 CMES, vol.93, no.6, pp.469-488, 2013

A Moving Kriging Interpolation Response Surface Method for Structural Reliability Analysis

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Abstract: In order to obtain reliable structural design, it is of extreme importance to evaluate the failure probability, safety levels of structure (reliability analysis) and the effect of a change in a variable parameter on structural safety (sensitivity analysis) when uncertainties are considered. With a computationally cheaper approximation of the limit state function, various response surface methods (RSMs) have emerged as a convenient tool to solve this especially for complex problems. However, the traditional RSMs may produce large errors in some conditions especially for those highly non-linear limit state functions. Instead of the traditional least squares approximation, in the present paper, a new RSM is proposed which employs moving Kriging interpolation, based on the axial experimental points selected from the region where the most probable failure point (MPFP) is likely to exist to construct the substitute response surface. The proposed method is illustrated by comparing with the results obtained from first order reliability method (FORM) and other conventional RSMs with reference to specific structural reliability analysis problems. The results show that the proposed method improves the accuracy of the reliability analysis with a reasonable computational cost, and could more quickly approach the exact solution.

Keywords: Response surface method; Moving Kriging interpolation; Limit state function; Structural reliability

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1 Introduction

Up to now, structural reliability analysis has received a great deal of attention to guarantee the structural safety due to uncertainties in material properties, geometry, boundary conditions, as well as in loads that should be considered in the structural design. The failure probability or reliability index in some kind of limit state is calculated as quantitative measures to cope with these uncertainties [Ditlevsen and Madsen (1996); Kami and Szafran (2012)]. As is well known, simulation techniques such as Monte Carlo method (MCS), moment methods including the first order reliability method (FORM) and the second order reliability method (SORM) provide a good methodology to perform structural reliability analysis. However, MCS will consume tremendous computational efforts especially for lower failure probability, and the most probable failure point (MPFP), parameter sensitivities (the failure probability with respect to variables) cannot be directly obtained, which results in difficulty to combine with optimization in the case of reliability-based design optimization (RBDO) problems [Santos, Matioli and Beck (2012)]. The FORM and SORM are another successful attempt to enhance the computational efficiency and can be easily applied in RBDO problems, which are also proved to consume too much computation time when a large number of random variables involved, and the performance of FORM and SORM will be affected for implicit limit state functions since the explicit formulation of those required gradients with respect to the basic variables namely their direct or analytical differentiations are not available [Kiureghian, Lin and Hwang (1987); Liu and Kiureghian (1991)].

The response surface method (RSM), as an alternative, is developed for this solution. This method approximates the actual limit state function of the structure through a number of deterministic structural analyses and then evaluates the failure probability by the FORM/SORM with the obtained explicit limit state function. The approximated points (composed of structural parameter values and the responding value of limit state function) are commonly chosen according to the experimental design method, and the traditional response surface approximation is constructed on finite polynomial basis functions, with the result that it rarely fits the actual limit state function exactly. The second best is as far as possible to approximate the actual limit state function near the design point, where its contribution to the total failure probability plays an important role. The design point is initially unknown, thus serials of iterations including constructing the substitute limit state function and the FORM/SORM procedure are required. Even this, for the acute nonlinearity problem, the traditional RSM doesn't show satisfactory performance in computational accuracy and efficiency [Rajashekhar and Ellingwood (1993); Guan and Melchers (2001); Bucher and Bourgund (1990)]. Many RSM techniques are dedicated to reducing the number of structural analysis or im-

proving the accuracy such as artificial neural network-based RSM [Cheng, Li and Xiao (2008)], Kriging-based RSM [Kaymaz (2005)], moving least square (MLS) approximation-based RSM [Kang, Koh and Choo (2010)], support vector regression (SVR) based RSM [Zhao, Liu and Ye (2011)] and etc.

Herein another efficient way to approximate the points, moving Kriging (MK) interpolation method is employed to construct the equivalent limit state function for the implicit response surface, which was introduced by Gu (2003) for the first time. The Kriging interpolation is a well-known geostatical technique for spatial interpolation in geology and mining, and the application of the MK interpolation is still in its early stages [Matheron (1963)]. Both MK interpolation method and MLS method belong to some kind of 'moving' method which means response surface is constructed on the varying sub-domain, but different from MLS method, which is an approximant and does not pass through the interpolation points, MK interpolation method satisfies the Kronecker's delta property, and also its derivatives with respect to the basic variables can be easily obtained at the same time as calculating the function values. Kang and his coworkers applied the MLS approximation to the RSM, which gave higher weight to the experimental points closer to the MPFP through weigh functions so as to allow the response surface function to be closer to the limit state function at the MPFP. A response surface based on linear polynomial basis functions was constructed at first and a response surface based on quadratic basis functions was formed using the axial experimental points selected from the reduced region where the MPFP is likely to exist [Kang, Koh and Choo (2010)]. In this paper, an equivalent limit state function is generated by MK interpolation method. In contrast with MLS method, it gives higher weight to the experimental points closer to the MPFP through correlation functions. The proposed procedure provides reasonably good results compared to those obtained by the conventional LS-RSM, FORM, SVR and MLS approximation-based RSM.

