phi(2^jx-k+7) & k\in[4,\ 2^j-4]\\ \sum\limits_{i=2^j+1}^{2^j+6}T_{1,2^j-k}(\frac{i}{2^j})\phi(2^jx-i+7)+\phi(2^jx-k+7) & k\in[2^j-3,\ 2^j]\end{cases}. \tag{4}$$

In Eq. (4), $\phi(x)$ is the generalized Coiflet-type orthogonal scaling function developed by Wang [Wang (2001)], and

$$T_{0,k}(x)=\sum_{i=0}^{3}\frac{p_{0,i,k}}{i!}x^i,\quad T_{1,k}=\sum_{i=0}^{3}\frac{p_{1,i,k}}{i!}(x-1)^i \tag{5}$$

in which the numerical differentiation coefficients $p_{0,i,k}$ and $p_{1,i,k}$ are assigned as

$$\mathbf{P_0}=\begin{bmatrix}1&0&0&0\\-11/6&3&-3/2&1/3\\2&-5&4&-1\\-1&3&-3&1\end{bmatrix},\quad \mathbf{P}_1=\begin{bmatrix}1&0&0&0\\11/6&-3&3/2&-1/3\\2&-5&4&-1\\1&-3&3&-1\end{bmatrix} \tag{6}$$

through relations $\mathbf{P}$=$\{2^{-ij}p_{0,i,k}\}$ and $\mathbf{P}_1$=$\{2^{-ij}p_{1,i,k}\}$, and i, k=0, 1, 2, 3.

The accuracy of the wavelet approximation (3) of the function $f(x)\in L^2[0,1]$, which is dependent on the number of vanishing moment γ=6 of the wavelet function associated with the scaling function $\phi(x)$ in Eq. (4) and the decomposition level j has been determined by [Liu, Wang and Zhou (2013); Wang (2001); Resnikoff and Wells (1998)]

$$||f(x)-P^jf(x)||_{L^2[0,\ 1]}\leq C_1 2^{-j\gamma} \tag{7}$$

and for the approximation of its derivatives, we similarly have

$$||\frac{d^mf(x)}{dx^m}-\frac{d^mP^jf(x)}{dx^m}||_{L^2[0,\ 1]}\leq C_2 2^{-j(\gamma-m)} \tag{8}$$

where constants C_1 and C_2 depend on the smoothness and boundary extension property of $f(x)$, m is a positive integer satisfying $m<\gamma$.

To meet homogeneous boundary conditions in Eq. (1) the coefficients $p_{0,j_{i,s}^0,k}$ and $p_{1,j_{i,s}^1,k}$ (k=0, 1, 2, 3) in matrix (6) should be set as zeros, and keeping all other elements unchanged [Liu, Zhou, Wang and Wang (2013)] Then the modified scaling basis $\varphi_{j,k}(x)$ in Eq. (4) will be specified accordingly, which is re-denoted as $h^i_{j,k}(x)$. Thus the unknown function $u_i(x)$ can be approximated by Eq. (3) as

$$u_i(x,t) \approx \sum_{k=0}^{2^j} u_i(k/2^j,t)h^i_{j,k}(x),\ i=1,2,...,M. \tag{9}$$

And following the nonlinear operational property for the wavelet approximation (3) [Liu, Zhou, Wang and Wang (2013)], the nonlinear term $\mathbf{N}_{i,l}[u_1(x,t),...,u_M(x,t),x,t]$ in Eq. (1) can be expressed as

$$\mathbf{N}_{i,l}[u_1(x,t),...,u_M(x,t),x,t] \approx \sum_{k=0}^{2^j} \mathbf{N}_{i,l}[u_1(\frac{k}{2^j},t),...,u_M(\frac{k}{2^j},t),\frac{k}{2^j},t]\varphi_{j,k}(x) \tag{10}$$

and the function $f_i(x,\ t)$ can be written as

$$f_i(x,t) \approx \sum_{k=0}^{2^j} f_i(k/2^j,t)\varphi_{j,k}(x) \tag{11}$$

where the modified scaling basis $\varphi_{j,k}(x)$ is denoted in Eq. (4) with coefficients $p_{0,i,k}$ and $p_{1,i,k}$ specified in Eq. (6). Substituting Eqs. (9) (10) and (11) into Eq. (1), yields

$$\begin{array}{l} \sum\limits_{k=0}^{2^j} \frac{du_i(k/2^j,t)}{dt} h^i_{j,k}(x) \approx \sum\limits_{l=1}^{M}\sum\limits_{k=0}^{2^j} u_l(k/2^j,t)\mathbf{L}^0_{i,l}h^l_{j,k}(x) + \sum\limits_{k=0}^{2^j} f_i(k/2^j,t)\varphi_{j,k}(x) \\ +\sum\limits_{l=1}^{M_2}\sum\limits_{k=0}^{2^j} \mathbf{N}_{i,l}[u_1(\frac{k}{2^j},t),...,u_M(\frac{k}{2^j},t),\frac{k}{2^j},t]\mathbf{L}^1_{i,l}\varphi_{j,k}(x),\ i=1,2,...,M \end{array}. \tag{12}$$

Multiplying both sides of the i_{th} equation in Eq. (12) by $h^i_{j,l}(x)$, $l=0,\ 1,\ 2,\ ...,\ 2^j$, respectively and perform integration over the interval [0, 1], gives

$$\mathbf{A}_i d\mathbf{U}_i(t)/dt \approx \sum_{n=1}^{M} \mathbf{B}_{i,n}\mathbf{U}_n(t) + \sum_{l=n}^{M_2} \mathbf{C}_{i,n}\mathbf{D}_{i,n}(t) + \mathbf{E}_i\mathbf{F}_i(t) \tag{13}$$

in which

matrixes $\mathbf{A}_i = \{a^i_{lk} = \int_0^1 h^i_{j,k}(x)h^i_{j,l}(x)dx\}$ and $\mathbf{B}_{i,n} = \{b^{i,n}_{lk} = \int_0^1 \mathbf{L}^0_{i,n}h^n_{j,k}(x)h^i_{j,l}(x)dx\}$

matrixes $\mathbf{C}_{i,n} = \{c^{i,n}_{lk} = \int_0^1 \mathbf{L}^1_{i,n}\varphi_{j,k}(x)h^i_{j,l}(x)dx\}$ and $\mathbf{E}_i = \{e^i_{lk} = \int_0^1 \varphi_{j,k}(x)h^i_{j,l}(x)dx\}$

vectors $\mathbf{U}_i(t) = \{u^i_k = u_i(k/2^j,t)\}^T$ and $\mathbf{F}_i(t) = \{f^i_k = f_i(k/2^j,t)\}^T$

vector $\mathbf{D}_{i,n}(t) = \{d^{i,n}_k = \mathbf{N}_{i,n}[u_1(k/2^j,t),...,u_M(k/2^j,t),k/2^j,t]\}^T$,

where the subscripts $i=1,\ 2,\ ...,\ M$, and $k,\ l=0,\ 1,\ 2,\ ...,\ 2^j$. And the generalized connection coefficients a^i_{lk}, $b^{i,n}_{lk}$ $c^{i,n}_{lk}$ and e^i_{lk} can be obtained exactly by using the procedure suggested by Wang [Wang (2001)], and the expression of the modified scaling basis have been given by Eq. (4).

In the system of ODEs (13), the numbers of both equations and unknown functions $u_i(k/2^j,t)$ ($i = 1, 2, ..., M$, and $k = 0, 1, 2, ..., 2^j$) are $M(2^j+1)$. Thus we can obtain the nodal values of unknown function $u(k/2^j,n\Delta t)$ ($k = 0,1, 2, ..., 2^j$, and $n = 1,2,3,...$), by directly using the time-integration methods with time step Δt.