2 Moving Kriging approximation [Gu (2003)]

As MLS method does, the MK interpolation method approximates the distribution function $u(\mathbf{x})$ within a sub-domain $\Omega_{\mathbf{x}}$ ($\Omega_{\mathbf{x}} \in \Omega$, Ω is the whole definition domain). Thus these values can be interpolated based on all nodal values of $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ within the sub-domain, and n is the total number of the nodes in $\Omega_{\mathbf{x}}$. The MK interpolation $u^h(\mathbf{x})$ postulates a combination of a linear regression model a stochastic error

$$u^h(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x}) a_j + z(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{a} + z(\mathbf{x}) \tag{1}$$

where $p_j(\mathbf{x})$ is a known basis function of $\mathbf{x}$ (usually polynomial). Coefficients a_j are regression parameters required to be determined, and $z(\mathbf{x})$ is supposed to be a stationary stochastic process with mean zero, variance σ^2, and non-zero covariance. The covariance matrix of $z(\mathbf{x})$ is defined as

$$cov\{z(\mathbf{x}_i), z(\mathbf{x}_j)\} = \sigma^2 \mathbf{R}[R(\mathbf{x}_i, \mathbf{x}_j)] \tag{2}$$

where σ^2 is a given scale factor. $\mathbf{R}[R(\mathbf{x}_i, \mathbf{x}_j)]$ is the correlation matrix, and $R(\mathbf{x}_i, \mathbf{x}_j)$ is the predefined correlation function between any two of the n nodes $\mathbf{x}_i$ and $\mathbf{x}_j$. A widely used correlation function is a Gaussian function

$$R(\mathbf{x}_i, \mathbf{x}_j) = \exp(-\theta r_{ij}^2) \tag{3}$$

where

$$r_{ij} = \left\| \mathbf{x}_i - \mathbf{x}_j \right\| \tag{4}$$

and $\theta > 0$ is a given correlation parameter.

Given a set of nodes $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ and function values at these nodes $\mathbf{U}_s = \{u(\mathbf{x}_1), u(\mathbf{x}_2), \cdots, u(\mathbf{x}_n)\}^T$, $u(\mathbf{x})$ at any $\mathbf{x}(\mathbf{x} \in \Omega)$ can be estimated using the linear (or polynomial) predictor

$$\hat{u}(\mathbf{x}) = \mathbf{c}^T(\mathbf{x})\mathbf{U}_s = \{c_1(\mathbf{x}), c_2(\mathbf{x}), \cdots, c_n(\mathbf{x})\} \begin{Bmatrix} u(\mathbf{x}_1) \\ u(\mathbf{x}_2) \\ \vdots \\ u(\mathbf{x}_n) \end{Bmatrix} \tag{5}$$

At the same time, according to the linear regression model Eq. (1), $\mathbf{U}_s$ is determined by

$$\mathbf{U}_s = \mathbf{P}_m \mathbf{a} + \mathbf{Z} \tag{6}$$

where $\mathbf{P}_m$ is an $n \times m$matrix of $p_j(\mathbf{x})$ $(j = 1, \cdots, m)$ values of the given nodes $\{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n\}$ and $\mathbf{Z}$ is an $n \times 1$ matrix of the corresponding errors in the linear regression model , i.e., Eq. (1). They are evaluated as

$$\mathbf{P}_m = \begin{Bmatrix} \mathbf{p}(\mathbf{x}_1) \\ \mathbf{p}(\mathbf{x}_2) \\ \vdots \\ \mathbf{p}(\mathbf{x}_n) \end{Bmatrix} = \begin{bmatrix} p_1(\mathbf{x}_1) & p_2(\mathbf{x}_1) & \cdots & p_m(\mathbf{x}_1) \\ p_1(\mathbf{x}_2) & p_2(\mathbf{x}_2) & \cdots & p_m(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(\mathbf{x}_n) & p_2(\mathbf{x}_n) & \cdots & p_m(\mathbf{x}_n) \end{bmatrix} \tag{7}$$

$$\mathbf{Z} = \{z_1(\mathbf{x}), z_2(\mathbf{x}), \cdots, z_n(\mathbf{x})\}^T \tag{8}$$

where

$$\mathbf{p}(\mathbf{x}) = [p_1(\mathbf{x}), p_2(\mathbf{x}), \cdots, p_m(\mathbf{x})] \tag{9}$$

and $\mathbf{p}(\mathbf{x})$ is a $1 \times n$ vector of the known m basis functions in Eq. (1). In general, a linear basis namely $\mathbf{p}(\mathbf{x})$ in one dimension is given by

$$\mathbf{p}(\mathbf{x}) = \{1, x\},\ m = 2 \tag{10}$$

and a quadratic basis by

$$\mathbf{p}(\mathbf{x}) = \{1, x, x^2\},\ m = 3 \tag{11}$$

whereas a linear basis in two dimensions is provided by

$$\mathbf{p}(\mathbf{x}) = \{1, x, y\},\ m = 3 \tag{12}$$

Thus, for any point $\mathbf{x} \in \Omega_{\mathbf{x}}$, the error $\phi(\mathbf{x})$ produced by the linear (or polynomial) predictor upon the given nodes can be calculated as

$$\begin{aligned} \phi(\mathbf{x}) &= \hat{u}(\mathbf{x}) - u^h(\mathbf{x}) = \mathbf{c}^T(\mathbf{x})\mathbf{U}_s - u^h(\mathbf{x}) = \mathbf{c}^T(\mathbf{x})(\mathbf{P}_m\mathbf{a} + \mathbf{Z}) - [\mathbf{p}^T(\mathbf{x})\mathbf{a} + z(\mathbf{x})] \\ &= \mathbf{c}^T(\mathbf{x})\mathbf{Z} - z(\mathbf{x}) + \{\mathbf{P}_m^T\mathbf{c}(\mathbf{x}) - \mathbf{p}(\mathbf{x})\}^T\mathbf{a} \end{aligned} \tag{13}$$