3 Solution of coupled viscous Burgers' equations

In this section, we will apply the proposed method to solve the coupled viscous Burgers' equations (2). Firstly, we set the nodal values of unknown functions $u(0/2^j,t) = u_0(t)$, $u(2^j/2^j,t) = u_1(t)$, $v(0/2^j,t) = v_0(t)$ and $v(2^j/2^j,t) = v_1(t)$ to meet the boundary conditions in Eq. (2), without any change of matrix (6). Thus, the unknown functions and nonlinear terms in Eq. (2) can be expressed as

$$u(x,t) \approx \sum_{k=0}^{2^j} u(\frac{k}{2^j},t)\varphi_{j,k}(x), \quad v(x,t) \approx \sum_{k=0}^{2^j} v(\frac{k}{2^j},t)\varphi_{j,k}(x) \tag{14}$$

$$u^2(x,t) \approx \sum_{k=0}^{2^j} u^2(\frac{k}{2^j},t)\varphi_{j,k}(x), \quad v^2(x,t) \approx \sum_{k=0}^{2^j} v^2(\frac{k}{2^j},t)\varphi_{j,k}(x) \tag{15}$$

$$u(x,t)v(x,t) \approx \sum_{k=0}^{2^j} u(\frac{k}{2^j},t)v(\frac{k}{2^j},t)\varphi_{j,k}(x) \tag{16}$$

$$p(x,t) \approx \sum_{k=0}^{2^j} p(\frac{k}{2^j},t)\varphi_{j,k}(x), \quad q(x,t) \approx \sum_{k=0}^{2^j} q(\frac{k}{2^j},t)\varphi_{j,k}(x). \tag{17}$$

Substituting Eqs. (14-17) into Eq. (2) and consider boundary conditions, we have

$$\begin{aligned}
&\sum_{k=1}^{2^j-1} \frac{du_{j,k}(t)}{dt}\varphi_{j,k}(x) \approx \varsigma \sum_{k=1}^{2^j-1} u_{j,k}(t)\frac{d^2\varphi_{j,k}(x)}{dx^2} - \sum_{k=1}^{2^j-1}[\frac{\eta u_{j,k}^2(t)}{2} + \alpha u_{j,k}(t)v_{j,k}(t)]\frac{d\varphi_{j,k}(x)}{dx} \\
&+\sum_{k=0}^{2^j} p_{j,k}(t)\varphi_{j,k}(x) - \frac{du_0(t)}{dt}\varphi_{j,0}(x) - \frac{du_1(t)}{dt}\varphi_{j,2^j}(x) + \varsigma[u_0(t)\frac{d^2\varphi_{j,0}(x)}{dx^2} + u_1(t)\frac{d^2\varphi_{j,2^j}(x)}{dx^2}] \\
&-[\frac{\eta u_0^2(t)}{2} + \alpha u_0(t)v_0(t)]\frac{d\varphi_{j,0}(x)}{dx} - [\frac{\eta u_1^2(t)}{2} + \alpha u_1(t)v_1(t)]\frac{d\varphi_{j,2^j}(x)}{dx}
\end{aligned} \tag{18a}$$

$$\sum_{k=1}^{2^j-1}\frac{dv_{j,k}(t)}{dt}\varphi_{j,k}(x)\approx\xi\sum_{k=1}^{2^j-1}v_{j,k}(t)\frac{d^2\varphi_{j,k}(x)}{dx^2}-\sum_{k=1}^{2^j-1}[\frac{\varepsilon v_{j,k}^2(t)}{2}+\beta u_{j,k}(t)v_{j,k}(t)]\frac{d\varphi_{j,k}(x)}{dx}$$
$$+\sum_{k=0}^{2^j}q_{j,k}(t)\varphi_{j,k}(x)-\frac{dv_0(t)}{dt}\varphi_{j,0}(x)-\frac{dv_1(t)}{dt}\varphi_{j,2^j}(x)+\xi[v_0(t)\frac{d^2\varphi_{j,0}(x)}{dx^2}+v_1(t)\frac{d^2\varphi_{j,2^j}(x)}{dx^2}]$$
$$-[\frac{\varepsilon v_0^2(t)}{2}+\beta u_0(t)v_0(t)]\frac{d\varphi_{j,0}(x)}{dx}-[\frac{\varepsilon v_1^2(t)}{2}+\beta u_1(t)v_1(t)]\frac{d\varphi_{j,2^j}(x)}{dx} \tag{18b}$$

Multiplying both sides of Eq. (18) by $\varphi_{j,l}(x)$, $l=1,\ 2,\ \ldots,\ 2^j-1$, respectively and perform integration over the interval [0, 1], yields

$$\begin{cases} d\mathbf{U}/dt\approx\varsigma\mathbf{A}^{-1}\mathbf{BU}-\mathbf{A}^{-1}\mathbf{CE}+\mathbf{A}^{-1}\mathbf{DP}-\mathbf{A}^{-1}\mathbf{G} \\ d\mathbf{V}/dt\approx\xi\mathbf{A}^{-1}\mathbf{BV}-\mathbf{A}^{-1}\mathbf{CF}+\mathbf{A}^{-1}\mathbf{DQ}-\mathbf{A}^{-1}\mathbf{H} \end{cases} \tag{19}$$

where the matrixes $\mathbf{A}=\{a_{lk}=\Gamma_{l,k}^{0,0}\}$, $\mathbf{B}=\{b_{lk}=\Gamma_{l,k}^{0,2}\}$, $\mathbf{C}=\{c_{lk}=\Gamma_{l,k}^{0,1}\}$ and $\mathbf{D}=\{d_{li}=\Gamma_{l,i}^{0,0}\}$, and the vectors $\mathbf{U}=\{u_k=u_{j,k}(t)\}^T$, $\mathbf{V}=\{v_k=v_{j,k}(t)\}^T$, $\mathbf{E}=\{e_k=\eta u_{j,k}^2(t)/2+\alpha u_{j,k}(t)v_{j,k}(t)\}^T$, $\mathbf{P}=\{p_i=p_{j,i}(t)\}^T$, $\mathbf{Q}=\{q_i=q_{j,i}(t)\}^T$, and $\mathbf{F}=\{f_k=\varepsilon v_{j,k}^2(t)/2+\beta u_{j,k}(t)v_{j,k}(t)\}^T$, and vectors $\mathbf{G}=\{g_l=\frac{du_0}{dt}\Gamma_{l,0}^{0,0}+\frac{du_1}{dt}\Gamma_{l,2^j}^{0,0}-\varsigma u_0\Gamma_{l,0}^{0,2}-\varsigma u_1\Gamma_{l,2^j}^{0,2}+(\frac{\eta u_0^2}{2}+\alpha u_0v_0)\Gamma_{l,0}^{0,1}+(\frac{\eta u_1^2}{2}+\alpha u_1v_1)\Gamma_{l,2^j}^{0,1}\}^T$ and $\mathbf{H}=\{h_l=\frac{dv_0}{dt}\Gamma_{l,0}^{0,0}+\frac{dv_1}{dt}\Gamma_{l,2^j}^{0,0}-\xi v_0\Gamma_{l,0}^{0,2}-\xi v_1\Gamma_{l,2^j}^{0,2}+(\frac{\varepsilon v_0^2}{2}+\beta u_0v_0)\Gamma_{l,0}^{0,1}+(\frac{\varepsilon v_1^2}{2}+\beta u_1v_1)\Gamma_{l,2^j}^{0,1}\}^T$. Here, the generalized connection coefficients $\Gamma_{l,k}^{0,m}=\int_0^1 d^m\varphi_{j,k}(x)/dx^m\varphi_{j,l}(x)dx$, the subscripts l, k=1, 2, …, 2^j-1 and i=0, 1, …, 2^j.

In this study, the classical fourth-order explicit Runge-Kutta method with the time step Δt is employed to solve the system of ODEs (19).