Assuming $\hat{u}(\mathbf{x})$ as random, to obtain the best estimation, $\hat{u}(\mathbf{x})$ should be the unbiased estimation of $u^h(\mathbf{x})$. In view of the generality of coefficient $\mathbf{a}$, the following constraints should be satisfied

$$\mathbf{P}_m^T\mathbf{c}(\mathbf{x}) - \mathbf{p}(\mathbf{x}) = \mathbf{0} \tag{14}$$

and the mean squared error (MSE) of the predictor averaged over the random process can be evaluated. It is obtained by picking $n \times 1$ vector $\mathbf{c}(\mathbf{x})$ to minimize

$$\begin{aligned} MSE[\hat{u}(\mathbf{x})] &= E[\phi(\mathbf{x})^2] = E[\mathbf{c}^T(\mathbf{x})\mathbf{U}_s - u^h(\mathbf{x})]^2 \\ &= E[\mathbf{c}^T(\mathbf{x})\mathbf{Z} - z(\mathbf{x})]^2 \end{aligned} \tag{15}$$

where $E[\bullet] = \int_\Omega \bullet d\mathbf{x} \big/ \int_\Omega d\mathbf{x}$ is the average of $\bullet$.
To implement the best linear unbiased predictor at $\mathbf{x}$, we use the notation

$$\mathbf{R}_Q = \begin{bmatrix} 1 & R(\mathbf{x}_1, \mathbf{x}_2) & \cdots & R(\mathbf{x}_1, \mathbf{x}_n) \\ R(\mathbf{x}_2, \mathbf{x}_1) & 1 & \cdots & R(\mathbf{x}_2, \mathbf{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ R(\mathbf{x}_n, \mathbf{x}_1) & R(\mathbf{x}_n, \mathbf{x}_2) & \cdots & 1 \end{bmatrix} \tag{16}$$

for the $n \times n$ matrix of correlation between the $z's$ at the given nodes, and

$$\mathbf{r}(\mathbf{x}) = \{R(\mathbf{x}, \mathbf{x}_1), \cdots, R(\mathbf{x}, \mathbf{x}_n)\} \tag{17}$$

for the $1 \times n$ matrix of correlation between the given nodes and $\mathbf{x}$. With these definitions, we can solve the optimization problem. The goal is to minimize formula (15) with respect to $\mathbf{c}(\mathbf{x})$ subject to the constraint Eq. (14). Introducing Lagrange multipliers as the ordinary Kriging interpolation method does, the constrained optimization problem can be converted to an unconstrained problem. According to the regulations of functional analysis and optimization, the following moving Kriging interpolation model can be obtained and more details can refer to the related Ref. [Gu (2003)]

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\hat{\mathbf{a}} + \mathbf{r}^T(\mathbf{x})\mathbf{R}_Q^{-1}(\mathbf{U}_s - \mathbf{P}_m\hat{\mathbf{a}}) \tag{18}$$

where

$$\hat{\mathbf{a}} = [\mathbf{P}_m^T\mathbf{R}_Q^{-1}\mathbf{P}]^{-1}\mathbf{P}_m^T\mathbf{R}_Q^{-1}\mathbf{U}_s \tag{19}$$

For convenience of presentation, we define the following notations

$$\mathbf{S}_{\hat{\mathbf{a}}} = (\mathbf{P}_m^T\mathbf{R}_Q^{-1}\mathbf{P})^{-1}\mathbf{P}_m^T\mathbf{R}_Q^{-1} \tag{20}$$

$$\mathbf{S}_{\hat{\mathbf{b}}} = \mathbf{R}_Q^{-1}(\mathbf{I} - \mathbf{P}_m\mathbf{S}_{\hat{\mathbf{a}}}) \tag{21}$$

where $\mathbf{I}$ is an $n \times n$ identity matrix. And thus Eq. (18) can be rewritten as

$$\mathbf{u}^h(\mathbf{x}) = [\mathbf{p}^T(\mathbf{x})\mathbf{S}_{\hat{\mathbf{a}}} + \mathbf{r}^T(\mathbf{x})\mathbf{S}_{\hat{\mathbf{b}}}]\mathbf{U}_s \tag{22}$$

or

$$u^h(\mathbf{x}) = \Phi(\mathbf{x})\mathbf{U}_s = \sum_{k=1}^{n} \varphi_k(\mathbf{x})u_k \tag{23}$$

where $\phi_k(\mathbf{x})$ and $\Phi(\mathbf{x})$ are the shape function and its responding matrix respectively, which are given by

$$\Phi(\mathbf{x}) = \{\phi_1(\mathbf{x}), \phi_2(\mathbf{x}), \cdots, \phi_n(\mathbf{x})\} \tag{24}$$

$$\phi_k(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x})S_{\hat{a}jk} + \sum_{i=1}^{n} r_k(\mathbf{x})S_{\hat{b}ik} \tag{25}$$

where $S_{\hat{a}jk}$ is the element at the j-th row and k-th column of $\mathbf{S}_{\hat{\mathbf{a}}}$, $S_{\hat{b}ik}$ is the element at the i-th row and k-th column of $\mathbf{S}_{\hat{\mathbf{b}}}$. The partial derivatives of shape function $\phi_k(\mathbf{x})$ against X_i (in one dimension X_i is x, and in two dimension is x or y, see the above definition of $\mathbf{p}(\mathbf{x})$) then can be calculated as

$$\phi_{k,X_i} = \frac{\partial \phi_k(\mathbf{x})}{\partial X_i} = \sum_{j=1}^{m} \frac{\partial p_j(\mathbf{x})}{\partial X_i} S_{\hat{a}jk} + \sum_{j=1}^{m} \frac{\partial r_k(\mathbf{x})}{\partial X_i} S_{\hat{b}jk} \tag{26}$$