4 Numerical results and discussions

To effectively evaluate the performance of the present wavelet method, we consider the maximum error L_{max}, the relative L_2 errors, and the order of convergence R, which are respectively defined as [Mitttal and Arora (2011)]

$$L_{\max}=\max_k\{|u_{\text{num}}(x_k,t)-u_{\text{exact}}(x_k,t)|\} \tag{20}$$

$$L_2=\sqrt{\sum_k|u_{\text{num}}(x_k,t)-u_{\text{exact}}(x_k,t)|^2/\sum_k|u_{\text{exact}}(x_k,t)|^2} \tag{21}$$

$$R=\frac{\log[Error(N_1)/Error(N_2)]}{\log(N_2/N_1)} \tag{22}$$

in which $Error(N_1)$ and $Error(N_2)$ are the L_{max} errors for number of spatial partitions $N = N_1$ and N_2, respectively

As a numerical test, we consider the coupled viscous Burgers' equation (2) with constants ζ=ξ=1/4π^2, η=ε=-1/π and α=β=1/2π, initial conditions $u(x,\ 0)$=$v(x,\ 0)$=-sin(2πx), boundary conditions $u(0,t) = u(1,t) = v(0,t) = v(1,t)$=0, and functions $p(x,t) = q(x,t)$=0 [Mitttal and Arora (2011); Mittal and Jiwari (2012); Srivastava, Awsthi and Tamsir (2013); Zhang, Yu and Zhao (2011)]. The exact solutions of this problem are $u(x,t) = v(x,t) = e^{-t}\sin(2\pi x\text{-}\pi)$.

Table 1: Maximum error L_{max} of $u(x,t)$ for time setp Δt=0.01, at different time t

t	Present, N=64	Mittal and Arora (2011), N=400	Srivastava et al. (2013), N=400	Zhang et al. (2011), N=64	Rashid and Ismail (2009), N=64
0.1	2.02E-07	1.86E-06	4.80E-05	—	—
0.5	1.36E-07	6.22E-06	1.62E-04	1.38E-05	—
1.0	8.25E-08	7.56E-06	1.98E-04	—	1.16E-05

Table 2: Relative L_2 errors of $u(x,t)$ for time setp Δt=0.01, at different time t

t	Present, N=64	Mittal and Arora (2011), N=400	Srivastava et al. (2013), N=400	Rashid and Ismail (2009), N=64
0.1	5.68E-08	2.05E-06	5.30E-05	—
0.5	5.71E-08	1.02E-05	2.67E-04	—
1.0	5.75E-08	2.04E-05	5.38E-04	2.91E-05

The maximum errors L_{max} and relative L_2 errors under Δt= 0.01 for different time t are listed in Table 1 and Table 2, respectively. It can been seen from Table 1 and Table 2 that the present wavelet method with much less number of spatial partitions N has a better numerical accuracy than many existing methods, including the B-spline collocation [Mitttal and Arora (2011)], the finite difference method [Srivastava, Awsthi and Tamsir (2013)], the conventional Galerkin method [Zhang, Yu and Zhao (2011)] and the Fourier pseudospectral method [Rashid and Ismail (2009)]. In Table 3 and Fig.1, we have shown the relationship between the maximum errors L_{max} and the number of spatial partitions N. From Table 3 and Fig.1 we can find out that the order of convergence R of the proposed wavelet method is about 5, which obviously exceeds the order of convergence of the B-spline collocation [Mitttal and Arora (2011)] and the conventional Galerkin method [Zhang, Yu and Zhao (2011)]. It clearly demonstrates the accuracy and efficiency of the present wavelet method.

Table 3: Maximum error L_{max} and order of convergence R for $u(x, t)$ at time t=0.5

N	Present		Mittal and Arora (2011)		Zhang et al. (2011)	
	L_{max}	R	L_{max}	R	L_{max}	R
8	3.05E-03	—	—	—	—	—
16	1.339E-04	4.511	—	—	—	—
32	4.336E-06	4.949	9.748E-04	—	1.106E-04	—
64	1.360E-07	4.995	2.436E-04	2.005	1.379E-05	3.003
128	—	—	6.090E-05	2.001	1.722E-06	3.001
256	—	—	1.522E-05	2.001	2.153E-07	3.000

As the other example, we study the following problem [Mitttal and Arora (2011); Mittal and Jiwari (2012); Srivastava, Awsthi and Tamsir (2013); Zhang, Yu and Zhao (2011)]:

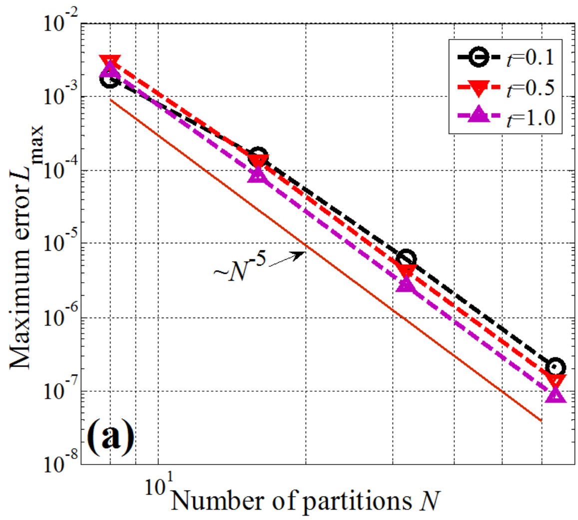
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{400}\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{20}\dfrac{\partial u^2}{\partial x} - \dfrac{\alpha}{20}\dfrac{\partial (uv)}{\partial x} \\ \dfrac{\partial v}{\partial t} = \dfrac{1}{400}\dfrac{\partial^2 v}{\partial x^2} - \dfrac{1}{20}\dfrac{\partial v^2}{\partial x} - \dfrac{\beta}{20}\dfrac{\partial (uv)}{\partial x} \end{cases}, \ x \in (0,\ 1),\ t > 0 \tag{23}$$

with the initial and boundary conditions which are taken from the travelling wave solution

$$\begin{cases} u(x,t) = a_0\{1 - \tanh[A(20x - 2At - 10)]\} \\ v(x,t) = a_0\left\{\dfrac{2\beta - 1}{2\alpha - 1} - \tanh[A(20x - 2At - 10)]\right\} \end{cases}, \ A = \frac{a_0}{2}\frac{4\alpha\beta - 1}{2\alpha - 1} \tag{24}$$

in which α and β are arbitrary constants, anda_0=0.05.

We present a comparison between the numerical solutions of Eq. (23) obtained by the proposed wavelet method and those obtained by other existing methods in Tables 4-7, which display the maximum error and the relative L_2 error for various values of t, α, and β As can be seen from Tables 4-7, the present wavelet solutions can achieve a similar accuracy comparing to those obtained by many other methods [Khater, Temsah and Hassan (2008); Rashid and Ismail (2009); Mitttal and Arora (2011); Srivastava, Awsthi and Tamsir (2013)], with much less number of spatial



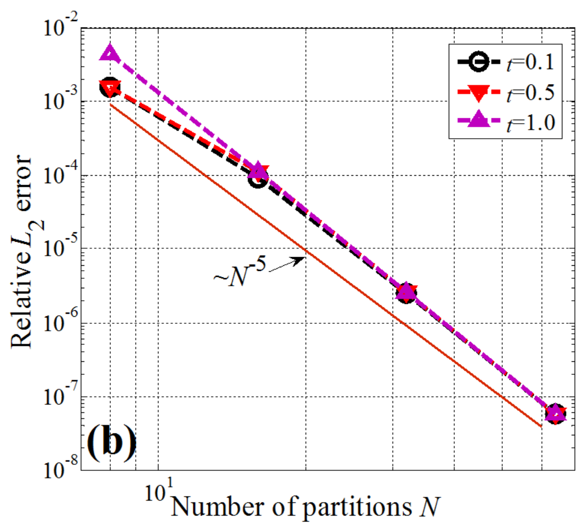


Figure 1: Errors of numerical solutions $u(x,t)$ under time setp Δt=0.01 for different time t, as a function of the number spatial partitions N: (a) Maximum error L_{max}, and (b) Relative L_2 errors.

partitions N, implying that the proposed wavelet algorithm has a much higher efficiency than these methods [Khater, Temsah and Hassan (2008); Rashid and Ismail (2009); Mitttal and Arora (2011); Srivastava, Awsthi and Tamsir (2013)].