Kronecker's delta property

The delta property is the inherent characteristic of the MK shape functions which cannot be found in general MLS approximations. Replacing $\mathbf{x}$ with any node of interest $\mathbf{x}_j$ in Eq. (25), where $j = 1, \cdots, n$, it will lead to the following Kronecker's delta property

$$\phi_k(\mathbf{x}_j) = \delta_{kj} \tag{27}$$

Choice of the correlation

The parameter θ has a significant effect on the Kriging quality. The 'optimal' value of θ depend strongly on the data $\mathbf{U}_s$. For simplicity, θ was chosen in our work as Ref. [Gu (2003)] does to satisfy the following condition

$$1 \times 10^{-6} \leq |R| \leq 1 \times 10^{-1} \tag{28}$$

3 MK interpolation based response surface

3.1 Response surface method [Rajashekhar and Ellingwood (1993); Guan and Melchers (2001)]

The limit state function subjected to a given condition is often defined as

$$g(\mathbf{X}) = \mathrm{g}(\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n}) = 0 \tag{29}$$

where $x_i (i = 1, 2, \cdots, n)$ is the basic random variables, n is the number of random variables, $\mathbf{X} = (\mathrm{x}_1, \mathrm{x}_2, \cdots, \mathrm{x}_\mathrm{n})$ is the vector of the random variables. In general, $g(X)$ is described in implicit forms. The response surface method is applied in which the original limit state function $g(\mathbf{X})$ is replaced by an explicit polynomial function $\widetilde{g}(\mathbf{X})$, a common used form as Eq. (30) on the basis of basic random variables. a,b_i and c_i are the $2n+1$ unknown coefficients, which can be obtained by regression methods such as least squares estimates, evaluated using the information obtained at the sampling points chosen in the vicinity of the mean values of basic random variables , that is, the sampling points are selected to be located at $\boldsymbol{\mu}$ and $\boldsymbol{\mu} \pm \mathbf{k} \bullet \sigma$, where $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_n)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_n)$ are vectors of the means and the standard deviations of random variables $\mathbf{X}$, $\mathbf{k} = (k_1, k_2, \cdots, k_n)$, $k_i = 0$ or $k (i = 1, 2, \cdots, n)$ and only one element of $\mathbf{k}$ equals k, k is an arbitrary factor, and $\bullet$ is the dot product. In order to improve the accuracy and efficiency of RSM, many sampling schemes are explored to choose the fitted points. Amongst the factorial experimental design and fully saturated experimental design method are widely used.

$$\widetilde{g}(\mathbf{X}) \approx \mathrm{a} + \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{b_i x_i} + \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{c_i x_i^2} \tag{30}$$

In order to best arrive at the original limit state surface, the fitted points are commonly chosen as an adaptive interpolation scheme. A simple adaptive scheme as the fully saturated experimental design method does involves a set of iterations:

1. Use the Hasofer-Lind algorithm [Madsen, Krenk and Lind (1986); Hasofer and Lind (1974)] or other FORMs to obtain the design point $\mathbf{X}_{\mathrm{D}}$ on the initial estimated response surface $\widetilde{g}(\mathbf{X})$;

2. Update the location of experimental design points. Their new centre point $\mathbf{X}_{\mathrm{M}}$ is obtained by line interpolation as Eq. (31) through the design point $(\mathbf{X}_{\mathrm{D}}, \mathrm{g}(\mathbf{X}_{\mathrm{D}}))$and the mean vector point $(\boldsymbol{\mu}, g(\boldsymbol{\mu}))$. The new centre point $\mathbf{X}_{\mathrm{M}}$ is thus located on the original limit state surface $g(\mathbf{X}) = 0$.

$$\mathbf{X}_M = \boldsymbol{\mu} + (\mathbf{X}_D - \boldsymbol{\mu}) \frac{g(\boldsymbol{\mu})}{g(\boldsymbol{\mu}) - g(\mathbf{X}_D)} \tag{31}$$

Then a new surface using $\mathbf{X}_{\mathrm{M}}$as the centre point is obtained and the above procedure is repeated till convergence criterion on the reliability index is satisfied.

3.2 Moving Kriging interpolation based response surface

Considering the delta property is the inherent characteristic of the MK shape functions which cannot be found in general MLS approximations, a moving Kriging interpolation based response surface method (MK-RSM) maybe make more accuracy. The MK-RSM is thus explored in this paper. It generates the points in the similar way as the basic response surface method does. In addition, to avoid great numbers dominate those small ones and eliminate numerical calculation difficulties and the requiring of MK interpolation, all the random variables are transformed to standard normal random variables before performing MK -RSM procedure. More details will be given in the next section.

In order to obtain a better fit of the response surface, the value of the parameter k is lowered as subsequent updating of the centre point in subsequent cycles of response surface updating, and gradually chosen from the set such as $\{4.0, 3.0, 2.0, 1.0, 0.5, 0.2, 0.1, \cdots\}$. According to Eq. (23), a linear basis is chosen to construct the substitute limit state function as

$$\widetilde{g}(\mathbf{X}) = \Phi(\mathbf{X})\mathbf{U}_s = \sum_{k=1}^{n} \phi_k(\mathbf{X}) g_k \tag{32}$$

The Gaussian function is chosen as the correlation function in this work. As the distance of fitted points is varying as the updating, the coefficient of correlation function is also varying as the sampling points and chosen in terms of Eq. (28).

4 Calculation of reliability index and its sensitivity

4.1 Calculation of reliability index

The FORM and the SORM have been proved to provide a rational measure of the reliability. The FORM often has adequate accuracy and is widely applied in reliability analysis. In the FORM, the reliability analysis requires a transformation **T** from the original random variable vector **X** to an equivalent system of independent and standard normal random variable vector **U**, and then the limit state function $g(\mathbf{X})$in the X space can be mapped into $g(\mathbf{U})$ in the U space.

For a component of the random variable vector **X**, x_i, in the limit state function, if it is an independent normal random variable, it can be transformed into a standard normal random variable u_i as the following

$$u_i = T(x_i) = \frac{x_i - \mu_i}{\sigma_i} \ (i = 1, 2, \cdots, n) \tag{33}$$

where μ_i and σ_i are the mean and standard deviation of the random variable x_i.

When a non-normal random variable $x_i$is involved, the transformation into a standard normal random variable u_i can be conducted by the following Rackwitz-Fiessler transformation [Rackwitz and Fiessler (1978)]:

$$F_X(x_i) = \Phi(u_i) \tag{34}$$

where $F_X(\cdot)$ is the non-normal cumulative probability and $\Phi(\cdot)$ is the standard normal cumulative probability; that is to say, u_i can be formulated as

$$u_i = T(x_i) = \Phi^{-1}(F_X(x_i)) \tag{35}$$

In our application, the substitute limit state function $\tilde{g}(\mathbf{X})$ is simultaneously mapped into the standard normal space

$$\begin{aligned} \tilde{g}(\mathbf{X}) &= \tilde{g}[R(\mathbf{X}), Q(\mathbf{X})] \\ &= \tilde{g}\{R[T^{-1}(\mathbf{u})], Q[T^{-1}(\mathbf{u})]\} = \tilde{g}(\mathbf{u}) \end{aligned} \tag{36}$$

where R represents the resistance and Q represents the load effect.

After that, all the random variables are transformed into statistically independent standard normal ones and herein grouped as a vector $\mathbf{u} = (u_1, u_2, \cdots, u_n)^T$. In the uncorrelated normal space **u**, only the linear Taylor-expansion items of the limit state equation at the design point are considered, then the reliability index β is the shortest distance in U space from the origin to the limit state surface given by $\tilde{g}(\mathbf{u}) = 0$. A design point $\mathbf{u}^*$ nearest to the origin on the limit state surface could

be found to calculate the reliability index. It is a minimization problem described as the following

$$\underset{\tilde{g}(\mathbf{u})=0}{\text{minimize}} \quad \|\mathbf{u}\| \tag{37}$$

and then the reliability index can be calculated by

$$\beta = \sqrt{\mathbf{u}^{*T}\mathbf{u}^{*}} \tag{38}$$

Any MPFP or design point search algorithm developed for first-order reliability analysis or general optimization procedure can be used to solve Eq. (37). In this paper, the Hasofer-Lind algorithm is employed to conduct the reliability analysis because of its simplicity and efficiency. The iterative Hasofer-Lind algorithm [Madsen, Krenk and Lind (1986); Hasofer and Lind (1974)] is formulated as

$$\mathbf{u}_{i+1} = \left(\mathbf{u}_i^T \mathbf{n}_i + \frac{\tilde{g}(\mathbf{u}_i)}{\|\nabla \tilde{g}(\mathbf{u}_i)\|}\right)\mathbf{n}_i \tag{39}$$

where

$$\nabla \tilde{g}(\mathbf{u}) = \left(\frac{\partial \tilde{g}(\mathbf{u})}{\partial u_1}, \frac{\partial \tilde{g}(\mathbf{u})}{\partial u_2}, \cdots, \frac{\partial \tilde{g}(\mathbf{u})}{\partial u_n}\right)^T \tag{40}$$

and

$$\mathbf{n} = -\frac{\nabla \tilde{g}(\mathbf{u})}{\|\nabla \tilde{g}(\mathbf{u})\|} \tag{41}$$

where the vector $\mathbf{n}$ is defined as the normalized steepest descent direction of $\tilde{g}(\mathbf{u})$at $\mathbf{u}$. Considering Eq. (40) and according to Eqs. (33)-(36), $\nabla \tilde{g}(\mathbf{u})$ can be calculated by

$$\frac{\partial \tilde{g}(\mathbf{u})}{\partial u_j} = \frac{\partial \tilde{g}}{\partial x_j}\frac{\partial x_j}{\partial u_j} = \frac{\partial \tilde{g}}{\partial x_j}\frac{\partial T^{-1}}{\partial u_j} \qquad (j = 1, 2, \cdots, n) \tag{42}$$

where, according to Eq. (32), $\frac{\partial \tilde{g}}{\partial x_j}$can be computed as

$$\widetilde{g}_{x_j}(\mathbf{X}) = \Phi_{x_j}(\mathbf{X})\mathbf{U}_s = \sum_{k=1}^{n} \phi_{k,x_j}(\mathbf{X}) g_k \tag{43}$$

Substituting Eq. (43) into Eqs. (39) and (40), iterating until the reliability index β satisfies a convergent criterion, the design point $\mathbf{u}^*$ will be simultaneously obtained.

4.2 Sensitivity estimation

Sensitivity analysis of the random variables is conducted to determine to what extent each random variable has effect on the reliability index. The sensitivity index is defined as a measure to quantify the influence of each basic random variable, which is related with reliability index and the standard normal random variables and given by [Chakraborty and Bhar (2006)]

$$\alpha_i = \frac{\partial \beta}{\partial u_i} = \frac{a_i \sigma_{u_i}}{\sigma_G} \tag{44}$$

In the first order reliability analysis method, $\widetilde{g}(\mathbf{u})$, in fact, is handled as some kind of a linear approximation $\widetilde{\widetilde{g}}(\mathbf{u})$ of the basic standard normal random variables in the following form

$$\widetilde{\widetilde{g}}(\mathbf{u}) = a_0 + a_1 u_1 + \cdots + a_n u_n \tag{45}$$

Considering $\sigma_{u_i} = 1 (i = 1, 2, \cdots, n)$ in the Uspace, the sensitivity index can be rewritten as

$$\boldsymbol{\alpha} = (\alpha_1, \cdots \alpha_n)^T = (\frac{\partial \beta}{\partial u_1}, \cdots, \frac{\partial \beta}{\partial u_n})^T = (\frac{a_1}{\sigma_g}, \cdots, \frac{a_n}{\sigma_g}) = \frac{\nabla \tilde{\tilde{g}}}{\left\| \nabla \tilde{\tilde{g}} \right\|} \tag{46}$$

Contrasting with Eq. (41), it can be found that

$$\boldsymbol{\alpha} = -\mathbf{n} \tag{47}$$

Thus, along with the iterative Hasofer-Lind algorithm, the sensitivity indices can be simultaneously obtained.