Table 4: Maximum error L_{max} for $u(x, t)$ at different time t

t	α	β	Present, N=8	Khater et al. (2008), N=20	Rashid and Ismail (2009), N=16	Mittal and Arora (2011), N=100	Srivastava et al. (2013), N=100
0.5	0.1	0.3	4.59E-05	4.38E-05	3.25E-05	4.17E-05	4.12E-05
	0.3	0.03	4.60E-05	4.58E-05	2.73E-05	4.59E-05	4.31E-05
1	0.1	0.3	8.61E-05	8.66E-05	2.41E-05	8.26E-05	8.15E-05
	0.3	0.03	9.16E-05	9.16E-05	2.83E-05	9.18E-05	8.54E-05

Table 5: Relative L_2 errors for $u(x, t)$ at different time t

t	α	β	Present, N=8	Khater et al. (2008), N=20	Rashid and Ismail (2009), N=16	Mittal and Arora (2011), N=100	Srivastava et al. (2013), N=100
0.5	0.1	0.3	7.21E-04	1.44E-03	9.62E-04	6.74E-04	6.63E-04
	0.3	0.03	8.03E-04	6.68E-04	4.31E-04	7.33E-04	6.90E-04
1	0.1	0.3	1.42E-03	1.27E-03	1.15E-03	1.33E-03	1.30E-03
	0.3	0.03	1.59E-03	1.30E-03	1.27E-03	1.45E-03	1.36E-03

Table 6: Maximum error L_{max} for $v(x, t)$ at different time t

t	α	β	Present, N=8	Khater et al. (2008), N=20	Rashid and Ismail (2009), N=16	Mittal and Arora (2011), N=100	Srivastava et al. (2013), N=100
0.5	0.1	0.3	2.38E-05	4.99E-05	2.75E-05	1.48E-04	2.13E-05
	0.3	0.03	1.81E-04	1.81E-04	2.45E-04	5.73E-04	4.91E-05
1	0.1	0.3	4.45E05	9.92E-05	3.75E-05	4.77E-05	4.11E-05
	0.3	0.03	3.61E-04	3.62E-04	4.53E-04	3.62E-04	9.78E-05

Table 7: Relative L_2 errors for $v(x, t)$ at different time t

t	α	β	Present, N=8	Khater et al. (2008), N=20	Rashid and Ismail (2009), N=16	Mittal and Arora (2011), N=100	Srivastava et al. (2013), N=100
0.5	0.1	0.3	5.40E-04	5.42E-04	3.33E-04	9.06E-04	4.89E-04
	0.3	0.03	1.41E-03	1.20E-03	1.15E-03	1.59E-03	7.06E-04
1	0.1	0.3	1.06E-03	1.29E-03	1.16E-03	1.25E-03	9.53E-04
	0.3	0.03	2.80E-03	2.35E-03	1.64E-03	2.25E-03	1.39E-03

5 Conclusion

In this paper, we have proposed a wavelet scheme for numerically solving the nonlinear time-dependent partial differential equations. The accuracy and efficiency of this wavelet method have been demonstrated by taking the coupled viscous Burgers' equations as test examples. The numerical results show that the proposed wavelet method has a much better accuracy and efficiency than many other numerical methods, and the order of convergence of such a wavelet method is about 5, better than other methods

Acknowledgement: This research is supported by grants from the National Natural Science Foundation of China (11032006, 11072094, 11121202), a grant from the Ph.D. Program Foundation of Ministry of Education of China (20100211110022), National Key Project of Magneto-Constrained Fusion Energy Development Program (2013GB110002), the Fundamental Research Funds for the Central Universities (lzujbky-2012-202, lzujbky-2013-1) and the Scholarship Award for Excellent Doctoral Student granted by Lanzhou University.

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