4.3 Procedure of the proposed method

Summing up the above MK interpolation-based reliability analysis method, we present the following procedure to perform the reliability analysis

1. Select the means of random variables $\mathbf{X}$ as the initial centre point, and select another $2n$ experimental points around the centre point totally $2n+1$ points as depicted in Section 3.1, where $k = 4$.

2. Transform the experimental points from $\mathbf{X}$ space to the $\mathbf{U}$ space as a normal vector $\mathbf{u}$ using the Rackwitz-Fiessler transformation Eq. (34). According to the average distance of the obtained $2n+1$ experimental points and Eq. (28), choose the correlation parameter θ. Construct the linear basis MK interpolation response surface $\widetilde{g}(\mathbf{u})^{(K)}$ of the limit state function, where K denotes the K-th outer iteration, and one outer iteration corresponds to one set of sampling experimental points.

3. In accordance with Hasofer-Lind algorithm, compute the partial derivatives of current $\widetilde{g}(\mathbf{u})^{(K)}$ according to Eq. (42), substitute them into Eqs. (39)-(41) and iterative until satisfying a converging criterion, e.g. $|\mathbf{u}_{i+1}^{*} - \mathbf{u}_{i}^{*}|/|\mathbf{u}_{i}^{*}| < 1e-3$, obtain the reliability index $\beta^{(K)}$, its sensitivities and the design point $\mathbf{u}^{*(K)}$, transform it back to the original space as $\mathbf{X}^{*(K)}$.

4. Determine whether the outer iteration is convergent to a criterion such as $\left|\mathbf{u}^{*(K)} - \mathbf{u}^{*(K-1)}\right| / |\mathbf{u}^{*(K-1)}| < 1e-3$, if it is satisfied, stop the outer iteration and the result is the reliability index and its sensitivities in the last outer iteration. Otherwise, return to Step 1, change the value of k, select $\mathbf{X}^{*(K)}$ as the new centre point, update the experimental points and repeat Steps (1)-(4). As indicated in Section 3.2, the value of k generally decreases successively, e.g., in the second outer iteration, a reasonable value of k is 2, and in the following outer iterations, $k = 1$.

5 Numerical examples

Three numerical examples borrowed from Ref. [Kang, Koh and Choo (2010); Melchers and Ahammed (2004)] are used to demonstrate the accuracy and the computational effectiveness of the MK interpolation RSM. For comparison, the reliability analysis results calculated by LS-RSM, MLS-RSM, FORM or SVR-RSM are presented and the FORM results are referred to exact ones.

The first example concerns non-linear limit state function in two variables. The second example is related with the reliability analysis of a finite element problem with an implicit limit state. The third example pays attention to a highly non-linear limit state involved six non-normal variables.

Example 1: a nonlinear limit state function

The limit state function is formulated as

$$G(\mathbf{u}) = \exp[0.4(u_1 + 2) + 6.2] - \exp(0.3u_2 + 5) - 200 \tag{48}$$

where u_1 and u_2 are independent and standard normal random variables, i.e., $u_1 \sim N(0,1)$, $u_2 \sim N(0,1)$.

The solutions given by FORM, LS-RSM, MLS-RSM are referred from Ref. [Kang, Koh and Choo (2010)], and results obtained by FORM are supposed to be the exact solution. We compare and analyze the results of MK-RSM and conventional RSMs. For the limit state function Eq. (48) expressed in explicit form, it is also considered as an implicit function to construct the substitute response surface in the RSM.

As Ref. [Kang, Koh and Choo (2010)] does, the histories of the reliability index obtained in each outer iteration of the above methods are plotted in Fig.1 and compared. It can be seen that the reliability index provided by MK-RSM converges faster to the exact value than the LS-RSM and MLS-RSM. Among the errors of the reliability index calculated by these methods in the first outer iteration, the maximum is obtained by the LS-RSM and the minimum is MK-RSM. Although the LS-RSM and MK-RSM start the iteration with the same number initial experimental points ($5 = 1 + 2n$), the MK-RSM exhibits less error.

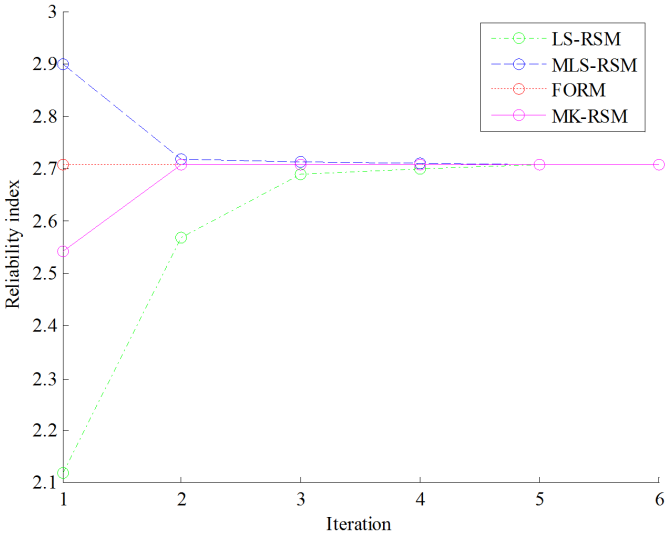

Figure 1: Example 1: comparisons of the histories of the reliability index per iteration

The reliability index, its sensitivities to each random variable and the MPFP calculated by the LS-RSM, MLS-RSM and the proposed MK-RSM are presented and compared in Table 1 to check the accuracy of these methods. As a measure of the efficiency of the method, the number of function evaluations (NFE), the errors with respect to the exact results obtained by the FORM are also listed in Table 1.

All the reliability indices resulted from the three methods gradually approach the exact solution as the iterations increase, and the MK-RSM obtains nearly the same value as the FORM does, whereas the LS-RSM produces the maximum error (0.048 %). On the MPFP and sensitivities of reliability index to the random variables, compared to the reliability index itself, the three methods make greater errors and they exhibit different precision. The LS-RSM, MLS-RSM and MK-RSM produces

Table 1: Comparison of analysis results of example 1

	Exact solution (FORM)	LS-RSM	LS-RSM error	MLS-RSM	MLS-RSM error	MK-RSM	MK-RSM error
Reliability							
index	2.7099	2.7112	0.048%	2.7100	0.004%	2.7099	0.00%
MPFP							
u_1^*	-2.5398	-2.5725	1.271%	-2.5411	0.051%	-2.5437	0.154%
u_2^*	0.9450	0.8562	10.371%	0.9417	0.350%	0.9344	1.122%
Sensitivity							
$\frac{\partial \beta}{\partial u_1}$	-0.9372	-0.9488	1.223%	-0.9377	0.053%	-0.9387	0.160%
$\frac{\partial \beta}{\partial u_2}$	0.3487	0.3158	10.418%	0.3475	0.345%	0.3448	1.118%
NFE	27	30		12		15	

a maximum error of 10.371%, 0.35% and 1.122% respectively on the MPFP and a maximum sensitivity error of 10.418%, 0.345% and 1.118% respectively, thus MLS-RSM and MK-RSM are more accurate than the conventional LS-RSM. On the efficiency of the above methods, the MLS-RSM needs 12 $(= (1+n)+(1+2n)+1\times 4)$ NFE to find the solution. The LS-RSM needs a total of 6 iterations leading to 30 $(= 6\times(1+2n))$ NFE, which is larger than the 27 NFE required by FORM, whereas the MK-RSM needs only 15 $(= 3\times(1+2\times 2))$ NFE and converges more rapidly than the LS-RSM methods. Its reliability indices of second and third iteration are 2.7088 and 2.7099 respectively, which is fast approaching the exact value. The obtained reliability index in the first iteration is 2.5437, which is closer to the exact solution than the other two methods. This first nonlinear example proves that the proposed MK-RSM provides nearly the same results as the exact solution in terms of the accuracy of the reliability index, and the close computational efficiency and accuracy of the MPFP and sensitivities to the random variables within a reduced number of structural analyses as MLS-RSM.

Example 2: 10-bar truss problem

A 10-bar truss problem (Fig. 2) widely used in the field of optimization and reported in a number of papers is presented, which involves finite element analysis and implicit limit function.

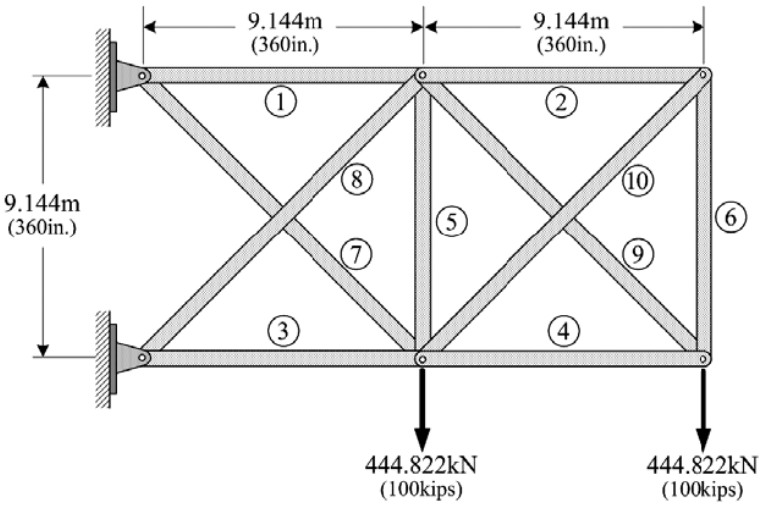

Figure 2: A 10-bar truss structure

The limit state function of element 1 in Fig. 2 is given by

$$G(\mathbf{A}) = \sigma_{allow} - |\sigma(\mathbf{A})| \tag{49}$$

with the following characteristics

$$A_i \sim N(10, 0.5^2)(in.^2) \quad \text{or} \quad A_i \sim N(64.52, 1.27^2)(cm^2)$$

$$\sigma_{allow} = 2.5 \times 10^4 (psi) \quad \text{or} \quad 172.4MPa$$

where the section areas of the 10 truss elements are normally distributed random variables [Kang, Koh and Choo (2010)].

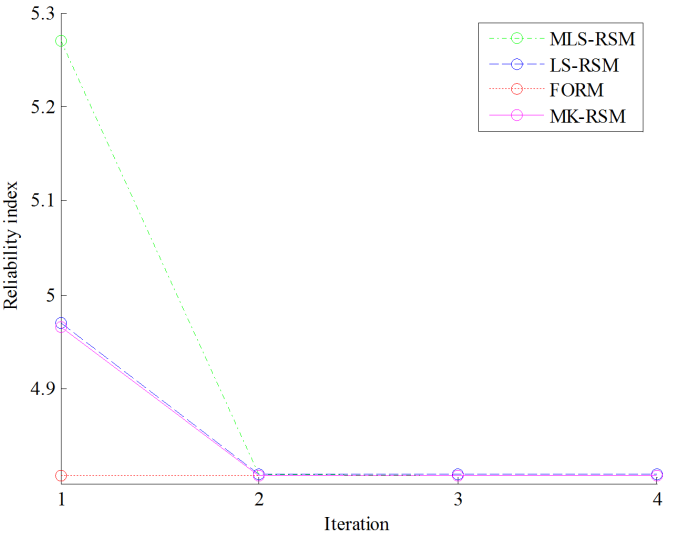

Figure 3: Example 2: comparisons of the histories of the reliability index per iteration

As Example 1 does, Fig. 3 shows and compares the historical results of each method until the convergence criterion is satisfied within 4 iterations. The reliability index, MPFP and sensitivities calculated by the LS-RSM, MLS-RSM and proposed MK-RSM are listed and compared in Table 2. The reliability indices and sensitivities obtained by the three methods are gradually approach the exact solution as the iterations increase, and the MK-RSM produces highest accuracy in terms of the reliability index, MPFP and sensitivities compared with the other two methods. At the same time, the MK-RSM rapidly converges to the exact solution with a

series of the reliability index values of 4.9669, 4.8080 and 4.8083 in the first three iterations. The LS-RSM is less efficient than the MLS-RSM and MK-RSM, since it requires 84 NFE, whereas the other two needs 34 NFE and 63 NFE respectively.

Table 2: Comparison of analysis results of example 2

	Exact solution (FORM)	LS-RSM	MLS-RSM	MK-RSM
Reliability index	4.8083	4.8084	4.8089	4.8083
MPFP	7.6658,9.9949 9.7302,10.0075 10.0350,9.9949 9.5916,10.2997 10.0212,9.9855	7.6636,9.9949 9.7326,10.0075 10.0346, 9.9949 9.5986,10.2946 10.0209,9.9857	7.6651, 9.9949 9.7300, 10.0075 10.0354, 9.9949 9.5928, 10.2981 10.0212, 9.9855	7.6655,9.9949 9.7305,10.0075 10.0350,9.9949 9.5924,10.2990 10.0212,9.9855
Sensitivity	-1.9418,-0.0043 -0.2244,0.0063 0.0291,-0.0043 -0.3397,0.2493 0.0177,-0.0121	-1.9436,-0.0042 -0.2225,0.0062 0.0287,-0.0042 -0.3339,0.2451 0.0174,-0.0119	-1.9422, -0.0043 -0.2245, 0.0063 0.0294, -0.0043 -0.3387, 0.2479 0.0176, -0.0120	-1.9420,-0.0043 -0.2242,0.0063 0.0291,-0.0043 -0.3391,0.2487 0.0176,-0.0121
NFE	90	84	34	63

Example 3: A highly nonlinear limit state function with non-normal random variables

A highly nonlinear limit state function shown in Eq. (50) with non-normal random variables of various distributions borrowed from Ref. [Melchers and Ahammed (2004)] is presented. The random variable parameters are listed in Table 3.

Table 3: Random variables and their parameters

Variable	Mean	Standard deviation	Distribution
X_1	4.0	0.1	Weibull
X_2	25000	2000	Lognormal
X_3	0.875	0.1	Gumbel
X_4	20.0	1.0	Uniform
X_5	100.0	100.0	Exponential
X_6	150.0	10.0	Normal

Table 4: Comparison of analysis results of example 3

	Exact solution (FORM)	SVR-RSM	MK-RSM
Reliability index	2.6697	2.67	2.6701
MPFP	4.0054, 24205 0.8227, 19.581 514.42, 155.86	4.006,24234 0.824,19.59 517.40,155.68	4.0053,24204 0.8204,19.617 513.86,155.85
Sensitivity	-0.0404, -0.1365 -0.1609, -0.1151 0.9447, 0.2193	-0.0395,-0.131 -0.156,-0.112 0.948,0.213	-0.0408,-0.1367 -0.1725,-0.1052 0.9438,0.2190
NFE	12	156	65

$$g(X) = X_1X_2X_3X_4 - X_5X_6^2/8 = 0 \tag{50}$$

Considering the strong adaptability of SVR-RSM in approximating the highly nonlinear limit state function, the results produced by the FORM, SVR-RSM and MK-RSM are presented and compared in Table 4. The accuracy and efficiency of the results obtained by the MK-RSM is similarly demonstrated in this example, which rapidly approaches and makes a good agreement with the exact solution. Especially, contrasting with the SVR-RSM, with less NFE (65 NFE against 156 NFE of SVR-RSM), MK-RSM produces more accuracy in terms of the MPFP and sensitivities, which will play an important role in the successive RBDO.

6 Conclusions

The RSMs have emerged as a computationally cheap tool to solve complex reliability analysis problems, whose induced errors in terms of reliability index, MPFP and sensitivities will determine the success of reliability analysis and the following RBDO. Thus, a novel RSM method, based on MK interpolation, is proposed in this paper. The MK interpolation has the characteristic of Kronecker's delta property, which means its good approximation for the limit state function. The results of the numerical examples have demonstrated that the MK-RSM improves the accuracy of the reliability index, MPFP and sensitivities, and more quickly approaches the exact solution. The deficiency is that the NFE required in the MK-RSM is greater that that in the MLS-RSM, but smaller than in the LS-RSM, therefore, further ef-

forts should be made to develop new sampling strategy, which will help to shorten the computational time.

Acknowledgement

This work is jointly supported by the NSFC (10972241, 11202138, 51208230 and 11032005), New Teacher Fund for Doctor Station, the Ministry of Education (20114401120012) and FDYT (LYM10022)

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